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Research article

Regularity criteria for a two dimensional Erying-Powell fluid flowing in a MHD porous medium

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Abstract: The intention and novelty in the presented study were to develop the regularity analysis for a parabolic equation describing a type of Eyring-Powell fluid flow in two dimensions. We proved that, under certain general conditions involving the space of bounded mean oscillation (BMO) and the Lebesgue space L^2, there exist bounded and regular velocity solutions under the L^2 space scope. This conclusion was additionally supplemented by the condition of a finite square integrable initial data (also some of the obtained expressions involved the gradient and the laplacian of the initial velocity distribution). To make our results further general, the proposed analysis was extended to cover regularity results in L^p $(p > 2)$ spaces. As a remarkable conclusion, we highlight that the solutions to the two dimensional Eyring-Powell fluid flow did not exhibit blow up behaviour.

Keywords: regularity criteria; porous medium; Powell-Fluid flow; two dimensional fluid; evolutionary flow

1. Introduction

A non-Newtonian description of a fluid may be required to model complex processes in physics and engineering. The constitutive equations, particularly the momentum, of such a non-Newtonian fluid give rise to mathematical concerns due to the additional rheological parameters in the constitutive relations. Unlike in the classical Newtonian viscous fluids, there is not a single constitutive equation for non-Newtonian fluids, which can describe in a general basis, their rheological properties. As a consequence, the non-Newtonian fluids are considered under particular descriptions supported by the observations and/or by theoretical arguments. One of this, that has attracted much attention in the last decades, is known as Eyring-Powell fluid. The diffusion operator associated to this kind of fluids emerges from the application of the kinetic theory of liquids, instead of particular experimental principles about viscosity (refer to [1, 2] and the studies cited therein).

Few recent attempts, focused on the solutions of Eyring-Powell fluid flow equations, can be mentioned through the studies from [1–10]. It should be noted that the cited references can be understood as representative of the state of the art. Therein, the reader can find remarkable applications of the mentioned fluid together with dedicated solutions obtained under analytical and numerical procedures. Nonetheless, the solutions are provided with no additional explorations about their regularity in the appropriate general mathematical spaces. On the contrary, such regularity principles are currently available in the literature for Navier-Stokes equations (see the references [11–16] for some analysis on the matter).

We focus our interest now on the discussions about some additional and remarkable studies involving analysis in fluid flows. These discussions may be regarded as additional justifications of the model to come. From an analytical perspective, we shall mention that the authors in [17] introduced a characterization of perfect fluid spacetimes based on Ricci solitons, gradient A-Einstein solitons, gradient Ricci solitons and gradient Schouten solitons. In [18], the authors obtained regularity results in L^p for the highest order derivative in an elliptic set of equations with coefficients in the space of Sarason vanishing mean oscillation (VMO). Some additional regularity conditions in fluids are given in [19], where the authors studied a blow-up condition for the set of solutions to a MHD fluid involving the Lebesgue space L^3. In [20], the authors study the heat dissipation of a stretching sheet under a MHD mixed convective flow of Eyring-Powell fluid. The solutions are obtained based a shooting method with a Fourth-order Runge-Kutta approach. In addition, in [21], the authors provide a comparative study between a nanofluid and water and the exploration of solutions complying with the oscillatory pattern of the boundary layer equations via a Perturbation Technique. In [22], a theoretical analysis, about the physical characteristics of an unsteady nanofluid flow, is provided based on a Parametric Continuation Method (PCM) validated with the Matlab numerical routine bvp4c. Another interesting analysis, involving the application of non-Newtonian fluids, is given in [23]; wherein the authors studied a mucus fluid transportation with the Laplace transform technique and the use of programming. In [24], the authors analyze the problem of a two dimensional laminar flow by converting the set of PDE into nonlinear and higher-order ODEs. The new system of ODEs is treated with a fourth-order Runge-Kutta method combined with shooting approach.

Supported by the fact that none of the mentioned references provides a general frame dealing with regularity of solutions, the purpose of the presented study is to develop the global regularity conditions for the solutions of a two-dimensional flow of Eyring-Powell fluid type arising in Magnetohydrodynamic (MHD) of porous materials. To make our analysis tractable, it should be noted that the induced magnetic field is considered to be negligible for small magnetic Reynolds numbers. This means that the electromagnetic interaction between charges is assumed to be small in comparison with the external applied magnetic field that induces the charged particles to move.

1.1. Model description

Mathematically, we consider a two-dimensional Eyring-Powell fluid with boundary layers and with, at least, a preliminary generalized solution belonging to the bounded mean oscillation (BMO) space. The existence of such a kind of solution is not a hard hypothesis, on the contrary, the references [2–10] provide different forms of solutions constructed by analytical and numerical means. Hence and given the existing vast literature dealing with solutions, it seems to us that the existence of such solutions is not a big issue at this stage, but the regularity and boundedness of them is.

Note that the fluid is considered to be incompressible and flowing through a porous medium. In this case, the flow velocity vector $\mathbf{V}$, the continuity and dynamical equations for the unsteady two-dimensional Eyring-Powell fluid flow with nano-boundary conditions (see [25]) are given by (note that we develop the analysis for the velocity component u. Similarly, it can be done for the other velocity component v)

$$\mathbf{V} = (u(x,y,t), v(x,y,t), 0) \text{ and } \mathrm{div}\mathbf{V} = 0. \tag{1.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (\nu + \frac{1}{\rho\beta d})\frac{\partial^2 u}{\partial y^2} - \frac{1}{2\rho\beta d^3}(\frac{\partial u}{\partial y})^2\frac{\partial^2 u}{\partial y^2} - \frac{\phi}{k}\left((\nu + \frac{1}{\rho\beta d}) - \frac{1}{6\rho\beta d^3}(\frac{\partial u}{\partial y})^2\right)u. \tag{1.2}$$

The related boundary conditions are expressed as follows

$$u(x,y,t) = Ux, \quad v(x,y,t) = 0 \quad \text{at } y = 0$$

$$u(x,y,t) = 0, \quad v(x,y,t) = 0 \text{ at } y \to \infty.$$

Note that at $y = 0$, the velocity component u is requested to satisfy an stretching condition with the constant U. This is introduced to ensure that the fluid flow complies with the continuity equation. Indeed any variation of the velocity component v shall comply with $\frac{\partial v}{\partial y} = -U$. Typically, this last value can be regarded as arbitrarily small. In the region where the fluid flow is fully developed (outside the stretching area), the stretching condition is considered as negligible. Assume that such a region is defined in the $y-$interval, (y_m, ∞), where $y_m > 0$. Then we can assume that the kinematic Dirichlet condition in the nanoboundary applies as follows

$$u(x,y,t) = 0 \text{ at } x = 0 \quad \text{and} \quad u(x,y,t) = 0 \text{ at } x = L.$$

In addition, the initial conditions is given by

$$u(x,y,0) = u_0(x,y) \quad \text{with} \quad \|u_0(x,y)\|_{L^p} < \infty, \quad p \geq 2.$$

It should be noted that u and v are the first and second velocity components respectively, while U is a constant. Recall that $u_0(x,y)$ is the initial velocity profile, ν is the kinematic viscosity of a fluid, defined as a ratio of dynamic viscosity μ and fluid density ρ, and β, d, ϕ and k refer to some fluid parameters that shall be obtained in accordance with the physical phenomenon to model (see a dedicated example in [25]).

2. Summary of results

Our results are stated as follows:

Theorem 2.1. Assume $u_0 \in H^1(\Omega)$, with $\left(u, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}\right) \in L^2(0,T,BMO)$, then the Eq (1.2) has solutions, $u(x,y,t)$, complying with a regularity condition in $[0,T]$. Note that here $\Omega = [y_m, L] \times [0,\infty)$.

Theorem 2.2. Suppose that $u_0 \in L^P(\Omega)$, with $\left(u^{\frac{P-2}{2}}\frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\right) \in L^2(0, T, BMO)$, then the Eq (1.2) has solutions, $u(x, y, t)$, complying with a regularity condition in $[0, T]$. Note that here $\Omega = [y_m, L] \times [0, \infty)$.

Theorem 2.3. Assuming $u_0 \in H^2(\Omega)$, together with $\left(\frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial \nabla u}{\partial y}, \Delta u, \frac{\partial \Delta u}{\partial y}\right) \in L^2(0, T, BMO)$ and $\left(\Delta v. \frac{\partial \nabla v}{\partial y}\right) \in L^2(\Omega)$. In addition, assume that $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ are of the same sign. Then the Eq (1.2) has solutions, $u(x, y, t)$, complying with a regularity condition in $[0, T]$. Note that here $\Omega = [y_m, L] \times [0, \infty)$.

3. Preliminaries

In this section, we introduce some notations and collect some preliminary results that will be used in the coming analysis. We introduce the well known functional space $L_p(\Omega)$ with the norm $\|\cdot\|_{L^p}$:

$$\|f\|_{L^p} = \left\{ \begin{array}{l} \left(\int\limits_{\Omega} |f(x)|^p \, dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty \\ ess\ \sup_{x \in \Omega} |f(x)|, \quad p = \infty. \end{array} \right\}.$$

The usual Sobolev space of order m is defined by

$$H^m(\Omega) = \left\{u \in L^2(\Omega) : \ \nabla^m(u) \in L^2(\Omega), \right\}$$

with the norm

$$\|u\|_{H^m} = \left(\|u\|_{L^2}^2 + \|\nabla^m u\|_{L^2}^2\right)^{\frac{1}{2}}.$$

In addition, we make use of the homogeneous space of bounded mean oscillation whose norm is defined as (see [26])

$$\|g\|_{BMO} = \sup_{R^n, r>0} \left(\frac{1}{|B_r(x)|}\right) \int_{B_r(x)} \left| g(y) - \left(\frac{1}{|B_r(y)|}\right) \int_{B_r(y)} g(z) \right| dy.$$

Lemma 3.1. (See [27]) Let us consider $1 < b < a < \infty$; then,

$$\|u_1\|_{L^a} \leq \|u_1\|_{BMO}^{1-\frac{b}{a}} \|u_1\|_{L^b}^{\frac{b}{a}}.$$

Lemma 3.2. The following anisotropic Sobolev inequality holds:
Let $f, g, h \in C_c^\infty\left(\mathbb{R}^3\right)$

$$\iiint\limits_{\Omega} |\, fgh \,| \ dxdydz \leq \ \overline{C}\, \|f\|_{L^2}^{\frac{1}{2}} \left\|\frac{\partial f}{\partial y}\right\|_{L^s}^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \left\|\frac{\partial g}{\partial x}\right\|_{L^2}^{\frac{1}{\alpha}} \|h\|_{L^2}.$$

Note that this last inequality is employed in $\mathbb{R}^2$ in this analysis.

4. Proof of Theorem 2.1

For proving Theorem 2.1, the following Propositions are firstly shown

Proposition 1. *(General bound and regularity in the first spatial derivative in the flow direction)* Given the initial data $u_0 \in L^2(\Omega)$, then any solution of Eq (1.2) satisfies

$$\sup_{0 \leq t \leq T} \|u\|_{L^2}^2 + \eta \int_0^T \left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2 dt \ \leq \psi \, \|u_0\|_{L^2}^2,$$

where η depends on a suitable constant to be introduced, namely C_5. In addition, the bounding constant ψ depends on another constant C_6 and the existence time T.

Proof. Multiplying the Eq (1.2) with u and applying integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + I_1 = -\left(v + \frac{1}{\rho\beta d}\right)\iint\limits_{\Omega}\left(\frac{\partial u}{\partial y}\right)^2 dxdy + \iint\limits_{\Omega}\left(\frac{\partial u}{\partial y}\right)^4 dxdy$$

$$-\frac{\phi}{k}\left(v + \frac{1}{\rho\beta d}\right)\iint\limits_{\Omega} u^2 dxdy + I_2$$

$$= -\left(v + \frac{1}{\rho\beta d}\right)\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2 + \frac{1}{6\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^4 - \frac{\phi}{k}\left(v + \frac{1}{\rho\beta d}\right)\|u\|_{L^2}^2 + I_2, \tag{4.1}$$

where

$$I_1 = \iint\limits_{\Omega}\left(u(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y})\right)dxdy, \qquad I_2 = \frac{\phi}{6k\rho\beta d^3}\iint\limits_{\Omega} u^2(\frac{\partial u}{\partial y})^2 dxdy.$$

For I_1, we have

$$I_1 = \iint\limits_{\Omega} u^2\frac{\partial u}{\partial x}dxdy + \iint\limits_{\Omega} uv\frac{\partial u}{\partial y}dxdy.$$

Applying integration by parts

$$I_1 = -\iint\limits_{\Omega}\frac{\partial v}{\partial y}\frac{u^2}{2}dxdy = \frac{1}{2}\iint\limits_{\Omega}\frac{\partial u}{\partial x}u^2 dxdy,$$

where we used Eq (1.1) . After integration, we get $I_1 = 0$.

For I_2, we consider the Young inequality, so that:

$$I_2 \le \frac{\phi}{12k\rho\beta d^3}\iint\limits_{\Omega} u^4 dxdy + \frac{\phi}{12k\rho\beta d^3}\iint\limits_{\Omega}\left(\frac{\partial u}{\partial y}\right)^4 dxdy$$

$$= \frac{\phi}{12k\rho\beta d^3}\|u\|_{L^4}^4 + \frac{\phi}{12k\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^4.$$

Introducing the values of I_1 and I_2 in Eq (4.1) we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 \le -\left(v + \frac{1}{\rho\beta d}\right)\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2 + \frac{1}{6\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^4 - \frac{\phi}{k}\left(v + \frac{1}{\rho\beta d}\right)\|u\|_{L^2}^2$$

$$+\frac{\phi}{12k\rho\beta d^3}\|u\|_{L^4}^4 + \frac{\phi}{12k\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^4$$

$$\le -\left(v + \frac{1}{\rho\beta d}\right)\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2 + \frac{C_1}{6\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial u}{\partial y}\right\|_{BMO}^2 - \frac{\phi}{k}\left(v + \frac{1}{\rho\beta d}\right)\|u\|_{L^2}^2$$

$$+\frac{C_1\phi}{12k\rho\beta d^3}\|u\|_{L^2}^2\|u\|_{BMO}^2+\frac{C_1\phi}{12k\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial u}{\partial y}\right\|_{BMO}^2,$$

where we used Lemma 3.1. Since $\left(\frac{\partial u}{\partial y},u\right)\in L^2(0,T,BMO)$

$$\frac{d}{dt}\|u\|_{L^2}^2+2\left(\nu+\frac{1}{\rho\beta d}-\frac{C_2}{6\rho\beta d^3}-\frac{C_4\phi}{12k\rho\beta d^3}\right)\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2$$
$$\leq 2\left|\frac{C_3\phi}{12k\rho\beta d^3}-\frac{\phi}{k}\left(\nu+\frac{1}{\rho\beta d}\right)\right|\|u\|_{L^2}^2,$$

which implies that

$$\frac{d}{dt}\|u\|_{L^2}^2+C_5\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2\leq C_6\|u\|_{L^2}^2,$$

where $\nu+\frac{1}{\rho\beta d}>\frac{C_2}{6\rho\beta d^3}+\frac{C_4\phi}{12k\rho\beta d^3}$. Therefore

$$C_5=2\left(\nu+\frac{1}{\rho\beta d}-\frac{C_2}{6\rho\beta d^3}-\frac{C_4\phi}{12k\rho\beta d^3}\right)>0$$
$$C_6=2\left|\frac{C_3\phi}{12k\rho\beta d^3}-\frac{\phi}{k}\left(\nu+\frac{1}{\rho\beta d}\right)\right|.$$

The Gronwall inequality yields to

$$\|u\|_{L^2}^2+\eta\int_0^T\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2dt\leq\psi\|u_0\|_{L^2}^2,$$

where η depends on C_5 and ψ depends on C_6 and T.

Proposition 2. *(Regularity in the first and second spatial derivatives in the flow direction)* Assume that the initial condition satisfies $\|\frac{\partial u_0}{\partial y}\|_{L^2}<\infty$ in Ω and that any solution of Eq (1.2) complies with $\left(\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial y^2}\right)\in L^2(0,T,BMO)$; then,

$$\sup_{0\leq t\leq<T}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2+\eta\int_0^T\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2dt\leq\phi\left\|\frac{\partial u_0}{\partial y}\right\|_{L^2}^2,$$

where η depends on the constant C_{10} (to be introduced) and ϕ depends on another constant C_{11} and the time T, such that in the interval $(0,T)$ the solutions are considered to exist.

Proof. Multiplying Eq (1.2) with $-\frac{\partial^2 u}{\partial y^2}$ and then solving by standard integration, we have

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2-I_3=-(\nu+\frac{1}{\rho\beta d})\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2+I_4$$
$$-\frac{\phi}{k}\left(\nu+\frac{1}{\rho\beta d}\right)\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2+\frac{\phi}{18k\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^4, \tag{4.2}$$

where

$$I_3 = \iint_\Omega \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial y^2}dxdy,$$

and

$$I_4 = \frac{1}{2\rho\beta d^3}\iint_\Omega \left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial y^2}\right)^2 dxdy.$$

Applying Lemma 3.1 on Eq (4.2) , we have

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2 - I_3 \le -(v + \frac{1}{\rho\beta d})\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2 + I_4$$

$$-\frac{\phi}{k}(v + \frac{1}{\rho\beta d})\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2 + \frac{C_1\phi}{18k\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial u}{\partial y}\right\|_{BMO}^2.$$

Since $\frac{\partial u}{\partial y} \in L^2(0, T, BMO)$, the following holds

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2 - I_3 \le -(v + \frac{1}{\rho\beta d})\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2 + I_4$$

$$-\frac{\phi}{k}(v + \frac{1}{\rho\beta d})\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2 + \frac{C_7\phi}{18k\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2. \tag{4.3}$$

For I_3, using integration by parts, we have

$$I_3 = -\iint_\Omega \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x\partial y}\right)dxdy - \frac{1}{2}\iint_\Omega \frac{\partial v}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 dxdy.$$

From Eq (1.1), the following holds

$$I_3 = -\iint_\Omega \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial u}{\partial x}dxdy - \iint_\Omega u\frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial y}dxdy + \frac{1}{2}\iint_\Omega \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2 dxdy.$$

Integrating the second term on the right hand side with regards to x

$$I_3 = -\frac{1}{2}\iint_\Omega \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2 dxdy + \frac{1}{2}\iint_\Omega \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2 dxdy = 0.$$

Applying Young inequality on I_4,

$$I_4 \le \frac{1}{4\rho\beta d^3}\iint_\Omega \left(\frac{\partial u}{\partial y}\right)^4 dxdy + \frac{1}{4\rho\beta d^3}\iint_\Omega \left(\frac{\partial^2 u}{\partial y^2}\right)^4 dxdy$$

$$\le \frac{1}{4\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^4 + \frac{1}{4\rho\beta d^3}\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^4}^4$$

$$\leq \frac{C_1}{4\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial u}{\partial y}\right\|_{BMO}^2+\frac{C_1}{4\rho\beta d^3}\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{BMO}^2,$$

where we used Lemma 2.1. Since $\left(\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial y^2}\right)\in L^2(0,T,BMO)$; then,

$$I_4 \leq \frac{C_8}{4\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2+\frac{C_9}{4\rho\beta d^3}\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2.$$

Introducing the values of I_3 and I_4 in Eq (4.3):

$$\begin{aligned}
&\frac{d}{dt}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2+\left(\nu+\frac{1}{\rho\beta d}-\frac{C_9}{4\rho\beta d^3}\right)\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2\\
\leq\ &\left(\frac{C_8}{4\rho\beta d^3}-\frac{\phi}{k}\nu-\frac{C_7\phi}{18k\rho\beta d}\right)\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2\\
&\leq\left|\frac{C_8}{4\rho\beta d^3}-\frac{\phi}{k}\nu-\frac{C_7\phi}{18k\rho\beta d}\right|\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2.
\end{aligned}\tag{4.4}$$

Choosing $\nu+\frac{1}{\rho\beta d}>\frac{C_9}{4\rho\beta d^3}$; then,

$$C_{10}=\left(\nu+\frac{1}{\rho\beta d}-\frac{C_9}{4\rho\beta d^3}\right)>0,$$

and for simplicity we can choose the constant value in the right hand side of expression (4.4):

$$C_{11}=\left|\frac{C_8}{4\rho\beta d^3}-\frac{\phi}{k}\nu-\frac{C_7\phi}{18k\rho\beta d}\right|.$$

Then, the Eq (4.4) becomes

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2+C_{10}\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2\leq C_{11}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2.$$

Applying the Gronwall inequality,

$$\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2+\eta\int_0^T\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2dt\leq\psi\left\|\frac{\partial u_0}{\partial y}\right\|_{L^2}^2,$$

where η depends on C_{10} and ψ depends on C_{11} and T.

Finally, the Theorem 2.1 proof is completed by using Propositions 1 and 2, as both propositions provide evidences on the global bound of solutions and regularity of the fluid in the flowing direction in $\Omega\times(0,T]$.

5. Proof of Theorem 2.2

The Theorem 2.2 provides a general bound and regularity in the first spatial derivative in the flow direction within the space $L^p(\Omega)$, $p > 2$, and locally in the interval $(0, T]$.

To prove such a theorem, we start by multiplying the Eq (1.2) with $|u|^{P-2}\, u$. After integration, the following holds:

$$\frac{1}{P}\frac{d}{dt}\,\|u\|_{L^p}^p + I_5 = -\left(v + \frac{1}{\rho\beta d}\right)(P-1)\iint\limits_{\Omega} |u|^{P-2}\,\frac{\partial^2 u}{\partial y^2}dxdy + I_6$$

$$-\frac{\phi}{k}\left(v + \frac{1}{\rho\beta d}\right)\iint\limits_{\Omega} |u|^P\, dxdy + \frac{\phi}{6k\rho\beta d^3}\iint\limits_{\Omega} |u|^P\ (\frac{\partial u}{\partial y})^2 dxdy$$

$$= -\left(v + \frac{1}{\rho\beta d}\right)(P-1)\iint\limits_{\Omega}\left(u^{\frac{P-2}{2}}\frac{\partial u}{\partial y}\right)^2 dxdy + I_6$$

$$-\frac{\phi}{k}\left(v + \frac{1}{\rho\beta d}\right)\|u\|_{L^p}^p + \frac{\phi}{12\rho\beta d^3}\,\|u\|_{L^{2p}}^p\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^2$$

$$\leq -\left(v + \frac{1}{\rho\beta d}\right)(P-1)\left\|u^{\frac{P-2}{2}}\frac{\partial u}{\partial y}\right\|_{L^2}^2 + I_6$$

$$-\frac{\phi}{k}\left(v + \frac{1}{\rho\beta d}\right)\|u\|_{L^p}^p + \frac{C_1^2\phi}{12\rho\beta d^3}\,\|u\|_{L^2}^{\frac{p}{2}}\,\|u\|_{BMO}^{\frac{p}{2}}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}\left\|\frac{\partial u}{\partial y}\right\|_{BMO}$$

$$\leq -\left(v + \frac{1}{\rho\beta d}\right)(P-1)\left\|u^{\frac{P-2}{2}}\frac{\partial u}{\partial y}\right\|_{L^2}^2 + I_6$$

$$-\frac{\phi}{k}\left(v + \frac{1}{\rho\beta d}\right)\|u\|_{L^p}^p + \frac{C_{12}\phi}{12\rho\beta d^3}\,\|u\|_{L^2}^{\frac{p}{2}}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}, \tag{5.1}$$

where we used Lemma 3.1. In addition

$$I_5 = \iint\limits_{\Omega} |u|^P\,\frac{\partial u}{\partial x}dxdy + \iint\limits_{\Omega} |u|^{P-1}\,\frac{\partial u}{\partial y}\ v\ dxdy,$$

and

$$I_6 = -\frac{1}{2\rho\beta d^3}\iint\limits_{\Omega} |u|^{P-1}\ (\frac{\partial u}{\partial y})^2\frac{\partial^2 u}{\partial y^2}dxdy.$$

Applying integration by parts on I_5, we have

$$I_5 = -\frac{1}{P}\iint\limits_{\Omega} u^P\frac{\partial v}{\partial y}dxdy$$

By using Eq (1.1), we can write

$$I_5 = \frac{1}{P}\iint\limits_{\Omega} u^P \left(\frac{\partial u}{\partial x}\right) dxdy.$$

Integrating again we obtain $I_5 = 0$. Applyling integration by parts on I_6, we have

$$\begin{aligned}
I_6 &= \frac{p-1}{6\rho\beta d^3}\iint\limits_{\Omega} u^{p-2}\left(\frac{\partial u}{\partial y}\right)^4 dxdy \\
&= \frac{p-1}{6\rho\beta d^3}\iint\limits_{\Omega} \left(u^{\frac{p-2}{2}}\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial u}{\partial y}\right)^2 dxdy \\
&\leq \frac{p-1}{6\rho\beta d^3}\left\|u^{\frac{p-2}{2}}\frac{\partial u}{\partial y}\right\|_{L^4}^2\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^2 \\
&\leq \frac{C_1^2(p-1)}{6\rho\beta d^3}\left\|u^{\frac{p-2}{2}}\frac{\partial u}{\partial y}\right\|_{L^2}\left\|u^{\frac{p-2}{2}}\frac{\partial u}{\partial y}\right\|_{BMO}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}\left\|\frac{\partial u}{\partial y}\right\|_{BMO} \\
&\leq \frac{C_{13}(p-1)}{6\rho\beta d^3}\left\|u^{\frac{p-2}{2}}\frac{\partial u}{\partial y}\right\|_{L^2}\left\|\frac{\partial u}{\partial y}\right\|_{L^2},
\end{aligned}$$

where we used Holder inequality and Lemma 3.1. Since $\left(u^{\frac{p-2}{2}}\frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\right) \in L^2(0, T, BMO)$ and after using Proposition 2, we get

$$I_6 \leq C_{14}\left\|u^{\frac{p-2}{2}}\frac{\partial u}{\partial y}\right\|_{L^2} \leq C_{14}^2\left\|u^{\frac{p-2}{2}}\frac{\partial u}{\partial y}\right\|_{L^2}^2.$$

Introducing I_5 and I_6 in Eq (5.1) , we get

$$\begin{aligned}
\frac{1}{P}\frac{d}{dt}\|u\|_{L^p}^p \leq &-\left(v+\frac{1}{\rho\beta d}\right)(P-1)\left\|u^{\frac{P-2}{2}}\frac{\partial u}{\partial y}\right\|_{L^2}^2 + C_{14}\left\|u^{\frac{p-2}{2}}\frac{\partial u}{\partial y}\right\|_{L^2}^2 \\
&-\frac{\phi}{k}\left(v+\frac{1}{\rho\beta d}\right)\|u\|_{L^p}^p + C_{15}\|u\|_{L^p}^p,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{d}{dt}\|u\|_{L^p}^p + \left[P(P-1)\left(v+\frac{1}{\rho\beta d}\right) - C_{14}\right]\left\|u^{\frac{P-2}{2}}\frac{\partial u}{\partial y}\right\|_{L^2}^2 \\
&\leq P\left|C_{15} - \frac{\phi}{k}\left(v+\frac{1}{\rho\beta d}\right)\right|\|u\|_{L^p}^p, \qquad (5.2)
\end{aligned}$$

As $p > 2$ and $v + \frac{1}{\rho\beta d} > C_{14}$; then, $C_{16} = \left[P(P-1)\left(v+\frac{1}{\rho\beta d}\right) - C_{14}\right] > 0$ and $C_{17} = P\left[C_{15} - \frac{\phi}{k}\left(v+\frac{1}{\rho\beta d}\right)\right]$. Consequently, the Eq (5.2) becomes

$$\frac{d}{dt}\|u\|_{L^p}^p + C_{16}\left\|u^{\frac{P-2}{2}}\frac{\partial u}{\partial y}\right\|_{L^2}^2 \leq C_{17}\|u\|_{L^p}^p.$$

The Gronwall inequality yields to

$$\|u\|_{L^p}^p + \eta \int_0^T \left\| u^{\frac{P-2}{2}} \frac{\partial u}{\partial y} \right\|_{L^2}^2 dt \leq \psi \, \|u_0\|_{L^p}^p ,$$

where η depend on C_{16} and ψ depend on C_{17} and T.

6. Proof of Theorem 2.3

Firstly, the following Propositions are required.

Proposition 3. *(General bound and regularity is the spatial gradient)* Assume that the initial condition satisfies $\|\nabla u_0\|_{L^2}^2 < \infty$ in Ω and for any time in $(0, T]$. Consider that u is a solution to Eq (1.2) with $\left(\frac{\partial u}{\partial y}, \nabla u, \frac{\partial}{\partial y}(\nabla u)\right) \in L^2(0, T, BMO)$; then, $u(x, y, t)$ satisfies

$$\|\nabla u\|_{L^2}^2 + \eta \int_0^T \left\| \frac{\partial}{\partial y} (\nabla u) \right\|_{L^2}^2 \leq \psi \, \|\nabla u_0\|_{L^2}^2 ,$$

where η depends on other constants, namely $v, \rho, \beta, d, C_{18}$ and $\in$ and the bounding constant ψ depends on $C, \phi, k, v, \rho, \beta, d, C_{19}$ and the time T, such that in the interval $(0, T)$ the solutions are considered to exist.

Proof. Taking the inner product of Eq (1.2) with Δu and upon integration

$$\iint_\Omega \frac{\partial(\nabla u)}{\partial t} \cdot (\nabla u) dxdy = I_7 + -\left(v + \frac{1}{\rho\beta d}\right) \iint_\Omega \left(\frac{\partial \nabla u}{\partial y}\right)^2 dxdy + I_8$$
$$- \frac{\phi}{k}(v + \frac{1}{\rho\beta d}) \iint_\Omega (\nabla u)^2 \, dxdy + I_9, \tag{6.1}$$

where

$$I_7 = \iint_\Omega \Delta u (u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) dxdy,$$

$$I_8 = \frac{1}{2\rho\beta d^3} \iint_\Omega \Delta u (\frac{\partial u}{\partial y})^2 \frac{\partial^2 u}{\partial y^2} dxdy,$$

$$I_9 = \frac{\phi}{6\rho\beta d^3} \iint_\Omega \Delta u . u (\frac{\partial u}{\partial y})^2 dxdy.$$

Applying integration by parts on I_7,

$$I_7 = -\iint_\Omega \nabla (u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) \nabla u dxdy$$

$$= -\iint_\Omega (\frac{\partial u}{\partial x})^3 dxdy - \iint_\Omega u \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x^2} dxdy + \iint_\Omega \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} dxdy$$

$$-\iint_\Omega v\frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial x}dxdy-\iint_\Omega \frac{\partial u}{\partial x}(\frac{\partial u}{\partial y})^2dxdy-\iint_\Omega u\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}dxdy$$

$$-\iint_\Omega \frac{\partial v}{\partial y}(\frac{\partial u}{\partial y})^2dxdy-\iint_\Omega v\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}dxdy$$

$$=-\frac{1}{2}\iint_\Omega (\frac{\partial u}{\partial x})^3dxdy-\iint_\Omega \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}dxdy$$

$$-\frac{1}{2}\iint_\Omega \frac{\partial u}{\partial x}(\frac{\partial u}{\partial y})^2dxdy-\frac{1}{2}\iint_\Omega \frac{\partial v}{\partial x}(\frac{\partial u}{\partial y})^2dxdy.$$

From Eq (1.1), we have

$$I_7=-\frac{1}{2}\iint_\Omega (\frac{\partial u}{\partial x})^3dxdy+\iint_\Omega \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}dxdy.$$

Integrating the second term on the right hand side, we have

$$I_7=-\frac{1}{2}\iint_\Omega (\frac{\partial u}{\partial x})^3dxdy-\iint_\Omega u\frac{\partial^2 v}{\partial y^2}\frac{\partial v}{\partial x}dxdy-\iint_\Omega u\frac{\partial v}{\partial y}\frac{\partial v}{\partial x\partial y}dxdy.$$

Integrating the third term on the right hand side, the following holds

$$I_7=-\frac{1}{2}\iint_\Omega (\frac{\partial u}{\partial x})^3dxdy-\iint_\Omega u\frac{\partial^2 v}{\partial y^2}\frac{\partial v}{\partial x}dxdy+\frac{1}{2}\iint_\Omega \left(\frac{\partial v}{\partial y}\right)^2\frac{\partial u}{\partial x}dxdy$$

$$=-\iint_\Omega u\frac{\partial^2 u}{\partial x\partial y}\frac{\partial v}{\partial x}dxdy,$$

where we used Eq (1.1). Therefore I_7 becomes

$$I_7\le \|u\|_{L^4}\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^2}\left\|\frac{\partial v}{\partial x}\right\|_{L^4}.$$

Applying Young inequality, we have

$$I_7\le \frac{\epsilon}{2}\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^2}^2+\frac{1}{2\epsilon}\|u\|_{L^4}^2\left\|\frac{\partial v}{\partial x}\right\|_{L^4}^2$$

Integrating I_8 by parts, we obtain

$$I_8=\frac{1}{6\rho\beta d^3}\iint_\Omega (\frac{\partial u}{\partial y})^3\Delta(\frac{\partial u}{\partial y})dxdy$$

$$=\frac{1}{2\rho\beta d^3}\iint_\Omega (\frac{\partial u}{\partial y})^2(\frac{\partial \nabla u}{\partial y})^2dxdy.$$

From Young inequality, we obtain

$$I_8 \leq \frac{1}{4\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^4 + \frac{1}{4\rho\beta d^3}\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^4}^4$$

$$\leq \frac{C_1}{4\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial u}{\partial y}\right\|_{BMO}^2 + \frac{C_1}{4\rho\beta d^3}\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial \nabla u}{\partial y}\right\|_{BMO}^2.$$

Since $\frac{\partial \nabla u}{\partial y} \in L^2(0,T,BMO)$ and also applying Proposition 1, we can write

$$I_8 \leq \frac{C_{18}}{4\rho\beta d^3}\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^2}^2.$$

Applying integration by parts on I_9, we have

$$I_9 = -\frac{\phi}{6\rho\beta d^3}\iint_\Omega (\nabla u)^2(\frac{\partial u}{\partial y})^2 dxdy - \frac{2\phi}{6\rho\beta d^3}\iint_\Omega u\nabla u\frac{\partial u}{\partial y}\frac{\partial \nabla u}{\partial y}dxdy.$$

Integrating again, we have

$$I_9 = -\frac{\phi}{6\rho\beta d^3}\iint_\Omega (\nabla u)^2(\frac{\partial u}{\partial y})^2 dxdy + \frac{\phi}{6\rho\beta d^3}\iint_\Omega (\nabla u)^2\left(u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right)dxdy$$

$$= \frac{\phi}{6\rho\beta d^3}\iint_\Omega (\nabla u)^2 u\frac{\partial^2 u}{\partial y^2}dxdy.$$

$$\leq \frac{\phi}{12\rho\beta d^3}\|\nabla u\|_{L^4}^4 + \frac{\phi}{12\rho\beta d^3}\|u\|_{L^2}^2\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^4}^2$$

$$\leq \frac{\phi C_1}{12\rho\beta d^3}\|\nabla u\|_{L^2}^2\|\nabla u\|_{BMO}^2 + \frac{\phi C_1}{12\rho\beta d^3}\|u\|_{L^2}^2\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{BMO},$$

where we used Holder inequality, Young inequality and Lemma 3.1.

Since $(\nabla u, \frac{\partial^2 u}{\partial y^2}) \in L^2(0,T,BM0)$ the following holds

$$I_9 \leq \frac{C_{19}\phi}{12\rho\beta d^3}\|\nabla u\|_{L^2}^2 + \frac{C_{20}\phi}{12\rho\beta d^3}\|u\|_{L^2}^2\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}.$$

Introducing the values of I_7, I_8 and I_9 in Eq (6.1) , we get

$$\frac{d}{dt}\|(\nabla u)\|_{L^2}^2 + \left(v + \frac{1}{\rho\beta d} - \frac{C_{18}}{4\rho\beta d^3} - \frac{\in}{2}\right)\left\|\frac{\partial(\nabla u)}{\partial y}\right\|_{L^2}^2$$

$$\leq \left(-\frac{\phi}{k}(v + \frac{1}{\rho\beta d}) + \frac{C_{19}\phi}{12\rho\beta d^3}\right)\|\nabla u\|_{L^2}^2 + \frac{C_{20}\phi}{12\rho\beta d^3}\|u\|_{L^2}^2\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2} + \frac{1}{2\epsilon}\|u\|_{L^4}^2\left\|\frac{\partial v}{\partial x}\right\|_{L^4}^2,$$

since $\frac{\partial v}{\partial x} \in L^2(0,T)$, and after using Propositions 1 and 2

$$\frac{d}{dt}\|(\nabla u)\|_{L^2}^2 + \left(v + \frac{1}{\rho\beta d} - \frac{C_{18}}{4\rho\beta d^3} - \frac{\in}{2}\right)\left\|\frac{\partial(\nabla u)}{\partial y}\right\|_{L^2}^2$$
$$\leq C\left|-\frac{\phi}{k}(v + \frac{1}{\rho\beta d}) + \frac{C_{19}\phi}{12\rho\beta d^3}\right|\|\nabla u\|_{L^2}^2.$$

Applying Gronwall inequality, we get

$$\|\nabla u\|_{L^2}^2 + \eta\int_0^T\left\|\frac{\partial(\nabla u)}{\partial y}\right\|_{L^2}^2 dt \leq \psi\|\nabla u_0\|_{L^2}^2,$$

where $v + \frac{1}{\rho\beta d} > \frac{C_{18}}{4\rho\beta d^3} > \frac{\in}{2}$, and η depends on $v, \rho, \beta, d, C_{18}$ and $\in$ and ψ depends on $C, \phi, k, v, \rho, \beta, d, C_{19}$ and the time T, such that in the interval $(0, T)$ the solutions are considered to exist.

Proposition 4. *(General bound and regularity is the spatial laplacian derivatives)* Assume that the initial condition satisfies $\|\Delta u_0\|_{L^2}^2 < \infty$ in Ω. Then, the solution of Eq (1.2) with $\left(\frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial}{\partial y}(\nabla u), \nabla u, \Delta u, \frac{\partial}{\partial y}(\Delta u)\right) \in L^2(0, T, BMO)$ and $\left(\Delta v. \frac{\partial \nabla v}{\partial y}\right) \in L^2(0, T)$ satisfies

$$\|(\Delta u)\|_{L^2}^2 + \eta\int_0^T\left\|\frac{\partial(\Delta u)}{\partial y}\right\|_{L^2}^2 dt + \kappa\int_0^T\left\|\frac{\partial u}{\partial y}^{\frac{3}{2}}\frac{\partial \Delta u}{\partial y}^{\frac{1}{2}}\right\|_{L^2}^2 dt$$
$$+\zeta\left\|\nabla u\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2 \leq \psi\|\Delta u_0\|_{L^2}^2,$$

where η, κ and ζ depend on suitable constants to be introduced, and the bounding ψ depends on another constant and the time T, such that in the interval $(0, T)$ the solutions are considered to exist.

Proof. Applying the Δ operator to Eq (1.2) and testing with Δu

$$\frac{1}{2}\frac{d}{dt}\|(\Delta u)\|_{L^2}^2 = -I_{10} - \left(v + \frac{1}{\rho\beta d}\right)\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^2$$
$$+ I_{11} - \frac{\phi}{k}(v + \frac{1}{\rho\beta d})\|(\Delta u)\|_{L^2}^2 + I_{12}, \tag{6.2}$$

where

$$I_{10} = \int\int \Delta u.\Delta\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)dxdy$$
$$I_{11} = -\frac{1}{2\rho\beta d^3}\iint_{\Omega}\Delta u.\Delta\left((\frac{\partial u}{\partial y})^2\frac{\partial^2 u}{\partial y^2}\right)dxdy$$
$$I_{12} = \frac{1}{6\rho\beta d^3}\iint_{\Omega}\Delta u.\Delta\left(u(\frac{\partial u}{\partial y})^2\right)dxdy.$$

For $I_{10,}$ we have

$$I_{10} = \int\int \Delta\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\Delta u dxdy$$

$$= -\iint_{\Omega}\left(\frac{\partial u}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right)\Delta u dxdy$$

$$-\iint_{\Omega}\left(2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x\partial y} + 2\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}\right)\Delta u dxdy$$

$$-\iint_{\Omega}\left(u\frac{\partial^3 u}{\partial x^3} + u\frac{\partial^3 u}{\partial x\partial y^2} + v\frac{\partial^3 u}{\partial x^2\partial y} + v\frac{\partial^3 u}{\partial y^3}\right)\Delta u dxdy.$$

Applying integration by parts and the continuity relation we obtain

$$\iint_{\Omega}\left(u\frac{\partial^3 u}{\partial x^3} + u\frac{\partial^3 u}{\partial x\partial y^2} + v\frac{\partial^3 u}{\partial x^2\partial y} + v\frac{\partial^3 u}{\partial y^3}\right)\Delta u dxdy = 0.$$

Therefore I_{10} becomes

$$I_{10} = -\iint_{\Omega}\Delta u\frac{\partial u}{\partial x}\Delta u dxdy - \iint_{\Omega}\Delta u\frac{\partial u}{\partial y}\Delta v dxdy - 2\iint_{\Omega}\Delta u\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}dxdy$$

$$-2\iint_{\Omega}\Delta u\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}dxdy - 2\iint_{\Omega}u\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x\partial y}dxdy - 2\iint_{\Omega}\Delta u\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}dxdy$$

$$= k_1 + k_2 + k_3 + k_4 + k_5 + k_6. \tag{6.3}$$

In order to solve the above named six terms, we use the inequality in Lemma 3.2,

$$k_1 = -\iint_{\Omega}\Delta u\frac{\partial u}{\partial x}\Delta u dxdy \le \iint_{\Omega}\left|\Delta u\frac{\partial u}{\partial x}\Delta u\right| dxdy$$

$$\le C_o\,||\Delta u||_{L^2}\,||\Delta u||_{L^2}^{\frac{1}{2}}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial u}{\partial x}\right\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^2}^{\frac{1}{2}}$$

$$\le \epsilon\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^2 + C_\epsilon\,||\nabla u||_{L^2}^2\,||\Delta u||_{L^2}^2 + C_{1\epsilon}\,||\Delta u||_{L^2}^2$$

For k_2, we have

$$k_2 = -\iint_{\Omega}\Delta u\frac{\partial u}{\partial y}\Delta v dxdy \le \iint_{\Omega}\left|\Delta u\frac{\partial u}{\partial y}\Delta v\right| dxdy$$

$$\le C_o\,||\Delta v||_{L^2}\,||\Delta u||_{L^2}^{\frac{1}{2}}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial^2 u}{\partial x\partial y}\right\|_{L^2}^{\frac{1}{2}}$$

$$\leq \epsilon \left\| \frac{\partial \Delta u}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial u}{\partial y} \right\|_{L^2}^2 \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2}^2 + C_\epsilon \|\Delta v\|_{L^2}^2$$

$$\leq \epsilon \left\| \frac{\partial \Delta u}{\partial y} \right\|_{L^2}^2 + C_\epsilon \|\nabla u\|_{L^2}^2 \left\| \frac{\partial \nabla u}{\partial y} \right\|_{L^2}^2 + C_\epsilon \|\Delta v\|_{L^2}^2 .$$

After applying Eq (1.1) on k_3,we obtain

$$k_3 = -2 \iint_\Omega \Delta u \frac{\partial v}{\partial y} \frac{\partial^2 u}{\partial x^2} dxdy \leq 2 \iint_\Omega \left| \Delta u \frac{\partial v}{\partial y} \frac{\partial^2 u}{\partial x^2} \right| dxdy$$

$$\leq C_o \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^2} \|\Delta u\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \Delta u}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial v}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial^2 v}{\partial x \partial y} \right\|_{L^2}^{\frac{1}{2}}$$

$$\leq C_o \|\Delta u\|_{L^2}^{\frac{3}{2}} \left\| \frac{\partial \Delta u}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial u}{\partial x} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \nabla v}{\partial y} \right\|_{L^2}^{\frac{1}{2}}$$

$$\leq \epsilon \left\| \frac{\partial \Delta u}{\partial y} \right\|_{L^2}^2 + C_\epsilon \|\nabla u\|_{L^2}^{\frac{2}{3}} \|\Delta u\|_{L^2}^2 \left\| \frac{\partial \nabla v}{\partial y} \right\|_{L^2}^{\frac{2}{3}} .$$

For k_4, we have

$$k_4 = -2 \iint_\Omega \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} \Delta u dxdy \leq 2 \iint_\Omega \left| \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} \Delta u \right| dxdy$$

$$\leq C_o \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2} \|\Delta u\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \Delta u}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial u}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2}^{\frac{1}{2}}$$

$$\leq C_o \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \Delta u}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \nabla u}{\partial y} \right\|_{L^2}^{\frac{3}{2}}$$

$$\leq \epsilon \left\| \frac{\partial \Delta u}{\partial y} \right\|_{L^2}^2 + C_\epsilon \|\nabla u\|_{L^2}^2 \|\Delta u\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial \nabla u}{\partial y} \right\|_{L^2}^3 .$$

For k_5, we have

$$k_5 = -2 \iint_\Omega \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial y} \Delta u dxdy \leq 2 \iint_\Omega \left| \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial y} \Delta u \right| dxdy$$

$$\leq C_o \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L^2} \|\Delta u\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \Delta u}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial u}{\partial x} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^2}^{\frac{1}{2}}$$

$$\leq C_o \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2} \left\| \frac{\partial \Delta u}{\partial y} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial \nabla u}{\partial y} \right\|_{L^2}$$

$$\leq \epsilon \left\| \frac{\partial \Delta u}{\partial y} \right\|_{L^2}^2 + C_\epsilon \left\| \frac{\partial \nabla u}{\partial y} \right\|_{L^2}^2 \|\Delta u\|_{L^2}^2 + C_\epsilon \|\nabla u\|_{L^2}^2 .$$

For k_6, we have

$$k_6 = -2\iint\limits_{\Omega} \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\Delta u dxdy \leq 2\iint\limits_{\Omega} \left|\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\Delta u\right| dxdy$$

$$\leq C_o \left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2} \left\|\frac{\partial u}{\partial x}\right\|_{L^2}^{\frac{1}{2}} \left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} \left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^{\frac{1}{2}}$$

$$\leq C_o \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2} \left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^{\frac{1}{2}} \left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^2}$$

$$\leq \epsilon \left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^2 + C_\epsilon \left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^2}^2 \|\Delta u\|_{L^2}^2 + C_\epsilon \|\nabla u\|_{L^2}^2 .$$

Introducing the values of k_1, k_2, k_3, k_4, k_5 and k_6 in Eq (6.3) , considering the Propositions 2 and 3, and letting $\left(\Delta v. \frac{\partial \nabla v}{\partial y}\right) \in L^2(0, T)$

$$I_{10} \leq 6\epsilon \left\|\frac{\partial (\Delta u)}{\partial y}\right\|_{L^2}^2 + C_{21} \|\Delta u\|_{L^2}^2 .$$

For I_{12},we can write

$$I_{11} = \frac{1}{6\rho\beta d^3}\iint\limits_{\Omega} \frac{\partial (\Delta u)}{\partial y}\Delta\left(\frac{\partial u}{\partial y}\right)^3 dxdy$$

which implies that

$$I_{11} = \frac{1}{\rho\beta d^3}\iint\limits_{\Omega} \frac{\partial (\Delta u)}{\partial y}\frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 dxdy + \frac{1}{2\rho\beta d^3}\iint\limits_{\Omega} \frac{\partial (\Delta u)}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial x^2\partial y}dxdy$$

$$+\frac{1}{\rho\beta d^3}\iint\limits_{\Omega} \frac{\partial (\Delta u)}{\partial y}\frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial y^2}\right)^2 dxdy + \frac{1}{2\rho\beta d^3}\iint\limits_{\Omega} \frac{\partial (\Delta u)}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^3}dxdy$$

$$= k_7 + k_8 + k_9 + k_{10}, \tag{6.4}$$

where

$$k_7 = \frac{1}{\rho\beta d^3}\iint\limits_{\Omega} \frac{\partial (\Delta u)}{\partial y}\frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 dxdy$$

$$k_8 = \frac{1}{2\rho\beta d^3}\iint\limits_{\Omega} \frac{\partial (\Delta u)}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial x^2\partial y}dxdy$$

$$k_9 = \frac{1}{\rho\beta d^3}\iint\limits_{\Omega} \frac{\partial (\Delta u)}{\partial y}\frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial y^2}\right)^2 dxdy$$

$$k_{10} = \frac{1}{2\rho\beta d^3}\iint\limits_{\Omega} \frac{\partial (\Delta u)}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^3}dxdy$$

In order to solve above pointed terms, we use the Young inequality on k_7

$$k_7 \leq \frac{1}{2\rho\beta d^3}\iint\limits_{\Omega}\left(\frac{\partial(\Delta u)}{\partial y}\frac{\partial u}{\partial y}\right)^2 dxdy + \frac{1}{2\rho\beta d^3}\iint\limits_{\Omega}\left(\frac{\partial \nabla u}{\partial y}\right)^4 dxdy$$

$$= \frac{1}{2\rho\beta d^3}\left\|\frac{\partial(\Delta u)}{\partial y}\frac{\partial u}{\partial y}\right\|_{L^2}^2 + \frac{1}{2\rho\beta d^3}\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^4}^4$$

$$\leq \frac{1}{2\rho\beta d^3}\left\|\frac{\partial(\Delta u)}{\partial y}\frac{\partial u}{\partial y}\right\|_{L^2}^2 + \frac{C_1}{2\rho\beta d^3}\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial \nabla u}{\partial y}\right\|_{BMO}^2,$$

where we used Lemma 3.1. Since $\frac{\partial \nabla u}{\partial y} \in L^2(O, T, BMO)$:

$$\begin{aligned} k_7 &\leq \frac{1}{2\rho\beta d^3}\left\|\frac{\partial(\Delta u)}{\partial y}\frac{\partial u}{\partial y}\right\|_{L^2}^2 + \frac{C_{22}}{2\rho\beta d^3}\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^2}^2 \\ &\leq \cdot\frac{1}{2\rho\beta d^3}\left\|\frac{\partial(\Delta u)}{\partial y}\frac{\partial u}{\partial y}\right\|_{L^2}^2 + \frac{C_{22}}{2\rho\beta d^3}\|\Delta u\|_{L^2}^2 \end{aligned}$$

$$+\frac{1}{2\rho\beta d^3}\iint\limits_{\Omega}\frac{\partial(\Delta u)}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial x^2 \partial y}dxdy$$

Applying Young inequality on k_8 , we get

$$k_8 \leq \frac{1}{4\rho\beta d^3}\iint\limits_{\Omega}\left(\frac{\partial(\Delta u)}{\partial y}\right)^4 dxdy + \frac{1}{4\rho\beta d^3}\iint\limits_{\Omega}\left(\frac{\partial u}{\partial y}\right)^4 dxdy$$

$$= \frac{1}{4\rho\beta d^3}\left\|\frac{\partial(\Delta u)}{\partial y}\right\|_{L^4}^4 + \frac{1}{4\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^4$$

$$\leq \frac{C_1}{4\rho\beta d^3}\left\|\frac{\partial(\Delta u)}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial(\Delta u)}{\partial y}\right\|_{BMO}^2 + \frac{C_1}{4\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial u}{\partial y}\right\|_{BMO}^2,$$

where we used Lemma 3.1. Since$\left(\frac{\partial(\Delta u)}{\partial y}, \frac{\partial u}{\partial y}\right) \in L^2(O, T, BMO)$ it holds that

$$k_8 \leq \frac{C_{23}}{4\rho\beta d^3}\left\|\frac{\partial(\Delta u)}{\partial y}\right\|_{L^2}^2 + \frac{C_{24}}{4\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2.$$

Integrating k_9,we have

$$k_9 = -\frac{1}{\rho\beta d^3}\iint\limits_{\Omega}\frac{\partial(\nabla u)}{\partial y}\left(\frac{2\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 \nabla u}{\partial y^2} + \left(\frac{\partial^2 u}{\partial y^2}\right)^2\frac{\partial(\nabla u)}{\partial y}\right)dxdy$$

$$= -\frac{2}{\rho\beta d^3}\iint\limits_{\Omega}\frac{\partial(\nabla u)}{\partial y}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 \nabla u}{\partial y^2}dxdy - \frac{1}{\rho\beta d^3}\iint\limits_{\Omega}\left(\frac{\partial(\nabla u)}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial y^2}\right)^2 dxdy$$

Integrating again, we obtain

$$k_9 = \frac{1}{\rho\beta d^3}\iint\limits_{\Omega}\left(\frac{\partial(\nabla u)}{\partial y}\right)^2\left(\left(\frac{\partial^2 u}{\partial y^2}\right)^2+\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y^3}\right)dxdy-\frac{1}{\rho\beta d^3}\iint\limits_{\Omega}\left(\frac{\partial(\nabla u)}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial y^2}\right)^2 dxdy$$

$$\leq \frac{1}{\rho\beta d^3}\iint\limits_{\Omega}\left(\frac{\partial(\nabla u)}{\partial y}\right)^2\left|\frac{\partial u}{\partial y}\right|\left|\frac{\partial\Delta u}{\partial y}\right|dxdy.$$

Using Holder Inequality, we obtain

$$k_9 \leq \frac{1}{\rho\beta d^3}\left\|\frac{\partial(\nabla u)}{\partial y}\right\|_{L^4}^2\left\|\frac{\partial u}{\partial y}\right\|_{L^4}\left\|\frac{\partial\Delta u}{\partial y}\right\|_{L^4}$$

$$\leq \frac{1}{\rho\beta d^3}\left\|\frac{\partial(\nabla u)}{\partial y}\right\|_{L^4}^4+\frac{1}{\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^2\left\|\frac{\partial\Delta u}{\partial y}\right\|_{L^4}^2$$

$$\leq \frac{C_1}{\rho\beta d^3}\left\|\frac{\partial(\nabla u)}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial(\nabla u)}{\partial y}\right\|_{BMO}^2+\frac{1}{\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^4+\frac{1}{\rho\beta d^3}\left\|\frac{\partial\Delta u}{\partial y}\right\|_{L^4}^4$$

$$\leq \frac{C_1}{\rho\beta d^3}\left\|\frac{\partial(\nabla u)}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial(\nabla u)}{\partial y}\right\|_{BMO}^2+\frac{C_1}{\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial u}{\partial y}\right\|_{BMO}^2+\frac{C_1}{\rho\beta d^3}\left\|\frac{\partial\Delta u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial\Delta u}{\partial y}\right\|_{BMO}^2,$$

where we used Young inequality and Lemma 3.1. Since $\left(\frac{\partial(\nabla u)}{\partial y},\frac{\partial u}{\partial y},\frac{\partial\Delta u}{\partial y}\right)\in L^2(O,T,BMO)$, we can write

$$k_9 \leq \frac{C_{25}}{\rho\beta d^3}\left\|\frac{\partial(\nabla u)}{\partial y}\right\|_{L^2}^2+\frac{C_{26}}{\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2+\frac{C_{27}}{\rho\beta d^3}\left\|\frac{\partial\Delta u}{\partial y}\right\|_{L^2}^2.$$

For k_{10}, we have

$$k_{10} = \frac{1}{2\rho\beta d^3}\iint\limits_{\Omega}\frac{\partial(\Delta u)}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^3}dxdy$$

$$\leq \frac{1}{2\rho\beta d^3}\iint\limits_{\Omega}\left|\frac{\partial(\Delta u)}{\partial y}\right|^2\left|\frac{\partial u}{\partial y}\right|^2 dxdy.$$

Applying Young inequality, we get

$$k_{10} \leq \frac{1}{4\rho\beta d^3}\iint\limits_{\Omega}\left(\frac{\partial(\Delta u)}{\partial y}\right)^4 dxdy+\frac{1}{4\rho\beta d^3}\iint\limits_{\Omega}\left(\frac{\partial u}{\partial y}\right)^4 dxdy$$

$$= \frac{1}{4\rho\beta d^3}\left\|\frac{\partial(\Delta u)}{\partial y}\right\|_{L^4}^4+\frac{1}{4\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^4$$

$$\leq \frac{C_1}{4\rho\beta d^3}\left\|\frac{\partial(\Delta u)}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial(\Delta u)}{\partial y}\right\|_{BMO}^2+\frac{C_1}{4\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial u}{\partial y}\right\|_{BMO}^2,$$

where we used Lemma 3.1, since $\left(\frac{\partial u}{\partial y},\frac{\partial(\Delta u)}{\partial y}\right)\in L^2(O,T,BMO)$:

$$k_{10} \leq \frac{C_{28}}{4\rho\beta d^3}\left\|\frac{\partial(\Delta u)}{\partial y}\right\|_{L^2}^2+\frac{C_{29}}{4\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2.$$

Introducing the values of k_7, k_8, k_9,and k_{10} in Eq (6.4) , we get

$$I_{11} \leq \frac{1}{2\rho\beta d^3}\left\|\frac{\partial(\Delta u)}{\partial y}\frac{\partial u}{\partial y}\right\|_{L^2}^2 + \frac{C_{22}}{2\rho\beta d^3}\|\Delta u\|_{L^2}^2$$

$$+\left(\frac{C_{23}}{4\rho\beta d^3} + 2\frac{C_{27}}{\rho\beta d^3} + \frac{C_{28}}{4\rho\beta d^3}\right)\left\|\frac{\partial(\Delta u)}{\partial y}\right\|_{L^2}^2 + \frac{C_{25}}{\rho\beta d^3}\left\|\frac{\partial(\nabla u)}{\partial y}\right\|_{L^2}^2$$

$$+\left(\frac{2C_{24}}{4\rho\beta d^3} + \frac{C_{26}}{\rho\beta d^3} + \frac{C_{29}}{\rho\beta d^3}\right)\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2$$

For I_{12}, we have

$$I_{12} = \frac{1}{6\rho\beta d^3}\iint_{\Omega}\Delta u.\Delta\left(u(\frac{\partial u}{\partial y})^2\right)dxdy$$

$$= \frac{1}{6\rho\beta d^3}\iint_{\Omega}\left(\frac{\partial u}{\partial y}\right)^2\Delta u.\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)dxdy + \frac{2}{3\rho\beta d^3}\iint_{\Omega}\Delta u\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}dxdy$$

$$+\frac{1}{3\rho\beta d^3}\iint_{\Omega}u\Delta u\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 dxdy + \frac{1}{3\rho\beta d^3}\iint_{\Omega}u\Delta u\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial x^2\partial y}dxdy$$

$$+\frac{2}{3\rho\beta d^3}\iint_{\Omega}\Delta u\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}dxdy$$

$$+\frac{1}{3\rho\beta d^3}\iint_{\Omega}u\Delta u\left(\frac{\partial^2 u}{\partial y^2}\right)^2 dxdy + \frac{1}{3\rho\beta d^3}\iint_{\Omega}u\Delta u\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y^3}dxdy$$

$$= k_1' + k_2' + k_3' + k_4' + k_5' + k_6' + k_7'. \tag{6.5}$$

For k_1', applying Young inequality, we get

$$\begin{aligned} k_1' &\leq \frac{1}{3\rho\beta d^3}\iint_{\Omega}\left(\frac{\partial u}{\partial y}\right)^4 dx + \frac{1}{3\rho\beta d^3}\iint_{\Omega}(\Delta u)^4 dxdy \\ &= \frac{1}{3\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^4 + \frac{1}{3\rho\beta d^3}\|\Delta u\|_{L^4}^4 \\ &= \frac{C_1}{3\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial u}{\partial y}\right\|_{BMO}^2 + \frac{C_1}{3\rho\beta d^3}\|\Delta u\|_{L^2}^2\|\Delta u\|_{BMO}^2 \end{aligned}$$

where we used Lemma 3.1. Since $(\Delta u, \nabla u) \in L^2(O, T, BMO)$:

$$k_1' \leq \frac{C_{30}}{3\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2 + \frac{C_{31}}{3\rho\beta d^3}\|\Delta u\|_{L^2}^2.$$

Integration k_2', we have

$$k_2' = -\frac{1}{3\rho\beta d^3}\int\int\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\Delta u + \frac{\partial u}{\partial x}\frac{\partial \Delta u}{\partial x}\right)dxdy$$
$$= -\frac{1}{3\rho\beta d^3}\iint_{\Omega}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial x^2}\Delta u dxdy - \frac{1}{3\rho\beta d^3}\iint_{\Omega}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial u}{\partial x}\frac{\partial \Delta u}{\partial x}dxdy.$$

Integrating again , we have

$$k_2' = -\frac{1}{3\rho\beta d^3}\iint_{\Omega}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial x^2}\Delta u dxdy + \frac{2}{3\rho\beta d^3}\iint_{\Omega}\frac{\partial \nabla u}{\partial x}\frac{\partial \nabla u}{\partial y}\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}dxdy$$
$$+\frac{1}{3\rho\beta d^3}\iint_{\Omega}\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial \nabla u}{\partial x}\right)^2 dxdy$$
$$\leq \frac{1}{3\rho\beta d^3}\iint_{\Omega}\left(\frac{\partial u}{\partial y}\right)^2(\Delta u)^2 dxdy + \frac{2}{3\rho\beta d^3}\iint_{\Omega}(\Delta u)^2(\nabla u)^2 dxdy + \frac{1}{3\rho\beta d^3}\iint_{\Omega}\left(\frac{\partial u}{\partial y}\right)^2(\Delta u)^2 dxdy.$$

$$= \frac{2}{3\rho\beta d^3}\iint_{\Omega}\left(\frac{\partial u}{\partial y}\right)^2(\Delta u)^2 dxdy + \frac{2}{3\rho\beta d^3}\iint_{\Omega}(\Delta u)^2(\nabla u)^2 dxdy$$
$$\leq \frac{4}{3\rho\beta d^3}\iint_{\Omega}(\Delta u)^2(\nabla u)^2 dxdy$$

Applying Young inequality, we get

$$k_2' \leq \frac{2}{3\rho\beta d^3}\iint_{\Omega}(\Delta u)^4 dxdy + \frac{2}{3\rho\beta d^3}\iint_{\Omega}(\nabla u)^4 dxdy$$
$$= \frac{2}{3\rho\beta d^3}\|\Delta u\|_{L^4}^4 + \frac{2}{3\rho\beta d^3}\|\nabla u\|_{L^4}^4$$
$$\leq \frac{C_1}{3\rho\beta d^3}\|\Delta u\|_{L^2}^2\|\Delta u\|_{LBMO}^2 + \frac{C_1}{3\rho\beta d^3}\|\nabla u\|_{L^2}^2\|\nabla u\|_{LBMO}^2,$$

where we used Lemma 3.1. Since $(\Delta u, \nabla u) \in L^2(O, T, BMO)$:

$$k_2' \leq \frac{C_{32}}{3\rho\beta d^3}\|\Delta u\|_{L^2}^2 + \frac{C_{33}}{3\rho\beta d^3}\|\nabla u\|_{L^2}^2.$$

For k_3', applying integration by parts

$$k_3' = -\frac{1}{3\rho\beta d^3}\iint_\Omega \nabla u\left(\nabla u\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 + 2u\frac{\partial^2 u}{\partial x\partial y}\frac{\partial^2 \nabla u}{\partial x\partial y}\right)dxdy$$

$$= -\frac{1}{3\rho\beta d^3}\iint_\Omega (\nabla u)^2\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 dxdy + \frac{2}{3\rho\beta d^3}\iint_\Omega u\frac{\partial^2 u}{\partial x\partial y}\frac{\partial^2 \nabla u}{\partial x\partial y}\nabla u dxdy$$

$$\leq \frac{1}{3\rho\beta d^3}\iint_\Omega (\nabla u)^2\left(\frac{\partial \nabla u}{\partial y}\right)^2 dxdy + \frac{2}{3\rho\beta d^3}\iint_\Omega u|\frac{\partial \nabla u}{\partial y}|\frac{\partial \Delta u}{\partial y}\Delta u dxdy$$

$$\leq \frac{1}{3\rho\beta d^3}\|\nabla u\|_{L^4}^2\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^4}^2 + \frac{2}{3\rho\beta d^3}\|u\|_{L^4}\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^4}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^4}\|\nabla u\|_{L^4}.$$

Applying Young inequality

$$k_3' \leq \frac{1}{6\rho\beta d^3}\|\nabla u\|_{L^4}^4 + \frac{1}{6\rho\beta d^3}\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^4}^4 + \frac{1}{3\rho\beta d^3}\|u\|_{L^4}^2\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^4}^2 + \frac{1}{3\rho\beta d^3}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^4}^2\|\nabla u\|_{L^4}^2$$

$$\leq \frac{1}{3\rho\beta d^3}\|\nabla u\|_{L^4}^4 + \frac{1}{6\rho\beta d^3}\|u\|_{L^4}^4 + \frac{1}{3\rho\beta d^3}\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^4}^4 + \frac{1}{6\rho\beta d^3}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^4}^4$$

$$\leq \frac{C_1}{3\rho\beta d^3}\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial \nabla u}{\partial y}\right\|_{BMO}^2 + \frac{C_1}{6\rho\beta d^3}\|u\|_{L^2}^2\|u\|_{BMO}^2$$

$$+\frac{C_1}{6\rho\beta d^3}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial \Delta u}{\partial y}\right\|_{BMO}^2 + \frac{C_1}{3\rho\beta d^3}\|\nabla u\|_{L^2}^2\|\nabla u\|_{BMO}^2,$$

where we used Lemma 3.1, since $\left(u, \Delta u, \frac{\partial \Delta u}{\partial y}, \frac{\partial \Delta u}{\partial y}\right) \in L^2(O, T, BMO)$:

$$k_3' \leq \frac{C_{34}}{6\rho\beta d^3}\|u\|_{L^2}^2 + \frac{C_{35}}{3\rho\beta d^3}\left\|\frac{\partial \nabla u}{\partial y}\right\|_{L^2}^2 + \frac{C_{36}}{6\rho\beta d^3}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^2 + \frac{C_{37}}{3\rho\beta d^3}\|\nabla u\|_{L^2}^2.$$

For k_4', we have

$$k_4' \leq \frac{1}{3\rho\beta d^3}\iint_\Omega u|\nabla u|\left|\frac{\partial \Delta u}{\partial y}\right|\Delta u dxdy.$$

Applying Holder Inequality, we get

$$k_4' \leq \frac{1}{3\rho\beta d^3}\|u\|_{L^4}\|\nabla u\|_{L^4}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^4}\|\Delta u\|_{L^4}.$$

Applying Young inequality, we obtain

$$k_4' \leq \frac{1}{6\rho\beta d^3}\|u\|_{L^4}^2\|\nabla u\|_{L^4}^2 + \frac{1}{6\rho\beta d^3}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^4}^2\|\Delta u\|_{L^4}^2$$

$$\leq \frac{1}{12\rho\beta d^3}\|u\|_{L^4}^4 + \frac{1}{12\rho\beta d^3}\|\nabla u\|_{L^4}^4 + \frac{1}{12\rho\beta d^3}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^4}^4 + \frac{1}{12\rho\beta d^3}\|\Delta u\|_{L^4}^4$$

$$\leq \frac{C_1}{12\rho\beta d^3}\|u\|_{L^2}^2\|u\|_{BMO}^2+\frac{C_1}{12\rho\beta d^3}\|\nabla u\|_{L^2}^2\|\nabla u\|_{BMO}^2$$

$$+\frac{C_1}{12\rho\beta d^3}\left\|\frac{\partial\Delta u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial\Delta u}{\partial y}\right\|_{BMO}^2+\frac{C_1}{12\rho\beta d^3}\|\Delta u\|_{L^2}^2\|\Delta u\|_{BMO}^2,$$

where we used Lemma 3.1. Since $\left(u,\nabla u,\Delta u,\frac{\partial\Delta u}{\partial y}\right)\in L^2(O,T,BMO)$:

$$k_4'\leq \frac{C_{38}}{6\rho\beta d^3}\|u\|_{L^2}^2+\frac{C_{39}}{6\rho\beta d^3}\|\nabla u\|_2^2+\frac{C_{40}}{6\rho\beta d^3}\left\|\frac{\partial\Delta u}{\partial y}\right\|_{L^2}^2+\frac{C_{41}}{6\rho\beta d^3}\|\Delta u\|_{L^2}^2.$$

Integrating k_5', we obtain

$$k_5'=-\frac{2}{9\rho\beta d^3}\iint\limits_{\Omega}\left(\frac{\partial u}{\partial y}\right)^3\frac{\partial\Delta u}{\partial y}dxdy$$

$$=-\frac{2}{9\rho\beta d^3}\iint\limits_{\Omega}\left(\left(\frac{\partial u}{\partial y}\right)^{\frac{3}{2}}\left(\frac{\partial\Delta u}{\partial y}\right)^{\frac{1}{2}}\right)^2dxdy$$

$$=-\frac{2}{9\rho\beta d^3}\left\|\frac{\partial u}{\partial y}^{\frac{3}{2}}\frac{\partial\Delta u}{\partial y}^{\frac{1}{2}}\right\|_{L^2}^2.$$

By integrating k_6', we have

$$k_6'=-\frac{1}{3\rho\beta d^3}\iint\limits_{\Omega}\nabla u\left(\nabla u\left(\frac{\partial^2 u}{\partial y^2}\right)^2+2u\frac{\partial^2 u}{\partial y^2}\frac{\partial^2\nabla u}{\partial y^2}\right)dxdy$$

$$=-\frac{1}{3\rho\beta d^3}\iint\limits_{\Omega}(\nabla u)^2\left(\frac{\partial^2 u}{\partial y^2}\right)^2dxdy-\frac{2}{3\rho\beta d^3}\iint\limits_{\Omega}u\nabla u\frac{\partial^2 u}{\partial y^2}\frac{\partial^2\nabla u}{\partial y^2}$$

$$\leq -\frac{1}{3\rho\beta d^3}\left\|\nabla u\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2+\frac{2}{3\rho\beta d^3}\|u\|_{L^4}\|\nabla u\|_{L^4}\left\|\frac{\partial\Delta u}{\partial y}\right\|_{L^4}\|\Delta u\|_{L^4}$$

$$\leq -\frac{1}{3\rho\beta d^3}\left\|\nabla u\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2+\frac{1}{3\rho\beta d^3}\|u\|_{L^4}^2\|\nabla u\|_{L^4}^2+\frac{1}{3\rho\beta d^3}\left\|\frac{\partial\Delta u}{\partial y}\right\|_{L^4}^2\|\Delta u\|_{L^4}^2$$

$$\leq -\frac{1}{3\rho\beta d^3}\left\|\nabla u\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2+\frac{1}{6\rho\beta d^3}\|u\|_{L^4}^4+\frac{1}{6\rho\beta d^3}\|\nabla u\|_{L^4}^4+\frac{1}{6\rho\beta d^3}\left\|\frac{\partial\Delta u}{\partial y}\right\|_{L^4}^4+\frac{1}{6\rho\beta d^3}\|\Delta u\|_{L^4}^4$$

$$\leq -\frac{1}{3\rho\beta d^3}\left\|\nabla u\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2\frac{C_1}{6\rho\beta d^3}\|u\|_{L^2}^2\|u\|_{BMO}^2+\frac{C_1}{6\rho\beta d^3}\|\nabla u\|_{L^2}^2\|\nabla u\|_{BMO}^2$$

$$+\frac{C_1}{6\rho\beta d^3}\left\|\frac{\partial\Delta u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial\Delta u}{\partial y}\right\|_{BMO}^2+\frac{C_1}{6\rho\beta d^3}\|\Delta u\|_{L^2}^2\|\Delta u\|_{BMO}^2,$$

where we used Lemma 3.1. Since $\left(u,\nabla u,\Delta u,\frac{\partial\Delta u}{\partial y}\right)\in L^2(O,T,BMO)$:

$$k_6'\leq -\frac{1}{3\rho\beta d^3}\left\|\nabla u\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2+\frac{C_{42}}{6\rho\beta d^3}\|u\|_{L^2}^2+\frac{C_{43}}{6\rho\beta d^3}\|\nabla u\|_{L^2}^2+\frac{C_{44}}{6\rho\beta d^3}\left\|\frac{\partial\Delta u}{\partial y}\right\|_{L^2}^2+\frac{C_{45}}{6\rho\beta d^3}\|\Delta u\|_{L^2}^2$$

For k_7', we have

$$k_7' \leq \frac{1}{3\rho\beta d^3}\iint\limits_{\Omega} u\Delta u \frac{\partial u}{\partial y}\frac{\partial \Delta u}{\partial y} dxdy.$$

By Holder Inequality, we get

$$k_7' \leq \frac{1}{3\rho\beta d^3}\|u\|_{L^4}\|\Delta u\|_{L^4}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^4}.$$

Applying Young inequality, we obtain

$$k_7' \leq \frac{1}{6\rho\beta d^3}\|u\|_{L^4}^2\|\Delta u\|_{L^4}^2 + \frac{1}{6\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^2\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^2$$

$$\leq \frac{1}{12\rho\beta d^3}\|u\|_{L^4}^4 + \frac{1}{12\rho\beta d^3}\|\Delta u\|_{L^4}^4 + \frac{1}{12\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^4}^4 + \frac{1}{12\rho\beta d^3}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^4$$

$$\leq \frac{C_1}{12\rho\beta d^3}\|u\|_{L^2}^2\|u\|_{BMO}^2 + \frac{C_1}{12\rho\beta d^3}\|\Delta u\|_{L^2}^2\|\Delta u\|_{BMO}^2$$

$$+\frac{C_1}{12\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial u}{\partial y}\right\|_{BMO}^2 + \frac{C_1}{12\rho\beta d^3}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^2\left\|\frac{\partial \Delta u}{\partial y}\right\|_{BMO}^2,$$

where we used (1.1). since $\left(u,,\frac{\partial u}{\partial y},\Delta u,\frac{\partial \Delta u}{\partial y}\right) \in L^2(O,T,BMO)$:

$$K_7' \leq \frac{C_{46}}{12\rho\beta d^3}\|u\|_{L^2}^2 + \frac{C_{47}}{12\rho\beta d^3}\|\Delta u\|_{L^2}^2 + \frac{C_{48}}{12\rho\beta d^3}\left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2 + \frac{C_{49}}{12\rho\beta d^3}\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^2$$

Introducing the values of $k_1'k_2', k_3', k_4', k_5', k_6'$, and k_7' and from Propositions 1–3, the Eq (5.5) becomes

$$I_{12} \leq \left(\frac{C_{31}}{3\rho\beta d^3} + \frac{C_{32}}{3\rho\beta d^3} + \frac{C_{41}}{6\rho\beta d^3} + \frac{C_{45}}{6\rho\beta d^3} + \frac{C_{47}}{12\rho\beta d^3}\right)\|\Delta u\|_{L^2}^2$$

$$+\left(\frac{C_{36}}{6\rho\beta d^3} + \frac{C_{40}}{6\rho\beta d^3} + \frac{C_{44}}{6\rho\beta d^3} + \frac{C_{49}}{12\rho\beta d^3}\right)\left\|\frac{\partial \Delta u}{\partial y}\right\|_{L^2}^2$$

$$-\frac{2}{9\rho\beta d^3}\left\|\frac{\partial u}{\partial y}^{\frac{3}{2}}\frac{\partial \Delta u}{\partial y}^{\frac{1}{2}}\right\|_{L^2}^2 - \frac{1}{3\rho\beta d^3}\left\|\nabla u\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2.$$

Utilizing the values of I_{10}, I_{11}, and I_{12} in Eq (6.1) , and after using the Propositions 1–3

$$\frac{d}{dt}\|(\Delta u)\|_{L^2}^2 + 2\left(v + \frac{1}{\rho\beta d} - 6\epsilon - \frac{C_{36}}{6\rho\beta d^3} - \frac{C_{40}}{6\rho\beta d^3} - \frac{C_{44}}{6\rho\beta d^3} - \frac{C_{49}}{12\rho\beta d^3}\right)\left\|\frac{\partial(\Delta u)}{\partial y}\right\|_{L^2}^2$$

$$+\frac{4}{9\rho\beta d^3}\left\|\frac{\partial u}{\partial y}^{\frac{3}{2}}\frac{\partial \Delta u}{\partial y}^{\frac{1}{2}}\right\|_{L^2}^2 + \frac{2}{3\rho\beta d^3}\left\|\nabla u\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2 \leq$$

$$2C\left|C_{21} + \frac{C_{31}}{12\rho\beta d^3} + \frac{C_{32}}{3\rho\beta d^3} + \frac{C_{41}}{6\rho\beta d^3} + \frac{C_{45}}{6\rho\beta d^3} + \frac{C_{47}}{12\rho\beta d^3} - \frac{\phi}{k}\left(v + \frac{1}{\rho\beta d}\right)\right|\|\Delta u\|_{L^2}^2. \quad (6.6)$$

Taking

$$C_{50} = 2\left(v + \frac{1}{\rho\beta d} - 6\epsilon - \frac{C_{36}}{6\rho\beta d^3} - \frac{C_{40}}{6\rho\beta d^3} - \frac{C_{44}}{6\rho\beta d^3} - \frac{C_{49}}{12\rho\beta d^3}\right) > 0$$

$$C_{51} = \frac{4}{9\rho\beta d^3}, \qquad C_{52} = \frac{2}{3\rho\beta d^3}$$

$$C_{53} = 2C\left|C_{21} + \frac{C_{31}}{12\rho\beta d^3} + \frac{C_{32}}{3\rho\beta d^3} + \frac{C_{35}}{6\rho\beta d^3} + \frac{C_{41}}{6\rho\beta d^3} + \frac{C_{45}}{6\rho\beta d^3} + \frac{C_{47}}{12\rho\beta d^3} - \frac{\phi}{k}\left(v + \frac{1}{\rho\beta d}\right)\right|.$$

Then the Eq (6.6) becomes

$$\frac{d}{dt}\left\|(\Delta u)\right\|_{L^2}^2 + C_{50}\left\|\frac{\partial(\Delta u)}{\partial y}\right\|_{L^2}^2 + C_{51}\left\|\frac{\partial u}{\partial y}^{\frac{3}{2}}\frac{\partial \Delta u}{\partial y}^{\frac{1}{2}}\right\|_{L^2}^2$$

$$+C_{52}\left\|\nabla u\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2 \leq C_{53}\left\|\Delta u\right\|_{L^2}^2.$$

The Gronwall inequality yields to

$$\left\|(\Delta u)\right\|_{L^2}^2 + \eta\int_0^T\left\|\frac{\partial(\Delta u)}{\partial y}\right\|_{L^2}^2 dt + \kappa\int_0^T\left\|\frac{\partial u}{\partial y}^{\frac{3}{2}}\frac{\partial \Delta u}{\partial y}^{\frac{1}{2}}\right\|_{L^2}^2 dt$$

$$+\zeta\int_0^T\left\|\nabla u\frac{\partial^2 u}{\partial y^2}\right\|_{L^2}^2 dt \leq \psi\left\|\Delta u_0\right\|_{L^2}^2,$$

where η depends on C_{50}, κ depends on C_{51}, ζ depends on C_{52} and ψ depends on C_{53} and T.

Eventually, the Theorem 2.3 is proved by using Propositions 3 and 4, as both propositions provide evidenced on the global bound of spatial gradients and laplacian derivatives for solutions in $\Omega \times (0, T]$.

7. Conclusions

The proofs of the proposed Theorems have been provided together with the different supporting propositions required. The Theorems have permitted to show the regularity criteria for different functional spaces conditions. The assessed bounds and regularity results have been applied to an Eyring-Powell fluid in $\Omega = [0, L] \times [y_m, \infty)$, where $y_m > 0$ is the minimum value of y for which the kinematic Dirichlet conditions of the flow holds, and for $T \in (0, \infty)$, where T refers to the maximal time for existence of solutions.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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