

YAMABE SYSTEMS AND OPTIMAL PARTITIONS ON MANIFOLDS WITH SYMMETRIES

MÓNICA CLAPP*

Instituto de Matemáticas
Universidad Nacional Autónoma de México, Circuito Exterior, Ciudad Universitaria
04510 Coyoacán, Ciudad de México, Mexico

ANGELA PISTOIA

Dipartimento SBAI
Sapienza Università di Roma
Via Antonio Scarpa 16, 00161 Roma, Italy

Dedicated to Norman Dancer on the occasion of his 75th birthday

ABSTRACT. We prove the existence of regular optimal G -invariant partitions, with an arbitrary number $\ell \geq 2$ of components, for the Yamabe equation on a closed Riemannian manifold (M, g) when G is a compact group of isometries of M with infinite orbits. To this aim, we study a weakly coupled competitive elliptic system of ℓ equations, related to the Yamabe equation. We show that this system has a least energy G -invariant solution with nontrivial components and we show that the limit profiles of its components separate spatially as the competition parameter goes to $-\infty$, giving rise to an optimal partition. For $\ell = 2$ the optimal partition obtained yields a least energy sign-changing G -invariant solution to the Yamabe equation with precisely two nodal domains.

1. Introduction. This paper is concerned with the existence and asymptotic behavior of solutions to the weakly coupled competitive Yamabe system

$$\begin{cases} \mathcal{L}_g u_i := -\Delta_g u_i + \kappa_m S_g u_i = |u_i|^{2^*-2} u_i + \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \lambda_{ij} |u_j|^{\frac{2^*}{2}} |u_i|^{\frac{2^*}{2}} u_i & \text{on } M, \\ u_i \neq 0, & i = 1, \dots, \ell, \end{cases} \quad (1)$$

where (M, g) is a closed Riemannian manifold of dimension $m \geq 3$, S_g is its scalar curvature, $\Delta_g := \operatorname{div}_g \nabla_g$ is the Laplace-Beltrami operator, $\kappa_m := \frac{m-2}{4(m-1)}$, $2^* := \frac{2m}{m-2}$ is the critical Sobolev exponent, and $\lambda_{ij} = \lambda_{ji} < 0$. We assume that the quadratic form induced by the conformal Laplacian \mathcal{L}_g is coercive.

This system was recently studied by Clapp, Pistoia and Tavares [10]. Here we complement the results obtained in [10] by considering manifolds with symmetries.

2020 *Mathematics Subject Classification.* Primary: 35B38, 35J20, 35J47, 35J60; Secondary: 35R35, 49K20, 49Q10, 58J05.

Key words and phrases. Competitive elliptic system, Riemannian manifold, critical nonlinearity, optimal partition, free boundary problem, regularity, Yamabe equation, sign-changing solution.

M. Clapp was partially supported by CONACYT (Mexico) through the grant for project A1-S-10457. A. Pistoia was partially supported by Fondi di Ateneo Sapienza Università di Roma (Italy).

* Corresponding author: Mónica Clapp.

Let G be a compact group of isometries of M and let $Gp := \{\gamma p : \gamma \in G\}$ denote the G -orbit of a point $p \in M$. Recall that a subset X of M is said to be G -invariant if $Gp \subset X$ for every $p \in \Omega$ and a function $u : X \rightarrow \mathbb{R}$ is called G -invariant if it is constant on every G -orbit of Ω . We shall say that a solution (u_1, \dots, u_ℓ) to the system (1) is G -invariant if every component u_i is G -invariant.

We prove the following result.

Theorem 1.1. *If $1 \leq \dim(Gp) < m$ for every $p \in M$, then the system (1) has a least energy G -invariant solution and infinitely many G -invariant solutions.*

To describe the limit profile of least energy G -invariant solutions to the system (1) as $\lambda_{ij} \rightarrow -\infty$, we consider the Dirichlet problem

$$\begin{cases} \mathcal{L}_g u = |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \text{ is } G\text{-invariant,} \end{cases} \quad (2)$$

in a G -invariant open subset Ω of M , and set

$$c_\Omega^G := \inf \left\{ \frac{1}{m} \int_M |u|^{2^*} d\mu_g : u \neq 0, u \text{ solves (2)} \right\}. \quad (3)$$

Let $\mathcal{P}_\ell^G := \{\{\Omega_1, \dots, \Omega_\ell\} : \Omega_i \neq \emptyset \text{ is a } G\text{-invariant open subset of } M \text{ and } \Omega_i \cap \Omega_j = \emptyset \text{ if } i \neq j\}$. We shall say that $\{\Omega_1, \dots, \Omega_\ell\} \in \mathcal{P}_\ell^G$ is an *optimal (G, ℓ) -partition* for the Yamabe equation

$$\mathcal{L}_g u = |u|^{2^*-2}u \quad \text{on } M, \quad (4)$$

if

$$\sum_{i=1}^{\ell} c_{\Omega_i}^G = \inf_{\{\Theta_1, \dots, \Theta_\ell\} \in \mathcal{P}_\ell^G} \sum_{i=1}^{\ell} c_{\Theta_i}^G.$$

The relation between variational elliptic systems having large competitive interaction and optimal partition problems was observed by Conti, Terracini and Verzini in [13, 14] and Chang, Lin, Lin and Lin in [5], and has been extensively studied ever since. An ample list of references is given in [10]. Our next result describes this relation for the system (1).

Theorem 1.2. *Assume $1 \leq \dim(Gp) < m$ for every $p \in M$. Let $\lambda_n < 0$ be such that $\lambda_n \rightarrow -\infty$. For each $n \in \mathbb{N}$, let $(u_{n,1}, \dots, u_{n,\ell})$ be a least energy G -invariant solution to the system (1) with $\lambda_{ij} = \lambda_n$ for all $i \neq j$, such that $u_{n,i} > 0$ for all $n \in \mathbb{N}$ and $i = 1, \dots, \ell$. Then, after passing to a subsequence, we have that*

- (i) $u_{n,i} \rightarrow u_{\infty,i}$ strongly in $H_g^1(M) \cap C^{0,\alpha}(M)$ for every $\alpha \in (0, 1)$, where $u_{\infty,i} \geq 0$, $u_{\infty,i} \neq 0$, and $u_{\infty,i}|_{\Omega_i}$ is a least energy G -invariant solution to the problem (2) in $\Omega_i := \{p \in M : u_{\infty,i}(p) > 0\}$ for each $i = 1, \dots, \ell$. Moreover,

$$\int_M \lambda_n u_{n,i}^{\frac{2^*}{2}} u_{n,j}^{\frac{2^*}{2}} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{whenever } i \neq j.$$

- (ii) $u_{\infty,i}$ is G -invariant and $u_{\infty,i} \in C^{0,1}(M)$ for each $i = 1, \dots, \ell$.
- (iii) $\{\Omega_1, \dots, \Omega_\ell\} \in \mathcal{P}_\ell^G$ and it is an optimal (G, ℓ) -partition for the Yamabe equation on (M, g) . In particular, each Ω_i is connected.
- (iv) $M \setminus \bigcup_{i=1}^{\ell} \Omega_i = \mathcal{R} \cup \mathcal{S}$, where $\mathcal{R} \cap \mathcal{S} = \emptyset$, \mathcal{R} is an $(m-1)$ -dimensional $C^{1,\alpha}$ -submanifold of M and \mathcal{S} is a closed subset of M with Hausdorff measure

$\leq m - 2$. In particular, $M = \cup_{i=1}^{\ell} \overline{\Omega}_i$. Moreover, given $p_0 \in \mathcal{R}$ there exist $i \neq j$ such that

$$\lim_{p \rightarrow p_0^+} |\nabla_g u_i(p)|^2 = \lim_{p \rightarrow p_0^-} |\nabla_g u_j(p)|^2 \neq 0,$$

where $p \rightarrow p_0^{\pm}$ are the limits taken from opposite sides of \mathcal{R} , and for $p_0 \in \mathcal{S}$ we have

$$\lim_{p \rightarrow p_0} |\nabla_g u_i(p)|^2 = 0 \quad \text{for every } i = 1, \dots, \ell.$$

- (v) If $\ell = 2$, then $u_{\infty,1} - u_{\infty,2}$ is a least energy G -invariant sign-changing solution to the Yamabe equation (4).

In the nonsymmetric case, the existence of a solution to the system (1) was recently shown in [10, Theorem 1.2] under the additional assumptions that (M, g) has either dimension 3 and is not conformal to the standard 3-sphere, or (M, g) is not locally conformally flat and $\dim M \geq 9$. A nonsymmetric version of Theorem 1.2 was also established in that work, provided (M, g) is not locally conformally flat, $\dim M \geq 10$, and satisfies a further geometric condition whenever $\dim M = 10$; see [10, Theorem 1.2].

The existence of a least energy sign-changing solution to the Yamabe equation (4) was established by Ammann and Humbert in [1] for manifolds of dimension ≥ 11 that are not locally conformally flat, without any symmetry assumption, whereas the existence of infinitely many G -invariant sign-changing solutions to (4) when $1 \leq \dim(Gp) < m$ for every $p \in M$ was shown by Clapp and Fernández in [7, Theorem 1.1].

Theorems 1.1 and 1.2 apply to many manifolds that are not covered by the results in [10], for instance, to locally conformally flat manifolds like the standard sphere \mathbb{S}^m or the m -dimensional flat torus \mathbb{T}^m . The group $\Gamma := O(n_1) \times O(n_2)$ with $n_1, n_2 \geq 2$ and $n_1 + n_2 = m + 1$, acting on $\mathbb{S}^m \subset \mathbb{R}^{m+1} \cong \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ in the obvious way, satisfies the hypothesis of Theorems 1.1 and 1.2 for any $m \geq 3$. This example will be discussed in detail in Section 4. One may also consider the action of \mathbb{S}^1 on $\mathbb{S}^{2k-1} \subset \mathbb{C}^k$ given by complex multiplication on each complex coordinate. In this case, every orbit has dimension one.

An immediate consequence of Theorems 1.1 and 1.2 is the following result.

Theorem 1.3. *If $1 \leq \dim(Gp) < m$ for every $p \in M$, then for every $\ell \geq 2$ there exists an optimal (G, ℓ) -partition $\{\Omega_1, \dots, \Omega_\ell\}$ for the Yamabe equation on (M, g) with the properties stated in items (iii) and (iv) of Theorem 1.2.*

When $M = \mathbb{S}^m$ and Γ is the group defined above, the optimal partition admits a more accurate description, as shown by Clapp and Pistoia [8] and Clapp, Saldaña and Szulkin [11]. We give an account of this and other known results for the standard sphere in Section 4. Section 2 is devoted to the proof of Theorem 1.1 and Section 3 to that of Theorem 1.2.

2. Existence of multiple solutions. Let $H_g^1(M)$ be the Sobolev space of square integrable functions on M having square integrable first weak derivatives. We write $\langle \cdot, \cdot \rangle$ for the Riemannian metric in (M, g) and denote the induced norm by $|\cdot|$. For $u, v \in H_g^1(M)$ let

$$\langle u, v \rangle_g := \int_M (\langle \nabla_g u, \nabla_g v \rangle + \kappa_m S_g uv) \, d\mu_g \quad \text{and} \quad \|u\|_g := \sqrt{\langle u, u \rangle_g},$$

where ∇_g is the weak gradient. Since we are assuming that the conformal Laplacian \mathcal{L}_g is coercive, $\|\cdot\|_g$ is a norm in $H_g^1(M)$, equivalent to the standard one.

Let G be a closed subgroup of the group of isometries of M and set

$$H_g^1(M)^G := \{u \in H_g^1(M) : u \text{ is } G\text{-invariant}\}.$$

If $\dim(Gp) < m$ for $p \in M$ then, for any given $k \in \mathbb{N}$, we may choose $u_1, \dots, u_k \in H_g^1(M)^G$ having pairwise disjoint supports. Hence, $H_g^1(M)^G$ has infinite dimension. The following result will play a crucial role in the proof of Theorem 1.1.

Lemma 2.1. *If $\dim(Gp) \geq 1$ for every $p \in M$, then the embedding $H_g^1(M)^G \hookrightarrow L_g^{2^*}(M)$ is compact.*

Proof. See [24, Corollary 1]. \square

We assume from now on that $1 \leq \dim(Gp) < m$ for every $p \in M$. Set $\mathcal{H}^G := (H_g^1(M)^G)^\ell$ and let $\mathcal{J} : \mathcal{H}^G \rightarrow \mathbb{R}$ be given by

$$\mathcal{J}(u_1, \dots, u_\ell) := \frac{1}{2} \sum_{i=1}^{\ell} \|u_i\|_g^2 - \frac{1}{2^*} \sum_{i=1}^{\ell} \int_M |u_i|^{2^*} d\mu_g - \frac{1}{2^*} \sum_{\substack{i,j=1 \\ j \neq i}}^{\ell} \int_M \lambda_{ij} |u_j|^{\frac{2^*}{2}} |u_i|^{\frac{2^*}{2}} d\mu_g. \quad (5)$$

This functional is of class \mathcal{C}^1 and its partial derivatives are

$$\partial_i \mathcal{J}(u_1, \dots, u_\ell) v = \langle u_i, v \rangle_g - \int_M |u_i|^{2^*-2} u_i v d\mu_g - \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \int_M \lambda_{ij} |u_j|^{\frac{2^*}{2}} |u_i|^{\frac{2^*}{2}-2} u_i v d\mu_g.$$

So, by the principle of symmetric criticality [26], the critical points of $\mathcal{J} : \mathcal{H}^G \rightarrow \mathbb{R}$ are the G -invariant solutions to the system

$$\mathcal{L}_g u_i = |u_i|^{2^*-2} u_i + \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \lambda_{ij} |u_j|^{\frac{2^*}{2}} |u_i|^{\frac{2^*}{2}-2} u_i \quad \text{on } M, \quad i = 1, \dots, \ell.$$

We are interested in solutions (u_1, \dots, u_ℓ) such that every u_i is nontrivial. They belong to the set

$$\mathcal{N}^G := \{(u_1, \dots, u_\ell) \in \mathcal{H}^G : u_i \neq 0, \partial_i \mathcal{J}(u_1, \dots, u_\ell) u_i = 0, \forall i = 1, \dots, \ell\}. \quad (6)$$

Note that $\mathcal{J}(u_1, \dots, u_\ell) = \frac{1}{m} \sum_{i=1}^{\ell} \|u_i\|_g^2$ if $(u_1, \dots, u_\ell) \in \mathcal{N}^G$.

Lemma 2.2. *There exists $d_0 > 0$ such that $\|u_i\|_g^2 \geq d_0$ for every $(u_1, \dots, u_\ell) \in \mathcal{N}^G$ and $i = 1, \dots, \ell$. Therefore, \mathcal{N}^G is a closed subset of \mathcal{H}^G and $\inf_{\mathcal{N}^G} \mathcal{J} > 0$.*

Proof. As $\lambda_{ij} < 0$, the Sobolev embedding yields a positive constant C such that

$$\|u_i\|_g^2 \leq \int_M |u_i|^{2^*} d\mu_g \leq C \|u_i\|_g^{2^*} \quad \text{for every } (u_1, \dots, u_\ell) \in \mathcal{N}^G, \quad i = 1, \dots, \ell,$$

and our claims follow. \square

A solution $\bar{u} \in \mathcal{N}^G$ to the system (1) satisfying $\mathcal{J}(\bar{u}) = \inf_{\mathcal{N}^G} \mathcal{J}$ is called a *least energy G -invariant solution*.

The variational approach introduced in [12] can be immediately adapted to establish the existence of infinitely many fully nontrivial critical points of \mathcal{J} . We sketch this procedure.

Given $\bar{u} = (u_1, \dots, u_\ell)$ and $\bar{s} = (s_1, \dots, s_\ell) \in (0, \infty)^\ell$, we write $\bar{s}\bar{u} := (s_1 u_1, \dots, s_\ell u_\ell)$. Let $\mathcal{S}^G := \{u \in H_g^1(M)^G : \|u\|_g = 1\}$, $\mathcal{T}^G := (\mathcal{S}^G)^\ell$, and

$$\mathcal{U}^G := \{\bar{u} \in \mathcal{T}^G : \bar{s}\bar{u} \in \mathcal{N}^G \text{ for some } \bar{s} \in (0, \infty)^\ell\}.$$

Lemma 2.3. (i) Let $\bar{u} \in \mathcal{T}^G$. If there exists $\bar{s}_{\bar{u}} \in (0, \infty)^\ell$ such that $\bar{s}_{\bar{u}}\bar{u} \in \mathcal{N}^G$, then $\bar{s}_{\bar{u}}$ is unique and satisfies

$$\mathcal{J}(\bar{s}_{\bar{u}}\bar{u}) = \max_{\bar{s} \in (0, \infty)^\ell} \mathcal{J}(\bar{s}\bar{u}).$$

- (ii) \mathcal{U}^G is a nonempty open subset of the Hilbert manifold \mathcal{T}^G .
 (iii) The map $\mathcal{U}^G \rightarrow \mathcal{N}^G$ given by $\bar{u} \mapsto \bar{s}_{\bar{u}}\bar{u}$ is a homeomorphism.

Proof. The proof is exactly the same as that of [12, Proposition 3.1]. \square

Define $\Psi : \mathcal{U}^G \rightarrow \mathbb{R}$ by

$$\Psi(\bar{u}) := \mathcal{J}(\bar{s}_{\bar{u}}\bar{u}).$$

If Ψ is of class \mathcal{C}^1 the norm of $\Psi'(\bar{u})$ in the cotangent space $T_{\bar{u}}^*(\mathcal{T}^G)$ to \mathcal{T}^G at \bar{u} is defined as

$$\|\Psi'(\bar{u})\|_* := \sup_{\substack{\bar{v} \in T_{\bar{u}}(\mathcal{U}^G) \\ \bar{v} \neq 0}} \frac{|\Psi'(\bar{u})\bar{v}|}{\|\bar{v}\|_g},$$

where $T_{\bar{u}}(\mathcal{U}^G)$ is the tangent space to \mathcal{U}^G at \bar{u} .

A sequence (\bar{u}_n) in \mathcal{U}^G is called a $(PS)_c^G$ -sequence for Ψ if $\Psi(\bar{u}_n) \rightarrow c$ and $\|\Psi'(\bar{u}_n)\|_* \rightarrow 0$, and Ψ is said to satisfy the $(PS)_c^G$ -condition if every such sequence has a convergent subsequence. Similarly, a $(PS)_c^G$ -sequence for \mathcal{J} is a sequence (\bar{u}_n) in \mathcal{H}^G such that $\mathcal{J}(\bar{u}_n) \rightarrow 0$ and $\|\mathcal{J}'(\bar{u}_n)\|_{(\mathcal{H}^G)'} \rightarrow 0$. \mathcal{J} satisfies the $(PS)_c^G$ -condition if any such sequence has a convergent subsequence. As usual, $(\mathcal{H}^G)'$ stands for the dual space of \mathcal{H}^G .

Lemma 2.4. (i) $\Psi \in \mathcal{C}^1(\mathcal{U}^G, \mathbb{R})$,

$$\Psi'(\bar{u})\bar{v} = \mathcal{J}'(\bar{s}_{\bar{u}}\bar{u})[\bar{s}_{\bar{u}}\bar{v}] \quad \text{for all } \bar{u} \in \mathcal{U}^G \text{ and } \bar{v} \in T_{\bar{u}}(\mathcal{U}^G),$$

and there exists $d_0 > 0$ such that

$$d_0 \|\mathcal{J}'(\bar{s}_{\bar{u}}\bar{u})\|_{(\mathcal{H}^G)'} \leq \|\Psi'(\bar{u})\|_* \leq |\bar{s}_{\bar{u}}|_\infty \|\mathcal{J}'(\bar{s}_{\bar{u}}\bar{u})\|_{(\mathcal{H}^G)'} \quad \text{for all } \bar{u} \in \mathcal{U}^G,$$

where $|\bar{s}|_\infty = \max\{|s_1|, \dots, |s_q|\}$ if $\bar{s} = (s_1, \dots, s_q)$.

- (ii) Let $\bar{u}_n \in \mathcal{U}^G$. If (\bar{u}_n) is a $(PS)_c^G$ -sequence for Ψ , then $(\bar{s}_{\bar{u}_n}\bar{u}_n)$ is a $(PS)_c^G$ -sequence for \mathcal{J} .
 (iii) Let $\bar{u} \in \mathcal{U}^G$. Then, \bar{u} is a critical point of Ψ if and only if $\bar{s}_{\bar{u}}\bar{u}$ is a critical point of \mathcal{J} if and only if $\bar{s}_{\bar{u}}\bar{u}$ is a G -invariant solution of (1).
 (iv) If (\bar{u}_n) is a sequence in \mathcal{U}^G and $\bar{u}_n \rightarrow \bar{u} \in \partial(\mathcal{U}^G)$, then $\Psi(\bar{u}_n) \rightarrow \infty$.
 (v) $\bar{u} \in \mathcal{U}^G$ if and only if $-\bar{u} \in \mathcal{U}^G$, and $\Psi(\bar{u}) = \Psi(-\bar{u})$.

Proof. The proof of these statements is exactly the same as that of [12, Theorem 3.3]. \square

Lemma 2.5. Ψ satisfies the $(PS)_c^G$ -condition for every $c \in \mathbb{R}$.

Proof. Let (\bar{v}_n) be a $(PS)_c^G$ -sequence for \mathcal{J} with $\bar{v}_n \in \mathcal{N}^G$. A standard argument shows that this sequence is bounded. Then, using Lemma 2.1 as in [8, Proposition 3.6], one sees that (\bar{v}_n) contains a convergent subsequence. The claim now follows from Lemmas 2.4(ii) and 2.3(iii). \square

Let \mathcal{Z} be a nonempty subset of \mathcal{T}^G such that $\bar{u} \in \mathcal{Z}$ if and only if $-\bar{u} \in \mathcal{Z}$. Recall that the *genus* of \mathcal{Z} , denoted $\text{genus}(\mathcal{Z})$, is the smallest integer $k \geq 1$ such that there exists an odd continuous function $\mathcal{Z} \rightarrow \mathbb{S}^{k-1}$ into the unit sphere \mathbb{S}^{k-1} in \mathbb{R}^k . If no such k exists, we define $\text{genus}(\mathcal{Z}) := \infty$. We set $\text{genus}(\emptyset) := 0$.

Lemma 2.6. $\text{genus}(\mathcal{U}^G) = \infty$.

Proof. This is shown following the argument in [12, Lemma 4.5]. \square

Proof of Theorem 1.1. Lemma 2.4(iv) implies that \mathcal{U}^G is positively invariant under the negative pseudogradient flow of Ψ , so the usual deformation lemma holds true for Ψ in \mathcal{U}^G , see e.g. [27, Section II.3] or [30, Section 5.3]. As Ψ satisfies the $(PS)_c^G$ -condition for every $c \in \mathbb{R}$,

$$\inf_{\mathcal{U}^G} \Psi = \inf_{\mathcal{N}^G} \mathcal{J}$$

is attained, i.e., the system (1) has a least energy G -invariant solution. Moreover, since Ψ is even and $\text{genus}(\mathcal{U}^G) = \infty$, a standard variational argument shows that Ψ has an unbounded sequence of critical values. \square

3. The limit profile of minimizers. We assume throughout that $1 \leq \dim(Gp) < m$ for every $p \in M$. Let Ω be an open G -invariant subset of M , $H_{g,0}^1(\Omega)$ be the closure of $\mathcal{C}_c^\infty(\Omega)$ in $H_g^1(M)$ and $H_{g,0}^1(\Omega)^G$ be the space of G -invariant functions in $H_{g,0}^1(\Omega)$. The solutions of (2) are the critical points of the \mathcal{C}^2 -functional $J_\Omega : H_{g,0}^1(\Omega)^G \rightarrow \mathbb{R}$ given by

$$J_\Omega(u) := \frac{1}{2} \|u\|_g^2 - \frac{1}{2^*} \int_M |u|^{2^*} d\mu_g.$$

The nontrivial ones belong to the Nehari manifold

$$\mathcal{N}_\Omega^G := \{u \in H_{g,0}^1(\Omega)^G : u \neq 0 \text{ and } J'_\Omega(u)u = 0\},$$

which is a natural constraint for J_Ω . So, a minimizer for J_Ω over \mathcal{N}_Ω^G is a nontrivial solution of (2). We call it a *least energy G -invariant solution*. A standard argument using Lemma 2.1 shows that a minimizer does exist. Note that $J_\Omega(u) = \frac{1}{m} \|u\|_g^2 = \frac{1}{m} \int_M |u|^{2^*} d\mu_g$ if $u \in \mathcal{N}_\Omega^G$, so the quantity defined in (3) is

$$c_\Omega^G = \inf_{u \in \mathcal{N}_\Omega^G} J_\Omega(u).$$

Proof of Theorem 1.2. Let $\lambda_n \rightarrow -\infty$. We write \mathcal{J}_n and \mathcal{N}_n^G for the functional and the set defined in (5) and (6) with $\lambda_{ij} = \lambda_n$ for all $i \neq j$. Let

$$\mathcal{M}_\ell^G := \{(v_1, \dots, v_\ell) \in \mathcal{H}^G : v_i \neq 0, \|v_i\|_g^2 = \int_M |v_i|^{2^*} d\mu_g, v_i v_j = 0 \text{ on } M \text{ if } i \neq j\},$$

$$c_\ell^G := \inf_{(v_1, \dots, v_\ell) \in \mathcal{M}_\ell^G} \frac{1}{m} \sum_{i=1}^\ell \|v_i\|_g^2.$$

Since $\dim(Gp) < m$ for $p \in M$, the set \mathcal{M}_ℓ^G is nonempty and, so, $c_\ell^G < \infty$.

Let $\bar{u}_n = (u_{n,1}, \dots, u_{n,\ell}) \in \mathcal{N}_n^G$ be such that $\mathcal{J}_n(\bar{u}_n) = \inf_{\mathcal{N}_n^G} \mathcal{J}_n$ and $u_{n,i} > 0$ for every n and i . Noting that $\mathcal{M}_\ell^G \subset \mathcal{N}_n^G$ for every $n \in \mathbb{N}$ and recalling Lemma 2.2, we see that

$$0 < \inf_{\mathcal{N}_n^G} \mathcal{J}_n = \frac{1}{m} \sum_{i=1}^\ell \|u_{n,i}\|_g^2 \leq c_\ell^G < \infty \quad \text{for every } n \in \mathbb{N}.$$

Applying Lemma 2.1 and passing to a subsequence, we get that $u_{n,i} \rightharpoonup u_{\infty,i}$ weakly in $H_g^1(M)^G$, $u_{n,i} \rightarrow u_{\infty,i}$ strongly in $L_g^{2^*}(M)$ and $u_{n,i} \rightarrow u_{\infty,i}$ a.e. on M , for each $i = 1, \dots, \ell$. Hence, $u_{\infty,i} \geq 0$.

As $\bar{u}_n \in \mathcal{N}_n^G$, we have that

$$0 \leq \int_M |u_{n,j}|^{\frac{2^*}{2}} |u_{n,i}|^{\frac{2^*}{2}} d\mu_g \leq \frac{\int_M |u_{n,i}|^{2^*} d\mu_g}{-\lambda_n} \leq \frac{C}{-\lambda_n} \quad \text{for each pair } j \neq i,$$

and, letting $n \rightarrow \infty$, we obtain

$$\int_M |u_{\infty,j}|^{\frac{2^*}{2}} |u_{\infty,i}|^{\frac{2^*}{2}} d\mu_g = 0.$$

So $u_{\infty,j}u_{\infty,i} = 0$ a.e. on M whenever $i \neq j$. We also have that

$$0 < d_0 \leq \|u_{n,i}\|_g^2 \leq \int_M |u_{n,i}|^{2^*} d\mu_g \quad \text{for every } n \in \mathbb{N} \text{ and } i = 1, \dots, \ell,$$

with d_0 as in Lemma 2.2. Passing to the limit as $n \rightarrow \infty$, we see that $u_{\infty,i} \neq 0$ and

$$\|u_{\infty,i}\|_g^2 \leq \int_M |u_{\infty,i}|^{2^*} d\mu_g \quad \text{for every } i = 1, \dots, \ell.$$

Hence, there exists $t_i \in (0, 1]$ such that $\|t_i u_{\infty,i}\|_g^2 = \int_M |t_i u_{\infty,i}|^{2^*} d\mu_g$. Consequently, $(t_1 u_{\infty,1}, \dots, t_\ell u_{\infty,\ell}) \in \mathcal{M}_\ell^G$ and

$$c_\ell^G \leq \frac{1}{m} \sum_{i=1}^\ell \|t_i u_{\infty,i}\|_g^2 \leq \frac{1}{m} \sum_{i=1}^\ell \|u_{\infty,i}\|_g^2 \leq \frac{1}{m} \liminf_{n \rightarrow \infty} \sum_{i=1}^\ell \|u_{n,i}\|_g^2 \leq c_\ell^G.$$

It follows that $t_i = 1$ and $u_{n,i} \rightarrow u_{\infty,i}$ strongly in $H_g^1(M)$. But then, $\|u_{\infty,i}\|_g^2 = \int_M |u_{\infty,i}|^{2^*} d\mu_g$ and, passing to the limit in

$$\sum_{i=1}^\ell \|u_{n,i}\|_g^2 = \sum_{i=1}^\ell |u_{n,i}|_{g,2^*}^{2^*} + \sum_{\substack{i,j=1 \\ j \neq i}}^\ell \int_M \lambda_n |u_{n,j}|^{\frac{2^*}{2}} |u_{n,i}|^{\frac{2^*}{2}},$$

we obtain

$$\lim_{n \rightarrow \infty} \int_M \lambda_n |u_{\infty,j}|^{\frac{2^*}{2}} |u_{\infty,i}|^{\frac{2^*}{2}} = 0 \quad \text{for every } i \neq j.$$

Moreover, $(u_{\infty,1}, \dots, u_{\infty,\ell}) \in \mathcal{M}_\ell^G$ and

$$\frac{1}{m} \sum_{i=1}^\ell \|u_{\infty,i}\|_g^2 = c_\ell^G.$$

We have now all assumptions needed to apply [10, Lemmas 4.3 and 4.4] and conclude that $(u_{n,i})$ is uniformly bounded in the Hölder norm, i.e., for any $\alpha \in (0, 1)$ there exists $C_\alpha > 0$ such that

$$\|u_{n,i}\|_{C^{0,\alpha}(M)} \leq C_\alpha \quad \text{for every } n \in \mathbb{N} \text{ and } i = 1, \dots, \ell.$$

As a consequence, $u_{n,i} \rightarrow u_{\infty,i}$ in $C^{0,\alpha}(M)$ for every i . In particular, $u_{\infty,i}$ is continuous and G -invariant, so the set $\Omega_i := \{p \in M : u_{\infty,i}(p) > 0\}$ is open and G -invariant. Since $u_{\infty,i}u_{\infty,j} = 0$ if $i \neq j$, we have that $\Omega_i \cap \Omega_j = \emptyset$. This shows that $\{\Omega_1, \dots, \Omega_\ell\} \in \mathcal{P}_\ell^G$.

We claim that $u_{\infty,i}$ is a least energy G -invariant solution to (2) in Ω_i for all i . Otherwise, $J_{\Omega_i}(u_{\infty,i}) > c_{\Omega_i}^G$ for some i and there would exist $v_i \in \mathcal{N}_{\Omega_i}^G$ with

$c_{\Omega_i}^G < J_{\Omega_i}(v_i) < J_{\Omega_i}(u_{\infty,i})$. But then, setting $v_j := u_{\infty,j}$ for $j \neq i$, we would have that $(v_1, \dots, v_\ell) \in \mathcal{M}_\ell^G$ and

$$\frac{1}{m} \sum_{i=1}^{\ell} \|v_i\|_g^2 < \frac{1}{m} \sum_{i=1}^{\ell} \|u_{\infty,i}\|_g^2 = c_\ell^G,$$

contradicting the definition of c_ℓ^G . Hence, $J_{\Omega_i}(u_{\infty,i}) = c_{\Omega_i}^G$ for all $i = 1, \dots, \ell$, as claimed. A similar argument shows that Ω_i is connected.

As a consequence, if $\{\Theta_1, \dots, \Theta_\ell\} \in \mathcal{P}_\ell^G$, taking $w_i \in \mathcal{N}_{\Theta_i}^G$ with $J_{\Theta_i}(w_i) = c_{\Theta_i}^G$, we have that $(w_1, \dots, w_\ell) \in \mathcal{M}_\ell^G$ and, therefore,

$$\sum_{i=1}^{\ell} c_{\Omega_i}^G = \frac{1}{m} \sum_{i=1}^{\ell} \|u_{\infty,i}\|_g^2 = c_\ell^G \leq \frac{1}{m} \sum_{i=1}^{\ell} \|w_i\|_g^2 = \sum_{i=1}^{\ell} c_{\Theta_i}^G.$$

This shows that $\{\Omega_1, \dots, \Omega_\ell\}$ is an optimal (G, ℓ) -partition, and completes the proof of statements (i) and (iii).

Statements (ii) and (iv) are local. In local coordinates the system (1) with λ_{ij} replaced by λ_n becomes

$$-\operatorname{div}(A(x)\nabla v_i) = f_i(x, v_i) + a(x) \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \lambda_n |v_j|^{\frac{2^*}{2}} |v_i|^{\frac{2^*}{2}-2} v_i, \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^m$ is open and bounded, $a(x) := \sqrt{|g(x)|}$, $A(x) := \sqrt{|g(x)|}(g^{kl}(x))$, $f_i(x, s) := a(x)(|s|^{2^*-2}s - \kappa_m S_g(x)s)$ and, as usual, $|g|$ denotes the determinant of the metric $g = (g_{kl})$ in local coordinates and (g^{kl}) its inverse. Applying [10, Theorem C.1] we conclude that (ii) and (iv) are true locally on M , hence also globally.

A standard argument yields the proof of statement (v). Namely, the G -invariant sign-changing solutions to the Yamabe equation (4) belong to the set

$$\mathcal{E}_M^G := \{u \in \mathcal{N}_M^G : u^+ \in \mathcal{N}_M^G \text{ and } u^- \in \mathcal{N}_M^G\},$$

where $u^+ := \max\{u, 0\} \neq 0$ and $u^- := \min\{u, 0\} \neq 0$. Moreover, as shown in [4, Lemma 2.6], any minimizer of J_M on \mathcal{E}_M^G is a G -invariant sign-changing solution of (4). For every $u \in \mathcal{E}_M^G$, we have that $(u^+, u^-) \in \mathcal{M}_2^G$ and $J_M(u) = \frac{1}{m}(\|u^+\|_g^2 + \|u^-\|_g^2)$. Therefore,

$$\inf_{\mathcal{E}_M^G} J_M \geq c_2^G = \frac{1}{m}(\|u_{\infty,1}\|_g^2 + \|u_{\infty,2}\|_g^2).$$

As $u_{\infty,1} - u_{\infty,2} \in \mathcal{E}_M^G$, it is a minimizer of J_M on \mathcal{E}_M^G . This completes the proof. \square

4. The Yamabe system on the standard sphere. In this section we give an account of some known results for the system (1) on the standard sphere.

The stereographic projection $\sigma : \mathbb{S}^m \setminus \{N\} \rightarrow \mathbb{R}^m$ from the north pole N is a conformal diffeomorphism. The conformal invariance of the operator $\mathcal{L}_{\bar{g}}$ (see [23, Proposition 6.1.1]) allows to establish a one-to-one correspondence between solutions to the system (1) on the standard sphere \mathbb{S}^m and solutions to the system

$$\begin{cases} -\Delta v_i = |v_i|^{2^*-2} v_i + \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \lambda_{ij} |v_j|^{\frac{2^*}{2}} |v_i|^{\frac{2^*}{2}-2} v_i \\ v_i \in D^{1,2}(\mathbb{R}^m), \quad v_i \neq 0, \quad i = 1, \dots, \ell, \end{cases} \quad (7)$$

where $D^{1,2}(\mathbb{R}^m) := \{v \in L^{2^*}(\mathbb{R}^m) : \nabla v \in L^2(\mathbb{R}^m, \mathbb{R}^m)\}$; see [8, 11].

Let us first consider the case of the single equation

$$-\Delta u = |u|^{2^*-2}u, \quad u \in D^{1,2}(\mathbb{R}^m). \quad (8)$$

It is well known [2, 25, 29] that all positive solutions to (8) are the so-called *standard bubbles*

$$U_{\delta,y}(x) := c_m \frac{\delta^{\frac{m-2}{2}}}{(\delta^2 + |x-y|^2)^{\frac{m-2}{2}}}, \quad x, y \in \mathbb{R}^m, \delta > 0.$$

The existence of sign-changing solutions was first established by W.Y. Ding in [17]. He considered solutions that are invariant under the action of the group $\Gamma := O(n_1) \times O(n_2)$ with $n_1, n_2 \geq 2$ and $n_1 + n_2 = m + 1$, acting on $\mathbb{S}^m \subset \mathbb{R}^{m+1} \equiv \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ in the obvious way. Using a variational approach he proved the existence of infinitely many Γ -invariant sign-changing solutions to the Yamabe equation (4) on \mathbb{S}^m or, equivalently, to the equation (8) on \mathbb{R}^m .

In [15, 16], del Pino, Musso, Pacard and Pistoia exploited the symmetries of the sphere to build solutions to (8) which have large energy and concentrate along some special submanifolds of \mathbb{S}^m . In particular, for $m \geq 4$ they obtained sequences of solutions to the Yamabe equation whose energy concentrates along one great circle or finitely many great circles that are linked to each other (corresponding to Hopf links embedded in $\mathbb{S}^3 \times \{0\} \subset \mathbb{S}^m$), and for $m \geq 5$ they also obtained sequences of solutions whose energy concentrates along a two-dimensional Clifford torus in $\mathbb{S}^3 \times \{0\} \subset \mathbb{S}^m$. These solutions are built via gluing techniques (e.g., a Ljapunov-Schmidt procedure) and can be described as the superposition of the constant solution to (4) with a large number of copies of negative solutions of (4) which are highly concentrated at points evenly arranged along some special submanifolds of the sphere.

Recently, using ODE techniques Fernández and Petean in [19] established the existence of solutions to the Yamabe problem (4) on \mathbb{S}^m with precisely ℓ nodal domains for each $\ell \geq 2$, whose nodal set consists of isoparametric hypersurfaces.

Regarding the competitive system (where all $\lambda_{ij} < 0$), Guo, Li and Wei [21] established the existence of solutions for (7) with $\ell = 2$ and $\lambda_{12} < 0$ in \mathbb{R}^3 , and using the approach developed in [15] they built a sequence of positive nonradial solutions whose first component looks like the constant function and the second component resembles a large number of copies of positive solutions of (4) concentrated at points that are placed along a circle. The argument of their proof, which relies on the Ljapunov-Schmidt procedure, cannot be extended to higher dimensions because the coupling terms have linear (if $m = 4$) or sublinear (if $m \geq 5$) growth.

Following the variational approach presented in the previous sections, successively Clapp and Pistoia [8], Clapp and Szulkin [12] and Clapp, Saldaña and Szulkin [11] found Γ -invariant solutions to the competitive Yamabe system (1) on \mathbb{S}^m for the groups considered by Ding, and described the limit profile of least energy solutions as $\lambda_{ij} \rightarrow \infty$. In this particular case, Theorems 1.1 and 1.2 were proved in [8, 12] and [8, 11] respectively. In addition, a more accurate description of the optimal (Γ, ℓ) -partition of \mathbb{S}^m is provided in [8, 11]. Note that the Γ -orbit of a point $p \in \mathbb{S}^m$ is diffeomorphic to either \mathbb{S}^{n_1-1} , or $\mathbb{S}^{n_1-1} \times \mathbb{S}^{n_2-1}$, or \mathbb{S}^{n_2-1} , and that the map $q : \mathbb{S}^m \rightarrow [0, \pi]$ given by

$$q(x, y) := \arccos(|x|^2 - |y|^2), \quad \text{where } x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}, \quad (9)$$

is a quotient map identifying each Γ -orbit in \mathbb{S}^m to a single point. So q maps a (Γ, ℓ) -partition of \mathbb{S}^m onto a partition of $[0, \pi]$ by relatively open subintervals. This last partition can be ordered. Taking advantage of this fact, the following result was proved in [8, 11].

Theorem 4.1. *Let $\Gamma := O(n_1) \times O(n_2)$ with $n_1, n_2 \geq 2$ and $n_1 + n_2 = m + 1$ and let $M := \mathbb{S}^m$. Then, the optimal (Γ, ℓ) -partition $\{\Omega_1, \dots, \Omega_\ell\} \in \mathcal{P}_\ell^\Gamma$ of \mathbb{S}^m given by Theorem 1.2 has the following properties: $\Omega_1, \dots, \Omega_\ell$ are smooth and connected, $\overline{\Omega_1 \cup \dots \cup \Omega_\ell} = \mathbb{S}^m$ and, after reordering,*

- $\Omega_1 \cong \mathbb{S}^{n_1-1} \times \mathbb{B}^{n_2}$, $\Omega_i \cong \mathbb{S}^{n_1-1} \times \mathbb{S}^{n_2-1} \times (0, 1)$ if $i = 2, \dots, \ell - 1$, and $\Omega_\ell \cong \mathbb{B}^{n_1} \times \mathbb{S}^{n_2-1}$,
- $\overline{\Omega_i} \cap \overline{\Omega_{i+1}} \cong \mathbb{S}^{n_1-1} \times \mathbb{S}^{n_2-1}$ and $\overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset$ if $|j - i| \geq 2$,
- the function

$$u := \sum_{i=1}^{\ell} (-1)^{i-1} u_{\infty, i}$$

is a Γ -invariant sign-changing solution to the Yamabe problem (4) on \mathbb{S}^m with precisely ℓ nodal domains, and u has least energy among all such solutions.

As shown by Theorems 1.2 and 4.1 the competitive Yamabe system (1) gives rise to one or even multiple solutions to the Yamabe equation (4). The opposite question has been also considered in the literature. Let us call a solution $\mathbf{u} = (u_1, \dots, u_\ell)$ to the Yamabe system (7) *fully synchronized* if there exist $c_i \neq 0$ and a nontrivial solution u to the single equation (8) such that $u_i = c_i u$ for all $i = 1, \dots, \ell$. It is readily seen that $(c_1 u, \dots, c_\ell u)$ solves (7) for any solution u of (8) iff $\mathbf{c} = (c_1, \dots, c_\ell) \in \mathbb{R}^\ell$ solves the algebraic system

$$c_i = |c_i|^{2^*-2} c_i + \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \lambda_{ij} |c_j|^{\frac{2^*}{2}} |c_i|^{\frac{2^*}{2}-2} c_i, \quad c_i > 0, \quad \text{for every } i = 1, \dots, \ell. \quad (10)$$

There are several results concerning the solvability of (10). The easiest case is when $m = 4$ (i.e., $2^* = 4$) and $\ell = 2$. Indeed, a straightforward computation shows that a solution to (10) exists if and only if $\lambda_{12} > -1$ and $\lambda_{12} \neq 1$. Bartsch proved in [3, Proposition 2.1] that, if $2^* = 4$ and $\ell \geq 2$, a fully synchronized solution to (10) exists when $\lambda_{ij} := \lambda \neq 1$ for all $i \neq j$ and $\lambda > \bar{\lambda}$ for some $\bar{\lambda} < 0$, while Chen and Zou [6, Theorem 1.1] showed that if $m \geq 5$ (i.e., $2^* < 4$) and $\ell = 2$ a fully synchronized solution to (10) always exists provided $\lambda_{12} > 0$. Recently, Clapp and Pistoia complemented these results in [9] showing that, if the system is purely cooperative (i.e., $\lambda_{ij} \geq 0$ for all $i, j = 1, \dots, \ell$, $i \neq j$), there exists a solution to (10). We recall that in the purely cooperative case every positive solution of (7) with $\ell = 2$ is fully synchronized, as shown by Guo and Liu in [22]. On the other hand, it is shown in [8, 12] that there exists $\lambda^* < 0$ such that the system (7) does not have a fully synchronized solution if $\lambda_{ij} < \lambda^*$ for all pairs $i \neq j$.

System (7) with mixed couplings (i.e., λ_{ij} can be positive or negative) has been recently studied by Clapp and Pistoia [9]. Let $\lambda_{ii} = 1$ and assume the matrix (λ_{ij}) is symmetric and admits a block decomposition as follows: For some $1 < q < \ell$ there exist $0 = \ell_0 < \ell_1 < \dots < \ell_{q-1} < \ell_q = \ell$ such that, if we set

$$I_h := \{i \in \{1, \dots, \ell\} : \ell_{h-1} < i \leq \ell_h\},$$

$$\mathcal{I}_h := I_h \times I_h, \quad \mathcal{K}_h := \{(i, j) \in I_h \times I_k : k \in \{1, \dots, q\} \setminus \{h\}\},$$

then

$$\lambda_{ij} > 0 \text{ if } (i, j) \in \mathcal{I}_h \quad \text{and} \quad \lambda_{ij} \leq 0 \text{ if } (i, j) \in \mathcal{K}_h, \quad h = 1, \dots, q.$$

According to the above decomposition, we shall write a solution $\mathbf{u} = (u_1, \dots, u_\ell)$ to (7) in block-form as

$$\mathbf{u} = (\bar{u}_1, \dots, \bar{u}_q) \quad \text{with} \quad \bar{u}_h = (u_{\ell_{h-1}+1}, \dots, u_{\ell_h}).$$

\mathbf{u} is called *fully nontrivial* if every component u_i is different from zero. In [9] it is proved that if, either $m \geq 5$, or $m = 4$ and $\lambda_{ij} =: b_h > 1$ for all $i, j \in I_h$ with $i \neq j$, the system (7) has a fully nontrivial solution if $\max_{(i,j) \in \mathcal{K}_h} |\lambda_{ij}| < \Lambda$ for some $\Lambda > 0$. This solution is invariant under the conformal action on \mathbb{R}^m of the group Γ defined above.

Finally, we would like to mention a couple of results, one of them by Grossi, Gladiali and Troestler [20] where they give sufficient conditions on the matrix (λ_{ij}) to ensure the existence of solutions bifurcating from the bubble of the critical Sobolev equation, and another one by Druet and Hebey [18] where they study the stability of solutions to (7) under linear perturbation.

REFERENCES

- [1] B. Ammann and E. Humbert, [The second Yamabe invariant](#), *J. Funct. Anal.*, **235** (2006), 377–412.
- [2] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, *J. Differ. Geom.*, **11** (1976), 573–598.
- [3] T. Bartsch, [Bifurcation in a multicomponent system of nonlinear Schrödinger equations](#), *J. Fixed Point Theory Appl.*, **13** (2013), 37–50.
- [4] A. Castro, J. Cossio and J. M. Neuberger, [A sign-changing solution for a superlinear Dirichlet problem](#), *Rocky Mountain J. Math.*, **27** (1997), 1041–1053.
- [5] S.-M. Chang, C.-S. Lin, T.-C. Lin and W.-W. Lin, [Segregated nodal domains of two-dimensional multispecies Bose-Einstein condensates](#), *Phys. D*, **196** (2004), 341–361.
- [6] Z. Chen and W. Zou, [Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent: Higher dimensional case](#), *Calc. Var. Partial Differential Equations*, **52** (2015), 423–467.
- [7] M. Clapp and J. C. Fernández, [Multiplicity of nodal solutions to the Yamabe problem](#), *Calc. Var. Partial Differential Equations*, **56** (2017), 22pp.
- [8] M. Clapp and A. Pistoia, [Existence and phase separation of entire solutions to a pure critical competitive elliptic system](#), *Calc. Var. Partial Differential Equations*, **57** (2018), 20pp.
- [9] M. Clapp and A. Pistoia, [Fully nontrivial solutions to elliptic systems with mixed couplings](#), [arXiv:2106.01637](#), (2021).
- [10] M. Clapp, A. Pistoia and H. Tavares, [Yamabe systems, optimal partitions and nodal solutions to the Yamabe equation](#), Preprint, [arXiv:2106.00579](#), 2021.
- [11] M. Clapp, A. Saldaña and A. Szulkin, [Phase separation, optimal partitions and nodal solutions to the Yamabe equation on the sphere](#), *Int. Math. Res. Not.*, **2021** (2021), 3633–3652.
- [12] M. Clapp and A. Szulkin, [A simple variational approach to weakly coupled competitive elliptic systems](#), *Nonlinear Differential Equations Appl.*, **26** (2019), 21pp.
- [13] M. Conti, S. Terracini and G. Verzini, [Nehari’s problem and competing species systems](#), *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **19** (2002), 871–888.
- [14] M. Conti, S. Terracini and G. Verzini, [A variational problem for the spatial segregation of reaction-diffusion systems](#), *Indiana Univ. Math. J.*, **54** (2005), 779–815.
- [15] M. del Pino, M. Musso, F. Pacard and A. Pistoia, [Large energy entire solutions for the Yamabe equation](#), *J. Differential Equations*, **251** (2011), 2568–2597.
- [16] Ma. del Pino, M. Musso, F. Pacard and A. Pistoia, [Torus action on \$S_n\$ and sign-changing solutions for conformally invariant equations](#), *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **12** (2013), 209–237.
- [17] W. Y. Ding, [On a conformally invariant elliptic equation on \$R^n\$](#) , *Comm. Math. Phys.*, **107** (1986), 331–335.

- [18] O. Druet and E. Hebey, [Stability for strongly coupled critical elliptic systems in a fully inhomogeneous medium](#), *Anal. PDE*, **2** (2009), 305–359.
- [19] J. C. Fernández and J. Petean, [Low energy nodal solutions to the Yamabe equation](#), *J. Differential Equations*, **268** (2020), 6576–6597.
- [20] F. Gladiali, M. Grossi and C. Troestler, [A non-variational system involving the critical Sobolev exponent. The radial case](#), *J. Anal. Math.*, **138** (2019), 643–671.
- [21] Y. Guo, B. Li and J. Wei, [Entire nonradial solutions for non-cooperative coupled elliptic system with critical exponents in \$\mathbb{R}^3\$](#) , *J. Differential Equations*, **256** (2014), 3463–3495.
- [22] Y. Guo and J. Liu, [Liouville type theorems for positive solutions of elliptic system in \$\mathbb{R}^n\$](#) , *Comm. Partial Differential Equations*, **33** (2008), 263–284.
- [23] E. Hebey, Introduction à l’analyse non linéaire sur les variétés, *Diderot, Paris*, 1997.
- [24] E. Hebey and M. Vaigon, [Sobolev spaces in the presence of symmetries](#), *J. Math. Pures Appl.*, **76** (1997), 859–881.
- [25] M. Obata, The conjectures on conformal transformations of Riemannian manifolds, *J. Differential Geometry*, **6** (1971/72), 247–258.
- [26] R. S. Palais, [The principle of symmetric criticality](#), *Comm. Math. Phys.*, **69** (1979), 19–30.
- [27] M. Struwe, [Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems](#), 2nd edition, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, **34**, Springer-Verlag, Berlin, 1996.
- [28] N. Soave, H. Tavares, S. Terracini and A. Zilio, [Hölder bounds and regularity of emerging free boundaries for strongly competing Schrödinger equations with nontrivial grouping](#), *Nonlinear Anal.*, **138** (2016), 388–427.
- [29] G. Talenti, [Best constant in Sobolev inequality](#), *Ann. Mat. Pura Appl.*, **110** (1976), 353–372.
- [30] M. Willem, [Minimax Theorems](#), *Progress in Nonlinear Differential Equations and their Applications*, **24**, Birkhäuser Boston, Inc., Boston, MA, 1996.

Received July 2021; revised September 2021; early access October 2021.

E-mail address: monica.clapp@im.unam.mx

E-mail address: angela.pistoia@uniroma1.it