ELECTRONIC RESEARCH ARCHIVE
Volume **29**, Number **6**, December **2021**
doi:10.3934/era.2021086
pp. **4297–4314**

GLOBAL DYNAMICS IN A COMPETITIVE TWO-SPECIES AND TWO-STIMULI CHEMOTAXIS SYSTEM WITH CHEMICAL SIGNALLING LOOP

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(Communicated by Bin Liu)

Abstract. This paper deals with the following competitive two-species and two-stimuli chemotaxis system with chemical signalling loop

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u\nabla v) + \mu_1 u(1-u-a_1 w),\ x\in\Omega,\ t>0, \\ 0 = \Delta v - v + w,\ x\in\Omega,\ t>0, \\ w_t = \Delta w - \chi_2\nabla\cdot(w\nabla z) - \chi_3\nabla\cdot(w\nabla v) + \mu_2 w(1-w-a_2 u),\ x\in\Omega,\ t>0, \\ 0 = \Delta z - z + u,\ x\in\Omega,\ t>0, \end{cases}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 1$, where the parameters $a_1, a_2,\ \chi_1, \chi_2, \chi_3,\ \mu_1, \mu_2$ are positive constants. We first showed some conditions between $\frac{\chi_1}{\mu_1}$, $\frac{\chi_2}{\mu_2}$, $\frac{\chi_3}{\mu_2}$ and other ingredients to guarantee boundedness. Moreover, the large time behavior and rates of convergence have also been investigated under some explicit conditions.

1. Introduction.

In this paper, we consider the two-species chemotaxis-competition system with two chemicals

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u\nabla v) + \mu_1 u(1-u-a_1 w), & x\in\Omega, t>0, \\ 0 = \Delta v - v + w, & x\in\Omega, t>0, \\ w_t = \Delta w - \chi_2\nabla\cdot(w\nabla z) - \chi_3\nabla\cdot(w\nabla v) + \mu_2 w(1-w-a_2 u), & x\in\Omega, t>0, \\ 0 = \Delta z - z + u, & x\in\Omega, t>0, \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = \frac{\partial z}{\partial\nu} = 0, & x\in\partial\Omega, t>0, \\ (u,w)(x,0) = (u_0(x), w_0(x)), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\partial/\partial\nu$ denotes the derivative with respect to the outer normal of $\partial\Omega$; $a_1, a_2,\ \chi_1, \chi_2, \chi_3,\ \mu_1$ and μ_2 are positive constants.

Chemotaxis describes oriented movement of cells along the concentration gradient of a chemical signal produced by the cells, which is important in a large variety of fields within the life cycle of most multicellular organisms. Based on a well-known chemotaxis model for the chemotactic movement of one specie [11], a generalization of chemotaxis model for multi-species or multi-chemical-signal chemotaxis system was proposed [9, 24, 42]. System (1.1) describes the communication between breast

2020 *Mathematics Subject Classification.* Primary: 92C17, 35K35; Secondary: 35B35, 35A01.
Key words and phrases. Two-species chemotaxis, boundedness, exponential convergence, algebraical converge.

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tumor cells and macrophages in close proximity via a short-ranged chemical signaling loop according to the classical Lotka-Volterra dynamics [22], which was proposed by Knútsdóttir et al. [12]. The key physical variables in (1.1) are assumed to be the density of the macrophages (denoted by u) and the tumor cells (denoted by w), the concentration of the colony-stimulating factor 1 (CSF-1, denoted by v) and the epidermal growth factor (EGF, denoted by z). In their experiment, it was shown that in a short-term chemical signalling loop, the process is promoted by the macrophages. More precisely, the tumor cells w secrete CSF-1 v, which can bind to CSF-1 receptors on the macrophages w. This activates the macrophages to increasing concentrations of CSF-1 gradient and to secrete EGF z. Then the EGF can bind to receptors on tumor cells, continuing the chain of activation of them in return. Furthermore, activated tumor cells release more CSF-1 and partially direct their movement toward both concentration gradients of the EGF and the CSF-1, respectively.

From a mathematical point of view, the system (1.1) contains mainly two subsystems. When $u = z \equiv 0$ and $a_2 \equiv 0$, (1.1) is reduced to the chemotaxis-only system [11]:

$$\begin{cases} w_t = \Delta w - \chi_3 \nabla \cdot (w \nabla v) + \mu_2 w(1-w), & x \in \Omega, \quad t > 0, \\ 0 = \Delta v - v + w, & x \in \Omega, \quad t > 0, \end{cases} \tag{1.2}$$

which describe the aggregation of the paradigm species Dictyostelium discoideum. Model (1.2) has been extensively studied during past four decades. Boundedness of global solutions and unbounded solutions for (1.2) have been extensively investigated (see the surveys [2, 6, 8] and the references therein). When $\chi_3 \equiv 0$, (1.1) becomes the two-species chemotaxis system with two chemicals:

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla v) + \mu_1 u(1-u-a_1 w), & x \in \Omega, \quad t > 0, \\ 0 = \Delta v - v + w, & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - \chi_2 \nabla \cdot (w \nabla z) + \mu_2 w(1-w-a_2 u), & x \in \Omega, \quad t > 0, \\ 0 = \Delta z - z + u, & x \in \Omega, \quad t > 0, \end{cases} \tag{1.3}$$

which describes the spatio-temporal evolution of two populations which on the one hand proliferate and mutually compete with Lotka-Volterra kinetics and on the other hand the individuals shall move according to random diffusion and migrate toward a chemical signal produced by the opposing species. Without kinetic terms, that is, when $\mu_1 = \mu_2 = 0$, if $\chi_1, \chi_2 \in \{-1, 1\}$, global boundedness and finite time blow-up was constructed by Tao and Winkler [32]. In particular, when $n = 2$, global bounded solutions are proved if $\max\{m_1, m_2\} < C$ with some $C > 0$, where $m_1 := \int_\Omega u_0$ and $m_2 := \int_\Omega w_0$; if $\min\{m_1, m_2\} > 4\pi$, the solution may be blow up in finite time. These results were improved in [45]. When $\mu_i > 0$ $(i = 1, 2)$, global bounded solution and stabilization have been addressed by Tu et al. [36]. Recently, this results were improved by Wang et al. [38]. There are some another results on various the system (1.3) ([3, 5, 10, 18, 23, 26, 29, 27, 43, 47, 46]).

Compared with the chemotaxis-only system and the two-species chemotaxis system with two chemicals, the coupled a competitive two-species and two-stimuli chemotaxis system with chemical signalling loop the system (1.1) is much less understood. For the simplified version (1.1) in the unit disk $\Omega = B_R(0) \subset \mathbb{R}^2$ with $\mu_i = 0$ $(i = 1, 2)$ and the second and fourth equation replaced by

$$0 = \Delta v - \frac{m_2}{|\Omega|} + w \quad \text{and} \quad 0 = \Delta z - \frac{m_1}{|\Omega|} + u,$$

respectively, where $m_1 := \int_\Omega u_0$ and $m_2 := \int_\Omega w_0$, global boundedness and blow-up of solutions were constructed in [16]. In particular, a critical mass phenomenon has been found: solutions remain bounded if $2m_1 + \frac{\chi_3}{\chi_2} m_2 < \frac{8\pi}{\chi_2}$, whereas blow-up may occur if $\frac{8\pi}{\chi_1} m_1 + \frac{8\pi}{\chi_2} m_2 < 2m_1 m_2 + \frac{m_2^2 \chi_3}{\chi_2}$ and $\int_\Omega u_0 |x|^2$ and $\int_\Omega w_0 |x|^2$ are sufficiently small. Recently, (1.1) with $\mu_i = 0$ $(i = 1, 2)$ and $\Omega \subset \mathbb{R}^2$ has been considered in [17]: global boundedness was constructed when $\chi_1 \chi_2 m_1 m_2 < (\pi^\star - \chi_3 m_2)\,\pi^\star$, where $\pi^\star = 8\pi$ when $\Omega = B_R(0)$ is a disk, otherwise, $\pi^\star = 4\pi$; for the fully parabolic system (1.1) with $\mu_i = 0$ $(i = 1, 2)$ and $\Omega \subset \mathbb{R}^2$, global bounded solution was considered when $\chi_1 \chi_2 m_1 m_2 < \frac{\sqrt{1 - 4\chi_3 C_{GN}^4 m_2}}{4 C_{GN}^8}$, where $C_{GN} > 0$ is a constant. Moreover, gradient estimates, blow-up in finite time and asymptotic behavior have also been established. When $n = 3$ and $\mu_i > 0$ $(i = 1, 2)$ are sufficiently large, for the fully parabolic system (1.1), global bounded solutions were proved by Pan et al.[25]. Very recently, a symmetric model of (1.1) has been investigated [34, 35], i.e., replacing the first equation in (1.1) by $u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla v) - \chi_4 \nabla \cdot (u \nabla z) + \mu_1 u(1 - u - a_1 w)$.

In order to better understand (1.1), we should mention two biological species which compete for the resources and migrate towards a higher concentration of a chemical produced by themselves was proposed by Tello and Winkler [33]

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - a_2 u - v), & x \in \Omega, \quad t > 0, \\ 0 = \Delta w - w + u + v, & x \in \Omega, \quad t > 0, \end{cases} \tag{1.4}$$

this model has been extensively studied. When $a_1, a_2 \in (0, 1)$, global existence and the large time behavior were established under some conditions by Tello and Winkler [33], which was partially improved in [4], and by Stinner et al. [30] for the case of $a_1 \geq 1 > a_2$. For all $a_1, a_2 > 0$, global boundedness solutions were derived by Mizukami [20], which covers the case that $a_1, a_2 \geq 1$. Moreover, the convergence rates has been also obtained in the case that $a_1, a_2 \in (0, 1)$ and $a_1 \geq 1 > a_2$. For the fully parabolic version of model (1.4), which is obtained by replacing the equation with $w_t = \Delta w - w + u + v$, global existence and boundedness has been established for the space dimension does not exceed two by Bai and Winkler [1] and the n-dimensional setting by [14, 15, 44]. Moreover, the convergence rates has been established [1], this conditions were improved by Mizukami [19]. Recently, the conditions for asymptotic behavior in the case of $a_1, a_2 \in (0, 1)$ were once more improved by Mizukami [21].

The focus of this paper is to establish the global existence, large time behavior, and the rates of convergence of solution to (1.1). The first of our result asserts global existence of a bounded solution based on the comparison methods in [38, 37, 4, 30].

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary, and $a_1, a_2, \chi_1, \chi_2, \chi_3, \mu_1, \mu_2$ are positive constants. If one of the following conditions holds:*
(i) $\frac{\chi_1}{\mu_1} \leq a_1$ *and* $\frac{\chi_3}{\mu_2} < 1$;
(ii) $\frac{\chi_2}{\mu_2} \leq a_2$ *and* $\frac{\chi_3}{\mu_2} < 1$;
(iii) $\frac{\chi_1}{\mu_1} > a_1$, $\frac{\chi_2}{\mu_2} > a_2$ *and* $\frac{\chi_3}{\mu_2} < 1$ *as well as* $\frac{\chi_1 \chi_2}{\mu_1 \mu_2} + a_1 a_2 - a_1 \frac{\chi_2}{\mu_2} - a_2 \frac{\chi_1}{\mu_1} + \frac{\chi_3}{\mu_2} < 1$.
Then for the nonnegative initial data $(u_0, w_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$, the model (1.1) possesses a unique global classical solution (u, v, w, z) which is uniformly bounded in $\Omega \times (0, \infty)$. Moreover, the solutions u, v, w, z are the Höder continuous functions,

i.e., there exists some $\theta \in (0,1)$ and $M > 0$ such that

$$\begin{aligned}&\|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} + \|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} + \|w\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])}\\ &+ \|z\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \leq M \quad \text{for all } t \geq 1.\end{aligned} \tag{1.5}$$

The large time behavior and convergence rates of solutions to (1.1) is mathematically and biologically interesting. We first give the result of competitive coexistence case $a_1, a_2 \in (0,1)$.

Theorem 1.2. *Let $a_1, a_2 \in (0,1)$ and $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with smooth boundary. Assume that the nonnegative initial data $(u_0, w_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$ with $u_0 \not\equiv 0 \not\equiv w_0$ and (u, v, w, z) is a global bounded classical solution of (1.1). Let χ_1, χ_2, χ_3, μ_1 and μ_2 are positive constants and satisfy*

$$\mu_1\mu_2 a_1(1-a_1a_2) - \frac{a_2\mu_2\chi_1^2 u_*}{16} - \frac{w_*\mu_1 a_1\chi_3^2}{8} > 0 \tag{1.6}$$

and

$$\begin{aligned}\mu_1\mu_2^2 a_1 a_2(1-a_1a_2) - \frac{a_2^2\mu_2^2\chi_1^2 u_*}{16} - \frac{\mu_1\mu_2 a_1 a_2\chi_3^2 w_*}{8} - \frac{\mu_1\mu_2 a_1^2\chi_2^2 w_*}{8}&\\ + \frac{\mu_2 a_1 a_2\chi_1^2\chi_2^2 u_* w_*}{128} + \frac{\mu_1 a_1^2\chi_2^2\chi_3^2 w_*}{64} > 0,&\end{aligned} \tag{1.7}$$

then one can find $C > 0$ and $\mu > 0$ such that

$$\|u-u_*\|_{L^\infty(\Omega)} + \|v-v_*\|_{L^\infty(\Omega)} + \|w-w_*\|_{L^\infty(\Omega)} + \|z-z_*\|_{L^\infty(\Omega)} \leq Ce^{-\mu t}$$

for all $t > 0$, where

$$u_* = z_* = \frac{1-a_1}{1-a_1a_2},\ v_* = w_* = \frac{1-a_2}{1-a_1a_2}. \tag{1.8}$$

Remark 1. Since $a_1, a_2 \in (0,1)$, it is easy to check $f(\mu_1,\mu_2) := \mu_1\mu_2^2 a_1 a_2(1 - a_1a_2) - \frac{a_2^2\mu_2^2\chi_1^2 u_*}{16} - \frac{\mu_1\mu_2 a_1 a_2\chi_3^2 w_*}{8} - \frac{\mu_1\mu_2 a_1^2\chi_2^2 w_*}{8} + \frac{\mu_2 a_1 a_2\chi_1^2\chi_2^2 u_* w_*}{128} + \frac{\mu_1 a_1^2\chi_2^2\chi_3^2 w_*}{64}$ satisfies

$$\lim_{\mu_1\to+\infty,\mu_2\to+\infty} \frac{f(\mu_1,\mu_2)}{\mu_1\mu_2^2} = a_1a_2(1-a_1a_2) > 0.$$

Hence, there exist some constants $\mu_{10}, \mu_{20} > 0$ such that $f(\mu_1,\mu_2) > 0$ for all $[\mu_{10},\infty) \times [\mu_{20},\infty)$. Thanks to a continuity argument implies (1.7) holds for all $[\mu_{10},\infty) \times [\mu_{20},\infty)$.

The following result is on the competitive exclusion case $a_1 \geq 1 > a_2 > 0$.

Theorem 1.3. *Let $a_1 \geq 1 > a_2 > 0$, $\chi_1, \chi_2, \chi_3 > 0$, $\mu_1, \mu_2 > 0$, and let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with smooth boundary. Assume that the nonnegative initial data $(u_0, w_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$ with $w_0 \not\equiv 0$ and (u, v, w, z) is a global bounded solution of (1.1).*

(i) If $a_1 > 1$ and assume that for some $a_1' \in (1, a_1]$ such that $a_1' a_2 < 1$. Moreover, let

$$\mu_2^2 a_2(1-a_1'a_2) - \frac{a_2\mu_2\chi_3^2}{8} + \frac{a_1'\chi_2^2\chi_3^2}{64} - \frac{\mu_2 a_1'\chi_2^2}{8} > 0 \quad \text{and} \quad \mu_2 > \frac{\chi_3^2}{8(1-a_1'a_2)}, \tag{1.9}$$

there exist $C > 0$ and $\lambda > 0$ such that

$$\|u\|_{L^\infty(\Omega)} + \|v-1\|_{L^\infty(\Omega)} + \|w-1\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \quad \text{for all } t > 0.$$

(ii) If $a_1 = 1$ and (1.9) holds with $a_1' = 1$, there exist $C > 0$ and $\kappa > 0$ such that

$$\|u\|_{L^\infty(\Omega)} + \|v-1\|_{L^\infty(\Omega)} + \|w-1\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\Omega)} \le C(1+t)^{-\kappa} \quad \textit{for all } t > 0.$$

This paper is organized as follows. In Section 2, we show local existence of a solution to (1.1) and use comparison methods to prove global existence and boundedness of (1.1) (Theorems 1.1). Section 3 is devoted to the proof of asymptotic stability to (1.1) (Theorems 1.2 and 1.3).

2. Boundedness by comparison methods.

The local existence of solutions to (1.1) which can be achieved similarly by using well-established methods in [30, 31, 41].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \ge 1)$ be a bounded domain with smooth boundary, $a_1, a_2 > 0$, $\chi_1, \chi_2, \chi_3 > 0$, $\mu_1, \mu_2 > 0$. Then for the nonnegative initial data $(u_0, w_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$, there exists $T_{max} \in (0, \infty]$ such that (1.1) has a unique local non-negative classical solution*

$$u, v, w, z \in C(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})), \tag{2.1}$$

and which is such that either $T_{max} = \infty$ or $T_{max} < \infty$ and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \textit{ as } t \nearrow T_{max}.$$

Moreover, if the nonnegative initial data $(u_0, w_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$ with $u_0 \not\equiv 0 \not\equiv w_0$, then the solution of (1.1) satisfies $u > 0$,$v \ge 0$ $w > 0$ and $z \ge 0$ in $\Omega \times (0, T_{max})$.

According to comparison methods in [38, 37, 4, 30], we will prove boundedness of solution to model (1.1). First we let the parabolic operator $\mathcal{L}_i = \mathcal{L}_i(x,t)$ $(i = 1, 2)$ satisfy

$$\mathcal{L}_1 u := \Delta u - \chi_1 \nabla u \cdot \nabla v, \quad (x,t) \in \Omega \times (0, T_{max}) \tag{2.2}$$

and

$$\mathcal{L}_2 w := \Delta w - \chi_2 \nabla w \cdot \nabla z - \chi_3 \nabla w \cdot \nabla v, \quad (x,t) \in \Omega \times (0, T_{max}). \tag{2.3}$$

Thus from the model (1.1) shows that

$$\begin{aligned} u_t - \mathcal{L}_1 u &= u\{-\chi_1 \Delta v + \mu_1(1 - u - a_1 w)\} \\ &= u\{\mu_1 - \mu_1 u - \chi_1 v + (\chi_1 - a_1\mu_1) w\} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} w_t - \mathcal{L}_2 w &= w\{-\chi_2 \Delta z - \chi_3 \Delta v + \mu_2(1 - w - a_2 u)\} \\ &= w\{\mu_2 - (\mu_2 - \chi_3) w + (\chi_2 - a_2\mu_2) u - \chi_2 z - \chi_3 v\} \end{aligned} \tag{2.5}$$

for all $(x,t) \in \Omega \times (0, T_{max})$.

We first consider the conditions $\frac{\chi_1}{\mu_1} \le a_1$ and $\frac{\chi_3}{\mu_2} < 1$.

Lemma 2.2. *Let $\tau = 0$ and the nonnegative initial data $(u_0, w_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$. Assume that $\frac{\chi_1}{\mu_1} \le a_1$ and $\frac{\chi_3}{\mu_2} < 1$. Then $T_{max} = \infty$ and u, v, w, z are bounded in $\Omega \times (0, \infty)$.*

Proof. Since $\frac{\chi_1}{\mu_1} \le a_1$ and u, v, w, z are nonnegative, in view of (2.4) and (2.5), we have

$$\mathbf{p}_1 u := u_t - \mathcal{L}_1 u - u\{\mu_1 - \mu_1 u\} \le 0, \quad (x,t) \in \Omega \times (0, T_{max}) \tag{2.6}$$

and

$$\mathbf{p}_2 w := w_t - \mathcal{L}_2 w - w\{\mu_2 - (\mu_2 - \chi_3)w + (\chi_2 - a_2\mu_2)u\} \le 0, (x,t) \in \Omega \times (0, T_{max}), \tag{2.7}$$

where $\mathcal{L}_1 u$ and $\mathcal{L}_2 w$ are given in (2.2) and (2.3), respectively. Next, in order to prove Lemma 2.2, we have two cases to proceed as below:

Case 1. $\chi_2 - a_2\mu_2 \le 0$**.** We choose $A > 0$ and let $\xi > 0$ sufficiently large such that

$$\xi \ge \max\left\{\frac{\|u_0\|_{L^\infty(\Omega)}}{A}, \frac{1}{A}, \|w_0\|_{L^\infty(\Omega)}, \frac{\mu_2}{\mu_2 - \chi_3}\right\}. \tag{2.8}$$

Hence, we can define constant functions $\overline{u}$ and $\overline{w}$ as follows

$$\overline{u} = \overline{u}(x,t) := A\xi \ \text{ and } \ \overline{w} = \overline{w}(x,t) := \xi, \quad (x,t) \in \overline{\Omega} \times [0, T_{max}), \tag{2.9}$$

which satisfy

$$\overline{u}(x,0) = A\xi \ge u_0(x) \ \text{ and } \ \overline{w}(x,0) = \xi \ge w_0(x) \ \text{ for all } \ x \in \Omega. \tag{2.10}$$

Due to the definition of $\mathbf{p}_1$ in (2.6), using (2.8) and (2.9), we have

$$\mathbf{p}_1 \overline{u} = -A\xi\{\mu_1 - \mu_1 A\xi\} \ge 0, \quad (x,t) \in \Omega \times (0, T_{max}). \tag{2.11}$$

According to (2.6), (2.10) and (2.11), and from the comparison principle for classical (sub-/super-) solutions [28, Proposition 52.6], we infer that

$$u(x,t) \le \overline{u}(x,t) = A\xi \ \text{ for all } \ (x,t) \in \Omega \times (0, T_{max}). \tag{2.12}$$

Using the definition of $\mathbf{p}_2$ in (2.7), we deduce from (2.8), $\frac{\chi_3}{\mu_2} < 1$ and $\chi_2 - a_2\mu_2 \le 0$ that

$$\begin{aligned} \mathbf{p}_2 \overline{w} &= -\xi\{\mu_2 - (\mu_2 - \chi_3)\xi + (\chi_2 - a_2\mu_2)u\} \\ &\ge -\xi\{\mu_2 - (\mu_2 - \chi_3)\xi\} \ge 0, \quad (x,t) \in \Omega \times (0, T_{max}). \end{aligned} \tag{2.13}$$

According to (2.7), (2.10) and (2.13), and from the comparison principle for classical (sub-/super-) solutions [28, Proposition 52.6], we infer that

$$w(x,t) \le \overline{w}(x,t) = \xi \ \text{ for all } \ (x,t) \in \Omega \times (0, T_{max}). \tag{2.14}$$

Case 2. $\chi_2 - a_2\mu_2 > 0$. Due to $\frac{\chi_3}{\mu_2} < 1$ and $\chi_2 - a_2\mu_2 > 0$, we can find A fulfilling

$$0 < A < \frac{\mu_2 - \chi_3}{\chi_2 - a_2\mu_2}. \tag{2.15}$$

Let ξ large enough such that

$$\xi \ge \max\left\{\frac{\|u_0\|_{L^\infty(\Omega)}}{A}, \frac{1}{A}, \|w_0\|_{L^\infty(\Omega)}, \frac{\mu_2}{\mu_2 - \chi_3 - (\chi_2 - a_2\mu_2)A}\right\}. \tag{2.16}$$

We again let $\overline{u} = \overline{u}(x,t) := A\xi$ and $\overline{w} = \overline{w}(x,t) := \xi$ for all $(x,t) \in \overline{\Omega} \times [0, T_{max})$, as in (2.9), we deduce from (2.16)

$$\overline{u}(x,0) = A\xi \ge u_0(x) \ \text{ and } \ \overline{w}(x,0) = \xi \ge w_0(x) \ \text{ for all } \ x \in \Omega. \tag{2.17}$$

Similar to (2.11) and (2.12), we infer

$$u(x,t) \le \overline{u}(x,t) = A\xi \ \text{ for all } \ (x,t) \in \Omega \times (0, T_{max}). \tag{2.18}$$

Hence, using $\chi_2 - a_2\mu_2 > 0$, (2.7), (2.16) and (2.18) yields

$$\begin{aligned}\mathbf{p}_2\overline{w} &= -\xi\{\mu_2 - (\mu_2-\chi_3)\xi + (\chi_2 - a_2\mu_2)u\}\\ &\geq -\xi\{\mu_2 - (\mu_2-\chi_3)\xi + (\chi_2 - a_2\mu_2)A\xi\}\\ &= -\xi\{\mu_2 - [\mu_2-\chi_3 - (\chi_2 - a_2\mu_2)A]\xi\} > 0, \quad (x,t)\in\Omega\times(0,T_{max}),\end{aligned} \tag{2.19}$$

thus by (2.7) and (2.17) we have

$$w(x,t) \leq \overline{w}(x,t) = \xi \text{ for all } (x,t)\in\Omega\times(0,T_{max}). \tag{2.20}$$

Using (2.12), (2.14), (2.18) and (2.20) along with extensibility criterion in Lemma 2.1, this entails that $T_{max}=\infty$ and u,v,w,z are globally bounded. □

Next, we consider the conditions $\frac{\chi_2}{\mu_2} \leq a_2$ and $\frac{\chi_3}{\mu_2} < 1$.

Lemma 2.3. *Let $\tau = 0$ and the nonnegative initial data $(u_0,w_0)\in C^0(\overline{\Omega})\times C^1(\overline{\Omega})$. Assume that $\frac{\chi_2}{\mu_2} \leq a_2$ and $\frac{\chi_3}{\mu_2} < 1$. Then $T_{max}=\infty$ and u,v,w,z are bounded in $\Omega\times(0,\infty)$.*

Proof. Since $\frac{\chi_2}{\mu_2} \leq a_2$, $\frac{\chi_3}{\mu_2} < 1$ and u,v,w,z are nonnegative, in view of (2.4) and (2.5), we have

$$\mathbf{p}_3 u := u_t - \mathcal{L}_1 u - u\{\mu_1 - \mu_1 u + (\chi_1 - a_1\mu_1)w\} \leq 0, \quad (x,t)\in\Omega\times(0,T_{max}) \tag{2.21}$$

and

$$\mathbf{p}_4 w := w_t - \mathcal{L}_2 w - w\{\mu_2 - (\mu_2-\chi_3)w\} \leq 0, \quad (x,t)\in\Omega\times(0,T_{max}), \tag{2.22}$$

where $\mathcal{L}_1 u$ and $\mathcal{L}_2 w$ are given in (2.2) and (2.3), respectively. In the case of $\chi_1 - a_1\mu_1 \leq 0$ and $\frac{\chi_3}{\mu_2} < 1$, Lemma 2.2 showed (1.1) has global bounded solution. Hence, here we need only consider the case $\chi_1 - a_1\mu_1 > 0$. We can find $A > 0$ and $\xi > 0$ fulfilling

$$A > \frac{\chi_1 - a_1\mu_1 + \frac{\mu_1}{\xi}}{\mu_1} \tag{2.23}$$

and

$$\xi \geq \max\left\{\frac{\|u_0\|_{L^\infty(\Omega)}}{A}, \|w_0\|_{L^\infty(\Omega)}, \frac{\mu_2}{\mu_2-\chi_3}\right\}. \tag{2.24}$$

We again let $\overline{u} = \overline{u}(x,t) := A\xi$ and $\overline{w} = \overline{w}(x,t) := \xi$ for all $(x,t)\in\overline{\Omega}\times[0,T_{max})$, as in (2.9), we deduce

$$\overline{u}(x,0) = A\xi \geq u_0(x) \text{ and } \overline{w}(x,0) = \xi \geq w_0(x) \text{ for all } x\in\Omega. \tag{2.25}$$

Using (2.22) and (2.24) we conclude

$$\mathbf{p}_4\overline{w} = -\xi\{\mu_2 - (\mu_2-\chi_3)\xi\} > 0, \quad (x,t)\in\Omega\times(0,T_{max}), \tag{2.26}$$

so according to (2.22) and (2.25) imply

$$w(x,t) \leq \overline{w}(x,t) = \xi \text{ for all } (x,t)\in\Omega\times(0,T_{max}). \tag{2.27}$$

Hence, using $\chi_1 - a_1\mu_1 > 0$, (2.21), (2.23) and (2.27) yields

$$\begin{aligned}\mathbf{p}_3\overline{u} &= -A\xi\{\mu_1 - \mu_1 A\xi + (\chi_1 - a_1\mu_1)w\}\\ &\geq -A\xi\{\mu_1 - \mu_1 A\xi + (\chi_1 - a_1\mu_1)\xi\} \geq 0, \quad (x,t)\in\Omega\times(0,T_{max}).\end{aligned} \tag{2.28}$$

From (2.21), (2.25) and (2.28) and the comparison principle, we infer

$$u(x,t) \le \overline{u}(x,t) = A\xi \ \text{ for all } \ (x,t) \in \Omega \times (0, T_{max}). \tag{2.29}$$

Then using (2.27) and (2.29) along with extensibility criterion in Lemma 2.1, this entails that $T_{max} = \infty$ and u, v, w, z are globally bounded. □

Finally, we consider the conditions $\frac{\chi_1}{\mu_1} > a_1$, $\frac{\chi_2}{\mu_2} > a_2$ and $\frac{\chi_3}{\mu_2} < 1$.

Lemma 2.4. *Let $\tau = 0$ and the nonnegative initial data $(u_0, w_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$. Assume that $\frac{\chi_1}{\mu_1} > a_1$, $\frac{\chi_2}{\mu_2} > a_2$ and $\frac{\chi_3}{\mu_2} < 1$ as well as $\left(\frac{\chi_1}{\mu_1} - a_1\right)\left(\frac{\chi_2}{\mu_2} - a_2\right) + \frac{\chi_3}{\mu_2} < 1$. Then $T_{max} = \infty$ and u, v, w, z are bounded in $\Omega \times (0, \infty)$.*

Proof. The conditions $\frac{\chi_1}{\mu_1} > a_1$, $\frac{\chi_2}{\mu_2} > a_2$, $\frac{\chi_3}{\mu_2} < 1$ and $\left(\frac{\chi_1}{\mu_1} - a_1\right)\left(\frac{\chi_2}{\mu_2} - a_2\right) + \frac{\chi_3}{\mu_2} < 1$ imply that

$$\frac{\chi_1 - a_1\mu_1}{\mu_1} < \frac{\mu_2 - \chi_3}{\chi_2 - a_2\mu_2}. \tag{2.30}$$

Hence, we can choose $\xi > 0$ large enough such that

$$\xi \ge \max\left\{\frac{\mu_1}{\chi_1 - a_1\mu_1}\|u_0\|_{L^\infty(\Omega)}, \|w_0\|_{L^\infty(\Omega)}\right\} \tag{2.31}$$

and

$$\frac{\chi_1 - a_1\mu_1 + \frac{\mu_1}{\xi}}{\mu_1} < \frac{\mu_2 - \chi_3 - \frac{\mu_2}{\xi}}{\chi_2 - a_2\mu_2},$$

which enables us to choose $A > 0$ satisfying

$$\frac{\chi_1 - a_1\mu_1 + \frac{\mu_1}{\xi}}{\mu_1} < A < \frac{\mu_2 - \chi_3 - \frac{\mu_2}{\xi}}{\chi_2 - a_2\mu_2}. \tag{2.32}$$

Hence, (2.32) implies

$$\begin{cases} -A\xi\{\mu_1 - \mu_1 A\xi + (\chi_1 - a_1\mu_1)\xi\} > 0, \\ -\xi\{\mu_2 - (\mu_2 - \chi_3)\xi + (\chi_2 - a_2\mu_2)A\xi\} > 0. \end{cases} \tag{2.33}$$

We again let $\overline{u} = \overline{u}(x,t) := A\xi$ and $\overline{w} = \overline{w}(x,t) := \xi$ for all $(x,t) \in \overline{\Omega} \times [0, T_{max})$, as in (2.9), we deduce

$$\overline{u}(x,0) = A\xi > \frac{\chi_1 - a_1\mu_1}{\mu_1}\xi \ge u_0(x) \ \text{ and } \ \overline{w}(x,0) = \xi \ge w_0(x) \ \text{ for all } \ x \in \Omega. \tag{2.34}$$

Due to u, v, w, z are nonnegative, from (2.4) and (2.5), and in view of (2.33), (2.34) imply that

$$\begin{cases} u_t - \mathcal{L}_1 u - u\{\mu_1 - \mu_1 u + (\chi_1 - a_1\mu_1)w\} \le \overline{u}_t - \mathcal{L}_1\overline{u} \\ -\overline{u}\{\mu_1 - \mu_1\overline{u} + (\chi_1 - a_1\mu_1)\overline{w}\}, \\ w_t - \mathcal{L}_2 w - w\{\mu_2 - (\mu_2 - \chi_3)w + (\chi_2 - a_2\mu_2)u\} \le \overline{w}_t - \mathcal{L}_2\overline{w} \\ -\overline{w}\{\mu_2 - (\mu_2 - \chi_3)\overline{w} + (\chi_2 - a_2\mu_2)\overline{u}\} \end{cases}$$

in $\Omega \times (0, T_{max})$, by the comparison principle for this cooperative systems [28, Proposition 52.22], it follows that

$$u(x,t) \le \overline{u}(x,t) = A\xi \ \text{ and } \ w(x,t) \le \overline{w}(x,t) = \xi \ \text{ for all } \ (x,t) \in \Omega \times (0, T_{max}).$$

Therefore, using the extensibility criterion in Lemma 2.1, we have $T_{max} = \infty$ and u, v, w, z are globally bounded. □

Proof of Theorem 1.1. We only need to use Lemmas 2.2-2.4 to obtain the global bounded solutions of (1.1). Finally, the Hölder continuity of the solution (u, v, w, z) comes from standard parabolic regularity theory [13].

3. **Stabilization.** In this section, we will derive the asymptotic behavior of the solutions to the model (1.1), the ideas mainly come from [1, 21, 38, 39]. To achieve our goals, we first recall the following lemma which is important for the proof of Theorems 1.2 and 1.3 (see [7, Lemma 4.6] or [20, Lemma 3.1]).

Lemma 3.1. *Suppose that $\varphi(x,t) \in C^0(\overline{\Omega} \times [0,\infty))$ and there exist constant $C > 0$ and $\sigma > 0$ such that*

$$\|\varphi(x,t)\|_{C^{\sigma,\frac{\sigma}{2}}(\overline{\Omega}\times[t,t+1])} \le C \text{ for all } t \ge 1.$$

Moreover, assume that there exists some constant $M > 0$ such that

$$\int_0^\infty \int_\Omega (\varphi(x,t) - M)^2 \, dxdt < \infty.$$

Then

$$\varphi(\cdot,t) \to M \text{ in } C^0(\overline{\Omega}) \text{ as } t \to \infty.$$

This following lemma is a straightforward result from [40, Lemma 5.1].

Lemma 3.2. *Let $a_{11}, a_{22}, a_{33}, a_{44}, a_{13}, a_{14}, a_{23} \in \mathbb{R}$ and satisfy*

$$a_{11} > 0, a_{22} > 0, a_{11}a_{22}a_{33} - \frac{a_{22}a_{13}^2}{4} - \frac{a_{11}a_{23}^2}{4} > 0$$

and

$$a_{11}a_{22}a_{33}a_{44} - \frac{a_{22}a_{44}a_{13}^2}{4} - \frac{a_{11}a_{44}a_{23}^2}{4} - \frac{a_{22}a_{33}a_{14}^2}{4} + \frac{a_{14}^2a_{23}^2}{16} > 0.$$

Then there exists $\varepsilon > 0$ such that

$$\begin{aligned} &a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 + a_{13}x_1x_3 + a_{14}x_1x_4 + a_{23}x_2x_3 \\ &\ge \varepsilon(x_1^2 + x_2^2 + x_3^2 + x_4^2) \end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in \mathbb{R}$.

In order to prove stabilization of solutions to (1.1), we will divide the proof into two cases.

3.1. Competitive coexistence: $a_1, a_2 \in (0, 1)$.

In this subsection, we will study the asymptotic behavior of the solution of (1.1) with $a_1, a_2 \in (0, 1)$ based on the following energy functional

$$E_1(t) := a_2\mu_2 \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) + a_1\mu_1 \int_\Omega \left(w - w_* - w_* \ln \frac{w}{w_*}\right), \quad (3.1)$$

where (u_*, v_*, w_*, z_*) is the constant steady state defined by (1.8).

Lemma 3.3. *Under the assumptions of Theorem 1.2, then there exists $\varepsilon > 0$ such that*

$$E_1(t) \ge 0 \quad \text{and} \quad \frac{d}{dt}E_1(t) \le -\varepsilon F_1(t) \quad \text{for all } t > 0, \quad (3.2)$$

where the function $E_1(t)$ defined by (3.1) and $F_1(t)$ satisfies

$$F_1(t) := \int_\Omega (u-u_*)^2 + \int_\Omega (v-v_*)^2 + \int_\Omega (w-w_*)^2 + \int_\Omega (z-z_*)^2. \tag{3.3}$$

Proof. From Taylor's formula we see that $E_1(t) \geq 0$ for $t > 0$ (for more details, see [1, Lemma 3.2]). From straightforward calculations we infer that

$$\begin{aligned}
\frac{d}{dt}E_1(t) =& - u_*\mu_2 a_2 \int_\Omega \frac{|\nabla u|^2}{u^2} + u_*\mu_2 a_2\chi_1 \int_\Omega \frac{\nabla u\cdot\nabla v}{u} - w_*\mu_1 a_1 \int_\Omega \frac{|\nabla w|^2}{w^2} \\
&+ w_*\mu_1 a_1\chi_2 \int_\Omega \frac{\nabla w\cdot\nabla z}{w} + w_*\mu_1 a_1\chi_3 \int_\Omega \frac{\nabla w\cdot\nabla v}{w} \\
&+ \mu_1\mu_2 a_1 \int_\Omega (w-w_*)(1-w-a_2u) + \mu_1\mu_2 a_2 \int_\Omega (u-u_*)(1-u-a_1w) \\
=& - \mu_1\mu_2 a_2 \int_\Omega (u-u_*)^2 - \mu_1\mu_2 a_1 \int_\Omega (w-w_*)^2 - u_*\mu_2 a_2 \int_\Omega \frac{|\nabla u|^2}{u^2} \\
&- 2\mu_1\mu_2 a_1 a_2 \int_\Omega (u-u_*)(w-w_*) + u_*\mu_2 a_2\chi_1 \int_\Omega \frac{\nabla u\cdot\nabla v}{u} \\
&- w_*\mu_1 a_1 \int_\Omega \frac{|\nabla w|^2}{w^2} + w_*\mu_1 a_1\chi_2 \int_\Omega \frac{\nabla w\cdot\nabla z}{w} + w_*\mu_1 a_1\chi_3 \int_\Omega \frac{\nabla w\cdot\nabla v}{w}.
\end{aligned} \tag{3.4}$$

In order to obtain (3.2), we first deal with the last five parts on the right of (3.4), by a simple computation we conclude

$$\begin{aligned}
& - u_*\mu_2 a_2 \int_\Omega \frac{|\nabla u|^2}{u^2} + u_*\mu_2 a_2\chi_1 \int_\Omega \frac{\nabla u\cdot\nabla v}{u} \\
=& - u_*\mu_2 a_2 \int_\Omega \left(\frac{\nabla u}{u} - \frac{\chi_1}{2}\nabla v\right)^2 + \frac{u_*\mu_2 a_2\chi_1^2}{4}\int_\Omega |\nabla v|^2 \\
\leq& \frac{u_*\mu_2 a_2\chi_1^2}{4}\int_\Omega |\nabla v|^2
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
& - w_*\mu_1 a_1 \int_\Omega \frac{|\nabla w|^2}{w^2} + w_*\mu_1 a_1\chi_2 \int_\Omega \frac{\nabla w\cdot\nabla z}{w} + w_*\mu_1 a_1\chi_3 \int_\Omega \frac{\nabla w\cdot\nabla v}{w} \\
=& - w_*\mu_1 a_1 \int_\Omega \left(\frac{\nabla w}{w} - \frac{\chi_2}{2}\nabla z - \frac{\chi_3}{2}\nabla v\right)^2 + \frac{w_*\mu_1 a_1\chi_2^2}{4}\int_\Omega |\nabla z|^2 \\
&+ \frac{w_*\mu_1 a_1\chi_3^2}{4}\int_\Omega |\nabla v|^2 + \frac{w_*\mu_1 a_1\chi_2\chi_3}{2}\int_\Omega \nabla v\cdot\nabla z \\
\leq& \frac{w_*\mu_1 a_1\chi_2^2}{4}\int_\Omega |\nabla z|^2 + \frac{w_*\mu_1 a_1\chi_3^2}{4}\int_\Omega |\nabla v|^2 + \frac{w_*\mu_1 a_1\chi_2\chi_3}{2}\int_\Omega \nabla v\cdot\nabla z \\
\leq& \frac{w_*\mu_1 a_1\chi_2^2}{2}\int_\Omega |\nabla z|^2 + \frac{w_*\mu_1 a_1\chi_3^2}{2}\int_\Omega |\nabla v|^2.
\end{aligned} \tag{3.6}$$

Since $v_* = w_*$, by the second equation in (1.1) we have

$$\int_\Omega |\nabla v|^2 = -\int_\Omega (v-v_*)^2 + \int_\Omega (w-w_*)(v-v_*). \tag{3.7}$$

Similar to the fourth equation in (1.1) yields

$$\int_\Omega |\nabla z|^2 = -\int_\Omega (z-z_*)^2 + \int_\Omega (u-u_*)(z-z_*). \tag{3.8}$$

Inserting (3.5)-(3.8) into (3.4) we conclude

$$\begin{aligned}\frac{d}{dt}E_1(t) \le & -a_2\mu_1\mu_2\int_\Omega (u-u_*)^2 - a_1\mu_1\mu_2\int_\Omega (w-w_*)^2 \\ & -\left(\frac{a_2\mu_2\chi_1^2u_*}{4}+\frac{w_*\mu_1a_1\chi_3^2}{2}\right)\int_\Omega (v-v_*)^2 - \frac{a_1\mu_1\chi_2^2w_*}{2}\int_\Omega (z-z_*)^2 \\ & + \frac{a_1\mu_1\chi_2^2w_*}{2}\int_\Omega (u-u_*)(z-z_*) - 2a_1a_2\mu_1\mu_2\int_\Omega (u-u_*)(w-w_*) \\ & + \left(\frac{a_2\mu_2\chi_1^2u_*}{4}+\frac{w_*\mu_1a_1\chi_3^2}{2}\right)\int_\Omega (w-w_*)(v-v_*).\end{aligned} \tag{3.9}$$

To see (3.2), we will show that there exists $\varepsilon > 0$ such that

$$\begin{aligned}& -a_2\mu_1\mu_2\int_\Omega (u-u_*)^2 - a_1\mu_1\mu_2\int_\Omega (w-w_*)^2 - 2a_1a_2\mu_1\mu_2\int_\Omega (u-u_*)(w-w_*) \\ & -\left(\frac{a_2\mu_2\chi_1^2u_*}{4}+\frac{w_*\mu_1a_1\chi_3^2}{2}\right)\int_\Omega (v-v_*)^2 - \frac{a_1\mu_1\chi_2^2w_*}{2}\int_\Omega (z-z_*)^2 \\ & + \frac{a_1\mu_1\chi_2^2w_*}{2}\int_\Omega (u-u_*)(z-z_*) + \left(\frac{a_2\mu_2\chi_1^2u_*}{4}+\frac{w_*\mu_1a_1\chi_3^2}{2}\right)\int_\Omega (w-w_*)(v-v_*) \\ & \le -\varepsilon\left(\int_\Omega (u-u_*)^2 + \int_\Omega (v-v_*)^2 + \int_\Omega (w-w_*)^2 + \int_\Omega (z-z_*)^2\right).\end{aligned} \tag{3.10}$$

To confirm that the assumptions of Lemma 3.2 are satisfied, let

$$a_{11} := \mu_1\mu_2a_2,\ a_{22} := \frac{a_2\mu_2\chi_1^2u_*}{4}+\frac{w_*\mu_1a_1\chi_3^2}{2},\ a_{33} := \mu_1\mu_2a_1,\ a_{44} := \frac{a_1\mu_1\chi_2^2w_*}{2},$$

$$a_{13} := 2\mu_1\mu_2a_1a_2, \quad a_{14} := -\frac{a_1\mu_1\chi_2^2w_*}{2}, \quad a_{23} := -\frac{a_2\mu_2\chi_1^2u_*}{4}-\frac{w_*\mu_1a_1\chi_3^2}{2}$$

and

$$x_1 := u-u_*, \quad x_2 := v-v_*, \quad x_3 := w-w_*, \quad x_4 := z-z_*.$$

Since $\mu_1, \mu_2, a_2, \chi_1$ and u_* are positive constants, we obtain

$$a_{11} = \mu_1\mu_2a_2 > 0, \quad a_{22} = \frac{a_2\mu_2\chi_1^2u_*}{4}+\frac{w_*\mu_1a_1\chi_3^2}{2} > 0. \tag{3.11}$$

Thanks to (1.6) we have

$$\begin{aligned}& a_{11}a_{22}a_{33} - \frac{a_{22}a_{13}^2}{4} - \frac{a_{11}a_{23}^2}{4} = (\mu_1^2\mu_2^2a_1a_2 - \mu_1^2\mu_2^2a_1^2a_2^2)\left(\frac{a_2\mu_2\chi_1^2u_*}{4}+\frac{w_*\mu_1a_1\chi_3^2}{2}\right) \\ & -\frac{\mu_1\mu_2a_2}{4}\left(\frac{a_2\mu_2\chi_1^2u_*}{4}+\frac{w_*\mu_1a_1\chi_3^2}{2}\right)^2 \\ & = \mu_1\mu_2a_2\left(\frac{a_2\mu_2\chi_1^2u_*}{4}+\frac{w_*\mu_1a_1\chi_3^2}{2}\right)\left(\mu_1\mu_2a_1(1-a_1a_2) - \frac{a_2\mu_2\chi_1^2u_*}{16} - \frac{w_*\mu_1a_1\chi_3^2}{8}\right) \\ & > 0.\end{aligned} \tag{3.12}$$

Using (1.7), we obtain

$$\begin{aligned}
&a_{11}a_{22}a_{33}a_{44}-\frac{a_{22}a_{44}a_{13}^2}{4}-\frac{a_{11}a_{44}a_{23}^2}{4}-\frac{a_{22}a_{33}a_{14}^2}{4}+\frac{a_{14}^2a_{23}^2}{16}\\
&=\frac{\mu_1^2\mu_2a_1a_2\chi_2^2w_*}{2}\left(\frac{a_2\mu_2\chi_1^2u_*}{4}+\frac{w_*\mu_1a_1\chi_3^2}{2}\right)\Big(\mu_1\mu_2a_1(1-a_1a_2)\\
&-\frac{a_2\mu_2\chi_1^2u_*}{16}-\frac{w_*\mu_1a_1\chi_3^2}{8}\Big)\\
&+\frac{\mu_1^2a_1^2\chi_2^4w_*^2}{16}\left(\frac{a_2\mu_2\chi_1^2u_*}{4}+\frac{w_*\mu_1a_1\chi_3^2}{2}\right)\left(\frac{a_2\mu_2\chi_1^2u_*}{16}+\frac{w_*\mu_1a_1\chi_3^2}{8}-\mu_1\mu_2a_1\right)\\
&=\frac{\mu_1^2a_1\chi_2^2w_*}{2}\left(\frac{a_2\mu_2\chi_1^2u_*}{4}+\frac{w_*\mu_1a_1\chi_3^2}{2}\right)\Big\{\mu_1\mu_2^2a_1a_2(1-a_1a_2)-\frac{a_2^2\mu_2^2\chi_1^2u_*}{16}\\
&-\frac{\mu_1\mu_2a_1a_2\chi_3^2w_*}{8}+\frac{\mu_2a_1a_2\chi_1^2\chi_2^2u_*w_*}{128}+\frac{\mu_1a_1^2\chi_2^2\chi_3^2w_*}{64}-\frac{\mu_1\mu_2a_1^2\chi_2^2w_*}{8}\Big\}>0.
\end{aligned} \tag{3.13}$$

Hence, combining (3.11)-(3.13) and Lemma 3.2 we have (3.10), which concludes the proof of Lemma 3.3. □

Next, we will establish convergence rates for the solution to the model (1.1).

Lemma 3.4. *Under the assumptions of Theorem 1.2, then there exist $C,\mu>0$ such that*

$$\|u-u_*\|_{L^\infty(\Omega)}+\|v-v_*\|_{L^\infty(\Omega)}+\|w-w_*\|_{L^\infty(\Omega)}+\|z-z_*\|_{L^\infty(\Omega)}\le Ce^{-\mu t}$$

for all $t>0$.

Proof. The proof is similar to the corresponding proofs of [20, Lemmas 3.5 and 3.6] or [38, Lemmas 3.4 and 3.5], for the convenience of the readers, we give a sketch the proof. We divide the proof into two steps.

Step 1. We derive the large time behavior of solution to (1.1).

Integrating the second part of (3.2) and using (3.3) to see that

$$\int_1^\infty\int_\Omega(u-u_*)^2+\int_1^\infty\int_\Omega(v-v_*)^2+\int_1^\infty\int_\Omega(w-w_*)^2+\int_1^\infty\int_\Omega(z-z_*)^2\le\frac{E_1(1)}{\varepsilon}.$$

According to Lemma 3.1 and (1.5), we obtain

$$\|u-u_*\|_{L^\infty(\Omega)}+\|v-v_*\|_{L^\infty(\Omega)}+\|w-w_*\|_{L^\infty(\Omega)}+\|z-z_*\|_{L^\infty(\Omega)}\to0\text{ as }t\to\infty. \tag{3.14}$$

Step 2. We derive the convergence rates of solution to (1.1).

Let $H(s):=s-u_*\ln s$ for $s>0$, by L'Hôpital's theorem to see that

$$\lim_{s\to u_*}\frac{H(s)-H(u_*)}{(s-u_*)^2}=\lim_{s\to u_*}\frac{H'(s)}{2(s-u_*)}=\frac{1}{2u_*}. \tag{3.15}$$

Hence, using (3.14) and (3.15) and the definitions $E_1(t)$ in (3.1), there exists $t_1>0$ such that

$$\begin{aligned}
\min\left\{\frac{a_2\mu_2}{4u_*},\frac{a_1\mu_1}{4w_*}\right\}\left(\int_\Omega(u-u_*)^2+\int_\Omega(w-w_*)^2\right)&\le E_1(t)\\
&\le\max\left\{\frac{a_2\mu_2}{u_*},\frac{a_1\mu_1}{w_*}\right\}\left(\int_\Omega(u-u_*)^2+\int_\Omega(w-w_*)^2\right)
\end{aligned} \tag{3.16}$$

for all $t > t_1$. Using (3.16) and (3.2) and the definitions $F_1(t)$ in (3.3), one can find some $c_1 > 0$ such that

$$\frac{d}{dt}E_1(t) \le -c_1\varepsilon E_1(t) \quad \text{for all } t > t_1,$$

which implies

$$\|u-u_*\|_{L^2(\Omega)} + \|w-w_*\|_{L^2(\Omega)} \le c_2 e^{-c_1\varepsilon t} \quad \text{for all } t>0 \tag{3.17}$$

with some $c_2 > 0$. Then by the Gagliardo-Nirenberg inequality with some $C_{GN} > 0$

$$\|\varphi\|_{L^\infty(\Omega)} \le C_{GN}\|\varphi\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|\varphi\|_{L^2(\Omega)}^{\frac{2}{n+2}} \quad \text{for all } \varphi \in W^{1,\infty}(\Omega), \tag{3.18}$$

we can find some $c_3, \mu > 0$ such that

$$\|u-u_*\|_{L^\infty(\Omega)} + \|w-w_*\|_{L^\infty(\Omega)} \le c_3 e^{-\mu t} \text{ for all } t>0. \tag{3.19}$$

Then by the application of the elliptic maximum principle (see more details (3.21)-(3.23) in [38]) enables us to obtain

$$\|v-v_*\|_{L^\infty(\Omega)} + \|z-z_*\|_{L^\infty(\Omega)} \le c_4 e^{-\mu t} \text{ for all } t>0 \tag{3.20}$$

with some $c_4 > 0$. Hence, combining (3.19) and (3.20) we can complete the proof of this lemma. ☐

3.2. Competitive exclusion: $a_1 \ge 1 > a_2$.

When $a_1 \ge 1 > a_2$, we will prove the competitive exclusion will occur based on the following energy functional

$$E_2(t) := a_2\mu_2\int_\Omega u + a_1'\mu_1\int_\Omega (w-1-\ln w), \tag{3.21}$$

where $a_1' \in [1, a_1]$ satisfies $a_1' a_2 < 1$.

Lemma 3.5. *Under the assumptions of Theorem 1.3, then there exists $\varepsilon > 0$ such that*

$$E_2(t) \ge 0 \quad \textit{and} \quad \frac{d}{dt}E_2(t) \le -\varepsilon F_2(t) - \mu_1\mu_2 a_2(a_1'-1)\int_\Omega u \quad \textit{for all } t>0, \tag{3.22}$$

where $E_2(t)$ is defined by (3.21) and

$$F_2(t) := \int_\Omega u^2 + \int_\Omega (w-1)^2 + \int_\Omega z^2.$$

Proof. Using Taylor's formula to see that $\int_\Omega (w-1-\ln w) \ge 0$ for all $t>0$ (for more details, see [1, Lemma 3.2]), then using u is nonnegative, which implies $E_2(t) \ge 0$ for all $t > 0$. By the straightforward calculations we infer

$$\begin{aligned} \frac{d}{dt}\int_\Omega u =& \mu_1\int_\Omega u(1-u-a_1 w) \\ =& -\mu_1\int_\Omega u^2 - \mu_1 a_1'\int_\Omega u(w-1) - \mu_1(a_1'-1)\int_\Omega u - \mu_1(a_1-a_1')\int_\Omega uw. \end{aligned} \tag{3.23}$$

Using the third equation in (1.1), we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega (w-1-\ln w) =& -\int_\Omega \frac{|\nabla w|^2}{w^2} + \chi_2\int_\Omega \frac{\nabla w\cdot\nabla z}{w} + \chi_3\int_\Omega \frac{\nabla w\cdot\nabla v}{w} \\ & - \mu_2\int_\Omega (w-1)^2 - a_2\mu_2\int_\Omega u(w-1). \end{aligned} \tag{3.24}$$

Combining (3.23) and (3.24), we have

$$\begin{aligned}\frac{d}{dt}E_2(t) \le& -\mu_1\mu_2 a_2\int_\Omega u^2 - 2\mu_1\mu_2 a_1' a_2\int_\Omega u(w-1) - \mu_1\mu_2 a_2(a_1'-1)\int_\Omega u\\ &- \mu_1\mu_2 a_1'\int_\Omega (w-1)^2 - \mu_1 a_1'\int_\Omega \frac{|\nabla w|^2}{w^2} + \mu_1 a_1'\chi_2\int_\Omega \frac{\nabla w\cdot\nabla z}{w}\\ &+ \mu_1 a_1'\chi_3\int_\Omega \frac{\nabla w\cdot \nabla v}{w}.\end{aligned} \tag{3.25}$$

Using Young's inequality we infer that

$$\begin{aligned}\mu_1 a_1'\chi_2\int_\Omega \frac{\nabla w\cdot\nabla z}{w} + \mu_1 a_1'\chi_3\int_\Omega \frac{\nabla w\cdot\nabla v}{w} \le& \mu_1 a_1'\int_\Omega \frac{|\nabla w|^2}{w^2} + \frac{\mu_1 a_1'\chi_3^2}{2}\int_\Omega |\nabla v|^2\\ &+ \frac{\mu_1 a_1'\chi_2^2}{2}\int_\Omega |\nabla z|^2.\end{aligned} \tag{3.26}$$

Multiplying the second equation in (1.1) with $\frac{\mu_1 a_1'\chi_3^2}{2}(v-1)$, we have

$$\frac{\mu_1 a_1'\chi_3^2}{2}\int_\Omega |\nabla v|^2 = -\frac{\mu_1 a_1'\chi_3^2}{2}\int_\Omega (v-1)^2 + \frac{\mu_1 a_1'\chi_3^2}{2}\int_\Omega (w-1)(v-1). \tag{3.27}$$

Testing the fourth equation in (1.1) with $\frac{\mu_1 a_1'\chi_2^2}{2}z$, we see that

$$\frac{\mu_1 a_1'\chi_2^2}{2}\int_\Omega |\nabla z|^2 = -\frac{\mu_1 a_1'\chi_2^2}{2}\int_\Omega z^2 + \frac{\mu_1 a_1'\chi_2^2}{2}\int_\Omega uz. \tag{3.28}$$

Substituting (3.26)-(3.28) into (3.25), we have

$$\begin{aligned}\frac{d}{dt}E_2(t) \le& -\mu_1\mu_2 a_2\int_\Omega u^2 - 2\mu_1\mu_2 a_1' a_2\int_\Omega u(w-1) - \mu_1\mu_2 a_2(a_1'-1)\int_\Omega u\\ &- \mu_1\mu_2 a_1'\int_\Omega (w-1)^2 - \frac{\mu_1 a_1'\chi_3^2}{2}\int_\Omega (v-1)^2 + \frac{\mu_1 a_1'\chi_3^2}{2}\int_\Omega (w-1)(v-1)\\ &- \frac{\mu_1 a_1'\chi_2^2}{2}\int_\Omega z^2 + \frac{\mu_1 a_1'\chi_2^2}{2}\int_\Omega uz.\end{aligned} \tag{3.29}$$

In order to prove (3.22), we will show that there exists $\varepsilon > 0$ such that

$$\begin{aligned}&-\mu_1\mu_2 a_2\int_\Omega u^2 - 2\mu_1\mu_2 a_1' a_2\int_\Omega u(w-1) - \mu_1\mu_2 a_1'\int_\Omega (w-1)^2 - \frac{\mu_1 a_1'\chi_2^2}{2}\int_\Omega z^2\\ &- \frac{\mu_1 a_1'\chi_3^2}{2}\int_\Omega (v-1)^2 + \frac{\mu_1 a_1'\chi_3^2}{2}\int_\Omega (w-1)(v-1) + \frac{\mu_1 a_1'\chi_2^2}{2}\int_\Omega uz\\ &\le -\varepsilon\left(\int_\Omega u^2 + \int_\Omega (v-1)^2 + \int_\Omega (w-1)^2 + \int_\Omega z^2\right).\end{aligned} \tag{3.30}$$

Then using the same argument as in the proof of Lemma 3.4, we put

$$a_{11} := \mu_1\mu_2 a_2,\quad a_{22} := \frac{\mu_1 a_1'\chi_3^2}{2},\quad a_{33} := \mu_1\mu_2 a_1',\quad a_{44} := \frac{\mu_1 a_1'\chi_2^2}{2},$$

$$a_{13} := 2\mu_1\mu_2 a_1' a_2,\quad a_{14} := -\frac{\mu_1 a_1'\chi_2^2}{2},\quad a_{23} := -\frac{\mu_1 a_1'\chi_3^2}{2}$$

and

$$x_1 := u,\quad x_2 := v-1,\quad x_3 := w-1,\quad x_4 := z.$$

Since $\mu_1, \mu_2, a_1', a_2, \chi_3$ are positive constants, we obtain

$$a_{11} = \mu_1\mu_2 a_2 > 0, \quad a_{22} = \frac{\mu_1 a_1' \chi_3^2}{2} > 0. \tag{3.31}$$

Thanks to the second part of (1.9) we have

$$\begin{aligned} & a_{11}a_{22}a_{33} - \frac{a_{22}a_{13}^2}{4} - \frac{a_{11}a_{23}^2}{4} \\ =& (\mu_1^2\mu_2^2 a_1' a_2 - \mu_1^2\mu_2^2 a_1'^2 a_2^2)\frac{\mu_1 a_1'\chi_3^2}{2} - \frac{\mu_1\mu_2 a_2}{4}\left(\frac{\mu_1 a_1'\chi_3^2}{2}\right)^2 \\ =& \mu_1\mu_2 a_2 \left(\frac{\mu_1 a_1'\chi_3^2}{2}\right)\left(\mu_1\mu_2 a_1'(1-a_1'a_2) - \frac{\mu_1 a_1'\chi_3^2}{8}\right) > 0. \end{aligned} \tag{3.32}$$

By the first part of (1.9), we infer that

$$\begin{aligned} & a_{11}a_{22}a_{33}a_{44} - \frac{a_{22}a_{44}a_{13}^2}{4} - \frac{a_{11}a_{44}a_{23}^2}{4} - \frac{a_{22}a_{33}a_{14}^2}{4} + \frac{a_{14}^2 a_{23}^2}{16} \\ =& \mu_1\mu_2 a_2 \left(\frac{\mu_1^2 a_1'^2 \chi_2^2\chi_3^2}{4}\right)\left(\mu_1\mu_2 a_1'(1-a_1'a_2) - \frac{\mu_1 a_1'\chi_3^2}{8}\right) \\ &+ \frac{\mu_1^3 a_1'^3 \chi_2^4 \chi_3^2}{32}\left(\frac{\mu_1 a_1'\chi_3^2}{8} - \mu_1\mu_2 a_1'\right) \\ =& \frac{\mu_1^4 a_1'^3 \chi_2^2\chi_3^2}{4}\left\{\mu_2^2 a_2(1-a_1'a_2) - \frac{a_2\mu_2\chi_3^2}{8} + \frac{a_1'\chi_2^2\chi_3^2}{64} - \frac{\mu_2 a_1'\chi_2^2}{8}\right\} > 0. \end{aligned} \tag{3.33}$$

Then collecting (3.31)-(3.33) we have (3.30), which implies the end of the proof. □

According ideas come from [20, Lemmas 3.8, 3.9 and 3.10] or [38, Lemmas 3.7 and 3.8], we shall give the convergence rates for the case $a_1 \geq 1 > a_2$.

Lemma 3.6. *Under the assumptions of Theorem 1.3.*
(i) Let $a_1 > 1$ and $a_2 \in (0,1)$, then there exist $C > 0$ and $\lambda > 0$ such that

$$\|u\|_{L^\infty(\Omega)} + \|v-1\|_{L^\infty(\Omega)} + \|w-1\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \quad \textit{for all } t > 0.$$

(ii) Suppose that $a_1 = 1$ and $a_2 \in (0,1)$, then there exist $C > 0$ and $\kappa > 0$ such that

$$\|u\|_{L^\infty(\Omega)} + \|v-1\|_{L^\infty(\Omega)} + \|w-1\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\Omega)} \leq C(1+t)^{-\kappa} \quad \textit{for all } t > 0.$$

Proof. (i) This part can be proved by a similar proof in Lemma 3.4.

(ii) Since $a' = 1$, then the second part in (3.22) can be rewritten as

$$\frac{d}{dt}E_2(t) \leq -\varepsilon F_2(t) \quad \text{for all } t > 0. \tag{3.34}$$

Similar to (3.14), integrating (3.34) and using Lemma 3.1 enable us to obtain

$$\|u\|_{L^\infty(\Omega)} + \|v-1\|_{L^\infty(\Omega)} + \|w-1\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \tag{3.35}$$

Hence, by the definition of $E_2(t)$ in (3.21), using L'Hôpital's theorem and Hölder's inequality, we can find $t_1 > 0$ such that

$$\begin{aligned} E_2(t) \leq & a_2\mu_2|\Omega|^{\frac{1}{2}}\left(\int_\Omega u^2\right)^{\frac{1}{2}} + \mu_1|\Omega|^{\frac{1}{2}}\left(\int_\Omega (w-1)^2\right)^{\frac{1}{2}} \\ \leq & 2|\Omega|^{\frac{1}{2}}\max\{a_2\mu_2, \mu_1\}\left(\int_\Omega u^2 + \int_\Omega (w-1)^2\right)^{\frac{1}{2}} \quad \text{for all } t > t_1. \end{aligned} \tag{3.36}$$

Combining (3.34) and (3.36), using the definition of $F_2(t)$ in Lemma 3.5, one can find some $C_1 > 0$ such that

$$\frac{d}{dt}E_2(t) \leq -C_1\varepsilon E_2^2(t) \quad \text{for all } t > t_1,$$

which implies there exists $C_2 > 0$ such that

$$E_2(t) \leq \frac{C_2}{t+1} \quad \text{for all } t > t_1. \tag{3.37}$$

By L'Hôpital's theorem and (3.37), we can find $C_2 > 0$ such that

$$\int_\Omega u + \int_\Omega (w-1)^2 \leq \frac{C_2}{t+1} \quad \text{for all } t > 0.$$

By (1.5) and the Gagliardo-Nirenberg inequality, we can find $C_3, \kappa > 0$ such that

$$\|u\|_{L^\infty(\Omega)} + \|w-1\|_{L^\infty(\Omega)} \leq \frac{C_3}{(t+1)^\kappa} \quad \text{for all } t > 0.$$

Using the application of the elliptic maximum principle (see more details (3.21)-(3.23) in [38]) implies

$$\|v-1\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\Omega)} \leq \frac{C_4}{(t+1)^\kappa} \quad \text{for all } t > 0$$

with some $C_4 > 0$. Hence, the proof of Lemma 3.6 is completed. □

Proof of Theorem 1.2. The proof of Theorem 1.2 follows from Lemma 3.4.

Proof of Theorem 1.3. We only need to use Lemma 3.6 to obtain the proof of Theorem 1.3.

Acknowledgments. The authors are very grateful to the anonymous reviewers for their carefully reading and valuable suggestions which greatly improved this work.

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Received August 2021; revised September 2021; early access October 2021.

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