

CANONICAL MAPS OF GENERAL HYPERSURFACES IN ABELIAN VARIETIES

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Dedicated to Olivier Debarre on the occasion of his 60-th +ε birthday

ABSTRACT. The main theorem of this paper is that, for a general pair (A, X) of an (ample) hypersurface X in an Abelian Variety A , the canonical map Φ_X of X is birational onto its image if the polarization given by X is not principal (i.e., its Pfaffian d is not equal to 1).

We also easily show that, setting $g = \dim(A)$, and letting d be the Pfaffian of the polarization given by X , then if X is smooth and

$$\Phi_X : X \rightarrow \mathbb{P}^{N:=g+d-2}$$

is an embedding, then necessarily we have the inequality $d \geq g + 1$, equivalent to $N := g + d - 2 \geq 2 \dim(X) + 1$.

Hence we formulate the following interesting conjecture, motivated by work of the second author: if $d \geq g + 1$, then, for a general pair (A, X) , Φ_X is an embedding.

1. Introduction. Let A be an Abelian variety of dimension g , and let $X \subset A$ be a smooth ample hypersurface in A such that the Chern class $c_1(X)$ of the divisor X is a polarization of type $\vec{d} := (d_1, d_2, \dots, d_g)$, so that the vector space $H^0(A, \mathcal{O}_A(X))$ has dimension equal to the Pfaffian $d := d_1 \cdots d_g$ of $c_1(X)$.

The classical results of Lefschetz [13] say that the rational map associated to $H^0(A, \mathcal{O}_A(X))$ is a morphism if $d_1 \geq 2$, and is an embedding of A if $d_1 \geq 3$.

There have been several improvements in this direction, by work of several authors, for instance [15] showed that for $d_1 \geq 2$ we have an embedding except in a very special situation, and, for progress in the case $d_1 = 1$, see for instance [14], [9].

Now, by adjunction, the canonical sheaf of X is the restriction $\mathcal{O}_X(X)$, so a natural generalization of Lefschetz' theorems is to ask about the behaviour of the canonical systems of such hypersurfaces X . Such a question is important in birational geometry, but the results for the canonical maps can depend on the hypersurface X and not just on the polarization type only.

We succeed in this paper to find (respectively: conjecture) simple results for general such hypersurfaces.

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Our work was motivated by a theorem obtained by the first author in a joint work with Schreyer [4] on canonical surfaces: if we have a polarization of type $(1, 1, 2)$ then the image Σ of the canonical map Φ_X is in general a surface of degree 12 in \mathbb{P}^3 , birational to X , while for the special case where X is the pull-back of the Theta divisor of a curve of genus 3, then the canonical map has degree 2, and Σ has degree 6.

The connection of the above result with the Lefschetz theorems is, as we already said, provided by adjunction, we have the following folklore result, a proof of which can be found for instance in [6] (a referee pointed out that the proof in the case of a principal polarization appears in 2.10 of [12], and that of course Green's proof works in general)

Lemma 1.1. *Let X be a smooth ample hypersurface of dimension n in an Abelian variety A , such that the class of X is a polarization of type $\bar{d} := (d_1, d_2, \dots, d_{n+1})$.*

Let $\theta_1, \dots, \theta_d$ be a basis of $H^0(A, \mathcal{O}_A(X))$ such that $X = \{\theta_1 = 0\}$.

If z_1, \dots, z_g are linear coordinates on the complex vector space V such that A is the quotient of V by a lattice Λ , $A = V/\Lambda$, then the canonical map Φ_X is given by

$$\left(\theta_2, \dots, \theta_d, \frac{\partial \theta_1}{\partial z_1}, \dots, \frac{\partial \theta_1}{\partial z_g}\right).$$

Hence first of all the canonical map is an embedding if $H^0(A, \mathcal{O}_A(X))$ yields an embedding of A ; secondly, since a projection of Φ_X is the Gauss map of X , given by $(\frac{\partial \theta_1}{\partial z_1}, \dots, \frac{\partial \theta_1}{\partial z_g})$, by a theorem of Ziv Ran [18] it follows that the canonical system $|K_X|$ is base-point-free and Φ_X a finite morphism.

This is our main result:

Theorem 1.2. *Let (A, X) be a general pair, consisting of a hypersurface X of dimension $n = g - 1$ in an Abelian variety A , such that the class of X is a polarization of type $\bar{d} := (d_1, d_2, \dots, d_g)$ with Pfaffian $d = d_1 \dots d_g > 1$.*

Then the canonical map Φ_X of X is birational onto its image Σ .

The first observation is: the hypothesis that we take a general such pair, and not any pair, is necessary in view of the cited result of [4].

The second observation is that the above result extends to more general situations, using a result on openness of birationality (this will be pursued elsewhere). This allows another proof of the theorem, obtained studying pull-backs of Theta divisors of hyperelliptic curves (observe that for Jacobians the Gauss map of the Theta divisor is a rational map, see [7] for a study of its degree).

Here we use the following nice result by Olivier Debarre [8]:

Theorem 1.3. *Let $\Theta \subset A'$ be the Theta divisor of a general principally polarized Abelian variety A' : then the Gauss map of Θ , $\psi : \Theta \rightarrow P := \mathbb{P}^{g-1}$ factors through $Y := \Theta/\pm 1$, and yields a generic covering $\Psi : Y \rightarrow P$, meaning that*

1. *The branch divisor $\mathcal{B} \subset P$ of Ψ is irreducible and reduced;*
2. *The ramification divisor of Ψ , $\mathcal{R} \subset Y$, maps birationally to \mathcal{B} ;*
3. *the local monodromies of the covering at the general points of \mathcal{B} are transpositions;*
4. *the Galois group of Ψ (the global monodromy group of the covering Ψ) is the symmetric group \mathfrak{S}_N , where $N = (\frac{1}{2}g!)$.*

The next question to which the previous result paves the way is: when is Φ_X an embedding for (A, X) general?

An elementary application of the Severi double point formula [19] (see also [10], [2]) as an embedding obstruction, yields a necessary condition (observe that a similar argument was used by van de Ven in [20], in order to study the embeddings of Abelian varieties).

Theorem 1.4. *Let $X \subset A$ be a smooth ample hypersurface in an Abelian variety of dimension g , giving a polarization with Pfaffian d .*

If the canonical map Φ_X is an embedding of X , then necessarily

$$d \geq g + 1.$$

With some optimism (hoping for a simple result), but relying on the highly non-trivial positive result of the second author [6] concerning polarizations of type $(1, 2, 2)$ (here $g = 3, d = 4$), and on the result by Debarre and others [9] that this holds true for polarizations of type $(1, \dots, 1, d)$ with $d > 2^g$, we pose the following conjecture:

Conjecture 1. *Assume that (A, X) is a general pair of a smooth ample hypersurface $X \subset A$ in an Abelian variety A , giving a polarization with Pfaffian d satisfying $d \geq g + 1$.*

Then the canonical map Φ_X is an embedding of X .

We end the paper discussing the conjecture.

2. Proof of the main theorem 1.2. We give first a quick outline of the strategy of the proof.

The first step 2.1 reduces to the case where the Pfaffian d of the polarization type is a prime number p .

Step 2.2 considers the particular case where $d = p$ and where X is the pull-back of the Theta divisor Θ of a principally polarized Abelian variety. In this case the Gauss map of X factors through $Y := \Theta/\pm 1$.

Step 2.3 shows that if $g = \dim(A) = 2$, then Step 2.2 works and the theorem is proven.

Step 2.4, the Key Step, shows that if $g \geq 3$ and 2.2 does not work, then the image Σ of the canonical map lies birationally between X and Y , hence corresponds to a subgroup of the dihedral group D_p

Step 2.5 finishes the proof, showing that, in each of the two possible cases corresponding to the subgroups of D_p , the canonical map becomes birational onto its image for a general deformation of X .

2.1. Reduction to the case of a polarization of type $(1, \dots, 1, p)$, with p a prime number. We shall proceed by induction, basing on the following concept.

We shall say that a polarization type $\bar{d} := (d_1, d_2, \dots, d_g)$ is divisible by $\bar{\delta} := (\delta_1, \delta_2, \dots, \delta_g)$ if, for all $i = 1, \dots, g$, we have that δ_i divides $d_i, d_i = r_i \delta_i$.

Lemma 2.1. *Assume that the polarization type \bar{d} is divisible by $\bar{\delta}$. Then, if the main Theorem 1.2 holds true for $\bar{\delta}$, then it also holds true for \bar{d} .*

Proof. We let (A', X') be a general pair giving a polarization of type $\bar{\delta}$, so that $\Phi_{X'}$ is birational.

There exists an étale covering $A \rightarrow A'$, with kernel $\cong \oplus_i (\mathbb{Z}/r_i)$, such that the pull back of X' is a polarization of type \bar{d} .

By induction, we may assume without loss of generality that all numbers $r_i = 1$, with one exception r_j , which is a prime number p .

Then we have $\pi : X \rightarrow X'$, which is an étale quotient with group \mathbb{Z}/p . Moreover, since the canonical divisor K_X is the pull-back of $K_{X'}$, the composition $\Phi_{X'} \circ \pi$ factors through Φ_X , and a morphism $f : \Sigma \rightarrow \Sigma'$.

Since by assumption $\Phi_{X'}$ is birational, either Φ_X is birational, and there is nothing to prove, or Φ_X has degree p , $f : \Sigma \rightarrow \Sigma'$ is birational, hence Φ_X factors (birationally) through π .

To contradict the second alternative, it suffices to show that the canonical system $H^0(X, \mathcal{O}_X(K_X))$ separates the points of a general fibre of π . Since each fibre is an orbit for the group \mathbb{Z}/p , it suffices to show that there are at least two different eigenspaces in the vector space $H^0(X, \mathcal{O}_X(K_X))$, with respective eigenvalues $1, \zeta$, where ζ is a primitive p -th root of unity.

Since then we would have as projection of the canonical map a rational map $F : X \dashrightarrow \mathbb{P}^1$ such that, for $g \in \mathbb{Z}/p$,

$$F(x) = (x_0, x_1) \neq F(gx) = (x_0, \zeta x_1),$$

thereby separating the points of a general fibre.

Now, if there were only one non-zero eigenspace, the one for the eigenvalue 1 , since we have an eigenspace decomposition

$$\begin{aligned} H^0(X, \mathcal{O}_X(K_X)) &= H^0(X, \pi^*(\mathcal{O}_{X'}(K_{X'}))) = H^0(X', \pi_*\pi^*(\mathcal{O}_{X'}(K_{X'}))) = \\ &= \oplus_1^p H^0(X', \mathcal{O}_{X'}(K_{X'} + i\eta)), \end{aligned}$$

(here η is a nontrivial divisor of p -torsion), all eigenspaces would have dimension zero, except for the case $i = p$. In particular we would have that $H^0(X, \mathcal{O}_X(K_X))$ and $H^0(X', \mathcal{O}_{X'}(K_{X'}))$ have the same dimension.

But this is a contradiction, since the dimension $h^0(X, \mathcal{O}_X(K_X))$ equals by lemma 1.1 the sum of the Pfaffian of X with $g - 1$, and the Pfaffian of X is p -times the Pfaffian of X' . □

2.2. The special case where X is a pull-back of a Theta divisor. Here, we shall consider a similar situation, assuming that X is a polarization of type $(1, \dots, 1, p)$, and that X is the pull back of a Theta divisor $\Theta \subset A'$ via an isogeny $A \rightarrow A'$ with kernel $\cong \mathbb{Z}/p$.

We define $Y := \Theta / \pm 1$ and use the cited result of Debarre [8].

We consider the Gauss map of X , $f : X \rightarrow P := \mathbb{P}^{g-1}$, and observe that we have a factorization

$$f = \Psi \circ \phi, \quad \phi : X \rightarrow Y, \quad \Psi : Y \rightarrow P.$$

The essential features are that:

- (i) ϕ is a Galois quotient $Y = X/G$, where G is the dihedral group D_p .
- (ii) Ψ is a generic map, in particular with monodromy group equal to \mathfrak{S}_N , $N = g!/2$.

Either the theorem is true, or, by contradiction, we have a factorization of f through $\Phi_X : X \rightarrow \Sigma$, whose degree m satisfies $2pN > m \geq 2$.

2.3. Warm-up case: $g = 2$, X defines a polarization of type $(1, p)$ with p a prime number. In this case X is a \mathbb{Z}/p étale covering of a genus 2 curve $C = \Theta$, and $X/G = \mathbb{P}^1$.

That the canonical map of X is not birational means that X is hyperelliptic, and then $f : X \rightarrow \mathbb{P}^1$ factors through the quotient of the hyperelliptic involution ι , which centralizes G . If we set $\Sigma := X/\iota \cong \mathbb{P}^1$, we see that Σ corresponds to

a subgroup of order two of G : since ι centralizes G , and is not contained in \mathbb{Z}/p , follows that G is abelian, whence $p = 2$.

We prove here a result which might be known (but we could not find it in [1]; a referee points out that, under the stronger assumption that A is also general, such a result is proven in [17] and in [16]).

Lemma 2.2. *The general divisor in a linear system $|X|$ defining a polarization of type $(1, 2)$ is not hyperelliptic.*

Proof. In this case of a polarization of type $(1, 2)$, the linear system $|X|$ on A is a pencil of genus 3 curves with 4 distinct base points, as we now show.

We consider, as before, the inverse image D of a Theta divisor for $g = 2$, which will be here called C , so that $A' = J(C)$.

The curve D is hyperelliptic, being a Galois covering of \mathbb{P}^1 with group $G = (\mathbb{Z}/2)^2$ (a so-called bidouble cover), fibre product of two coverings respectively branched on 2, 4 points (hence $D = E \times_{\mathbb{P}^1} \mathbb{P}^1$, where E is an elliptic curve, and C is the quotient of D by the diagonal involution). The divisor cut by $H^0(A, \mathcal{O}_A(X))$ on X is the ramification divisor of $X \rightarrow E$, hence it consists of 4 distinct points.

The double covering $D \rightarrow E$ of an elliptic curve is also induced by the homomorphism $J(D) \rightarrow E$, with kernel isogenous to A' . Therefore, moving X in the linear system $|D|$, all these curves are a double cover of some elliptic curve E' , such that $J(X)$ is isogenous to $A \times E'$.

Hence $E' = X/\sigma$, where σ is an involution.

Assume now that X is hyperelliptic and denote by ι the hyperelliptic involution: since ι is central, ι and σ generate a group $G \cong (\mathbb{Z}/2)^2$, and we claim that the quotient of the involution $\iota \circ \sigma$ yields a genus 2 curve C' as quotient of X . In fact, the quotient $X/G \cong \mathbb{P}^1$, hence we have a bidouble cover of \mathbb{P}^1 with branch loci of respective degrees 2, 4 and the quotient C' is a double covering of \mathbb{P}^1 branched in 6 points.

Hence the hyperelliptic curves in $|D|$ are just the degree 2 étale coverings of genus 2 curves, and are only a finite number in $|D|$ (since $J(C')$ is isogenous to A). \square

2.4. We proceed with the Key Step: Showing that birationally Σ must lie between X and Y .

2.4.1. *To achieve this we need to recall some basic facts about covering spaces.* We shall use the Grauert-Remmert [11] extension of Riemann’s theorem, stating that finite coverings $X \rightarrow Y$ between normal varieties correspond to covering spaces $X^0 \rightarrow Y^0$ between respective Zariski open subsets of X , resp. Y .

Remark 1. Given a connected unramified covering $X \rightarrow Y$ between good spaces, we let \tilde{X} be the universal covering of X , and we write $X = H \backslash \tilde{X}$, $Y = \Gamma \backslash \tilde{X}$, hence $x \in X$ as $x = H \tilde{x}$, $y \in Y$ as $y = \Gamma \tilde{x}$, where both groups act on the left.

(1) Then the group of covering transformations

$$G := \text{Aut}(X \rightarrow Y) \cong N_H/H,$$

where N_H is the normalizer subgroup of H in Γ .

(2) The monodromy group $\text{Mon}(X \rightarrow Y)$, also called the Galois group of the covering in [8], is defined as $\Gamma/\text{core}(H)$, where $\text{core}(H)$ is the maximal normal subgroup of Γ contained in H ,

$$\text{core}(H) = \bigcap_{\gamma \in \Gamma} H\gamma = \bigcap_{\gamma \in \Gamma} (\gamma^{-1}H\gamma),$$

which is also called the normal core of H in Γ .

(3) The two actions of the two above groups on the fibre over $\Gamma\tilde{x}$, namely $H\backslash\Gamma \cong \{H\gamma\tilde{x}|\gamma \in \Gamma\}$, commute, since G acts on the left, and Γ acts on the right (the two groups coincide exactly when H is a normal subgroup of Γ , and we have what is called a normal covering space, or a Galois covering).

We have in fact an antihomomorphism

$$\Gamma \rightarrow \text{Mon}(X \rightarrow Y)$$

with kernel $\text{core}(H)$.

(4) Factorizations of the covering $X \rightarrow Y$ correspond to intermediate subgroups H' lying in between, $H \subset H' \subset \Gamma$.

2.4.2. *Our standard situation.* We consider the composition of finite coverings $X \rightarrow Y \rightarrow P$.

To simplify our notation, we consider the corresponding composition of unramified covering spaces of Zariski open sets, and the corresponding fundamental groups

$$1 \rightarrow K_1 \rightarrow H_1 \rightarrow \Gamma_1.$$

Then the monodromy group of the Gauss map of X is the quotient $\Gamma_1/\text{core}(K_1)$.

We shall now divide all the above groups by the normal subgroup $\text{core}(K_1)$, and obtain

$$1 \rightarrow K \rightarrow H \rightarrow \Gamma, \Gamma = \text{Mon}(X^0 \rightarrow P^0).$$

Since $X \rightarrow Y$ is Galois, and by Debarre's theorem $Y \rightarrow P$ is a generic covering, we have:

- we have a surjection of the monodromy group $\Gamma \rightarrow \mathfrak{S}_N := \mathfrak{S}(H\backslash\Gamma)$ with kernel $\text{core}(H)$, where $N = |H\backslash\Gamma| = \frac{g!}{2}$.
- H is the inverse image of a stabilizer, hence H maps onto \mathfrak{S}_{N-1} ,
- H is a maximal subgroup of Γ , since \mathfrak{S}_{N-1} is a maximal subgroup of \mathfrak{S}_N ;
- $H/K \cong G := D_p$,
- Γ acts on the fibre $M := K\backslash\Gamma$
- $K' := K \cap \text{core}(H)$ is normal in K , and $K'' := K/K'$ maps isomorphically to a normal subgroup of \mathfrak{S}_{N-1} , which is a quotient of G , hence it has index at most $2p$. Therefore there are only two possibilities:
 - (I) $K'' = \mathfrak{S}_{N-1}$, or
 - (II) $K'' = \mathfrak{A}_{N-1}$.

We consider now the case where there is a nontrivial factorization of the Gauss map $f, X \rightarrow \Sigma \rightarrow P$.

Define \widehat{H} to be the subgroup of the monodromy group Γ associated to Σ , so that

$$1 \rightarrow K \rightarrow \widehat{H} \rightarrow \Gamma,$$

and set:

$$H' := \widehat{H} \cap \text{core}(H), H'' := \widehat{H}/H', H'' \subset \mathfrak{S}_N.$$

Obviously we have $K'' \subset H''$: hence

- in case (I), where $K'' = \mathfrak{S}_{N-1}$, we have either
 - (Ia) $H'' = \mathfrak{S}_{N-1}$ or
 - (Ib) $H'' = \mathfrak{S}_N$, while
- in case (II), where $K'' = \mathfrak{A}_{N-1}$, either
 - (IIa) $H'' = \mathfrak{A}_{N-1}$,
 - (IIb) $H'' = \mathfrak{A}_N$, or
 - (IIc) $H'' = \mathfrak{S}_N$

We first consider cases b) and c) where the index of H'' in \mathfrak{S}_N is ≤ 2 . Hence the index of \hat{H} in Γ is at most 2 times the index of H' in $\text{core}(H)$. Since $K' \subset H'$, and $\text{core}(H)/K'$ is a quotient of H/K , we see that the index of \hat{H} in Γ divides $2p$ (in fact, in case (IIb) $K'' = \mathfrak{A}_{N-1}$, hence in this case the cardinality of $\text{core}(H)/K'$ equals p).

Lemma 2.3. *Cases (b) and (c), where the degree m of the covering $\Sigma \rightarrow P$ divides $2p$, are not possible.*

Proof. Observe in fact that m equals the index of \hat{H} in Γ . We have two factorizations of the Gauss map f :

$$X \rightarrow Y \rightarrow P, \quad X \rightarrow \Sigma \rightarrow P,$$

hence we have that the degree of f equals

$$(2p)N = m\left(N\frac{2p}{m}\right).$$

Consider now the respective ramification divisors $\mathcal{R}_f, \mathcal{R} = \mathcal{R}_Y, \mathcal{R}_\Sigma$ of the respective maps $f : X \rightarrow P, \Psi : Y \rightarrow P, \Sigma \rightarrow P$.

Since $X \rightarrow Y$ is quasi-étale (unramified in codimension 1), \mathcal{R}_f is the inverse image of \mathcal{R} , hence \mathcal{R}_f maps to the branch locus \mathcal{B} with mapping degree $2p$.

Since the branch locus is known to be irreducible, and reduced, and \mathcal{R}_Σ is non empty, it follows that $X \rightarrow \Sigma$ is quasi-étale, and \mathcal{R}_f is the inverse image of \mathcal{R}_Σ , so that \mathcal{R}_f maps to the branch locus \mathcal{B} with mapping degree $(N\frac{2p}{m})\delta$, where δ is the mapping degree of \mathcal{R}_Σ to \mathcal{B} , which is at most p .

From the equality $2p = \delta N(\frac{2p}{m})$ and since $N = g!/2$, we see that $g!$ divides $4p$, which is absurd for $g \geq 4$. For $g = 3$ it follows that $p = 3$, and either $\delta = 2, m = 6$, or $\delta = 1$, and $m = 3$.

To show that these special cases cannot occur, we can use several arguments.

For the case $m = 3$, then Σ is a nondegenerate surface of degree 3 in \mathbb{P}^4 (hence it is by the way the Segre variety $\mathbb{P}^2 \times \mathbb{P}^1$), and has a linear projection to \mathbb{P}^2 of degree 3: hence its branch locus is a curve of degree 6, contradicting that \mathcal{B} has degree 12 (see [4], \mathcal{B} is the dual curve of the plane quartic curve whose Jacobian is A').

For the case $m = 6$, we consider the normalization of the fibre product

$$Z := \Sigma \times_P Y,$$

so that there is a morphism of X into Z , which we claim to be birational. In fact, the degree of the map $Z \rightarrow P$ is 18, and each component Z_i of Z is a covering of Σ ; therefore, if Z is not irreducible, there is a component Z_i mapping to P with degree 6: and the conclusion is that $\Sigma \rightarrow P$ factors through Y , which is what we wanted to happen, but assumed not to happen.

If Z is irreducible, then X is birational to Z , hence the group G acts on Z , and trivially on Y : follows that the fibres of $X \rightarrow \Sigma$ are G -orbits, whence $Y = \Sigma$, again a contradiction.

Remark 2. Indeed, we know ([4]) that the monodromy group of $\Theta \rightarrow P$ is \mathfrak{S}_4 , and one could describe explicitly Γ in relation to the series of inclusions

$$1 \rightarrow K \rightarrow K_1 \rightarrow H \rightarrow \Gamma,$$

where K_1 is the subgroup associated to Θ .

At any rate, if $m = 3$, we can take the normalization of the fibre product $W := \Theta \times_P \Sigma$, and argue as before that if W is reducible, then Θ dominates Σ , which is only possible if $\Sigma = Y$.

While, if W is irreducible, $W = X$ and the fibres of $X \rightarrow \Sigma$ are made of $\mathbb{Z}/3$ -orbits, hence again Θ dominates Σ .

□

Excluded cases (b) and (c), we are left with case (a), where

$$H'' \subset \mathfrak{S}_{N-1} \Rightarrow \widehat{H} \subset H,$$

equivalently Σ lies between X and Y , hence it corresponds to an intermediate subgroup of $G = D_p$, either \mathbb{Z}/p and then $\Sigma = \Theta$, or $\mathbb{Z}/2$, and then $\Sigma = X/\pm 1$ for a proper choice of the origin in the Abelian variety A .

2.5. In the case where Σ lies between X and Y , for a general deformation of X the canonical map becomes birational. Here, we can soon dispense of the case $\Sigma = \Theta$: just using exactly the same argument we gave in lemma 2.1, that \mathbb{Z}/p does not act on $H^0(X, \mathcal{O}_X(K_X))$ as the identity, so the fibres of $X \rightarrow \Theta$ are separated.

For the case where $\Sigma = X/(\mathbb{Z}/2)$ we need to look at the vector space $H^0(X, \mathcal{O}_X(K_X))$, which is a p -dimensional representation of D_p , where we have a basis $\theta_1, \dots, \theta_p$ of eigenvectors for the different characters of \mathbb{Z}/p . In other words, for a generator r of \mathbb{Z}/p , we must have

$$r(\theta_i) = \zeta^i \theta_i.$$

If s is an element of order 2 in the dihedral group, then $r \circ s = s \circ r^{-1}$, hence

$$r(s(\theta_i)) = s(r^{-1}(\theta_i)) = s(\zeta^{-1} \theta_i) = \zeta^{-1} s(\theta_i),$$

hence we may assume without loss of generality that

$$s(\theta_i) = \theta_{-i}, \quad -i \in \mathbb{Z}/p.$$

It is then clear that s does not act as the identity unless we are in the special case $p = 2$.

In the special case $p = 2$, we proceed as in [4]. We have a basis θ_1, θ_2 of even functions, i.e., such that $\theta_i(-z) = \theta_i(z)$, and $\Theta = X/(\mathbb{Z}/2)^2$, where $(\mathbb{Z}/2)^2$ acts sending $z \mapsto \pm z + \eta$, where η is a 2-torsion point on A . Since the partial derivatives of θ_1 are invariant for $z \mapsto z + \eta$, and since $\theta_2(z + \eta) = -\theta_2(z)$, the canonical map Φ_X factors through the involution $\iota : X \rightarrow X$ such that

$$\iota(z) = -z + \eta.$$

If for a general deformation of X as a symmetric divisor the canonical map would factor through ι , then X would be ι -invariant; being symmetric, it would be $(\mathbb{Z}/2)^2$ -invariant, hence for all deformations X would remain the pull-back of a Theta divisor. This is a contradiction, since the Kuranishi family of X has higher dimension than the Kuranishi family of a Theta divisor Θ (see [4]).

3. Embedding obstruction.

Theorem 3.1. *Let X be an ample smooth divisor in an Abelian variety A of dimension $n + 1$.*

Assume moreover that X is not a hyperelliptic curve of genus 3 yielding a polarization of type $(1, 2)$.

If X defines a polarization of type (m_1, \dots, m_{n+1}) , then the canonical map Φ_X of X is a morphism, and it can be an embedding only if $p_g(X) := h^0(K_X) \geq 2n + 2$, which means that the Pfaffian $d := m_1 \cdot m_2 \cdots m_{n+1}$ satisfies the inequality

$$d \geq n + 2 = g + 1.$$

Proof. Assume the contrary, $d \leq n + 1$, so that X embeds in \mathbb{P}^{n+c} , with codimension $c \leq n$, $c = d - 1$.

Observe that the pull back of the hyperplane class of \mathbb{P}^{n+c} is the divisor class of X restricted to X , and that the degree m of X equals to the maximal self-intersection of X in A , namely $m = X^{n+1} = d(n + 1)!$.

The Severi double point formula yields see ([10], also [5])

$$m^2 = c_n(\Phi^*T_{\mathbb{P}^{2n}} - T_X),$$

where Φ is the composition of Φ_X with a linear embedding $\mathbb{P}^{n+c} \hookrightarrow \mathbb{P}^{2n}$.

By virtue of the exact sequence

$$0 \rightarrow T_X \rightarrow T_A|_X \rightarrow \mathcal{O}_X(X) \rightarrow 0,$$

we obtain

$$m^2 = [(1 + X)^{2n+2}]_n = \binom{2n + 2}{n} X^{n+1} \Leftrightarrow d(n + 1)! = m = \binom{2n + 2}{n}.$$

To have a quick proof, let us also apply the double point formula to the section of $X = \Phi(X)$ with a linear subspace of codimension $(n - d + 1)$, which is a variety Y of dimension and codimension $(d - 1)$ inside \mathbb{P}^{2d-2} .

In view of the exact sequence

$$0 \rightarrow T_Y \rightarrow T_A|_Y \rightarrow \mathcal{O}_Y(X)^{n-d+2} \rightarrow 0,$$

we obtain

$$m^2 = [(1 + X)^{n+d+1} X^{n-d+1}]_n = \binom{n + d + 1}{d - 1} X^{n+1}$$

equivalently,

$$d(n + 1)! = m = \binom{n + d + 1}{d - 1}.$$

Since, for $d \leq n + 1$, $\binom{n+d+1}{d-1} \leq \binom{2n+2}{n}$, equality holding if and only if $d = n + 1$, we should have $d = n + 1$ and moreover

$$(n + 1)(n + 1)! = \binom{2n + 2}{n} \Leftrightarrow (n + 2)! = \binom{2n + 2}{n + 1}.$$

We have equality for $n = 1$, but then when we pass from n to $n + 1$ the left hand side gets multiplied by $(n + 3)$, the right hand side by $\frac{(2n+4)(2n+3)}{(n+2)^2}$, which is a strictly smaller number since

$$(n + 3)(n + 2) = n^2 + 5n + 6 > 2(2n + 3) = 4n + 6.$$

We are done with showing the desired assertion since we must have $n = 1$, and $d = 2$, and in the case $n = 1$, $d = 2$ we have a curve in \mathbb{P}^2 of degree 4 and genus 3. □

4. Remarks on the conjecture. Recall Conjecture 1:

Conjecture 2. *Assume that (A, X) is a general pair of a smooth ample hypersurface $X \subset A$ in an Abelian variety A , giving a polarization with Pfaffian d satisfying $d \geq g + 1$.*

Then the canonical map Φ_X is an embedding of X .

The first observation is that we can assume $g \geq 3$, since for a curve the canonical map is an embedding if and only if it is birational onto its image, hence we may apply here our main Theorem 1.2.

The second remark is that we have a partial result which is similar to lemma 2.1

Lemma 4.1. *Assume that the polarization type \bar{d} is divisible by $\bar{\delta}$. Then, if the embedding Conjecture 1 holds true for $\bar{\delta}$, and the linear system $|X'|$ is base point free for the general element X' yielding a polarization of type $\bar{\delta}$, then the embedding conjecture also holds true for \bar{d} .*

Proof. As in lemma 2.1 we reduce to the following situation: we have $\pi : X \rightarrow X'$, which is an étale quotient with group \mathbb{Z}/p . Moreover, since the canonical divisor K_X is the pull-back of $K_{X'}$, the composition $\Phi_{X'} \circ \pi$ factors through Φ_X , and a morphism $f : \Sigma \rightarrow \Sigma'$.

Since by assumption $\Phi_{X'}$ is an embedding, Φ_X is a local embedding at each point, and it suffices to show that Φ_X separates all the fibres.

Recalling that

$$H^0(X, \mathcal{O}_X(K_X)) = \oplus_1^p H^0(X', \mathcal{O}_{X'}(K_{X'} + i\eta)),$$

(here η is a nontrivial divisor of p -torsion), this follows immediately if we know that for each point $p \in X'$ there are two distinct eigenspaces which do not have p as a base-point.

Under our strong assumption $|K_{X'} + i\eta|$ contains the restriction of $|X' + i\eta|$ to X' , but $|X' + i\eta|$ is a translate of $|X'|$, so it is base-point free. \square

Already in the case of surfaces ($n = 2, g = 3$) the result is not fully established, we want $d \geq 4$, and the case of a polarization of type $(1, 1, 4)$ is not yet written down ([6] treats the case of a polarization of type $(1, 2, 2)$, which is quite interesting for the theory of canonical surfaces in \mathbb{P}^5).

Were our conjecture too optimistic, then the question would arise about the exact range of validity for the statement of embedding of a general pair (X, A) .

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