Research article

An operational calculus formulation of fractional calculus with general analytic kernels

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Abstract: Fractional calculus with analytic kernels provides a general setting of integral and derivative operators that can be connected to Riemann–Liouville fractional calculus via convergent infinite series. We interpret these operators from an algebraic viewpoint, using Mikusiński's operational calculus, and utilise this algebraic formalism to solve some fractional differential equations.

Keywords: fractional integrals; fractional derivatives; operational calculus; fractional differential equations; Mikusiński's operational calculus

1. Introduction

Fractional calculus is a field of study, like many in mathematics, which arises from a type of generalisation starting from a basic concept: in this case, taking the concepts of differentiation and integration, and generalising them by allowing the orders of derivatives and integrals to take values outside of the set of integers. As well as this purely mathematical motivation, the study of such generalised operators is also useful for scientific applications. The theory of fractional calculus is introduced in textbooks such as [1–4] while the applications are summarised in survey works such as [5, 6].

Surprisingly, perhaps, there are many different definitions of fractional integrals and derivatives in the literature, with no single agreed-upon way to define what (for example) the "derivative to order one-half" should actually mean. Especially in the early 21st century, the number of possible definitions has been growing out of control, including many that are purely mathematical curiosities without any verified applications. In order to put some structure on this rapidly growing field, we return to the mathematical concept of generalisation: it must be possible to define some broader categories [7], with some unifying properties and behaviours, which many of the recently defined operators can be slotted into.

Many of the newly defined fractional integral operators have been created by replacing the power

function in the classical Riemann–Liouville integral by a different kernel function, which may have various properties such as singular or nonsingular, etc. For this reason, several researchers have defined general classes of fractional integral operators by taking convolutions with some more general function: for example, Sonine kernels [8,9], analytic kernels [10] and others [11] including some introduced from the viewpoint of applications [12]. We will focus, in this paper, on the general class of operators with analytic kernels, in which the fractional integral operator $^A I_{0+}^{\alpha,\beta}$ is defined for $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$ by

$$^A I_{0+}^{\alpha,\beta} f(x) = \int_0^x (x-t)^{\alpha-1} A((x-t)^\beta) f(t) \, \mathrm{d}t, \qquad x > 0, \tag{1.1}$$

where the kernel function A can be any analytic function, with a power series like the following:

$$A(x) = \sum_{r=0}^{\infty} a_r(\alpha,\beta) x^r. \tag{1.2}$$

Note that the coefficient a_r may, in general, depend on the parameters α and β. We believe that this definition is general enough to cover many useful cases with real-world applications, while also specific enough that the solutions can be expressed in a form which is not too abstract to compute easily in those special cases.

A powerful method for solving differential equations is the method of operational calculus [13], formally introduced by Heaviside and later made rigorous by mathematicians including Mikusiński. Briefly, the idea of Mikusiński's operational calculus is to introduce a type of generalised functions (different from those used in distribution theory), so-called "operators" which behave like functions but are not, and which give rise to a unified algebraic structure [14]. The results yielded by Mikusiński's methodology are formally similar to those yielded by Laplace transform methods, but Mikusiński's method has advantages [15] both in how easy it is to make rigorous (the Laplace transform is easy to apply but less easy to justify rigorously) and in how many problems it can apply to (the Laplace transform requires exponential boundedness while Mikusiński's method requires only continuity).

This method has been applied to ordinary differential equations [14], partial differential equations [16], and fractional-order differential equations [17] with fractional derivatives of Riemann–Liouville type [18], Caputo type [19], Hilfer type [20], Erdélyi–Kober type [21,22], Prabhakar type [23,24] and in general classes such as fractional calculus with Sonine kernels [25,26] and fractional calculus with respect to functions [27,28].

In our recent work applying Mikusiński's operational calculus to differential equations of Prabhakar type [23,24], we made use of the series formula for Prabhakar operators [29] to complete the work inside the same function spaces as those used for Riemann–Liouville and Caputo differential equations. The most general setting in which such a series formula can be used is the fractional calculus with general analytic kernels [10], and therefore we wish to focus on this general class and try to find a way to apply Mikusiński's method to differential equations in this model. Such a result would be valuable in providing a greater level of generality to the work on Mikusiński's operational calculus in fractional calculus: by going straight to the general setting, we eliminate the need for multiple papers solving smaller subproblems of this general one.

The structure of this paper is as follows. Section 2 provides key definitions and facts concerning Riemann–Liouville fractional calculus which will be used later. Section 3 is devoted to a development

of fractional calculus with analytic kernels in the setting of suitable function spaces, including mapping properties, composition relations, and series formulae for both fractional integral and derivative operators. Section 4 develops a Mikusiński-type operational calculus for these operators, using results of Section 3 and also the general theory of Mikusiński's operational calculus for fractional operators established in previous literature. Section 5 considers applications of the developed theory in solving some initial value problems, and Section 6 concludes the paper with some comparative discussion and pointers towards future work.

2. Preliminaries

The purpose of this section is to briefly introduce the key definitions and some important fundamental properties of Riemann–Liouville and Caputo fractional calculus.

Definition 2.1 ([2, 3]). For any $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, the Riemann–Liouville fractional integral to order α, with constant of integration 0, is the operator ${}^{RL}I_{0+}^{\alpha}$ defined by

$$
{}^{RL}I_{0+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t) \, \mathrm{d}t, \qquad x > 0, \tag{2.1}
$$

where f is any function locally L^1 on the positive axis.

For any $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) \geq 0$, say $n - 1 \leq \text{Re}(\alpha) < n \in \mathbb{N}$, the Riemann–Liouville and Caputo fractional derivatives to order α, with constant of integration 0, are the operators ${}^{RL}D_{0+}^{\alpha}$ and ${}^{C}D_{0+}^{\alpha}$ defined as follows:

$$
{}^{RL}D_{0+}^{\alpha}f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left({}^{RL}I_{0+}^{n-\alpha}f(x) \right), \qquad x > 0, \tag{2.2}
$$

$$
{}^{C}D_{0+}^{\alpha}f(x) = {}^{RL}I_{0+}^{n-\alpha} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) \right), \qquad x > 0, \tag{2.3}
$$

where f is any function such that these expressions are defined (for example, a function which is in the AC^n space on any compact positive interval).

An important fact about the Riemann–Liouville derivative [2] is that it is precisely the (unique) analytic continuation, in the complex variable α, of the Riemann–Liouville integral, under the convention that

$$
{}^{RL}I_{0+}^{\alpha} = {}^{RL}D_{0+}^{-\alpha}, \qquad \text{Re}(\alpha) < 0. \tag{2.4}
$$

This means that, in some mathematical sense, the Riemann–Liouville derivative (2.2) is the "natural" fractional derivative arising from the fractional integral (2.1). However, the Caputo derivative (2.3) is nowadays often preferred in applications, because Caputo fractional differential equations require initial conditions of classical type while Riemann–Liouville ones require fractional initial conditions.

There are several important relations involving the Riemann–Liouville integrals and derivatives ("differintegrals" is a word often used in fractional calculus to include both integrals and derivatives), and we now focus on the composition relations as these will be used later in our work below. The composition of two Riemann–Liouville differintegrals, following the convention (2.4) to allow any orders in $\mathbb{C}$, always has a semigroup property if the inner operator is a fractional integral:

$$
{}^{RL}I_{0+}^{\alpha} \left({}^{RL}I^{\beta} f(x) \right) = {}^{RL}I_{0+}^{\alpha+\beta} f(x), \qquad \alpha, \beta \in \mathbb{C}, \quad \text{Re}\,\beta > 0. \tag{2.5}
$$

On the other hand, if the inner operator is a fractional derivative, we have a modified semigroup property involving a finite sum of initial value terms, as follows:

$$
{}^{RL}D_{0+}^{\alpha}\left({}^{RL}D^{\beta}f(x)\right) = {}^{RL}D_{0+}^{\alpha+\beta}f(x) - \sum_{k=0}^{n-1} \frac{x^{k-n-\alpha}}{\Gamma(k-n-\alpha+1)} \cdot \left[{}^{RL}D_{0+}^{\beta-n+k}f(x)\right]_{x=c}, \tag{2.6}
$$

where $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}\,\beta \geq 0$ and the natural number n is defined to be $\lfloor \mathrm{Re}\,\beta \rfloor + 1$. For more details on these and other composition relations between Riemann–Liouville differintegrals, the reader may refer to classic works such as [3, §2.7] and [30, §2.3], as well as the recent survey of composition relations [31].

3. Fractional calculus with analytic kernels

3.1. Function spaces and integral operators

Let us begin by defining the important function spaces to be used throughout this paper. In the following definition, the space C_η comes from Dimovski [32], the space Ω_η^α from Hadid and Luchko [18], and the space C_η^n from Luchko and Gorenflo [19].

Definition 3.1 ([18, 19, 32]). Let $\eta \in \mathbb{R}$ and $\alpha \geq 0$ and $m \in \mathbb{Z}_0^+$ be fixed.

(a) C_η is the space of all functions expressible as $f(x) = x^p f_1(x)$ with $p > \eta$ in $\mathbb{R}$ and $f_1 \in C[0, \infty)$.

(b) Ω_η^α is the space of all functions $f \in C_\eta$ such that the Riemann–Liouville derivatives ${}^{RL}D_{0+}^\nu f$ are included in C_η for all real numbers ν with $0 \leq \nu \leq \alpha$.

(c) C_η^m is the space of all m times differentiable functions f such that the mth derivative $f^{(m)}$ is included in C_η.

Clearly, all three of these are vector spaces. An important feature of the C_η space is that, for $\eta \geq -1$, any function $f \in C_\eta$ must be continuous on $(0, \infty)$ and integrable on any compact $[0, X]$. They also have the following convolution property: if $f \in C_\eta$ and $g \in C_\mu$ and $\eta, \mu \geq -1$, then $f * g \in C_{\eta+\mu+1}$, where this is the Laplace-type convolution defined by

$$
(f * g)(x) = \int_0^x f(x-t)g(t)\,\mathrm{d}t, \qquad x > 0. \tag{3.1}
$$

It is clear that $\Omega_\eta^0 = C_\eta = C_\eta^0$ for any $\eta \in \mathbb{R}$. We also have the following inclusions:

$$
\begin{aligned}
C_\eta \subseteq C_\mu &\iff \eta \geq \mu; \\
\Omega_\eta^\alpha \subseteq \Omega_\nu^\alpha &\iff \eta \geq \nu; \\
\Omega_\eta^{\alpha_1} \subseteq \Omega_\eta^{\alpha_2} &\iff \alpha_1 \geq \alpha_2.
\end{aligned}
$$

Let us also write down the formal definition of the fractional integral operator with general analytic kernel, while leaving the corresponding derivative operators for the next subsection.

Definition 3.2 ([10]). Let $\alpha, \beta \in \mathbb{C}$ be any parameters with positive real parts, and let A be a function defined by a power series (1.2) with radius of convergence $R > X^{\mathrm{Re}\,\alpha}$. The fractional integral operator with kernel function A and power parameters α and β is defined by the formula (1.1) for any function f on the interval $[0, X]$ such that this integral is well-defined.

One possible function space for this operator is $L^1[0, X]$, as proved in [10]. The following theorem shows that C_{-1} is also a suitable function space for the same operator, assuming that the kernel function A is entire (infinite radius of convergence).

Theorem 3.1. *Let $\alpha, \beta \in \mathbb{R}^+$, and let A be a function as in Definition 3.2 which is entire. Then, for any given $\eta \geq -1$, the operator $^A I_{0+}^{\alpha,\beta}$ is a linear map of the space C_η into itself:*

$$^A I_{0+}^{\alpha,\beta} : C_\eta \to C_{\eta+\alpha} \subseteq C_\eta. \tag{3.2}$$

Also, for any function $f \in C_\eta$, the infinite series representation

$$^A I_{0+}^{\alpha,\beta} f(x) = \sum_{r=0}^{\infty} a_r(\alpha, \beta) \Gamma(r\beta + \alpha) \, ^{RL} I_{0+}^{r\beta+\alpha} f(x) \tag{3.3}$$

is locally uniformly convergent for $x \in (0, \infty)$.

Proof. The integral with analytic kernel is clearly a linear operator. For the C_η mapping, we write $f(x) = x^p f_1(x)$ with $p > \eta$ and $f_1 \in C[0, \infty)$, and set $t = x\tau$ in (1.1) to obtain

$$^A I_{0+}^{\alpha,\beta} f(x) = \int_0^1 (x - x\tau)^{\alpha-1} A((x - x\tau)^\beta)(x\tau)^p f_1(x\tau) x \, d\tau$$

$$= x^{p+\alpha} \int_0^1 \tau^p (1 - \tau)^{\alpha-1} A(x^\beta(1 - \tau)^\beta) f_1(x\tau) dt,$$

$$= x^{p+\alpha} f_2(x),$$

where the last integral converges, uniformly with respect to x in any closed interval $[0, X]$, which means $f_2 \in C[0, \infty)$ and $^A I_{0+}^{\alpha,\beta} f \in C_{\eta+\alpha}$. Now

$$^A I_{0+}^{\alpha,\beta} f(x) = \int_0^x (x - t)^{\alpha-1} A((x - t)^\beta) f(t) dt$$

$$= \int_0^x (x - t)^{\alpha-1} \sum_{r=0}^{\infty} a_r(\alpha, \beta)(x - t)^{r\beta} f(t) dt$$

$$= \int_0^x \sum_{r=0}^{\infty} a_r(\alpha, \beta)(x - t)^{r\beta+\alpha-1} f(t) dt$$

$$= \sum_{r=0}^{\infty} a_r(\alpha, \beta) \Gamma(r\beta + \alpha) \cdot \frac{1}{\Gamma(r\beta + \alpha)} \int_0^x (x - t)^{r\beta+\alpha-1} f(t) dt$$

$$= \sum_{r=0}^{\infty} a_r(\alpha, \beta) \Gamma(r\beta + \alpha) \, ^{RL} I_{0+}^{r\beta+\alpha} f(x),$$

where interchange of the summation and integration is justified by local uniform convergence of the analytic power series, which also guarantees local uniform convergence of the series formula (3.3). □

Proposition 3.1. *Let A be an entire kernel function, and fix $\eta \geq -1$. The fractional integral with kernel A has a semigroup property in the space C_η, of the form*

$$^A I_{0+}^{\alpha_1,\beta} \circ \, ^A I_{0+}^{\alpha_2,\beta} f = \, ^A I_{0+}^{\alpha_1+\alpha_2,\beta} f, \qquad f \in C_\eta, \tag{3.4}$$

for all $\alpha_1, \alpha_2, \beta \in \mathbb{C}$, if and only if

$$\sum_{r_1+r_2=k} a_{r_1}(\alpha_1, \beta) a_{r_2}(\alpha_2, \beta) B(\beta r_1 + \alpha_1, \beta r_2 + \alpha_2) = a_k(\alpha_1 + \alpha_2, \beta), \qquad k \in \mathbb{Z}_0^+.$$

Proof. Identical to the proof of [10, Theorem 2.10]. Indeed, the result is exactly the same except that here we have specified a function space for f. $\qquad\square$

3.2. Fractional derivative operators

Every fractional integral operator should have a corresponding fractional derivative operator. Usually, in line with the classical fundamental theorem of calculus, these operators would have inversion relations as follows: the fractional derivative is a left inverse to the fractional integral, while the fractional integral of the fractional derivative equals the original function minus some initial value terms.

The fractional derivative operators of Riemann–Liouville and Caputo type corresponding to (1.1) were originally defined [10, Definition 2.15] as follows:

$$_{RL}^{A}D_{0+}^{\alpha,\beta} f(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} \left(\int_0^x (x-t)^{m-\alpha-1} \bar{A} \left((x-t)^\beta \right) f(t)\,\mathrm{d}t, \right),$$

$$_{C}^{A}D_{0+}^{\alpha,\beta} f(x) = \int_0^x (x-t)^{m-\alpha-1} \overline{A} \left((x-t)^\beta \right) \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t)\,\mathrm{d}t,$$

where $m = \lfloor \operatorname{Re}\alpha \rfloor + 1$ and f is in some suitable function space in each case, and where $\overline{A}$ is defined by [10, Eq. (36)]:

$$\overline{A}_\Gamma \cdot A_\Gamma = 1, \tag{3.5}$$

where the Γ-transformed version of a function is defined by multiplying each coefficient in its Taylor series by a gamma function multiplier:

$$A(x) = \sum_{n=0}^\infty a_n x^n \implies A_\Gamma(x) = \sum_{n=0}^\infty a_n \Gamma(\beta n + \alpha) x^n.$$

The condition (3.5) is, as claimed in [10], sufficient to ensure that the Riemann–Liouville type fractional derivative $_{RL}^{A}D_{0+}^{\alpha,\beta}$ is a left inverse to the fractional integral operator $^A I^{\alpha,\beta}$.

However, there is more subtlety here than was realised in [10]. We must take into account the fact that the coefficients of the power series for A are allowed to depend on α and β. Indeed, a better notation, taking into account the dependence on the parameters, might be

$$A^{(\alpha,\beta)}(x) = \sum_{n=0}^\infty a_n(\alpha, \beta) x^n,$$

and then the condition (3.5) should be replaced by

$$\overline{A}_\Gamma^{(m-\alpha,\beta)} \cdot A_\Gamma^{(\alpha,\beta)} = 1, \qquad m = \lfloor \alpha \rfloor + 1, \tag{3.6}$$

and it is now apparent that this assumption is actually quite complicated and less natural than it originally seemed.

To simplify the discussion, and to focus attention on the case which is most often of interest, let us assume that $0 < \alpha < 1$, so that $m = 1$ and the relation (3.6) does not depend on a variable discrete natural number as well as variable real or complex parameters. We state the inversion relation rigorously, including relevant function spaces, in the following theorem.

Theorem 3.2. *Let $0 < \alpha < 1$ and $\beta > 0$ and $\eta \geq -1$, and let A be a function defined by (1.2).*

(i) The Riemann–Liouville type fractional derivative operator with kernel A, namely

$$\,_{RL}^{A}D_{0+}^{\alpha,\beta}f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x (x-t)^{-\alpha}\bar{A}\left((x-t)^\beta\right)f(t)\,\mathrm{d}t\right),$$

where $\overline{A}(x) = \sum_{n=0}^\infty \bar{a}_n x^n$ is defined by the identity

$$\left(\sum_{r=0}^\infty a_r(\alpha,\beta)\Gamma(\beta r + \alpha)x^r\right)\cdot\left(\sum_{n=0}^\infty \bar{a}_n(1-\alpha,\beta)\Gamma(\beta n + 1 - \alpha)x^n\right) = 1, \tag{3.7}$$

maps the function space Ω_η^α into C_η:

$$\,_{RL}^{A}D_{0+}^{\alpha,\beta} : \Omega_\eta^\alpha \to C_\eta. \tag{3.8}$$

Indeed, for any function $f \in \Omega_\eta^\alpha$, the infinite series

$$\sum_{n=0}^\infty \bar{a}_n(1-\alpha,\beta)\Gamma(n\beta + 1 - \alpha)\,^{RL}I_{0+}^{n\beta-\alpha}f(x) \tag{3.9}$$

converges in C_η, locally uniformly for all $x \in (0, \infty)$, to the fractional derivative $\,_{RL}^{A}D_{0+}^{\alpha,\beta}f(x)$.

(ii) The Riemann–Liouville type fractional derivative operator with kernel A is a left inverse of the fractional integral operator with kernel A, in the sense that, for any $f \in C_\eta$, we have $\,^{A}I_{0+}^{\alpha,\beta}f \in \Omega_\eta^\alpha$ and

$$\,_{RL}^{A}D_{0+}^{\alpha,\beta}\left(\,^{A}I_{0+}^{\alpha,\beta}f\right) = f \in C_\eta. \tag{3.10}$$

Proof. (i) We let $f \in \Omega_\eta^\alpha \subset C_\eta$ and prove the series formula and function space inclusion for $\,_{RL}^{A}D_{0+}^{\alpha,\beta}f \in C_\eta$. We can use the series formula for the fractional integral with analytic kernel, by Theorem 3.1:

$$\,_{RL}^{A}D_{0+}^{\alpha,\beta}f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^\infty \bar{a}_n(1-\alpha,\beta)\Gamma(n\beta + 1 - \alpha)\,^{RL}I_{0+}^{n\beta+1-\alpha}f(x)\right),$$

where the series inside the parentheses is locally uniformly convergent. The above series is equal to

$$\sum_{n=0}^\infty \frac{\mathrm{d}}{\mathrm{d}x}\left(\bar{a}_n(1-\alpha,\beta)\Gamma(n\beta + 1 - \alpha)\,^{RL}I_{0+}^{n\beta+1-\alpha}f(x)\right),$$

or equivalently

$$\sum_{n=0}^\infty \bar{a}_n(1-\alpha,\beta)\Gamma(n\beta + 1 - \alpha)\,^{RL}I_{0+}^{n\beta-\alpha}f(x),$$

as long as this new series is also locally uniformly convergent.

Note that the operator $^{RL}I_{0+}^{n\beta-\alpha}$ in the series is a fractional derivative if n is small enough that $\mathrm{Re}(n\beta - \alpha) \le 0$, while it is a fractional integral for all n large enough that $\mathrm{Re}(n\beta - \alpha) > 0$. So we split the series into two subseries according to the sign of $\mathrm{Re}(n\beta - \alpha)$. Let $N := \lfloor \alpha/\beta \rfloor$ denote the cutoff point, and we have:

$$_{RL}^{A}D_{0+}^{\alpha,\beta}f(x) = J_1 + J_2 = \sum_{n=0}^{N} \overline{a}_n(1-\alpha,\beta)\Gamma(n\beta + 1 - \alpha)\,^{RL}D_{0+}^{\alpha-n\beta}f(x)$$

$$+ \sum_{n=N+1}^{\infty} \overline{a}_n(1-\alpha,\beta)\Gamma(n\beta + 1 - \alpha)\,^{RL}I_{0+}^{n\beta-\alpha}f(x).$$

Now, since $f \in \Omega_\eta^\alpha$, the first sum J_1 is a finite sum of elements of C_η and therefore also in C_η. It remains to prove that the second sum J_2 is locally uniformly convergent to an element of C_η.

Since $N + 1 > \alpha/\beta$, we can choose $\varepsilon > 0$ so that $(N+1)\beta - \alpha - \varepsilon > 0$. Then, letting $g = \,_{0}^{RL}I_{x}^{(N+1)\beta-\alpha-\varepsilon}f \in C_\eta$, we have

$$J_2 = \sum_{n=0}^{\infty} \overline{a}_{n+N+1}(1-\alpha,\beta)\Gamma((n+N+1)\beta + 1 - \alpha)\,_{0}^{RL}I_{x}^{n\beta+\varepsilon}g(x)$$

$$= \left(\sum_{n=0}^{\infty} \frac{\Gamma(n\beta + (N+1)\beta + 1 - \alpha)}{\Gamma(n\beta + \varepsilon)} \cdot \overline{a}_{n+N+1}(1-\alpha,\beta)x^{n\beta+\varepsilon-1} \right) * g,$$

where this power series has the same radius of convergence as $\overline{A}$ due to [1, Chapter IV Theorem 1], and is therefore locally uniformly convergent and in C_{-1}. Its conjugation with g must then be in C_η, and we have proved both the mapping property of $_{RL}^{A}D_{0+}^{\alpha,\beta}$ and the series formula for it.

(ii) Now we let $f \in C_\eta$ and show that $^{A}I_{0+}^{\alpha,\beta}f \in \Omega_\eta^\alpha$ by using the series formula from Theorem 3.1. For any ν with $0 \le \nu < \alpha$, we have:

$$^{RL}D_{0+}^{\nu}\left(^{A}I_{0+}^{\alpha,\beta}f(x) \right) = \,^{RL}D_{0+}^{\nu}\left(\sum_{r=0}^{\infty} a_r(\alpha,\beta)\Gamma(r\beta + \alpha)\,^{RL}I_{0+}^{r\beta+\alpha}f(x) \right)$$

$$= \sum_{r=0}^{\infty} a_r(\alpha,\beta)\Gamma(r\beta + \alpha)\,^{RL}D_{0+}^{\nu}\left(^{RL}I_{0+}^{r\beta+\alpha}f(x) \right)$$

$$= \sum_{r=0}^{\infty} a_r(\alpha,\beta)\Gamma(r\beta + \alpha)\,^{RL}I_{0+}^{r\beta+\alpha-\nu}f(x)$$

$$= \left(\sum_{r=0}^{\infty} \frac{\Gamma(r\beta + \alpha)}{\Gamma(r\beta + \alpha - \nu)} \cdot a_r(\alpha,\beta)x^{r\beta+\alpha-\nu-1} \right) * f,$$

and this is the convolution of f with a function in C_{-1} (convergence of the power series is again guaranteed by [1, Chapter IV Theorem 1]), therefore in C_η as required. Setting $\nu = \alpha$, we get a very similar expression but the first term of the series must be treated differently:

$$^{RL}D_{0+}^{\alpha}\left(^{A}I_{0+}^{\alpha,\beta}f(x) \right) = a_0(\alpha,\beta)\Gamma(\alpha)f(x) + \left(\sum_{r=1}^{\infty} \frac{\Gamma(r\beta + \alpha)}{\Gamma(r\beta)} \cdot a_r(\alpha,\beta)x^{r\beta-1} \right) * f,$$

which is again in C_η. Now we have proved that $^A I_{0+}^{\alpha,\beta} f$ is in the space Ω_η^α defined in Definition 3.1(b).

To prove the inversion relation (3.10), the argument is the same as in [10, §2.2], and goes as follows:

$$
\begin{aligned}
{}_{RL}^A D_{0+}^{\alpha,\beta}\left({}^A I_{0+}^{\alpha,\beta} f(x) \right) &= \sum_{n=0}^{\infty} \overline{a}_n(1-\alpha,\beta)\Gamma(n\beta+1-\alpha) \, {}^{RL} I_{0+}^{n\beta-\alpha}\left(\sum_{r=0}^{\infty} a_r(\alpha,\beta)\Gamma(r\beta+\alpha) \, {}^{RL} I_{0+}^{r\beta+\alpha} f(x) \right) \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \overline{a}_n(1-\alpha,\beta)a_r(\alpha,\beta)\Gamma(n\beta+1-\alpha)\Gamma(r\beta+\alpha) \, {}^{RL} I_{0+}^{n\beta+r\beta} f(x) \\
&= f(x),
\end{aligned}
$$

using the inversion relation (3.7) between the A_Γ and $\overline{A}_\Gamma$ series.

$\square$

Theorem 3.3. *Let $0 < \alpha < 1$ and $\beta > 0$ and $\eta \geq -1$, and let A be a function defined by (1.2). Then the Caputo type fractional derivative operator with kernel A, namely*

$$
{}_C^A D_{0+}^{\alpha,\beta} f(x) = \int_0^x (x-t)^{-\alpha} \overline{A}\left((x-t)^\beta \right) f'(t)\, dt,
$$

where $\overline{A}(x) = \sum_{n=0}^{\infty} \overline{a}_n x^n$ is defined by (3.7), is well-defined on the space C_η^1 and maps this space into C_η:

$$
{}_C^A D_{0+}^{\alpha,\beta} : C_\eta^1 \to C_\eta. \tag{3.11}
$$

Further, for any function $f \in C_\eta^1$, the infinite series

$$
\sum_{n=0}^{\infty} \overline{a}_n(1-\alpha,\beta)\Gamma(n\beta+1-\alpha) \, {}^{RL} I_{0+}^{n\beta+1-\alpha} f'(x) \tag{3.12}
$$

converges in C_η, locally uniformly for all $x \in (0,\infty)$, to the fractional derivative ${}_C^A D_{0+}^{\alpha,\beta} f(x)$.

Proof. Let $f \in C_{-1}^1$. By Definition 3.1(c), $f' \in C_{-1}$. Then by the definition of the Caputo type operator, it is immediate that ${}_C^A D_{0+}^{\alpha,\beta} f = {}^{\overline{A}} I_{0+}^{1-\alpha,\beta} f' \in C_{-1}$, giving the desired inclusion. The series formula (3.12) is also an immediate consequence of Theorem 3.1. $\square$

Theorem 3.4. *Let $0 < \alpha < 1$ and $\beta > 0$ and $\eta \geq -1$, and let A be a function defined by (1.2). Then, for any $f \in \Omega_\eta^\alpha$, we have the following inversion relation:*

$$
{}^A I_{0+}^{\alpha,\beta} \, {}_{RL}^A D_{0+}^{\alpha,\beta} f(x) = f(x) - x^{\alpha-1} A^{(\alpha,\beta)}(x^\beta) \cdot \sum_{n=0}^{N} \overline{a}_n(1-\alpha,\beta)\Gamma(n\beta+1-\alpha) \, {}^{RL} D_{0+}^{\alpha-1-n\beta} f(0),
$$

where $N := \lfloor \alpha/\beta \rfloor$.

Proof. We start by employing the series formulae (3.3) and (3.9):

$$
{}^A I_{0+}^{\alpha,\beta} \, {}_{RL}^A D_{0+}^{\alpha,\beta} f(x) = \sum_{r=0}^{\infty} a_r(\alpha,\beta)\Gamma(r\beta+\alpha) \, {}^{RL} I_{0+}^{r\beta+\alpha}\left(\sum_{n=0}^{\infty} \overline{a}_n(1-\alpha,\beta)\Gamma(n\beta+1-\alpha) \, {}_0^{RL} I_x^{n\beta-\alpha} f(x) \right)
$$

$$= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} a_r(\alpha,\beta)\overline{a}_n(1-\alpha,\beta)\Gamma(r\beta+\alpha)\Gamma(n\beta+1-\alpha)\, ^{RL}D_{0+}^{-r\beta-\alpha}\, ^{RL}D_{0+}^{-n\beta+\alpha}f(x),$$

where we have made use of the notational convention (2.4) that fractional integrals to positive order are fractional derivatives to negative order. The outer operator $^{RL}D_{0+}^{-r\beta-\alpha}$ is a fractional integral for all values of r; the inner operator $^{RL}D_{0+}^{-n\beta+\alpha}$ in the series is a fractional derivative if n is small enough that $-n\beta+\alpha \geq 0$, while it is a fractional integral for all n large enough that $-n\beta+\alpha < 0$.

As discussed in Section 2 above, in Riemann–Liouville fractional calculus, any differintegral of an integral obeys a semigroup property (2.5), while differintegrals of derivatives satisfy instead a relation of the form (2.6). So we have

$$^{RL}D_{0+}^{-r\beta-\alpha}\, ^{RL}D_{0+}^{-n\beta+\alpha}f(x) = {}^{RL}D_{0+}^{-r\beta-n\beta}f(x)$$

if $-n\beta+\alpha < 0$, and

$$^{RL}D_{0+}^{-r\beta-n\beta}f(x) - \sum_{k=0}^{M_n-1} \frac{x^{k-(-r\beta-\alpha)-M_n}}{\Gamma(k-(-r\beta-\alpha)-M_n+1)}\, ^{RL}D_{0+}^{-n\beta+\alpha-M_n+k}f(0)$$

if $-n\beta+\alpha \geq 0$, where $M_n = \lfloor -n\beta+\alpha \rfloor + 1 \in \mathbb{Z}^+$. Splitting the inner series in the above double sum according to the sign of $-n\beta+\alpha$, and letting $N := \lfloor \alpha/\beta \rfloor$ denote the cutoff point, we have:

$$^{A}I_{0+}^{\alpha,\beta}\, ^{A}_{RL}D_{0+}^{\alpha,\beta}f(x) = \sum_{r=0}^{\infty}\sum_{n=0}^{N} a_r(\alpha,\beta)\overline{a}_n(1-\alpha,\beta)\Gamma(r\beta+\alpha)\Gamma(n\beta+1-\alpha)$$

$$\times \left({}^{RL}I_{0+}^{r\beta+n\beta}f(x) - \sum_{k=0}^{M_n-1} \frac{x^{r\beta+\alpha-M_n+k}}{\Gamma(r\beta+\alpha-M_n+k+1)}\, ^{RL}D_{0+}^{-n\beta+\alpha-M_n+k}f(0) \right)$$

$$+ \sum_{r=0}^{\infty}\sum_{n=N+1}^{\infty} a_r(\alpha,\beta)\overline{a}_n(1-\alpha,\beta)\Gamma(r\beta+\alpha)\Gamma(n\beta+1-\alpha)\, ^{RL}I_{0+}^{r\beta+n\beta}f(x)$$

$$= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} a_r(\alpha,\beta)\overline{a}_n(1-\alpha,\beta)\Gamma(r\beta+\alpha)\Gamma(n\beta+1-\alpha)\, ^{RL}I_{0+}^{(r+n)\beta}f(x)$$

$$- \sum_{n=0}^{N}\sum_{k=0}^{M_n-1} \overline{a}_n(1-\alpha,\beta)\Gamma(n\beta+1-\alpha)x^{\alpha-M_n+k}$$

$$\times \left[\sum_{r=0}^{\infty} \frac{a_r(\alpha,\beta)\Gamma(r\beta+\alpha)x^{r\beta}}{\Gamma(r\beta+\alpha-M_n+k+1)} \right] {}^{RL}D_{0+}^{-n\beta+\alpha-M_n+k}f(0)$$

Now, the double infinite sum reduces to a single term $f(x)$ by using the relation (3.7) between A and $\overline{A}$, while the infinite sum over r in square brackets is locally uniformly convergent by using [1, Chapter IV Theorem 1] again. So we have

$$^{A}I_{0+}^{\alpha,\beta}\, ^{A}_{RL}D_{0+}^{\alpha,\beta}f(x) = f(x)$$

$$- \sum_{n=0}^{N}\sum_{k=1}^{M_n} \overline{a}_n(1-\alpha,\beta)\Gamma(n\beta+1-\alpha)x^{\alpha-k} \left[\sum_{r=0}^{\infty} \frac{a_r(\alpha,\beta)\Gamma(r\beta+\alpha)x^{r\beta}}{\Gamma(r\beta+\alpha-k+1)} \right] {}^{RL}D_{0+}^{-n\beta+\alpha-k}f(0).$$

This looks like the final answer, but it can still be simplified further. As $0 < \alpha < 1$, by definition of N we have $0 \leq -n\beta + \alpha \leq \alpha < 1$ for all $n \leq N$. Therefore, $M_n = 1$ for all $n \leq N$, and the sum over k reduces to just a single term for every value of n, giving:

$$
\begin{aligned}
{}_0^A I_x^{\alpha,\beta}\, {}_{RL}^A D_x^{\alpha,\beta} f(x) &= f(x) - \sum_{n=0}^{N} \overline{a}_n(1-\alpha,\beta)\Gamma(n\beta+1-\alpha)x^{\alpha-1} \\
&\qquad\qquad \times \left[\sum_{r=0}^{\infty} \frac{a_r(\alpha,\beta)\Gamma(r\beta+\alpha)x^{r\beta}}{\Gamma(r\beta+\alpha)} \right] {}^{RL}D_{0+}^{-n\beta+\alpha-1} f(0) \\
&= f(x) - x^{\alpha-1}\left[\sum_{r=0}^{\infty} a_r(\alpha,\beta)x^{r\beta} \right] \sum_{n=0}^{N} \overline{a}_n(1-\alpha,\beta)\Gamma(n\beta+1-\alpha)\, {}^{RL}D_{0+}^{-n\beta+\alpha-1} f(0) \\
&= f(x) - x^{\alpha-1}A(x^{\beta}) \sum_{n=0}^{N} \overline{a}_n(1-\alpha,\beta)\Gamma(n\beta+1-\alpha)\, {}^{RL}D_{0+}^{-n\beta+\alpha-1} f(0),
\end{aligned}
$$

which completes the proof. □

Theorem 3.5. *Let $0 < \alpha < 1$ and $\beta > 0$ and $\eta \geq -1$, and let A be a function defined by (1.2). Then, for any $f \in C_\eta^1$, we have the following inversion relation:*

$$
\left({}^A I_{0+}^{\alpha,\beta}\, {}_C^A D_{0+}^{\alpha,\beta} f \right)(x) = f(x) - f(0), \qquad x \geq 0.
$$

Proof. By the definition of the Caputo-type fractional derivative with kernel A, and the relation (3.7) between A and $\overline{A}$, we have

$$
\begin{aligned}
\left({}^A I_{0+}^{\alpha,\beta}\, {}_C^A D_{0+}^{\alpha,\beta} f \right)(x) &= {}^{RL}I_{0+}^{\alpha} A_\Gamma \left({}^{RL}I_{0+}^{\beta} \right) \overline{A}_\Gamma \left({}^{RL}I_{0+}^{\beta} \right) {}^{RL}I_{0+}^{1-\alpha} f'(x) \\
&= {}^{RL}I_{0+}^{\alpha}\, {}^{RL}I_{0+}^{1-\alpha} f'(x) \\
&= {}^{RL}I_{0+}^{1} f'(x) = f(x) - f(0),
\end{aligned}
$$

where we have used $0 < \alpha < 1$ to get this simple form of the inversion relation with just a single initial value term. □

Remark 3.1. It is worth noting that the inversion relation (3.7) defining $\overline{A}$ bears considerable similarity to some series convolution relations discussed previously in the literature: by Sonine [33] following Eq (5) in his paper of 1883, and by Wick [34] in his paper of 1968, although these works both used ordinary power series, essentially taking $\beta = 1$ in our formulation, rather than having a fractional power inside the series.

Indeed, under certain assumptions on the parameters involved, the analytic kernel in (1.1) can be seen as a special case of the Sonine kernel. However, fractional calculus with analytic kernels and fractional calculus with Sonine kernels form two separate fields of study, because analytic kernels are not assumed to have a Sonine relationship and Sonine kernels are not assumed to have a power series. One of the purposes of this paper is to show how the series formula can give an advantage in dealing with fractional calculus with general kernels, leading to approaches which work for analytic kernels but would not work in general for Sonine kernels. It should also be noted that the operators considered here have two fractional orders, α and β, and thus it makes sense to speak about properties such as semigroup (Proposition 3.1). The Sonine kernels do not have any parameters (fractional orders), so they have inversion relations but no semigroup properties.

4. Mikusiński style algebraic interpretation

We recall from the classical work of Mikusiński [14], Dimovski [32] and Luchko [17] the following facts. The function space C_{-1} forms a commutative rng (ring without multiplicative identity) under the operations of addition and convolution in the form of (3.1). Because this rng has no zero divisors, due to a theorem of Titchmarsh, it can be extended to a field M_{-1}, its field of fractions. The elements of this field are abstract algebraic objects, which can be seen as operators or generalised functions, and are not necessarily functions themselves.

The operation of the fractional integral $^A I_{0+}^{\alpha,\beta}$ with analytic kernel A corresponds to the function $K_{\alpha,\beta} \in C_{-1}$ given by $K_{\alpha,\beta}(x) = x^{\alpha-1} A(x^\beta)$. We let $R_{\alpha,\beta}$ denote its algebraic inverse in M_{-1}, a field element which is not itself a function in the function space C_{-1}.

Theorem 4.1. *Let $0 < \alpha < 1$ and $\beta > 0$ and $\eta \geq -1$, and let A be a function defined by (1.2). The fractional differential operator with analytic kernel of Riemann–Liouville type may be represented in the field M_{-1} in the following form:*

$$_{RL}^{A}D_{0+}^{\alpha,\beta}f = R_{\alpha,\beta} * f - \left(\sum_{n=0}^{N} \Gamma(n\beta + 1 - \alpha) \, {}^{RL}_{0}D_x^{\alpha-1-n\beta}f(0) \right) \mathbb{I}, \qquad f \in \Omega_{-1}^{\alpha}, \tag{4.1}$$

where $N := \lfloor \alpha/\beta \rfloor$ and $\mathbb{I}$ denotes the multiplicative identity element in the field M_{-1}.

Proof. Let $f \in \Omega_{-1}^{\alpha}$. From Theorem 3.4, interpreting the result in the notation of the $K_{\alpha,\beta}$ function, we have

$$K_{\alpha,\beta} * {}_{RL}^{A}D_{0+}^{\alpha,\beta}f = f - \left(\sum_{n=0}^{N} \Gamma(n\beta + 1 - \alpha) \, {}^{RL}D_{0+}^{\alpha-1-n\beta}f(0) \right) K_{\alpha,\beta}.$$

Upon multiplying both sides by $R_{\alpha,\beta}$, the algebraic inverse of $K_{\alpha,\beta}$, we get the required result. $\qquad \square$

Theorem 4.2. *Let $0 < \alpha < 1$ and $\beta > 0$ and $\eta \geq -1$, and let A be a function defined by (1.2). The fractional differential operator with analytic kernel of Caputo type may be represented in the field M_{-1} in the following form:*

$$_{C}^{A}D_{0+}^{\alpha,\beta}f = R_{\alpha,\beta} * f - R_{\alpha,\beta} * [f(0)1], \qquad f \in C_{-1}^{1}, \tag{4.2}$$

where $f(0)1$ denotes the scalar $f(0)$ times the constant function 1.

Proof. Let $f \in C_{-1}^{m}$, and use the result of Theorem 3.5 with the notation of $K_{\alpha,\beta}$:

$$K_{\alpha,\beta} * {}_{RL}^{A}D_{0+}^{\alpha,\beta}f = f - f(0)1.$$

Upon multiplying both sides by $R_{\alpha,\beta}$, the algebraic inverse of $K_{\alpha,\beta}$, we get the required result. $\qquad \square$

In the remainder of this paper, we shall assume A is such that the integral operator $^A I_{0+}^{\alpha,\beta}$ obeys a semigroup property in the form of (3.4). Under this assumption, we can take convolutions of the function $K_{\alpha,\beta}$ with itself to find

$$K_{\alpha,\beta}^{n} := K_{\alpha,\beta} * \cdots * K_{\alpha,\beta} = K_{n\alpha,\beta}. \tag{4.3}$$

Following the same notion, we can then define arbitrary positive real powers of $K_{\alpha,\beta}$:

$$K_{\alpha,\beta}^{\lambda} = K_{\lambda\alpha,\beta}(x), \qquad \lambda > 0. \tag{4.4}$$

Making use also of the algebraic inverse $R_{\alpha,\beta}$, we can define arbitrary real powers:

$$K_{\alpha,\beta}^{\lambda} = \begin{cases} K_{\lambda\alpha,\beta}, & \lambda > 0; \\ identity, & \lambda = 0; \\ R_{\lambda\alpha,\beta}, & \lambda < 0, \end{cases}$$

and similarly for arbitrary real powers of $R_{\alpha,\beta}$ too. It is clear that these real powers satisfy a semigroup property:

$$R_{\alpha,\beta}^{\lambda} * R_{\alpha,\beta}^{\nu} = R_{\alpha,\beta}^{\lambda+\nu}, \qquad \lambda, \nu \in \mathbb{R}. \tag{4.5}$$

Example 4.1. Let $0 < \alpha < 1$ and $\beta > 0$ and $\eta \geq -1$, and let A be a function defined by (1.2) and satisfying the semigroup property (3.4). For any constants $c \in \mathbb{R}$ and $m \in \mathbb{N}$, we consider the field element

$$\frac{\mathbb{I}}{(R_{\alpha,\beta} - c\mathbb{I})^m},$$

and rewrite it as a sum of negative powers of $R_{\alpha,\beta}$ in order to express it as a function in C_{-1}:

$$\frac{\mathbb{I}}{(R_{\alpha,\beta} - c\mathbb{I})^m} = R_{\alpha,\beta}^{-m} \frac{\mathbb{I}}{(\mathbb{I} - cR_{\alpha,\beta})^m} = R_{\alpha,\beta}^{-m} \sum_{i=0}^{\infty} \frac{(m)_i}{i!} c^i R_{\alpha,\beta}^{-i} = \sum_{i=0}^{\infty} \frac{(m)_i}{i!} c^i R_{\alpha,\beta}^{-m-i} = \sum_{i=0}^{\infty} \frac{(m)_i}{i!} c^i K_{(m+i)\alpha,\beta},$$

where

$$\sum_{i=0}^{\infty} \frac{(m)_i}{i!} c^i K_{(m+i)\alpha,\beta}(x) = \sum_{i=0}^{\infty} \frac{(m)_i}{i!} c^i x^{(m+i)\alpha-1} A(x^\beta)$$

$$= \sum_{i=0}^{\infty} c^i \frac{(m)_i}{i!} x^{(m+i)\alpha-1} \sum_{r=0}^{\infty} a_r((m+i)\alpha, \beta) x^{r\beta}$$

$$= \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} a_r((m+i)\alpha, \beta) c^i \frac{(m)_i}{i!} x^{(m+i)\alpha+r\beta-1}. \tag{4.6}$$

We mention this here, both because it is an example of doing algebraic manipulations in the abstract field M_{-1} to obtain a function in C_{-1}, and because it will be useful in the next section when we study some fractional differential equations.

5. Applications and examples

In this section, we shall make use of the mathematical terminology and toolset developed above, to solve some fractional differential equations involving fractional derivatives with general analytic kernels. As in the last part of the preceding section, we shall assume the kernel A is such that the semigroup property (3.4) is always valid; this is necessary to obtain relatively simple forms of the solution functions for the differential equations below.

Example 5.1. We consider the following initial value problem of Riemann–Liouville type, where $0 < \alpha < 1$ and $\beta > 0$ and $\lambda \in \mathbb{R}$ are fixed while A is a function defined by (1.2) and satisfying the semigroup property (3.4).

$$_{RL}^{A}D_{0+}^{\alpha,\beta} y(x) - \lambda y(x) = f(x), \qquad x > 0; \tag{5.1}$$

$$^{RL}D_{0+}^{\alpha-1-n\beta}y(0) = c_n, \qquad 0 \le n \le N, \tag{5.2}$$

where $N := \lfloor \alpha/\beta \rfloor$ is a finite non-negative integer and $c_0, c_1, \cdots, c_N$ are fixed constants.

Making use of Theorem 4.1, we can rewrite this problem using the algebraic element $R_{\alpha,\beta}$ in the field M_{-1}:

$$R_{\alpha,\beta} * y - \lambda y = f + \left(\sum_{n=0}^{N} \Gamma(n\beta + 1 - \alpha) \, {}^{RL}_0 D_x^{\alpha-1-n\beta} f(0) \right) \mathbb{I}$$
$$= f + C\mathbb{I},$$

where $\mathbb{I}$ denotes the multiplicative identity in the field M_{-1} and the constant $C \in \mathbb{R}$ is given by

$$C := \sum_{n=0}^{N} \Gamma(n\beta + 1 - \alpha)c_n.$$

Now we have converted the differential equation to an algebraic equation, which can be solved in M_{-1} as follows:

$$y = \frac{\mathbb{I}}{R_{\alpha,\beta} - \lambda\mathbb{I}} * f + \frac{C\mathbb{I}}{R_{\alpha,\beta} - \lambda\mathbb{I}}. \tag{5.3}$$

Using what we found above (4.6) in the case $m = 1$, we find

$$\frac{\mathbb{I}}{R_{\alpha,\beta} - \lambda\mathbb{I}} = \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} a_r((1+i)\alpha, \beta)\lambda^i x^{i\alpha+r\beta+\alpha-1},$$

which means (5.3) becomes

$$y(x) = \int_0^x \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} a_r((1+i)\alpha, \beta)\lambda^i(x-t)^{i\alpha+r\beta+\alpha-1} f(t)\, dt + C \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} a_r((1+i)\alpha, \beta)\lambda^i x^{i\alpha+r\beta+\alpha-1},$$

So the solution function for the differential equation (5.1)–(5.2) can be written as

$$y(x) = \int_0^x \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} a_r((1+i)\alpha, \beta)\lambda^i(x-t)^{i\alpha+r\beta+\alpha-1} f(t)\, dt$$
$$+ \sum_{n=0}^{N} \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} c_n a_r((1+i)\alpha, \beta)\Gamma(n\beta + 1 - \alpha)\lambda^i x^{i\alpha+r\beta+\alpha-1},$$

this being not just an algebraic abstraction but an actual function in C_{-1}. In fact, our work here has proved that this is the unique solution to (5.1)–(5.2) in the function space C_{-1}.

Example 5.2. We now consider the following initial value problem of Caputo type, where $0 < \alpha < 1$ and $\beta > 0$ and $\lambda \in \mathbb{R}$ are fixed while A is a function defined by (1.2) and satisfying the semigroup property (3.4).

$$^A_C D_{0+}^{\alpha,\beta} y(x) - \lambda y(x) = f(x), \qquad x > 0; \tag{5.4}$$
$$y(0) = c, \tag{5.5}$$

where c_0 is a fixed constant.

Making use of Theorem 4.2, we can rewrite this problem using the algebraic element $R_{\alpha,\beta}$ in the field M_{-1}:

$$R_{\alpha,\beta} * y - \lambda y = f + R_{\alpha,\beta} * [y(0)1],$$

where 1 is the constant function $1 \in C_{-1}$.

Now we have converted the differential equation to an algebraic equation, which can be solved in M_{-1} as follows:

$$y = \frac{\mathbb{I}}{R_{\alpha,\beta} - \lambda\mathbb{I}} * f + \frac{R_{\alpha,\beta}}{R_{\alpha,\beta} - \lambda\mathbb{I}} * [y(0)1]. \tag{5.6}$$

We already know from the previous example, the $m = 1$ case of (4.6), that

$$\frac{\mathbb{I}}{R_{\alpha,\beta} - \lambda\mathbb{I}} = \sum_{i=0}^{\infty}\sum_{r=0}^{\infty} a_r((1 + i)\alpha, \beta)\lambda^i x^{i\alpha+r\beta+\alpha-1},$$

therefore

$$\frac{R_{\alpha,\beta}}{R_{\alpha,\beta} - \lambda\mathbb{I}} * [y(0)1] = \left(\mathbb{I} + \frac{\lambda\mathbb{I}}{R_{\alpha,\beta} - \lambda\mathbb{I}}\right) * [y(0)1]$$

$$= y(0) + \lambda y(0)\left(\sum_{i=0}^{\infty}\sum_{r=0}^{\infty} a_r((1 + i)\alpha, \beta)\lambda^i x^{i\alpha+r\beta+\alpha-1}\right) * \left(1\right)$$

$$= y(0) + y(0)\sum_{i=0}^{\infty}\sum_{r=0}^{\infty} \frac{a_r((1 + i)\alpha, \beta)\lambda^{i+1}}{i\alpha + r\beta + \alpha} x^{i\alpha+r\beta+\alpha}.$$

Making use of the initial condition too, we now have that the solution of the initial value problem (5.4)–(5.5) is

$$y(x) = c + \sum_{i=0}^{\infty}\sum_{r=0}^{\infty} \frac{c a_r((1 + i)\alpha, \beta)\lambda^{i+1}}{i\alpha + r\beta + \alpha} x^{i\alpha+r\beta+\alpha} + \int_0^x \sum_{i=0}^{\infty}\sum_{r=0}^{\infty} a_r((1 + i)\alpha, \beta)\lambda^i (x - t)^{i\alpha+r\beta+\alpha-1} f(t)\,\mathrm{d}t$$

this being not just an algebraic abstraction but an actual function in C_{-1}. In fact, our work here has proved that this is the unique solution to (5.4)–(5.5) in the function space C_{-1}.

6. Conclusions

This work represents an extension of the theory of Mikusiński's operational calculus to a new generalised setting, namely the setting of fractional integrals and derivatives with analytic kernel functions.

The most comparable previous works are the papers [25, 26], where Mikusiński's operational calculus was used to interpret fractional operators of Caputo and Riemann–Liouville types with Sonine kernels and generalised Sonine kernels. However, there are some key differences in the ideas needed. The operators with Sonine kernels do not have any "fractional order", any real or complex number that parametrises them, so the results must be interpreted purely in terms of the arbitrary kernel function. Here, we have two parameters labelling our operators, which means our results depend directly and explicitly on parameters α and β either of which can be interpreted as a fractional order. We also

have a series formula, relating our generalised operators to Riemann–Liouville fractional differinte-grals, which has been useful in a couple of ways. As we saw above in the proofs of Theorem 3.2 and Theorem 3.4, the series formula was instrumental in allowing us to establish the correct function space mappings for the operators. It also allowed us to obtain a more explicit form for the solutions of the associated fractional differential equations, depending on the analytic function's coefficients rather than on a general kernel function, and this explicit form may be easier to calculate numerically in specific examples. Thus, even though adding the semigroup assumption (3.4) makes the theory of fractional calculus with analytic kernels very similar to the theory of fractional calculus with Sonine kernels previously discussed in [25, 26], we do obtain a different theory here with new forms of the results, owing to the series formula which is missing in the general Sonine theory.

As part of our work here, we have also highlighted a sensitive issue in defining the fractional derivatives corresponding to the integral operators with analytic kernels. The formulae written in the original paper [10] are correct but more subtle than was explained there, given the potential dependence of the analytic function coefficients on the parameters. This also indicates some limitations of our results here, as well as potential future directions of extension: we have assumed that the parameter α is in the real interval $(0, 1)$ and that the fractional integral operators have a semigroup property, in order to get an elegant form of the results, but it will also be interesting to investigate what happens if these restrictions are removed.

Conflict of interest

The authors declare there are no conflicts of interest.

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