ELECTRONIC RESEARCH ARCHIVE
Volume **29**, Number **6**, December **2021**

doi:10.3934/era.2021085
pp. **4269–4296**

OPTIMAL CONTROL FOR THE COUPLED CHEMOTAXIS-FLUID MODELS IN TWO SPACE DIMENSIONS

Yunfei Yuan and Changchun Liu*

Department of Mathematics, Jilin University
Changchun 130012, China

(Communicated by Bin Liu)

Abstract. This paper deals with a distributed optimal control problem to the coupled chemotaxis-fluid models. We first explore the global-in-time existence and uniqueness of a strong solution. Then, we define the cost functional and establish the existence of Lagrange multipliers. Finally, we derive some extra regularity for the Lagrange multiplier.

1. **Introduction.** In this paper, we study the coupled chemotaxis-fluid models with the initial-bounary conditions

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla c) + \gamma n - \mu n^2, & \text{in } Q \equiv (0,T)\times\Omega,\\ c_t + u\cdot\nabla c = \Delta c - c + n + f, & \text{in } Q,\\ u_t + u\cdot\nabla u = \Delta u - \nabla\pi + n\nabla\varphi, & \text{in } Q,\\ \nabla\cdot u = 0, & \text{in } Q,\\ \dfrac{\partial n}{\partial\nu} = \dfrac{\partial c}{\partial\nu} = 0, \quad u = 0, & \text{on } (0,T)\times\partial\Omega,\\ n(x,0) = n_0(x), c(x,0) = c_0(x), u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. ν is the outward normal vector to $\partial\Omega$, and γ, μ are positive constants. n, c denote the bacterial density, the oxygen concentration, respectively. u, π are the fluid velocity and the associated pressure. Here, the function f denotes a control that acts on chemical concentration, which lies in a closed convex set $\mathcal{U}$. We observe that in the subdomains where $f \geq 0$ we inject oxygen, and conversely where $f \leq 0$ we extract oxygen.

In order to understand the development of system (1.1), let us mention some previous contributions in this direction. Jin [11] dealed with the time periodic problem of (1.1) in spatial dimension $n = 2, 3$. Jin [12] also obtained the existence of large time periodic solution in $\Omega \subset R^3$ without the term $u \cdot \nabla u$.

2020 *Mathematics Subject Classification.* Primary: 92C17, 49J20; Secondary: 49K20, 35K51.

Key words and phrases. Chemotaxis-fluid models, optimal control, Lagrange multipliers.

This work is supported by the Jilin Scientific and Technological Development Program (no. 20210101466JC).

* Corresponding author: Changchun Liu.

Espejo and Suzuki [6] discussed the chemotaxis-fluid model

$$n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) + n(\gamma - \mu n), \tag{1.2}$$

$$c_t + u \cdot \nabla c = \Delta c - c + n, \tag{1.3}$$

$$u_t = \Delta u - \nabla \pi + n \nabla \varphi, \tag{1.4}$$

$$\nabla \cdot u = 0, \tag{1.5}$$

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, \quad u = 0. \tag{1.6}$$

They proved the global existence of weak solution. Tao and Winkler [17] proved the existence of global classical solution and the uniform boundedness. Tao and Winkler [18] also obtained the global classical solution and uniform boundedness under the condition of $\mu > 23$.

The optimal control problems governed by the coupled partial differential equations is important. Colli et al. [4] studied the distributed control problem for a phase-field system of conserved type with a possibly singular potential. Liu and Zhang [14] considered the optimal control of a new mechanochemical model with state constraint. Chen et al. [3] studied the distributed optimal control problem for the coupled Allen-Cahn/Cahn-Hilliard equations. Recently, Guillén-González et al. [9] studied a bilinear optimal control problem for the chemo-repulsion model with the linear production term. The existence, uniqueness and regularity of strong solutions of this model are deduced. They also derived the first-order optimality conditions by using a Lagrange multipliers theorem. Frigeri et al. [8] studied an optimal control problem for two-dimensional nonlocal Cahn-Hilliard-Navier-Stokes systems with degenerate mobility and singular potential. Some other results can be found in [2, 5, 13, 15, 19].

In this paper, we discuss the optimal control problem for (1.1). We adjust the external source f, so that the bacterial density n, oxygen concentration c and fluid velocity u are as close as possible to a desired state n_d, c_d and u_d, and at the final moment T is as close as possible to a desired state n_Ω, c_Ω and u_Ω. The main difficulties for treating the problem are caused by the nonlinearity of $u \cdot \nabla u$. Our method is based on fixed point method and Simon's compactness results. We overcome the above difficulties and derive first-order optimality conditions by using a Lagrange multipliers theorem.

2. Basic estimates of linearized problem.

In this section, we will construct the existence and some priori estimates of the linearized problem for the chemotaxis-Navier-Stokes system in a bounded domain $\Omega \subset \mathbb{R}^2$. The proofs in this section will be established for a detailed framework.

In the following lemmas we will state the Gagliardo-Nirenberg interpolation inequality [7].

Lemma 2.1. *Let l and k be two integers satisfying $0 \leqslant l < k$. Suppose that $1 \leqslant q$, $r \leqslant \infty$, $p > 0$ and $\frac{l}{k} \leqslant a \leqslant 1$ such that*

$$\frac{1}{p} - \frac{l}{N} = a\left(\frac{1}{q} - \frac{k}{N}\right) + (1-a)\frac{1}{r}. \tag{2.1}$$

Then, for any $u \in W^{k,q}(\Omega) \cap L^r(\Omega)$, there exist two positive constants C_1 and C_2 depending only on Ω, q, k, r and N such that the following inequality holds

$$\left\|D^l u\right\|_{L^p} \leqslant c_1 \left\|D^k u\right\|_{L^q}^a \|u\|_{L^r}^{1-a} + c_2 \|u\|_{L^r}$$

with the following exception: If $1 < q < \infty$ *and* $k - l - \frac{N}{q}$ *is a non-negative integer, the (2.1) holds only for a satisfying* $\frac{l}{k} \leq a < 1$.

The following log-interpolation inequality has been proved by [1].

Lemma 2.2. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with smooth boundary. Then for all non-negative* $u \in H^1(\Omega)$*, there holds*

$$\|u\|_{L^3(\Omega)}^3 \leq \delta \|u\|_{H^1(\Omega)}^2 \|(u+1)\log(u+1)\|_{L^1(\Omega)} + p\left(\delta^{-1}\right)\|u\|_{L^1(\Omega)},$$

where δ *is any positive number, and* $p(\cdot)$ *is an increasing function.*

We first consider the existence of solutions to the linear problem of system (1.1). Assume functions $u_0 \in H^1(\Omega)$, $\hat{u} \in L^4\left(0,T;L^4(\Omega)\right), \hat{n} \in L^2\left(0,T;L^2(\Omega)\right)$, and consider

$$\begin{cases} u_t - \Delta u + \hat{u}\cdot\nabla u = -\nabla\pi + \hat{n}\nabla\varphi, & \text{in } Q,\\ \nabla\cdot u = 0, & \text{in } Q,\\ u = 0, & \text{on } \partial\Omega,\\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{2.2}$$

By using fixed point method, the existence of solutions can be easily obtained. Therefore, we ignore the process of proof and just give the regularity estimate.

Lemma 2.3. *Let* $u_0 \in H^1(\Omega)$, $\hat{u} \in L^4\left(0,T;L^4(\Omega)\right)$, $\hat{n} \in L^2\left(0,T;H^1(\Omega)\right), \nabla\varphi \in L^\infty(Q)$*, and u be the solution of the problem (2.2), then* $u \in L^\infty\left(0,T;H^1(\Omega)\right) \cap L^2\left(0,T;H^2(\Omega)\right)$ *and* $u_t \in L^2(0,T;L^2(\Omega))$.

Proof. Multiplying the first equation of (2.2) by u, and integrating it over Ω, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx + \int_\Omega |\nabla u|^2dx + \int_\Omega u^2dx\\ =&\int_\Omega \hat{n}\nabla\varphi\cdot udx + \int_\Omega u^2dx\\ \leq&\|\hat{n}\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2\\ \leq&C(\|\hat{n}\|_{L^2}^2 + \|u\|_{L^2}^2). \end{aligned}$$

By Gronwall's inequality, we have

$$\|u\|_{L^2}^2 + \int_0^T \|u\|_{H^1}^2 d\tau \leq C\left(\int_0^T \|\hat{n}\|_{L^2}^2 d\tau + \|u_0\|_{L^2}^2\right).$$

Operating the Helmholtz projection operator P to the first equation of (2.2), we know

$$u_t + Au + P(\hat{u}\cdot\nabla u) = P(\hat{n}\nabla\varphi),$$

where $A := -P\Delta$ is called Stokes operator, which is an unbounded self-adjoint positive operator in L^2 with compact inverse, for more properties of Stokes operator, we refer to [10]. Note that $\nabla\cdot u = 0$, that is $Pu = u$, $P\Delta u = \Delta u$, $Pu_t = u_t$. So, in following calculations, we ignore the projection operator P. Multiplying this equation by Δu, and integrating it over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2dx + \int_\Omega |\Delta u|^2dx + \int_\Omega |\nabla u|^2dx$$

$$= \int_\Omega P(\hat{u}\nabla u)\Delta u dx - \int_\Omega P(\hat{n}\nabla\varphi)\Delta u dx + \int_\Omega |\nabla u|^2 dx.$$

For the terms on the right, we have

$$\begin{aligned}
&\int_\Omega P(\hat{u}\nabla u)\Delta u dx - \int_\Omega P(\hat{n}\nabla\varphi)\Delta u dx + \int_\Omega |\nabla u|^2 dx \\
\le& \|\hat{u}\|_{L^4}\|\nabla u\|_{L^4}\|\Delta u\|_{L^2} + \|\hat{n}\|_{L^2}\|\Delta u\|_{L^2} + \|\nabla u\|_{L^2}^2 \\
\le& \|\hat{u}\|_{L^4}\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{3/2} + \|\hat{u}\|_{L^4}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2} + \|\hat{n}\|_{L^2}\|\Delta u\|_{L^2} + \|\nabla u\|_{L^2}^2 \\
\le& \frac{1}{2}\|\Delta u\|_{L^2}^2 + C\left(\|\hat{u}\|_{L^4}^4 + \|\hat{u}\|_{L^4}^2 + 1\right)\|\nabla u\|_{L^2}^2 + \|\hat{n}\|_{L^2}^2.
\end{aligned}$$

Therefore, we get

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 \le C\left(\|\hat{u}\|_{L^4}^4 + \|\hat{u}\|_{L^4}^2 + 1\right)\|\nabla u\|_{L^2}^2 + C\|\hat{n}\|_{L^2}^2 + C.$$

By Gronwall's inequality, we derive

$$\|\nabla u\|_{L^2}^2 + \int_0^T \|\nabla u\|_{H^1}^2 d\tau \le C.$$

Multiplying the first equation of (2.2) by u_t, and combining with above inequality, we have

$$\int_0^T \int_\Omega |u_t|^2 dx dt \le C.$$

Summing up, we complete the proof. □

For the above solution u, we consider the following linear problem

$$\begin{cases} c_t - \Delta c + u\cdot\nabla c + c = \hat{n}_+ + f, & \text{in } Q, \\ \dfrac{\partial c}{\partial \nu} = 0, & \text{on } (0,T)\times\partial\Omega, \\ c(x,0) = c_0(x), & \text{in } \Omega. \end{cases} \tag{2.3}$$

Along with fixed point method, the existence of solutions can be easily obtained. Thus we omit the proof and only give the regularity estimate.

Lemma 2.4. *Let $c_0 \in H^2(\Omega)$, $\hat{n} \in L^2\left(0,T;H^1(\Omega)\right)$, $f \in L^2\left(0,T;H^1(\Omega)\right)$, u be the solution of the problem (2.2), and c be the solution of (2.3). Then $c \in L^\infty\left((0,T),H^2(\Omega)\right) \cap L^2\left((0,T),H^3(\Omega)\right)$ and $c_t \in L^2(0,T;L^2(\Omega))$.*

Proof. Multiplying the first equation of (2.3) by c, and integrating it over Ω, we infer from $\int_\Omega c(u\cdot\nabla c) = -\frac{1}{2}\int_\Omega c^2\nabla\cdot u dx = 0$ that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega c^2 dx + \int_\Omega |\nabla c|^2 dx + \int_\Omega c^2 dx \le \|\hat{n}\|_{L^2}\|c\|_{L^2} + \|f\|_{L^2}\|c\|_{L^2}.$$

Therefore, we have

$$\|c\|_{L^2}^2 + \|c\|_{H^1}^2 \le C(\|c_0\|_{L^2}^2 + \int_0^t (\|\hat{n}\|_{L^2}^2 + \|f\|_{L^2}^2) d\tau).$$

Multiplying the first equation of (2.3) by $-\Delta c$, and integrating it over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 dx + \int_\Omega |\Delta c|^2 dx + \int_\Omega |\nabla c|^2 dx \\
=& \int_\Omega u\nabla c\Delta c dx - \int_\Omega \Delta c\hat{n} dx - \int_\Omega \Delta c f dx.
\end{aligned}$$

Using the Young inequality and the Hölder inequality, we obtain

$$
\begin{aligned}
&\int_\Omega u\nabla c\Delta c dx - \int_\Omega \Delta c\hat{n} dx - \int_\Omega \Delta c f dx \\
\leq &\|u\|_{L^4}\|\nabla c\|_{L^4}\|\Delta c\|_{L^2} + \|\hat{n}\|_{L^2}\|\Delta c\|_{L^2} + \|f\|_{L^2}\|\Delta c\|_{L^2} \\
\leq &C\|u\|_{H^1}(\|\nabla c\|_{L^2}^{\frac{1}{2}}\|\Delta c\|_{L^2}^{\frac{1}{2}} + \|\nabla c\|_{L^2})\|\Delta c\|_{L^2} + \|\hat{n}\|_{L^2}\|\Delta c\|_{L^2} + \|f\|_{L^2}\|\Delta c\|_{L^2} \\
= &C\|u\|_{H^1}\|\nabla c\|_{L^2}^{\frac{1}{2}}\|\Delta c\|_{L^2}^{\frac{3}{2}} + C\|\nabla c\|_{L^2}\|\Delta c\|_{L^2} + \|\hat{n}\|_{L^2}\|\Delta c\|_{L^2} + \|f\|_{L^2}\|\Delta c\|_{L^2} \\
\leq &\frac{1}{2}\|\Delta c\|_{L^2}^2 + C\|u\|_{H^1}^4\|\nabla c\|_{L^2}^2 + C(\|\hat{n}\|_{L^2}^2 + \|f\|_{L^2}^2).
\end{aligned}
$$

Combining this and above inequalities, we conclude

$$
\frac{d}{dt}\|\nabla c\|_{L^2}^2 + \|\nabla c\|_{H^1}^2 \leq C\|u\|_{H^1}^4\|\nabla c\|_{L^2}^2 + C(\|\hat{n}\|_{L^2}^2 + \|f\|_{L^2}^2).
$$

We therefore verify that

$$
\|\nabla c\|_{L^2}^2 + \int_0^t \|\nabla c\|_{H^1}^2 \leq C(\int_0^t \|\hat{n}\|_{L^2}^2 d\tau + \int_0^t \|f\|_{L^2}^2 d\tau).
$$

Applying ∇ to the first equation of (2.3), multiplying it by $\nabla\Delta c$, and integrating over Ω give

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta c|^2 dx + \int_\Omega |\nabla\Delta c|^2 dx + \int_\Omega |\Delta c|^2 dx \\
= &\int_\Omega \nabla(u\nabla c)\nabla\Delta c dx - \int_\Omega \nabla\hat{n}_+\nabla\Delta c dx - \int_\Omega \nabla f\nabla\Delta c dx.
\end{aligned}
$$

For the terms on the right, we obtain

$$
\begin{aligned}
&\int_\Omega \nabla(u\nabla c)\nabla\Delta c dx - \int_\Omega \nabla\hat{n}_+\nabla\Delta c dx - \int_\Omega \nabla f\nabla\Delta c dx \\
\leq &\|\nabla\Delta c\|_{L^2}\left(\|u\|_{L^4}\|\Delta c\|_{L^4} + \|\nabla u\|_{L^4}\|\nabla c\|_{L^4}\right) + \|\nabla\hat{n}\|_{L^2}\|\nabla\Delta c\|_{L^2} \\
&+ \|\nabla f\|_{L^2}\|\nabla\Delta c\|_{L^2} \\
\leq &\|\nabla\Delta c\|_{L^2}(\|u\|_{L^4}\|\Delta c\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta c\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^4}\|\Delta c\|_{L^2} \\
&+ \|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{L^2}^{\frac{1}{2}}\|\Delta c\|_{L^2}^{\frac{1}{2}} + \|\nabla u\|_{L^2}\|\nabla c\|_{L^2}^{\frac{1}{2}}\|\Delta c\|_{L^2}^{\frac{1}{2}} \\
&+ \|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla c\|_{L^2}) + \|\nabla\hat{n}\|_{L^2}\|\nabla\Delta c\|_{L^2} \\
&+ \|\nabla f\|_{L^2}\|\nabla\Delta c\|_{L^2} \\
\leq &\frac{1}{2}\|\nabla\Delta c\|_{L^2}^2 + C(1 + \|\Delta c\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|\nabla\hat{n}\|_{L^2}^2 + \|\nabla f\|_{L^2}^2).
\end{aligned}
$$

Straightforward calculations yield

$$
\|\Delta c\|_{L^2}^2 + \int_0^t \|\Delta c\|_{H^1}^2 d\tau \leq C\left(1 + \int_0^t \|\hat{n}\|_{H^1}^2 d\tau + \int_0^t \|f\|_{H^1}^2 d\tau\right).
$$

Multiplying the first equation of (2.3) by c_t, and combining with above inequality, we have

$$
\int_0^T \int_\Omega |c_t|^2 dx dt \leq C,
$$

and thereby precisely arrive at the conclusion. □

With above solutions u and c in hand, we deal with the following linear problem.

$$\begin{cases} n_t - \Delta n + u \cdot \nabla n + n = -\nabla \cdot (n \nabla c) + (1+\gamma)\hat{n}_+ - \mu \hat{n}_+ n, & \text{in } Q, \\ \dfrac{\partial n}{\partial \nu}\Big|_{\partial \Omega} = 0, \\ n(x,0) = n_0(x), & \text{in } \Omega. \end{cases} \tag{2.4}$$

By a similar argument as the above two problems, the existence of solutions can be easily obtained. Therefore, we only give the regularity estimate.

Lemma 2.5. *Suppose $0 \le n_0 \in H^1(\Omega)$, $\hat{n} \in L^2\left(0,T;H^1(\Omega)\right) \cap L^4\left(0,T;L^4(\Omega)\right)$, and u, c, n are the solutions of the problem (2.2), (2.3) and (2.4), respectively. Then $n \ge 0$, $n \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ and $n_t \in L^2(0,T;L^2(\Omega))$.*

Proof. Firstly, we verify the nonnegativity of n. We examine the set $A(t) = \{x : n(x,t) < 0\}$. Along with (2.4), we get

$$\frac{d}{dt}\int_{A(t)} n dx - \int_{\partial A(t)} \frac{\partial n}{\partial \nu} ds + \int_{A(t)} n dx = (1+\gamma)\int_{A(t)} \hat{n}_+ dx - \mu \int_{A(t)} \hat{n}_+ n dx.$$

Since $\frac{\partial n}{\partial \nu} \ge 0$ on $\partial\{n<0\}$, from this we deduce that the right hand side is nonnegative. Integrating this equality on $[0,t]$ gives

$$\int_{A(t)} n dx d\tau + \int_0^t \int_{A(t)} n dx d\tau = 0.$$

Then, we get $n \ge 0$.

Next, multiplying the first equation of (2.4) by n, and integrating it over Ω, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega n^2 dx + \int_\Omega \left(n^2 + |\nabla n|^2\right) dx + \mu \int_\Omega \hat{n}_+ n^2 dx \\ =& \int_\Omega n \nabla c \nabla n dx + (1+\gamma)\int_\Omega n \hat{n}_+ dx \\ \le& \|n\|_{L^4}\|\nabla c\|_{L^4}\|\nabla n\|_{L^2} + (1+\gamma)\|\hat{n}\|_{L^2}\|n\|_{L^2} \\ \le& C(\|n\|_{L^2}^{\frac{1}{2}}\|\nabla n\|_{L^2}^{\frac{1}{2}} + \|n\|_{L^2})\|c\|_{H^2}\|\nabla n\|_{L^2} + (1+\gamma)\|\hat{n}\|_{L^2}\|n\|_{L^2} \\ \le& C(\|n\|_{L^2}^2\|c\|_{H^2}^4 + \|n\|_{L^2}^2\|c\|_{H^2}^2 + \|\hat{n}\|_{L^2}) + \frac{1}{2}\|n\|_{H^1}^2. \end{aligned}$$

So, we derive that

$$\|n\|_{L^2}^2 + \int_0^T \|n\|_{H^1}^2 dt \le C\left(1 + \int_0^T \|\hat{n}\|_{L^2}^2 dt\right).$$

Multiplying the first equation of (2.4) by $-\Delta n$, and integrating it over Ω, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla n|^2 dx + \int_\Omega |\Delta n|^2 dx + \int_\Omega |\nabla n|^2 dx \\ =& \int_\Omega u \nabla n \Delta n dx + \int_\Omega \left(\nabla \cdot (n\nabla c)\Delta n - (1+\gamma)\hat{n}_+ \Delta n + \mu \hat{n}_+ n \Delta n\right) dx \\ \le& \|u\|_{L^4}\|\nabla n\|_{L^4}\|\Delta n\|_{L^2} + \|n\|_{L^4}\|\Delta c\|_{L^4}\|\Delta n\|_{L^2} + \|\nabla n\|_{L^4}\|\nabla c\|_{L^4}\|\Delta n\|_{L^2} \\ &+ (1+\gamma)\|\hat{n}\|_{L^2}\|\Delta n\|_{L^2} + \mu\|n\|_{L^4}\|\hat{n}\|_{L^4}\|\Delta n\|_{L^2} \\ \le& C\|u\|_{H^1}(\|\nabla n\|_{L^2}^{\frac{1}{2}}\|\Delta n\|_{L^2}^{\frac{1}{2}} + \|\nabla n\|_{L^2})\|\Delta n\|_{L^2} \end{aligned}$$

$$
\begin{aligned}
&+\|n\|_{L^4}\left(\|\Delta c\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta c\|_{L^2}^{\frac{1}{2}}+\|\Delta c\|_{L^2}\right)\|\Delta n\|_{L^2}+\mu\|n\|_{L^4}\|\hat{n}\|_{L^4}\|\Delta n\|_{L^2}\\
&+(\|\nabla n\|_{L^2}^{\frac{1}{2}}\|\Delta n\|_{L^2}^{\frac{1}{2}}+\|\nabla n\|_{L^2})\|\nabla c\|_{H^1}\|\Delta n\|_{L^2}+(1+\gamma)\|\hat{n}\|_{L^2}\|\Delta n\|_{L^2}\\
\le&\frac{1}{2}\|\Delta n\|_{L^2}^2+C\left(\|\nabla n\|_{L^2}^2+\|n\|_{L^4}^4+\|\Delta c\|_{L^2}^4+\|\nabla\Delta c\|_{L^2}^2+\|\hat{n}\|_{L^2}^2+\|\hat{n}\|_{L^4}^4\right)\\
\le&\frac{1}{2}\|\Delta n\|_{L^2}^2+C(1+\|\nabla n\|_{L^2}^2+\|n\|_{L^2}^4+\|n\|_{L^2}^2\|\nabla n\|_{L^2}^2+\|\nabla\Delta c\|_{L^2}^2\\
&+\|\hat{n}\|_{L^2}^2+\|\hat{n}\|_{L^4}^4).
\end{aligned}
$$

Straightforward calculations yield

$$\|\nabla n\|_{L^2}^2+\int_0^T\int_\Omega\left(|\Delta n|^2+|\nabla n|^2+\hat{n}_+|\nabla n|^2\right)dxdt\le C.$$

Multiplying the first equation of (2.4) by n_t, and combining with above inequality, we have

$$\int_0^T\int_\Omega|n_t|^2dxdt\le C.$$

The proof is complete. □

Introduce the spaces

$$
\begin{aligned}
&X_u=L^4\left(0,T;L^4(\Omega)\right),\quad X_n=L^4\left(0,T;L^4(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right),\\
&Y_u=L^\infty\left(0,T;H^1(\Omega)\right)\cap L^2\left(0,T;H^2(\Omega)\right),\\
&Y_n=L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega)).
\end{aligned}
$$

Define a map

$$
\begin{aligned}
&\mathcal{F}:X_u\times X_n\to X_u\times X_n,\\
&\mathcal{F}(\hat{u},\hat{n})=(u,n),
\end{aligned}
$$

where the (u,n) is the solution of the decoupled linear problem

$$
\begin{cases}
n_t-\Delta n+u\cdot\nabla n+n=-\nabla\cdot(n\nabla c)+(1+\gamma)\hat{n}_+-\mu\hat{n}_+n, & \text{in } (0,T)\times\Omega\equiv Q,\\
c_t-\Delta c+u\cdot\nabla c+c=\hat{n}_++f, & \text{in } (0,T)\times\Omega\equiv Q,\\
u_t-\Delta u+\hat{u}\cdot\nabla u=-\nabla\pi+\hat{n}\nabla\varphi, & \text{in } (0,T)\times\Omega\equiv Q,\\
\nabla\cdot u=0, & \text{in } (0,T)\times\Omega\equiv Q,\\
\dfrac{\partial n}{\partial\nu}=\dfrac{\partial c}{\partial\nu}=0,\quad u=0, & \text{on } (0,T)\times\partial\Omega,\\
n(x,0)=n_0(x),c(x,0)=c_0(x),u(x,0)=u_0(x), & \text{in } \Omega.
\end{cases}
$$

Next, we use fixed point method to prove the local existence of solutions of the problem (1.1).

Lemma 2.6. *The map $\mathcal{F}:X_u\times X_n\to X_u\times X_n$ is well defined and compact.*

Proof. Let $(\hat{n},\hat{u})\in X_u\times X_n$, by Lemmas 2.3, 2.4, 2.5 we deduce that $(n,u)=\mathcal{F}(\hat{n},\hat{u})$ is bounded in $Y_u\times Y_n$. Note that the embeddings $H^2(\Omega)\hookrightarrow H^1(\Omega)$ is compact and interpolating between $L^\infty(0,T;H^1(\Omega))$ and $L^2(0,T;H^2(\Omega))$. It is easy to get that u is bounded in $L^4\left(0,T;L^4(\Omega)\right)$ and n is bounded in $L^4\left(0,T;L^4(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right)$. Therefore, the operator $\mathcal{F}:X_u\times X_n\to X_u\times X_n$ is a compact operator. □

3. **Existence and uniqueness of strong solution of system.** From Lemma 2.6, $(n,u)\in Y_n\times Y_u$ satisfies pointwisely a.e. in Q the following problem

$$\begin{cases} n_t-\Delta n+u\cdot\nabla n+n=-\nabla\cdot(n\nabla c) & \\ +\alpha(1+\gamma)n-\mu n^2, & \text{in } Q,\\ c_t-\Delta c+u\cdot\nabla c+c=n+\alpha f, & \text{in } Q,\\ u_t-\Delta u+u\cdot\nabla u=-\nabla\pi+\alpha n\nabla\varphi, & \text{in } Q,\\ \nabla\cdot u=0, & \text{in } Q,\\ \dfrac{\partial n}{\partial\nu}=\dfrac{\partial c}{\partial\nu}=0,\quad u=0, & \text{on } (0,T)\times\partial\Omega,\\ n(x,0)=n_0(x),c(x,0)=c_0(x),u(x,0)=u_0(x), & \text{in } \Omega.\end{cases}\tag{3.1}$$

In order to prove the existence of solution, we first give some a priori estimates.

Lemma 3.1. *Let (n,c,u) be a local solution to (3.1). Then, it holds that*

$$\|n\|_{L^1}+\int_0^t(\|n\|_{L^1}+\|n\|_{L^2})d\tau\le C,\tag{3.2}$$

$$\|\nabla u\|_{L^2}^2+\int_0^t\|\nabla u\|_{H^1}^2d\tau\le C,\tag{3.3}$$

$$\|\nabla c\|_{L^2}^2+\int_0^t\|\nabla c\|_{H^1}^2d\tau\le C.\tag{3.4}$$

Proof. With Lemma 2.5 in hand, we get $n\ge0$. Integrating the first equation of (3.1) over Ω, we see that

$$\frac{d}{dt}\int_\Omega ndx+\int_\Omega ndx+\mu\int_\Omega n^2dx=\alpha(1+\gamma)\int_\Omega ndx\le\frac{\mu}{2}\int_\Omega n^2dx+C.$$

Solving this differential inequality, we obtain that

$$\|n\|_{L^1}+\int_0^t(\|n\|_{L^1}+\|n\|_{L^2})d\tau\le C.$$

Multiplying the third equation of (3.1) by u, and integrating it over Ω, we get

$$\begin{aligned}\frac12\frac{d}{dt}\int_\Omega u^2dx+\int_\Omega|\nabla u|^2dx+\int_\Omega u^2dx=&\alpha\int_\Omega n\nabla\varphi\cdot udx+\int_\Omega u^2dx\\ \le&\|n\|_{L^2}\|u\|_{L^2}+\|u\|_{L^2}^2\le C(\|n\|_{L^2}^2+\|u\|_{L^2}^2).\end{aligned}$$

Therefore, we see that

$$\|u\|_{L^2}^2+\int_0^t\|u\|_{H^1}d\tau\le C.$$

By the Gagliardo-Nirenberg interpolation inequality, we deduce that

$$\begin{aligned}\int_0^t\|u\|_{L^4}^4d\tau\le&C\int_0^t(\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2d+\|u\|_{L^2}^2)\tau\\ \le&\|u\|_{L^2}^2\int_0^t\|\nabla u\|_{L^2}^2d\tau+\int_0^t\|u\|_{L^2}^2d\tau\\ \le&C.\end{aligned}$$

Multiplying the third equation of (3.1) by Δu, and integrating it over Ω, we get

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2+\|\nabla u\|_{H^1}^2\le C\left(\|u\|_{L^4}^4+\|u\|_{L^4}^2+1\right)\|\nabla u\|_{L^2}^2+C\|n\|_{L^2}^2+C.$$

Thus, we know

$$\|\nabla u\|_{L^2}^2 + \int_0^t \|\nabla u\|_{H^1}^2 d\tau \le C.$$

Multiplying the second equation of (3.1) by c, and integrating it over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega c^2 dx + \int_\Omega |\nabla c|^2 dx + \int_\Omega c^2 dx \le \|n\|_{L^2}\|c\|_{L^2} + \alpha\|f\|_{L^2}\|c\|_{L^2}.$$

Then, we have

$$\|c\|_{L^2} + \int_0^t \|c\|_{H^1} d\tau \le C.$$

Multiplying the second equation of (3.1) by $-\Delta c$, and integrating it over Ω, we get

$$\frac{d}{dt}\|\nabla c\|_{L^2}^2 + \|\nabla c\|_{H^1}^2 \le C\|u\|_{H^1}^4\|\nabla c\|_{L^2}^2 + C(\|n\|_{L^2}^2 + \|f\|_{L^2}^2).$$

Further, we have

$$\|\nabla c\|_{L^2}^2 + \int_0^t \|\nabla c\|_{H^1}^2 d\tau \le C.$$

The proof is complete. □

Lemma 3.2. *Let (n, c, u) be a local solution to (3.1). Then, it holds that*

$$\|(n+1)\ln(n+1)\|_{L^1} + \|\nabla c\|_{L^2}^2 + \|\nabla c\|_{H^1}^2 \le C. \tag{3.5}$$

Proof. We rewrite the first equation of (3.1) as

$$\begin{aligned}&\frac{d}{dt}(n+1) + u\cdot\nabla(n+1) - \Delta(n+1)\\ =& -\nabla\cdot((n+1)\cdot\nabla c) + \Delta c + \alpha(1+\gamma)n - \mu n^2.\end{aligned}$$

Multiplying the above equation by $\ln(n+1)$ and integrating the equation, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (n+1)\ln(n+1)dx + 4\int_\Omega |\nabla\sqrt{n+1}|^2 dx\\ \le& \int_\Omega \nabla(n+1)\cdot\nabla c dx + \int_\Omega \Delta c\ln(n+1)dx + \alpha(1+\gamma)\int_\Omega n\ln(n+1)dx\\ =& I_1 + I_2 + I_3.\end{aligned}$$

For I_1, integrating by parts and using Young's inequality with small δ, we get

$$I_1 = -\int_\Omega n\Delta c dx \le \|n\|_{L^2}\|\Delta c\|_{L^2} \le \delta\|\Delta c\|_{L^2}^2 + C\|n\|_{L^2}^2.$$

For the term I_2, we have

$$\begin{aligned}I_2 =& \int_\Omega \Delta c\ln(n+1)dx \le \delta\|\Delta c\|_{L^2}^2 + C\|\ln(n+1)\|_{L^2}^2\\ \le& \delta\|\Delta c\|_{L^2}^2 + C\int_\Omega (n+1)\ln(n+1)dx.\end{aligned}$$

For the rest term I_3, straightforward calculations yield

$$I_3 = \alpha(1+\gamma)\int_\Omega n\ln(n+1)dx \le (1+\gamma)\int_\Omega (n+1)\ln(n+1)dx.$$

Combining I_1, I_2 with I_3, we conduct that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (n+1)\ln(n+1)dx + 4\int_\Omega |\nabla\sqrt{n+1}|^2 dx\\ \le&\delta\|\Delta c\|_{L^2}^2 + C\int_\Omega (n+1)\ln(n+1)dx + C\|n\|_{L^2}^2.\end{aligned}\tag{3.6}$$

Multiplying the second equation of (3.1) by Δc, and integrating it over Ω, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 dx + \int_\Omega |\Delta c|^2 dx + \int_\Omega |\nabla c|^2 dx\\ =&\int_\Omega u\nabla c\Delta c dx - \int_\Omega \Delta c n dx - \alpha\int_\Omega \Delta c f dx.\end{aligned}$$

Straightforward calculations yield

$$\frac{d}{dt}\|\nabla c\|_{L^2}^2 + \|\nabla c\|_{H^1}^2 \le C\|\nabla c\|_{L^2}^2 + C(\|n\|_{L^2}^2 + \|f\|_{L^2}^2).\tag{3.7}$$

Combing (3.6) and (3.7), it follows that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (n+1)\ln(n+1)dx + \frac{d}{dt}\|\nabla c\|_{L^2}^2 + (1-\delta)\|\nabla c\|_{H^1}^2 + 4\int_\Omega |\nabla\sqrt{n+1}|^2 dx\\ \le&C\int_\Omega (n+1)\ln(n+1)dx + C(\|f\|_{L^2}^2 + \|n\|_{L^2}^2).\end{aligned}$$

Taking δ small enough, and solving this differential inequality, we obtain that

$$\|(n+1)\ln(n+1)\|_{L^1} + \|\nabla c\|_{L^2}^2 + \|\nabla c\|_{H^1}^2 \le C.$$

The proof is complete. □

Lemma 3.3. *Assume $f\in L^2(0,T;H^1(\Omega))$, let (n,c,u) be a local solution to (3.1). Then, it holds that*

$$\|n\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 + \int_0^t \|n\|_{H^1} d\tau + \int_0^t \|\Delta c\|_{H^1} d\tau \le C.\tag{3.8}$$

Proof. Taking the L^2-inner product with n for the first equation of (3.1) implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega n^2 dx + \int_\Omega \left(n^2 + |\nabla n|^2\right)dx + \mu\int_\Omega n^3 dx\\ =&\int_\Omega n\nabla c\nabla n dx + \alpha(1+\gamma)\int_\Omega n^2 dx\\ =&-\frac{1}{2}\int_\Omega n^2\Delta c dx + \alpha(1+\gamma)\int_\Omega n^2 dx.\end{aligned}$$

Here, we note that

$$\begin{aligned}\left|\int_\Omega n^2\Delta c dx\right| \le&\|n\|_{L^3}^2\|\Delta c\|_{L^3}\\ \le&C\|n\|_{L^3}^2(\|\nabla\Delta c\|_{L^2}^{\frac{2}{3}}\|\nabla c\|_{L^2}^{\frac{1}{3}} + \|\nabla c\|_{L^2})\\ \le&C\|n\|_{L^3}^2(\|\nabla\Delta c\|_{L^2}^{\frac{2}{3}} + 1).\end{aligned}$$

From Lemma 2.2 and (3.2), it follows that

$$-\frac{\chi}{2}\int_\Omega n^2\Delta c dx$$

$$
\begin{aligned}
&\le C\left(\delta\|n\|_{H^1}^2\|(n+1)\log(n+1)\|_{L^1}+p\left(\delta^{-1}\right)\|n\|_{L^1}\right)^{\frac{2}{3}}\left(\|\nabla\Delta c\|_{L^2}^{\frac{2}{3}}+1\right)\\
&\le C\left(\delta\|n\|_{H^1}^2+p\left(\delta^{-1}\right)\right)^{\frac{2}{3}}\left(\|\nabla\Delta c\|_{L^2}^{\frac{2}{3}}+1\right)\\
&\le C(\delta\|n\|_{H^1}^{\frac{4}{3}}\|\nabla\Delta c\|_{L^2}^{\frac{2}{3}}+\delta\|n\|_{H^1}^{\frac{4}{3}}+p^{\frac{2}{3}}\left(\delta^{-1}\right)\|\nabla\Delta c\|_{L^2}^{\frac{2}{3}}+p^{\frac{2}{3}}\left(\delta^{-1}\right))\\
&\le \delta\|\nabla\Delta c\|_{L^2}^2+C\delta^{\frac{1}{2}}\|n\|_{H^1}^2+C_\delta^{-1/2}p\left(\delta^{-1}\right).
\end{aligned}
$$

As an immediate consequence

$$
\frac{d}{dt}\|n\|_{L^2}^2+\|n\|_{H^1}^2\le\delta\|\nabla\Delta c\|_{L^2}^2+C\delta^{\frac{1}{2}}\|n\|_{H^1}^2+C\|n\|_{L^2}^2. \tag{3.9}
$$

Applying ∇ to the first equation of (3.1), multiplying it by $\nabla\Delta c$, and integrating over Ω give

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|\Delta c|^2dx+\int_\Omega|\nabla\Delta c|^2dx+\int_\Omega|\Delta c|^2dx\\
=&\int_\Omega\nabla(u\nabla c)\nabla\Delta c dx-\int_\Omega\nabla n\nabla\Delta c dx-\int_\Omega\nabla f\nabla\Delta c dx=I_4+I_5.
\end{aligned}
$$

For I_4, by using the Gagliardo-Nirenberg interpolation inequality, we get

$$
\begin{aligned}
I_4=&\int_\Omega\nabla(u\nabla c)\nabla\Delta c dx\\
\le&\|\nabla\Delta c\|_{L^2}\left(\|u\|_{L^4}\|\Delta c\|_{L^4}+\|\nabla u\|_{L^4}\|\nabla c\|_{L^4}\right)\\
\le&\|\nabla\Delta c\|_{L^2}(\|u\|_{L^4}\|\Delta c\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta c\|_{L^2}^{\frac{1}{2}}+\|u\|_{L^4}\|\Delta c\|_{L^2}\\
&+\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{L^2}^{\frac{1}{2}}\|\Delta c\|_{L^2}^{\frac{1}{2}}+\|\nabla u\|_{L^2}\|\nabla c\|_{L^2}^{\frac{1}{2}}\|\Delta c\|_{L^2}^{\frac{1}{2}}\\
&+\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{L^2}+\|\nabla u\|_{L^2}\|\nabla c\|_{L^2})\\
\le&\frac{1}{4}\|\nabla\Delta c\|_{L^2}^2+C(1+\|\Delta c\|_{L^2}^2+\|\Delta u\|_{L^2}^2).
\end{aligned}
$$

For the term I_5, we have

$$
\begin{aligned}
I_5=&-\int_\Omega\nabla n\nabla\Delta c dx-\int_\Omega\nabla f\nabla\Delta c dx\\
\le&C(\|\nabla n\|_{L^2}^2+\|\nabla f\|_{L^2}^2)+\frac{1}{4}\|\nabla\Delta c\|_{L^2}^2.
\end{aligned}
$$

Along with I_4 and I_5, we conclude

$$
\begin{aligned}
&\frac{d}{dt}\|\Delta c\|_{L^2}^2+\|\nabla\Delta c\|_{L^2}^2+\|\Delta c\|_{L^2}^2\\
\le&C(1+\|\Delta c\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\|\nabla n\|_{L^2}^2+\|\nabla f\|_{L^2}^2).
\end{aligned} \tag{3.10}
$$

Combining (3.9) and (3.10), it follows that

$$
\begin{aligned}
&\frac{d}{dt}(\|n\|_{L^2}^2+\|\Delta c\|_{L^2}^2)+\|\Delta c\|_{L^2}^2+(1-C\delta^{\frac{1}{2}})\|n\|_{H^1}^2+(1-\delta)\|\nabla\Delta c\|_{L^2}^2\\
\le&C(1+\|\Delta c\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\|\nabla n\|_{L^2}^2+\|\nabla f\|_{L^2}^2).
\end{aligned}
$$

By choosing δ small enough and using (3.3) and (3.5), we have

$$
\|n\|_{L^2}^2+\|\Delta c\|_{L^2}^2+\int_0^t\|n\|_{H^1}d\tau+\int_0^t\|\Delta c\|_{H^1}d\tau\le C.
$$

The proof is complete. □

Lemma 3.4. *Assume $f \in L^2(0,T;H^1(\Omega))$, let (n,c,u) be a local solution to (3.1). Then, it holds that*

$$\|\nabla n\|_{L^2}^2 + \int_0^t \|n\|_{H^2}^2 d\tau \le C. \tag{3.11}$$

Proof. Taking the L^2-inner product with $-\Delta n$ for the first equation of (3.1) implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla n|^2 dx + \int_\Omega |\Delta n|^2 dx + \int_\Omega |\nabla n|^2 dx\\ =&\int_\Omega u\nabla n\Delta n dx + \int_\Omega \nabla\cdot(n\nabla c)\Delta n dx + (1+\gamma)\int_\Omega |\nabla n|^2 dx + \mu\int_\Omega n^2\Delta n dx\\ =&I_6 + I_7 + I_8.\end{aligned}$$

For the term I_6, with the estimate (3.3), we have

$$\begin{aligned}I_6 =&\int_\Omega u\nabla n\Delta n dx = -\frac{1}{2}\int_\Omega \nabla u(\nabla n)^2 dx \le \|\nabla u\|_{L^2}\|\nabla n\|_{L^4}^2\\ \le&\|\nabla u\|_{L^2}(\|\nabla n\|_{L^2}^{\frac{1}{2}}\|\Delta n\|_{L^2}^{\frac{1}{2}} + \|\nabla n\|_{L^2})^2\\ \le&\delta\|\Delta n\|_{L^2}^2 + C\|\nabla n\|_{L^2}^2.\end{aligned}$$

For the term I_7, taking (3.8) into considering, we conduct that

$$\begin{aligned}I_7 =&\int_\Omega \nabla\cdot(n\nabla c)\Delta n dx\\ =&\int_\Omega (\nabla n\nabla c + n\Delta c)\Delta n dx\\ \le&\|\Delta n\|_{L^2}\left(\|\nabla n\|_{L^3}\|\nabla c\|_{L^6} + \|n\|_{\mathcal{C}}\|\Delta c\|_{L^2}\right)\\ \le&C\|\Delta n\|_{L^2}\left(\|\nabla n\|_{H^{\frac{1}{3}}}\|\nabla c\|_{H^1} + \|n\|_{H^{\frac{4}{3}}}\|\Delta c\|_{L^2}\right)\\ \le&C\|n\|_{H^2}\|n\|_{H^{\frac{4}{3}}}\|c\|_{H^2} \le C\|n\|_{H^2}^{\frac{5}{3}}\|n\|_{L^2}^{\frac{1}{3}}\|c\|_{H^2}\\ \le&\delta\|n\|_{H^2}^2 + C(\delta)\|n\|_{L^2}^2\|c\|_{H^2}^6 \le \delta\|n\|_{H^2}^2 + C.\end{aligned}$$

For the term I_8, thanks to the nonnegativity of n, we see that

$$\begin{aligned}I_8 =&(1+\gamma)\int_\Omega |\nabla n|^2 dx + \mu\int_\Omega n^2\Delta n dx\\ =&(1+\gamma)\int_\Omega |\nabla n|^2 dx - 2\mu\int_\Omega |\nabla n|^2 n dx\\ \le&(1+\gamma)\|\nabla n\|_{L^2}^2.\end{aligned}$$

Combine the estimates about I_6, I_7 and I_8, it follows that

$$\frac{d}{dt}\|\nabla n\|_{L^2}^2 + (1-4\delta)\|n\|_{H^2}^2 \le C\|\nabla n\|_{L^2}^2 + C.$$

By taking δ small enough, we get

$$\|\nabla n\|_{L^2}^2 + \int_0^t \|n\|_{H^2}^2 d\tau \le C.$$

Therefore, this proof is complete. □

Lemma 3.5. *The operator $\mathcal{F}: X_u \times X_n \to X_u \times X_n$, is continuous.*

Proof. Let $\{(\hat{n}_m, \hat{u}_m)\}_{m\in\mathbb{N}}$ be a sequence of $X_u \times X_n$, Then, with Lemmas 2.3, 2.4 and 2.5 in hand, we conduct that $\{(n_m, u_m) = \mathcal{F}(\hat{n}_m, \hat{u}_m)\}_{m\in\mathbb{N}}$ is bounded in $Y_u \times Y_n$. Taking the compactness of $Y_u \times Y_n$ in $X_u \times X_n$ into consider, we see that $\mathcal{F}$ is a compact operator, which means there exists a subsequence of $\{\mathcal{F}(\hat{n}_m, \hat{u}_m)\}_{m\in\mathbb{N}}$, for convenience, still denoted as $\{\mathcal{F}(\hat{n}_m, \hat{u}_m)\}_{m\in\mathbb{N}}$, and exists an element $(\hat{n}, \hat{u})$ in $Y_u \times Y_n$ such that

$$\mathcal{F}(\hat{n}_m, \hat{u}_m) \to (\hat{n}, \hat{u}) \text{ weakly in } Y_u \times Y_n \text{ and strongly in } X_u \times X_n.$$

Let $m \to \infty$ and take the limit, it is clear that $(n, u) = \mathcal{F}(\hat{n}_m, \hat{u}_m)$ and $(\hat{n}_m, \hat{u}_m) = (\hat{n}, \hat{u})$, this means that $\mathcal{F}(\hat{n}_m, \hat{u}_m) = (\hat{n}_m, \hat{u}_m)$. Since uniqueness of limit, the map $\mathcal{F}$ is continuous. ☐

Theorem 3.1. *Let $u_0 \in H^1(\Omega)$, $n_0 \in H^1(\Omega)$, $c_0 \in H^2(\Omega)$ with $n_0 \geq 0$ in Ω, and $f \in L^2(0,T;H^1(\Omega))$, then (1.1) exists unique strong solution (n, c, u). Moreover, there exists a positive C constant such that*

$$\begin{aligned}
&\|n\|_{L^\infty(0,T;H^1(\Omega))} + \|n\|_{L^2(0,T;H^2(\Omega))} + \|n_t\|_{L^2(0,T;L^2(\Omega))} + \|c\|_{L^\infty(0,T;H^2(\Omega))} \\
&\quad + \|c\|_{L^2(0,T;H^3(\Omega))} + \|c_t\|_{L^2(0,T;L^2(\Omega))} + \|u\|_{L^\infty(0,T;H^1(\Omega))} \\
&\quad + \|u\|_{L^2(0,T;H^2(\Omega))} + \|u_t\|_{L^2(0,T;L^2(\Omega))} \leq C. \qquad (3.12)
\end{aligned}$$

Proof. From Lemmas 3.1, 3.3 and 3.4, it is easy to verify the existence of solution and (3.11). Therefore, we will prove the uniqueness of the solution in the following part. For convenience, we set $n = n_1 - n_2$, $c = c_1 - c_2$ and $u = u_1 - u_2$, where (n_i, c_i, u_i) is the strong solution of the system, where $i = 1, 2$. Thus, we obtain the following system

$$\begin{aligned}
&n_t - \Delta n + u_1 \cdot \nabla n + u\nabla n_2 = -\nabla \cdot (n_1 \nabla c) \\
&-\nabla(n\nabla c_2) + \gamma n - \mu n(n_1 + n_2), && \text{in } (0,T)\times\Omega \equiv Q, && (3.13)\\
&c_t - \Delta c + u_1 \cdot \nabla c + u\nabla c_2 + c = n, && \text{in } (0,T)\times\Omega \equiv Q, && (3.14)\\
&u_t - \Delta u + u_1 \cdot \nabla u + u \cdot \nabla u_2 = n\nabla\varphi, && \text{in } (0,T)\times\Omega \equiv Q, && (3.15)\\
&\nabla \cdot u = 0, && \text{in } (0,T)\times\Omega \equiv Q, && (3.16)\\
&\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, \quad u = 0, && \text{on } (0,T)\times\partial\Omega, && (3.17)\\
&n_0(x) = c_0(x) = u_0(x) = 0, && \text{in } \Omega. && (3.18)
\end{aligned}$$

Taking the L^2-inner product with n for the (3.13) implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega n^2 dx + \int_\Omega |\nabla n|^2 dx + \int_\Omega n^2 dx \\
\leq& -\int_\Omega u\nabla n_2 n dx + \int_\Omega n_1 \nabla c \nabla n dx + \int_\Omega n\nabla c_2 \nabla n dx + (1+\gamma)\int_\Omega n^2 dx \\
=& I_9 + I_{10} + I_{11} + I_{12}.
\end{aligned}$$

For the term I_9, due to the estimates (3.3) and (3.8), we have

$$\begin{aligned}
I_9 =& -\int_\Omega u\nabla n_2 n dx \leq \|\nabla n_2\|_{L^2}\|u\|_{L^4}\|n\|_{L^4} \\
\leq& C\|\nabla n_2\|_{L^2}\|u\|_{H^1}(\|n\|_{L^2}^{\frac{1}{2}}\|\nabla n\|_{L^2}^{\frac{1}{2}} + \|n\|_{L^2}) \\
\leq& \frac{\delta}{3}\|\nabla n\|_{L^2}^2 + C\|n\|_{L^2}^2.
\end{aligned}$$

For the term I_{10}, with the estimate (3.8) and (3.11), we get

$$\begin{aligned}I_{10} =&\int_\Omega n_1\nabla c\nabla n dx \le \|\nabla n\|_{L^2}\|n_1\|_{L^4}\|\nabla c\|_{L^4}\\ \le& C\|\nabla n\|_{L^2}\|n_1\|_{H^1}\|\nabla c\|_{H^1}\\ \le&\frac{\delta}{3}\|\nabla n\|_{L^2}^2 + C.\end{aligned}$$

For the term I_{11},

$$\begin{aligned}I_{11} =&\int_\Omega n\nabla c_2\nabla n dx \le \|\nabla n\|_{L^2}\|\nabla c_2\|_{L^4}\|n\|_{L^4}\\ \le&\|\nabla n\|_{L^2}\|\nabla c_2\|_{H^1}\|n\|_{H^1}\\ \le&\frac{\delta}{3}\|\nabla n\|_{L^2}^2 + C.\end{aligned}$$

With the use of estimates $I_i (i = 9, 10, 11, 12)$, we have

$$\frac{d}{dt}\|n\|_{L^2}^2 + \|n\|_{H^1} \le \delta\|\nabla n\|_{L^2}^2 + C\|n\|_{L^2}^2 + C. \tag{3.19}$$

Taking the L^2-inner product with c for the (3.14) implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega c^2 dx + \int_\Omega |\nabla c|^2 dx + \int_\Omega c^2 dx\\ =& -\int_\Omega u_1\nabla c c dx - \int_\Omega u\nabla c_2 c dx + \int_\Omega nc dx\\ \le&\|c\|_{L^4}^2\|\nabla u_1\|_{L^2} + \|u\|_{L^2}\|\nabla c_2\|_{L^4}\|c\|_{L^4} + \|n\|_{L^2}\|c\|_{L^2}\\ \le& C(\|c\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{L^2}^{\frac{1}{2}} + \|c\|_{L^2})^2\|\nabla u_1\|_{L^2} + (\|c\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{L^2}^{\frac{1}{2}} + \|c\|_{L^2})\|u\|_{L^2}\|\nabla c_2\|_{H^1}\\ &+ \|n\|_{L^2}\|c\|_{L^2}\\ \le&\delta\|\nabla c\|_{L^2}^2 + C\|c\|_{L^2}^2.\end{aligned}$$

Then, we get

$$\frac{d}{dt}\|c\|_{L^2}^2 + \|c\|_{H^1} \le \delta\|\nabla c\|_{L^2}^2 + C\|c\|_{L^2}^2. \tag{3.20}$$

Taking the L^2-inner product with c for the (3.15) implies

$$\frac{1}{2}\int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx = \int_\Omega n\nabla\varphi u dx.$$

Straightforward calculations yield

$$\frac{d}{dt}\|u\|_{L^2}^2 + \|u\|_{H^1} \le C(\|u\|_{L^2}^2 + \|n\|_{L^2}^2). \tag{3.21}$$

Then, a combination of (3.19), (3.20) and (3.21) yields

$$\begin{aligned}&\frac{d}{dt}(\|n\|_{L^2}^2 + \|c\|_{L^2}^2 + \|u\|_{L^2}^2) + (\|n\|_{H^1} + \|c\|_{H^1} + \|u\|_{H^1})\\ \le&\delta(\|\nabla n\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) + (\|n\|_{L^2}^2 + \|c\|_{L^2}^2 + \|u\|_{L^2}^2) + C.\end{aligned}$$

By choosing δ small enough, we get

$$\frac{d}{dt}(\|n\|_{L^2}^2 + \|c\|_{L^2}^2 + \|u\|_{L^2}^2) \le C(\|n\|_{L^2}^2 + \|c\|_{L^2}^2 + \|u\|_{L^2}^2) + C.$$

Applying Gronwall's lemma to the resulting differential inequality, we finally obtain the uniqueness of the solution. □

4. **Existence of an optimal control.** In this section, we will prove the existence of the optimal solution of control problem. The method we use for treating this problem was inspired by some ideas of Guillén-González et al [9]. Assume $\mathcal{U} \subset L^2(0,T;H^1(\Omega_c))$ is a nonempty, closed and convex set, where control domain $\Omega_c \subset \Omega$, and $\Omega_d \subset \Omega$ is the observability domain. We adjust the external source f, so that the bacterial density n, oxygen concentration c and fluid velocity u are as close as possible to a desired state n_d, c_d and u_d, and at the final moment T is as close as possible to a desired state n_Ω, c_Ω and u_Ω. We consider the optimal control problem as follows

Minimize the cost functional

$$\begin{aligned} J(n,c,u,f) =& \frac{\beta_1}{2}\|n-n_d\|^2_{L^2(Q_d)} + \frac{\beta_2}{2}\|c-c_d\|^2_{L^2(Q_d)} + \frac{\beta_3}{2}\|u-u_d\|^2_{L^2(Q_d)} \\ &+ \frac{\beta_4}{2}\|n(T)-n_\Omega\|^2_{L^2(\Omega_d)} + \frac{\beta_5}{2}\|c(T)-c_\Omega\|^2_{L^2(\Omega_d)} \\ &+ \frac{\beta_6}{2}\|u(T)-u_\Omega\|^2_{L^2(\Omega_d)} + \frac{\beta_7}{2}\|f(x,t)\|^2_{L^2(Q_c)}, \end{aligned} \tag{4.1}$$

subject to the system (1.1). Moreover, the nonnegative constants $\beta_i, i = 1, 2, \cdots, 7$ are given but not all zero, the functions n_d, c_d, u_d represents the desired states satisfying

$$\begin{aligned} &n_d \in L^2(Q_d), c_d \in L^2(Q_d), u_d \in L^2(Q_d), \\ &n_\Omega \in L^2(\Omega_c), c_\Omega \in L^2(\Omega_c), u_\Omega \in L^2(\Omega_c),\ f \in \mathcal{U}. \end{aligned}$$

The set of admissible solutions of optimal control problem (4.1) is defined by

$$\mathcal{S}_{ad} = \{s = (n,c,u,f) \in \mathcal{H} : s \text{ is a strong solution of } (1.1)\}.$$

The function space $\mathcal{H}$ is given by

$$\mathcal{H} = Y_n \times Y_c \times Y_u \times \mathcal{U},$$

where $Y_c = L^\infty\left(0,T;H^2(\Omega)\right) \cap L^2\left(0,T;H^3(\Omega)\right)$.

Now, we prove the existence of a global optimal control for problem (1.1).

Theorem 4.1. *Suppose $f \in \mathcal{U}$ is satisfied, and $n_0 \geq 0$, then the optimal control problem (4.1) admits a solution $(\bar{n},\bar{c},\bar{u},\bar{f}) \in \mathcal{S}_{ad}$.*

Proof. Along with Theorem 3.1, we conduct that $\mathcal{S}_{ad} \neq \emptyset$, then there exists the minimizing sequence $\{(n_m,c_m,u_m,f_m)\}_{m\in\mathbb{N}} \in \mathcal{S}_{ad}$ such that

$$\lim_{m\to+\infty} J(n_m,c_m,u_m,f_m) = \inf_{(n,c,u,f)\in\mathcal{S}_{ad}} J(n,c,u,f). \tag{4.2}$$

According to the definition of $\mathcal{S}_{ad}$, for each $m \in \mathbb{N}$ there exists (n_m,c_m,u_m,f_m) satisfying

$$\begin{cases} n_{mt} + u_m\cdot\nabla n_m = \Delta n_m - \nabla\cdot(n_m\cdot\nabla c_m) + \gamma n_m - \mu n_m^2, & \text{in } Q, \\ c_{mt} + u_m\cdot\nabla c_m = \Delta c_m - c_m + n_m + f_m, & \text{in } Q, \\ u_{mt} + u_m\cdot\nabla u_m = \Delta u_m - \nabla\pi + n_m\nabla\varphi, & \text{in } Q, \\ \nabla\cdot u_m = 0, & \text{in } Q, \\ \dfrac{\partial n_m}{\partial\nu}\Big|_{\partial\Omega} = \dfrac{\partial c_m}{\partial\nu}\Big|_{\partial\Omega} = 0, \quad u_m|_{\partial\Omega} = 0, & \\ n_m(0) = n_0, c_m(0) = c_0, u_m(0) = u_0, & \text{in } \Omega. \end{cases} \tag{4.3}$$

Observing that $\mathcal{U}$ is a closed convex subset of $L^2(0,T;H^1(\Omega_c))$. According to the definition of $\mathcal{S}_{ad}$, we deduce that there exists $(\bar{n},\bar{c},\bar{u},\bar{f})$ bounded in $\mathcal{H}$ such that, for subsequence of $(n_m,c_m,u_m,f_m)_{m\in\mathbb{N}}$, for convenience, still denoted by (n_m,c_m,u_m,f_m), as $m\to+\infty$

$$\begin{aligned}
&n_m \to \bar{n}, \text{ weakly in } L^2\left(0,T;H^2(\Omega)\right) \text{ and weakly* in } L^\infty\left(0,T;H^1(\Omega)\right),\\
&c_m \to \bar{c}, \text{ weakly in } L^2\left(0,T;H^3(\Omega)\right) \text{ and weakly* in } L^\infty\left(0,T;H^2(\Omega)\right),\\
&u_m \to \bar{u}, \text{ weakly in } L^2\left(0,T;H^2(\Omega)\right) \text{ and weakly* in } L^\infty\left(0,T;H^1(\Omega)\right),\\
&f_m \to \bar{f}, \text{ weakly in } L^2(0,T;H^1(\Omega_c)), \text{ and } \tilde{f}\in\mathcal{U}.
\end{aligned}$$

According to the Aubin-Lions lemma [16] and the compact embedding theorems, we obtain

$$\begin{aligned}
n_m &\to \bar{n}, \quad \text{strongly in } C\left([0,T];L^2(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right),\\
c_m &\to \bar{c}, \quad \text{strongly in } C\left([0,T];H^1(\Omega)\right)\cap L^2\left(0,T;H^2(\Omega)\right),\\
u_m &\to \bar{u}, \quad \text{strongly in } C\left([0,T];L^2(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right).
\end{aligned}$$

Since $\nabla\cdot(n_m\nabla c_m)=\nabla n_m\cdot\nabla c_m+n_m\Delta c_m$ is bounded in $L^2(0,T;L^2(\Omega))$, then

$$\nabla\cdot(n_m\nabla c_m)\to\chi, \text{ weakly in } L^2(0,T;L^2(\Omega)).$$

Recalling that

$$n_m\nabla c_m\to\bar{n}\nabla\bar{c}, \text{ weakly in } L^\infty(0,T;L^2(\Omega)).$$

Therefore, we get that $\chi=\nabla(\bar{n}\nabla\bar{c})$. Owing to $(\bar{n},\bar{c},\bar{u},\bar{f})\in\mathcal{H}$, we see that $(\bar{n},\bar{c},\bar{u},\bar{f})$ is solution of the system (1.1), along with (4.2) implies that

$$\lim_{m\to+\infty} J(n_m,c_m,u_m,f_m)=\inf_{(u,c,u,f)\in\mathcal{S}_{ad}} J(u,c,u,f)\leq J(\bar{n},\bar{c},\bar{u},\bar{f}).$$

On the other hand, we deduce from the weak lower semicontinuity of the cost functional

$$J(\bar{n},\bar{c},\bar{u},\bar{f})\leq\liminf_{m\to+\infty} J(n_m,c_m,u_m,f_m).$$

Therefore, this implies that $(\bar{n},\bar{c},\bar{u},\bar{f})$ is an optimal pair for problem (1.1). □

5. **The first-order necessary optimality condition.** In order to derive the first-order necessary optimality conditions for a local optimal solution of problem (4.1). To this end, we will use a result on existence of Lagrange multipliers in Banach spaces ([20]). First, we discuss the following problem

$$\min J(s) \text{ subject to } s\in\mathcal{S}=\{s\in\mathcal{H}:G(s)\in\mathcal{N}\}, \tag{5.1}$$

where $J:X\to\mathbb{R}$ is a functional, $G:X\to Y$ is an operator, X and Y are Banach spaces, and nonempty closed convex set $\mathcal{H}$ is subset of X and nonempty closed convex cone $\mathcal{N}$ with vertex at the origin in Y.

A^+ denotes its polar cone

$$A^+=\{\rho\in X':\langle\rho,a\rangle_{X'}\geq 0,\forall a\in A\}.$$

We consider the following Banach spaces

$$\begin{aligned}
&X=V_n\times V_c\times V_u\times L^2(0,T;H^1(\Omega_c)),\\
&Y=L^2(Q)\times L^2(0,T;H^1(\Omega))\times L^2(Q)\times H^1(\Omega)\times H^2(\Omega)\times H^1(\Omega),
\end{aligned}$$

where

$$V_n = \{n \in Y_n : \frac{\partial n}{\partial \nu} \text{ on } (0,T) \times \partial\Omega\},$$
$$V_c = \{n \in Y_c : \frac{\partial c}{\partial \nu} \text{ on } (0,T) \times \partial\Omega\},$$
$$V_u = \{n \in Y_u : u = 0 \text{ on } (0,T) \times \partial\Omega \text{ and } \nabla \cdot u = 0 \text{ in } (0,T) \times \Omega\}$$

and the operator $G = (G_1, G_2, G_3, G_4, G_5, G_6) : X \to Y$, where

$$G_1 : X \to L^2(Q), \qquad G_2 : X \to L^2(0,T; H^1(\Omega)), \qquad G_3 : X \to L^2(Q),$$
$$G_4 : X \to H^1(\Omega), \qquad G_5 : X \to H^2(\Omega), \qquad G_6 : X \to H^1(\Omega),$$

which are defined at each point $s = (n, c, u, f) \in X$ by

$$\begin{cases} G_1 = n_t + u \cdot \nabla n - \Delta n + \nabla \cdot (n \cdot \nabla c) - \gamma n + \mu n^2, \\ G_2 = c_t + u \cdot \nabla c - \Delta c + c - n - f, \\ G_3 = u_t + u \cdot \nabla u - \Delta u + \nabla \pi - n \nabla \varphi, \\ G_4 = n(0) - n_0, \\ G_5 = c(0) - c_0, \\ G_6 = u(0) - u_0. \end{cases} \tag{5.2}$$

The function spaces are given as follows

$$\mathcal{H} = V_n \times V_c \times V_u \times \mathcal{U}.$$

We see that $\mathcal{H}$ is a closed convex subset of X and $\mathcal{N} = \{0\}$, and rewrite the optimal control problem

$$\min J(s) \text{ subject to } s \in \mathcal{S}_{ad} = \{s \in \mathcal{H} : G(s) = 0\}. \tag{5.3}$$

Taking the differentiability of J and G into consider, it follows that

Lemma 5.1. *The functional $J : X \to R$ is Fréchet differentiable and the Fréchet derivative of J in $\bar{s} = (\bar{n}, \bar{c}, \bar{u}, \bar{f}) \in X$ in the direction $r = (\tilde{n}, \tilde{c}, \tilde{u}, \tilde{f})$ is given by*

$$\begin{aligned} J'(\bar{s})[r] =& \beta_1 \int_0^T \int_{\Omega_d} (\bar{n} - n_d)\tilde{n} dx dt + \beta_2 \int_0^T \int_{\Omega_d} (\bar{c} - c_d)\tilde{c} dx dt \\ &+ \beta_3 \int_0^T \int_{\Omega_d} (\bar{u} - u_d)\tilde{u}(T) dx dt + \beta_4 \int_{\Omega_d} (\bar{n}(T) - n_\Omega)\tilde{n}(T) dx \\ &+ \beta_5 \int_{\Omega_d} (\bar{c}(T) - c_\Omega)\tilde{c} dx + \beta_6 \int_{\Omega_d} (\bar{u}(T) - u_\Omega)\tilde{u}(T) dx \\ &+ \beta_7 \int_0^T \int_{\Omega_d} \bar{f}\tilde{f} dx dt. \end{aligned} \tag{5.4}$$

Lemma 5.2. *The operator $G : X \to Y$ is continuous-Fréchet differentiable and the Fréchet derivative of J in $\bar{s} = (\bar{n}, \bar{c}, \bar{u}, \bar{f}) \in X$ in the direction $r = (\tilde{n}, \tilde{c}, \tilde{u}, \tilde{f})$, is the linear operator*

$$G'(\bar{s})[r] = (G_1'(\bar{s})[r], G_2'(\bar{s})[r], G_3'(\bar{s})[r], G_4'(\bar{s})[r], G_5'(\bar{s})[r], G_6'(\bar{s})[r])$$

defined by

$$\begin{cases} G_1'(\bar{s})[r] = \tilde{n}_t - \Delta\tilde{n} + \bar{u}\cdot\nabla\tilde{n} + \tilde{u}\nabla\bar{n} + \nabla\cdot(\bar{n}\nabla\tilde{c}) \\ \quad + \nabla(\tilde{n}\nabla\bar{c}) - \gamma\tilde{n} + 2\mu\tilde{n}\bar{n}, & in\ Q, \\ G_2'(\bar{s})[r] = \tilde{c}_t - \Delta\tilde{c} + \bar{u}\cdot\nabla\tilde{c} + \tilde{u}\cdot\nabla\bar{c} + \tilde{c} - \tilde{n} - \tilde{f}, & in\ Q, \\ G_3'(\bar{s})[r] = \tilde{u}_t - \Delta\tilde{u} + \bar{u}\cdot\nabla\tilde{u} + \tilde{u}\cdot\nabla\bar{u} - \tilde{n}\nabla\varphi, & in\ Q, \\ \nabla\cdot\tilde{u} = 0, & in\ Q, \\ \dfrac{\partial\tilde{n}}{\partial\nu} = \dfrac{\partial\tilde{c}}{\partial\nu} = 0, \tilde{u} = 0, & on\ (0,T)\times\partial\Omega, \\ \tilde{n}(0) = \tilde{n}_0, \tilde{c}(0) = \tilde{c}_0, \tilde{u}(0) = \tilde{u}_0, & in\ \Omega. \end{cases}$$

Lemma 5.3. *Let $\bar{s} = (\bar{n}, \bar{c}, \bar{u}, \bar{f}) \in \mathcal{S}_{ad}$, then $\bar{s}$ is a regular point.*

Proof. For any fixed $(\bar{n}, \bar{c}, \bar{u}, \bar{f}) \in \mathcal{S}_{ad}$, we set $(g_n, g_c, g_u, \tilde{n}_0, \tilde{c}_0, \tilde{u}_0) \in Y$. Since $0 \in \mathcal{C}(\bar{f})$, it suffices to show the existence of $(\tilde{n}, \tilde{c}, \tilde{u}) \in Y_n \times Y_c \times Y_u$ such that

$$\begin{cases} \tilde{n}_t - \Delta\tilde{n} + \bar{u}\cdot\nabla\tilde{n} + \tilde{u}\nabla\bar{n} + \nabla\cdot(\bar{n}\nabla\tilde{c}) \\ \quad + \nabla(\tilde{n}\nabla\bar{c}) - \gamma\tilde{n} + 2\mu\tilde{n}\bar{n} = g_n, & \text{in } Q, \\ \tilde{c}_t - \Delta\tilde{c} + \bar{u}\cdot\nabla\tilde{c} + \tilde{u}\cdot\nabla\bar{c} + \tilde{c} - \tilde{n} = g_c, & \text{in } Q, \\ \tilde{u}_t - \Delta\tilde{u} + \bar{u}\cdot\nabla\tilde{u} + \tilde{u}\cdot\nabla\bar{u} - \tilde{n}\nabla\varphi = g_u, & \text{in } Q, \\ \nabla\cdot\tilde{u} = 0, & \text{in } Q, \\ \dfrac{\partial\tilde{n}}{\partial\nu} = \dfrac{\partial\tilde{c}}{\partial\nu} = 0, \tilde{u} = 0, & \text{on } (0,T)\times\partial\Omega, \\ \tilde{n}(0) = \tilde{n}_0, \tilde{c}(0) = \tilde{c}_0, \tilde{u}(0) = \tilde{u}_0, & \text{in } \Omega. \end{cases} \tag{5.5}$$

Next, we use Leray-Schauder's fixed point method to prove the existence of solutions of the problem (5.5), the operator $T : (\dot{n}, \dot{u}) \in X_n \times X_u \to (\tilde{n}, \tilde{u}) \in Y_n \times Y_u$ with $(\tilde{n}, \tilde{c}, \tilde{u})$ solving the decoupled problem:

$$\begin{cases} \tilde{n}_t - \Delta\tilde{n} + \bar{u}\cdot\nabla\tilde{n} + \tilde{u}\nabla\bar{n} + \nabla\cdot(\bar{n}\nabla\tilde{c}) \\ \quad + \nabla(\tilde{n}\nabla\bar{c}) - \gamma\tilde{n} + 2\mu\dot{n}\bar{n} = g_n, & \text{in } Q, \\ \tilde{c}_t - \Delta\tilde{c} + \bar{u}\cdot\nabla\tilde{c} + \tilde{u}\cdot\nabla\bar{c} + \tilde{c} - \dot{n} = g_c, & \text{in } Q, \\ \tilde{u}_t - \Delta\tilde{u} + \bar{u}\cdot\nabla\tilde{u} + \dot{u}\cdot\nabla\bar{u} - \dot{n}\nabla\varphi = g_u, & \text{in } Q. \end{cases} \tag{5.6}$$

The system (5.6) is complemented by the corresponding Neumann boundary and initial conditions. Similar to the proof of Lemmas 2.3, 2.4, 2.5 and 2.6, we conduct that operator $T : X_n \times X_u \to X_n \times X_u$ is well-defined and compact.

Similar to the proof of Theorem 3.1, $(\tilde{n}, \tilde{u})$ solves the coupled problem $(\bar{n}, \bar{c}, \bar{u}, \bar{f}) \in \mathcal{S}_{ad}$, and we set $(g_n, g_c, g_u, \tilde{n}_0, \tilde{c}_0, \tilde{u}_0) \in Y$. Since $0 \in \mathcal{C}(\bar{f})$, it suffices to show the existence of $(\tilde{n}, \tilde{c}, \tilde{u}) \in Y_n \times Y_c \times Y_u$ such that

$$\begin{cases} \tilde{n}_t - \Delta\tilde{n} + \tilde{n} = -\bar{u}\cdot\nabla\tilde{n} - \tilde{u}\cdot\nabla\bar{n} - \nabla\cdot(\bar{n}\nabla\tilde{c}) \\ \quad - \nabla(\tilde{n}\nabla\bar{c}) + \alpha(\gamma+1)\tilde{n} - 2\mu\tilde{n}\bar{n} + \alpha g_n, & \text{in } Q, \\ \tilde{c}_t - \Delta\tilde{c} + \tilde{c} = -\bar{u}\cdot\nabla\tilde{c} - \tilde{u}\cdot\nabla\bar{c} + \alpha\tilde{n} + \alpha g_c, & \text{in } Q, \\ \tilde{u}_t - \Delta\tilde{u} = -\bar{u}\cdot\nabla\tilde{u} - \tilde{u}\cdot\nabla\bar{u} + \alpha\tilde{n}\nabla\varphi + \alpha g_u, & \text{in } Q, \end{cases} \tag{5.7}$$

complemented by the corresponding Neumann boundary and initial conditions.

Taking the L^2-inner product with $\tilde{u}$ for the third equation of (5.7) implies

$$\frac{1}{2}\int_\Omega \tilde{u}^2 dx + \int_\Omega |\nabla\tilde{u}|^2 dx = \alpha\int_\Omega \tilde{n}\nabla\varphi\tilde{u}dx + \alpha\int_\Omega \tilde{u}g_u dx.$$

By the Poincaré inequality and Young's inequality, we have

$$\frac{d}{dt}\|\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{H^1}^2 \le C(\|\tilde{n}\|_{L^2}^2 + \|g_u\|_{L^2}^2) + C\|\tilde{u}\|_{L^2}^2. \tag{5.8}$$

Taking the L^2-inner product with $\tilde{c}$ for the second equation of (5.7) implies

$$\begin{aligned}&\frac{1}{2}\int_\Omega \tilde{c}^2 dx + \int_\Omega |\nabla\tilde{c}|^2 dx + \int_\Omega \tilde{c}^2 dx\\ =&\int_\Omega \tilde{u}\nabla\bar{c}\tilde{c}dx + \alpha\int_\Omega \tilde{n}\tilde{c}dx + \alpha\int_\Omega g_c\tilde{c}dx.\end{aligned}$$

With the Poincaré inequality and Young's inequality in hand, we see that

$$\frac{d}{dt}\|\tilde{c}\|_{L^2}^2 + \|\tilde{c}\|_{H^1}^2 \le C(\|\tilde{n}\|_{L^2}^2 + \|g_c\|_{L^2}^2) + C\|\tilde{c}\|_{L^2}^2. \tag{5.9}$$

Taking the L^2-inner product with $-\Delta\tilde{c}$ for the second equation of (5.7) implies

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\nabla\tilde{c}|^2 dx + \int_\Omega |\Delta\tilde{c}|^2 dx + \int_\Omega |\nabla\tilde{c}|^2 dx\\ =&\int_\Omega \tilde{u}\nabla\bar{c}\Delta\tilde{c}dx + \int_\Omega \bar{u}\nabla\tilde{c}\Delta\tilde{c}dx - \alpha\int_\Omega \tilde{n}\Delta\tilde{c}dx - \alpha\int_\Omega g_c\Delta\tilde{c}dx\\ =&J_1 + J_2 + J_3.\end{aligned}$$

For the term J_1

$$\begin{aligned}J_1 =&\int_\Omega \tilde{u}\nabla\bar{c}\Delta\tilde{c}dx \le \|\Delta\tilde{c}\|_{L^2}\|\nabla\bar{c}\|_{L^4}\|\tilde{u}\|_{L^4}\\ \le&\frac{1}{6}\|\Delta\tilde{c}\|_{L^2}^2 + C\|\nabla\bar{c}\|_{H^1}^2\|\tilde{u}\|_{H^1}^2.\end{aligned}$$

For the term J_2, we see that

$$\begin{aligned}J_2 =&\int_\Omega \bar{u}\nabla\tilde{c}\Delta\tilde{c}dx = -\frac{1}{2}\int_\Omega \nabla\bar{u}|\nabla\tilde{c}|^2 dx\\ \le&\|\nabla\bar{u}\|_{L^2}\|\nabla\tilde{c}\|_{L^4}^2\\ \le&\|\nabla\bar{u}\|_{L^2}(\|\nabla\tilde{c}\|_{L^2}^{\frac{1}{2}}\|\Delta\tilde{c}\|_{L^2}^{\frac{1}{2}} + \|\nabla\tilde{c}\|_{L^2})\\ \le&\frac{1}{6}\|\Delta\tilde{c}\|_{L^2}^2 + C\|\nabla\tilde{c}\|_{L^2}^2.\end{aligned}$$

For the term J_3, we get

$$\begin{aligned}J_3 =&-\alpha\int_\Omega \tilde{n}\Delta\tilde{c}dx - \alpha\int_\Omega g_c\Delta\tilde{c}dx\\ \le&\frac{1}{6}\|\Delta\tilde{c}\|_{L^2}^2 + C(\|\tilde{n}\|_{L^2}^2 + \|g_c\|_{L^2}^2).\end{aligned}$$

Therefore, combining J_1, J_2 and J_3, we have

$$\frac{d}{dt}\|\nabla\tilde{c}\|_{L^2}^2 + \|\nabla\tilde{c}\|_{H^1}^2 \le C\|\nabla\tilde{c}\|_{L^2}^2 + C(\|\tilde{n}\|_{L^2}^2 + \|g_c\|_{L^2}^2). \tag{5.10}$$

Taking the L^2-inner product with $\tilde{n}$ for the first equation of (5.7) implies

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \tilde{n}^2 dx + \int_\Omega |\nabla\tilde{n}|^2 dx + \int_\Omega \tilde{n}^2 dx\\ =&-\int_\Omega \tilde{u}\nabla\bar{n}\tilde{n}dx + \int_\Omega \nabla\tilde{n}\bar{n}\nabla\tilde{c}dx + \int_\Omega \nabla\tilde{n}\tilde{n}\nabla\bar{c}dx + \alpha(\gamma+1)\int_\Omega \tilde{n}^2 dx\end{aligned}$$

$$
\begin{aligned}
&+2\mu\int_\Omega \bar{n}\tilde{n}^2dx+\alpha\int_\Omega \tilde{n}g_ndx\\
=&J_4+J_5+J_6+J_7.
\end{aligned}
$$

For the term J_4, by Gagliardo-Nirenberg interpolation inequality, we have

$$
\begin{aligned}
J_4=&-\int_\Omega \tilde{u}\nabla\bar{n}\tilde{n}dx\le\|\tilde{u}\|_{L^4}\|\nabla\bar{n}\|_{L^2}\|\tilde{n}\|_{L^4}\\
\le&C(\|\nabla\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\tilde{u}\|_{L^2}^{\frac{1}{2}}+\|\tilde{u}\|_{L^2})\|\nabla\bar{n}\|_{L^2}\|\tilde{n}\|_{H^1}\\
\le&\delta\|\tilde{n}\|_{H^1}^2+C\|\nabla\tilde{u}\|_{L^2}\|\tilde{u}\|_{L^2}+C\|\tilde{u}\|_{L^2}^2\\
\le&\delta\|\tilde{n}\|_{H^1}^2+\delta\|\nabla\tilde{u}\|_{L^2}^2+C\|\tilde{u}\|_{L^2}^2.
\end{aligned}
$$

For the term J_5,

$$
\begin{aligned}
J_5=&\int_\Omega \nabla\tilde{n}\bar{n}\nabla\tilde{c}dx\le\|\nabla\tilde{n}\|_{L^2}\|\bar{n}\|_{L^4}\|\nabla\tilde{c}\|_{L^4}\\
\le&\|\nabla\tilde{n}\|_{L^2}\|\bar{n}\|_{H^1}(\|\nabla\tilde{c}\|_{L^2}^{\frac{1}{2}}\|\Delta\tilde{c}\|_{L^2}^{\frac{1}{2}}+\|\nabla\tilde{c}\|_{L^2})\\
\le&\delta\|\nabla\tilde{n}\|_{L^2}^2+\|\nabla\tilde{c}\|_{L^2}\|\Delta\tilde{c}\|_{L^2}+C\|\nabla\tilde{c}\|_{L^2}^2\\
\le&\delta\|\nabla\tilde{n}\|_{L^2}^2+\delta\|\Delta\tilde{c}\|_{L^2}+C\|\nabla\tilde{c}\|_{L^2}^2.
\end{aligned}
$$

For the term J_6,

$$
\begin{aligned}
J_6=&\int_\Omega \nabla\tilde{n}\tilde{n}\nabla\bar{c}dx\le\|\tilde{n}\|_{L^4}^2\|\Delta\bar{c}\|_{L^2}\\
\le&(\|\tilde{n}\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{n}\|_{L^2}^{\frac{1}{2}}+\|\tilde{n}\|_{L^2})\|\Delta\bar{c}\|_{L^2}\\
\le&\delta\|\nabla\tilde{n}\|_{L^2}^2+C\|\tilde{n}\|_{L^2}^2+C.
\end{aligned}
$$

For the term J_7,

$$
\begin{aligned}
J_7=&\alpha(\gamma+1)\int_\Omega \tilde{n}^2dx+2\mu\int_\Omega \bar{n}\tilde{n}^2dx+\alpha\int_\Omega \tilde{n}g_ndx\\
\le&(\gamma+1)\|\tilde{n}\|_{L^2}^2+\|g_n\|_{L^2}\|\tilde{n}\|_{L^2}+\|\bar{n}\|_{L^2}\|\tilde{n}\|_{L^4}^2\\
\le&(\gamma+1)\|\tilde{n}\|_{L^2}^2+\|g_n\|_{L^2}\|\tilde{n}\|_{L^2}+\|\bar{n}\|_{L^2}(\|\tilde{n}\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{n}\|_{L^2}^{\frac{1}{2}}+\|\tilde{n}\|_{L^2})\\
\le&\delta\|\nabla\tilde{n}\|_{L^2}+C\|\tilde{n}\|_{L^2}^2+C\|g_n\|_{L^2}^2.
\end{aligned}
$$

Therefore, by choosing δ small enough, from J_4, J_5, J_6 and J_7, it follows that

$$
\begin{aligned}
&\frac{d}{dt}\|\tilde{n}\|_{L^2}^2+\|\tilde{n}\|_{H^1}^2\\
\le&C(\|\tilde{n}\|_{L^2}^2+\|\nabla\tilde{c}\|_{L^2}^2+\|\tilde{u}\|_{L^2}^2)+\delta\|\Delta\tilde{c}\|_{L^2}+\delta\|\nabla\tilde{u}\|_{L^2}^2+C\|g_n\|_{L^2}^2.
\end{aligned}
\tag{5.11}
$$

By choosing δ small enough and combining (5.8)-(5.11), we get

$$
\begin{aligned}
&\frac{d}{dt}(\|\tilde{n}\|_{L^2}^2+\|\tilde{c}\|_{H^1}^2+\|\tilde{u}\|_{L^2}^2)+\|\tilde{n}\|_{H^1}^2+\|\tilde{c}\|_{H^2}^2+\|\tilde{u}\|_{H^1}^2\\
\le&C(\|g_n\|_{L^2}^2+\|g_c\|_{L^2}^2+\|g_u\|_{L^2}^2)+C(\|\tilde{n}\|_{L^2}^2+\|\tilde{c}\|_{H^1}^2+\|\tilde{u}\|_{L^2}^2).
\end{aligned}
$$

Applying Gronwall's lemma to the resulting differential inequality, we obatin

$$
\|\tilde{n}\|_{L^2}^2+\|\tilde{c}\|_{H^1}^2+\|\tilde{u}\|_{L^2}^2+\int_0^t\|\tilde{n}\|_{H^1}^2d\tau+\int_0^t\|\tilde{c}\|_{H^2}^2d\tau+\int_0^t\|\tilde{u}\|_{H^1}^2d\tau\le C. \tag{5.12}
$$

Taking the L^2-inner product with $-\Delta\tilde{u}$ for the third equation of (5.7) implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\tilde{u}|^2dx+\int_\Omega|\Delta\tilde{u}|^2dx\\=&\int_\Omega\bar{u}\cdot\nabla\tilde{u}\Delta\tilde{u}dx+\int_\Omega\tilde{u}\cdot\nabla\bar{u}\Delta\tilde{u}dx-\alpha\int_\Omega\tilde{n}\nabla\varphi\Delta\tilde{u}dx-\alpha\int_\Omega g_u\Delta\tilde{u}dx\\=&J_8+J_9+J_{10}.\end{aligned}$$

With the use of the Gagliardo-Nirenberg interpolation inequality, we derive

$$\begin{aligned}J_8=&\int_\Omega\bar{u}\cdot\nabla\tilde{u}\Delta\tilde{u}dx\le\|\bar{u}\|_{L^4}\|\nabla\tilde{u}\|_{L^4}\|\Delta\tilde{u}\|_{L^2}\\\le&\|\bar{u}\|_{H^1}(\|\nabla\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\Delta\tilde{u}\|_{L^2}^{\frac{1}{2}}+\|\nabla\tilde{u}\|_{L^2})\|\Delta\tilde{u}\|_{L^2}\\\le&\delta\|\Delta\tilde{u}\|_{L^2}^2+C\|\nabla\tilde{u}\|_{L^2}^2\end{aligned}$$

and

$$\begin{aligned}J_9=&\int_\Omega\tilde{u}\cdot\nabla\bar{u}\Delta\tilde{u}dx\le\|\Delta\tilde{u}\|_{L^2}\|\nabla\bar{u}\|_{L^4}\|\tilde{u}\|_{L^4}\\\le&C\|\Delta\tilde{u}\|_{L^2}\|\nabla\bar{u}\|_{H^1}(\|\nabla\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\tilde{u}\|_{L^2}^{\frac{1}{2}}+\|\tilde{u}\|_{L^2})\\\le&\delta\|\Delta\tilde{u}\|_{L^2}^2+C\|\nabla\tilde{u}\|_{L^2}^2.\end{aligned}$$

For the term J_{10}, we deduce

$$\begin{aligned}J_{10}=&\alpha\int_\Omega\tilde{n}\nabla\varphi\Delta\tilde{u}dx-\alpha\int_\Omega g_u\Delta\tilde{u}dx\\\le&\delta\|\Delta\tilde{u}\|_{L^2}^2+C(\|\tilde{n}\|_{L^2}^2+\|g_u\|_{L^2}^2).\end{aligned}$$

By choosing δ small enough, with the estimates J_8, J_9 and J_{10}, we have

$$\frac{d}{dt}\|\nabla\tilde{u}\|_{L^2}^2+\|\Delta\tilde{u}\|_{L^2}^2\le C\|\nabla\tilde{u}\|_{L^2}^2+C\|g_u\|_{L^2}^2. \tag{5.13}$$

Applying ∇ to the first equation of (5.7), multiplying it by $\nabla\Delta\tilde{c}$, and integrating over Ω give

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\Delta c|^2dx+\int_\Omega|\nabla\Delta c|^2dx+\int_\Omega|\Delta c|^2dx\\=&-\int_\Omega\nabla(\bar{u}\nabla\tilde{c})\nabla\Delta\tilde{c}dx-\int_\Omega\nabla(\tilde{u}\nabla\bar{c})\nabla\Delta\tilde{c}dx+\alpha\int_\Omega\nabla\tilde{n}\nabla\Delta\tilde{c}dx\\&+\alpha\int_\Omega\nabla g_c\nabla\Delta\tilde{c}dx\\=&J_{11}+J_{12}+J_{13}.\end{aligned}$$

For the first term J_{11}, we have

$$\begin{aligned}J_{11}=&-\int_\Omega\nabla(\bar{u}\nabla\tilde{c})\nabla\Delta\tilde{c}dx=-\int_\Omega\nabla\bar{u}\nabla\tilde{c}\nabla\Delta\tilde{c}dx-\int_\Omega\bar{u}\Delta\tilde{c}\nabla\Delta\tilde{c}dx\\\le&\|\nabla\Delta\tilde{c}\|_{L^2}\|\nabla\bar{u}\|_{L^4}\|\nabla\tilde{c}\|_{L^4}+\|\nabla\Delta\tilde{c}\|_{L^2}\|\bar{u}\|_{L^4}\|\Delta\tilde{c}\|_{L^4}\\\le&\|\nabla\Delta\tilde{c}\|_{L^2}(\|\nabla\bar{u}\|_{L^2}^{\frac{1}{2}}\|\Delta\bar{u}\|_{L^2}^{\frac{1}{2}}+\|\nabla\bar{u}\|_{L^2})(\|\nabla\bar{c}\|_{L^2}^{\frac{1}{2}}\|\Delta\bar{c}\|_{L^2}^{\frac{1}{2}}+\|\nabla\bar{c}\|_{L^2})\\&+\|\nabla\Delta\tilde{c}\|_{L^2}\|\bar{u}\|_{H^1}(\|\nabla\Delta\tilde{c}\|_{L^2}^{\frac{1}{2}}\|\Delta\tilde{c}\|_{L^2}^{\frac{1}{2}}+\|\Delta\tilde{c}\|_{L^2})\\\le&\delta\|\nabla\Delta\tilde{c}\|_{L^2}^2+C\|\Delta\bar{u}\|_{L^2}^2+C\|\Delta\tilde{c}\|_{L^2}^2.\end{aligned}$$

Similarly, for the term J_{12},

$$\begin{aligned} J_{12} =& -\int_\Omega \nabla(\tilde{u}\nabla\bar{c})\nabla\Delta\tilde{c}dx = -\int_\Omega \nabla\tilde{u}\nabla\bar{c}\nabla\Delta\tilde{c}dx - \int_\Omega \tilde{u}\Delta\bar{c}\nabla\Delta\tilde{c}dx \\ \leq& \|\nabla\Delta\tilde{c}\|_{L^2}\|\nabla\tilde{u}\|_{L^4}\|\nabla\bar{c}\|_{L^4} + \|\tilde{u}\|_{L^4}\|\Delta\bar{c}\|_{L^4}\|\nabla\Delta\tilde{c}\|_{L^2} \\ \leq& C\|\nabla\Delta\tilde{c}\|_{L^2}(\|\nabla\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\Delta\tilde{u}\|_{L^2}^{\frac{1}{2}} + \|\nabla\tilde{u}\|_{L^2})\|\nabla\bar{c}\|_{H^1} \\ &+ (\|\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{u}\|_{L^2}^{\frac{1}{2}} + \|\tilde{u}\|_{L^2})(\|\Delta\bar{c}\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta\bar{c}\|_{L^2}^{\frac{1}{2}} + \|\Delta\bar{c}\|_{L^2})\|\nabla\Delta\tilde{c}\|_{L^2} \\ \leq& \delta\|\nabla\Delta\tilde{c}\|_{L^2}^2 + \delta\|\Delta\tilde{u}\|_{L^2}^2 + C\|\nabla\Delta\bar{c}\|_{L^2}^2 + C\|\nabla\tilde{u}\|_{L^2}^2. \end{aligned}$$

For the rest term J_{13}, we see

$$\begin{aligned} J_{13} =& \alpha\int_\Omega \nabla\tilde{n}\nabla\Delta\tilde{c}dx + \alpha\int_\Omega \nabla g_c\nabla\Delta\tilde{c}dx \\ \leq& \delta\|\nabla\Delta\tilde{c}\|_{L^2}^2 + C(\|\nabla\tilde{n}\|_{L^2}^2 + \|\nabla g_c\|_{L^2}^2). \end{aligned}$$

By choosing δ small enough, we get

$$\begin{aligned} &\frac{d}{dt}\|\Delta\tilde{c}\|_{L^2}^2 + \|\Delta\tilde{c}\|_{H^1}^2 \\ \leq& C(\|\nabla\tilde{n}\|_{L^2}^2 + \|\Delta\tilde{c}\|_{L^2}^2 + \|\nabla\tilde{u}\|_{L^2}^2) + C\|\Delta\bar{u}\|_{L^2}^2 + \delta\|\Delta\tilde{u}\|_{L^2}^2 \\ &+ C\|\nabla\Delta\bar{c}\|_{L^2}^2 + C\|\nabla g_c\|_{L^2}^2. \end{aligned} \tag{5.14}$$

From (5.13) and (5.14), along with δ small enough, it follows that

$$\begin{aligned} &\frac{d}{dt}(\|\nabla\tilde{u}\|_{L^2}^2 + \|\Delta\tilde{c}\|_{L^2}^2) + \|\Delta\tilde{u}\|_{L^2}^2 + \|\Delta\tilde{c}\|_{H^1}^2 \\ \leq& C(\|\nabla\tilde{u}\|_{L^2}^2 + \|\Delta\tilde{c}\|_{L^2}^2) + (\|\nabla\tilde{n}\|_{L^2}^2 + \|\Delta\bar{u}\|_{L^2}^2 + \|\nabla\Delta\bar{c}\|_{L^2}^2 + \|\nabla g_c\|_{L^2}^2) + C\|g_u\|_{L^2}^2. \end{aligned}$$

Applying Gronwall's lemma to the resulting differential inequality, we know

$$\|\nabla\tilde{u}\|_{L^2}^2 + \|\Delta\tilde{c}\|_{L^2}^2 + \int_0^t \|\Delta\tilde{u}\|_{L^2}^2 d\tau + \int_0^t \|\Delta\tilde{c}\|_{H^1}^2 d\tau \leq C.$$

Taking the L^2-inner product with $-\Delta\tilde{n}$ for the first equation of (5.7) implies

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\tilde{n}|^2dx + \int_\Omega |\Delta\tilde{n}|^2dx + \int_\Omega |\nabla\tilde{n}|^2dx \\ =& -\int_\Omega \bar{u}\cdot\nabla\tilde{n}\Delta\tilde{n}dx - \int_\Omega \tilde{u}\cdot\nabla\bar{n}\Delta\tilde{n}dx - \int_\Omega \nabla(\tilde{n}\nabla\bar{c})\Delta\tilde{n}dx - \int_\Omega \nabla(\bar{n}\nabla\tilde{c})\Delta\tilde{n}dx \\ &- \alpha(1+\gamma)\int_\Omega \tilde{n}\Delta\tilde{n}dx + 2\mu\int_\Omega \tilde{n}\bar{n}\Delta\tilde{n}dx - \alpha\int_\Omega g_n\Delta\tilde{n}dx \\ =& J_{14} + J_{15} + J_{16} + J_{17} + J_{18}. \end{aligned}$$

With the Gagliardo-Nirenberg interpolation inequality in hand, we can estimate J_{14} as follows

$$\begin{aligned} J_{14} =& -\int_\Omega \bar{u}\cdot\nabla\tilde{n}\Delta\tilde{n}dx \leq \|\bar{u}\|_{L^4}\|\nabla\tilde{n}\|_{L^4}\|\Delta\tilde{n}\|_{L^2} \\ \leq& C\|\bar{u}\|_{H^1}(\|\nabla\tilde{n}\|_{L^2}^{\frac{1}{2}}\|\Delta\tilde{n}\|_{L^2}^{\frac{1}{2}} + \|\nabla\tilde{n}\|_{L^2})\|\Delta\tilde{n}\|_{L^2} \\ \leq& \delta\|\Delta\tilde{n}\|_{L^2}^2 + C\|\nabla\tilde{n}\|_{L^2}^2. \end{aligned}$$

Similar to above estimates, we see

$$J_{15} = -\int_\Omega \tilde{u}\cdot\nabla\bar{n}\Delta\tilde{n}dx \leq \|\tilde{u}\|_{L^4}\|\nabla\bar{n}\|_{L^4}\|\Delta\tilde{n}\|_{L^2}$$

$$\begin{aligned}&\le C\|\tilde{u}\|_{H^1}\|\nabla\bar{n}\|_{H^1}\|\Delta\tilde{n}\|_{L^2}\\&\le\delta\|\Delta\tilde{n}\|_{L^2}+C\|\nabla\bar{n}\|_{H^1}^2.\end{aligned}$$

Similarly, we derive

$$\begin{aligned}J_{16}=&-\int_\Omega\nabla(\tilde{n}\nabla\bar{c})\Delta\tilde{n}dx=-\int_\Omega\nabla\tilde{n}\nabla\bar{c}\Delta\tilde{n}dx-\int_\Omega\tilde{n}\Delta\bar{c}\Delta\tilde{n}dx\\\le&\|\nabla\tilde{n}\|_{L^4}\|\nabla\bar{c}\|_{L^4}\|\Delta\tilde{n}\|_{L^2}+\|\tilde{n}\|_{L^4}\|\Delta\bar{c}\|_{L^4}\|\Delta\tilde{n}\|_{L^2}\\\le&(\|\nabla\tilde{n}\|_{L^2}^{\frac12}\|\Delta\tilde{n}\|_{L^2}^{\frac12}+\|\nabla\tilde{n}\|_{L^2})\|\nabla\bar{c}\|_{H^1}\|\Delta\tilde{n}\|_{L^2}\\&+(\|\tilde{n}\|_{L^2}^{\frac12}\|\nabla\tilde{n}\|_{L^2}^{\frac12}+\|\tilde{n}\|_{L^2})(\|\Delta\bar{c}\|_{L^2}^{\frac12}\|\nabla\Delta\bar{c}\|_{L^2}^{\frac12}+\|\Delta\bar{c}\|_{L^2})\|\Delta\tilde{n}\|_{L^2}\\\le&\delta\|\Delta\tilde{n}\|_{L^2}^2+C\|\nabla\tilde{n}\|_{L^2}^2+C\|\nabla\Delta\bar{c}\|_{L^2}^2+C\end{aligned}$$

and

$$\begin{aligned}J_{17}=&-\int_\Omega\nabla(\bar{n}\nabla\tilde{c})\Delta\tilde{n}dx=-\int_\Omega\nabla\bar{n}\nabla\tilde{c}\Delta\tilde{n}dx-\int_\Omega\nabla\bar{n}\Delta\tilde{c}\Delta\tilde{n}dx\\\le&\|\nabla\bar{n}\|_{L^4}\|\nabla\tilde{c}\|_{L^4}\|\Delta\tilde{n}\|_{L^2}+\|\bar{n}\|_{L^4}\|\Delta\tilde{c}\|_{L^4}\|\Delta\tilde{n}\|_{L^2}\\\le&(\|\nabla\bar{n}\|_{L^2}^{\frac12}\|\Delta\bar{n}\|_{L^2}^{\frac12}+\|\nabla\bar{n}\|_{L^2})\|\nabla\tilde{c}\|_{H^1}\|\Delta\tilde{n}\|_{L^2}\\&+\|\bar{n}\|_{H^1}(\|\Delta\tilde{c}\|_{L^2}^{\frac12}\|\nabla\Delta\tilde{c}\|_{L^2}^{\frac12}+\|\Delta\tilde{c}\|_{L^2})\|\Delta\tilde{n}\|_{L^2}\\\le&\delta\|\Delta\tilde{n}\|_{L^2}^2+C\|\nabla\Delta\tilde{c}\|_{L^2}^2+C.\end{aligned}$$

For the rest terms, we know

$$\begin{aligned}J_{18}=&-\alpha(1+\gamma)\int_\Omega\tilde{n}\Delta\tilde{n}dx+2\mu\int_\Omega\tilde{n}\bar{n}\Delta\tilde{n}dx-\alpha\int_\Omega g_n\Delta\tilde{n}dx\\\le&(1+\gamma)\|\tilde{n}\|_{L^2}\|\Delta\tilde{n}\|_{L^2}+2\mu\|\tilde{n}\|_{L^4}\|\bar{n}\|_{L^4}\|\Delta\tilde{n}\|_{L^2}+\|g_n\|_{L^2}\|\Delta\tilde{n}\|_{L^2}\\\le&(1+\gamma)\|\tilde{n}\|_{L^2}\|\Delta\tilde{n}\|_{L^2}+C(\|\tilde{n}\|_{L^2}^{\frac12}\|\nabla\tilde{n}\|_{L^2}^{\frac12}+\|\tilde{n}\|_{L^2})\|\bar{n}\|_{H^1}\|\Delta\tilde{n}\|_{L^2}\\&+\|g_n\|_{L^2}\|\Delta\tilde{n}\|_{L^2}\\\le&\delta\|\Delta\tilde{n}\|_{L^2}^2+C\|\nabla\tilde{n}\|_{L^2}^2+C\|g_n\|_{L^2}^2.\end{aligned}$$

Therefore, Taking δ small enough and together with $J_{14}-J_{18}$, we see that

$$\begin{aligned}&\frac{d}{dt}\|\nabla\tilde{n}\|_{L^2}^2+\|\nabla\tilde{n}\|_{H^1}^2\\\le&C(\|\nabla\tilde{n}\|_{L^2}^2+\|\nabla\bar{n}\|_{H^1}^2+\|\nabla\Delta\bar{c}\|_{L^2}^2+\|\nabla\Delta\tilde{c}\|_{L^2}^2+\|g_n\|_{L^2}^2)+C.\end{aligned}$$

Applying Gronwall's lemma to the resulting differential inequality, we know

$$\|\nabla\tilde{n}\|_{L^2}^2+\int_0^t\|\nabla\tilde{n}\|_{H^1}^2d\tau\le C.$$

Therefore, from Leray-Schauder theorem, we derive the existence of solution for (5.5). Along with the regularity of $(\tilde{n},\tilde{c},\tilde{u})$, the uniqueness of solution can easily get, so we omit the process. □

Theorem 5.1. *Assume that $\bar{s}=(\bar{n},\bar{c},\bar{u},\bar{f})\in\mathcal{S}_{ad}$ be an optimal solution for the control problem (5.3). Then, there exist Lagrange multipliers $(\lambda,\eta,\rho,\xi,\varphi,\omega)\in L^2(Q)\times(L^2(0,T;H^1(\Omega)))'\times L^2(Q)\times(H^1(\Omega))'\times(H^2(\Omega))'\times(H^1(\Omega))'$ such that for all $(\tilde{n},\tilde{c},\tilde{u},\tilde{f})\in V_n\times V_c\times V_u\times\mathcal{C}(\bar{f})$ has*

$$\beta_1\int_0^T\int_{\Omega_d}(\bar{n}-n_d)\tilde{n}dxdt+\beta_2\int_0^T\int_{\Omega_d}(\bar{c}-c_d)\tilde{c}dxdt+\beta_3\int_0^T\int_{\Omega_d}(\bar{u}-u_d)\tilde{u}dxdt$$

$$
\begin{aligned}
&+ \beta_4 \int_{\Omega_d} (\bar{n}(T) - n_\Omega)\tilde{n}(T)dx + \beta_5 \int_{\Omega_d} (\bar{c}(T) - c_\Omega)\tilde{c}(T)dx \\
&- \int_0^T \int_\Omega (\tilde{n}_t - \Delta\tilde{n} + \bar{u}\cdot\nabla\tilde{n} + \tilde{u}\cdot\nabla\bar{n} + \nabla\cdot(\bar{n}\nabla\tilde{c}) + \nabla(\tilde{n}\nabla\bar{c}) - \gamma\tilde{n} + 2\mu\tilde{n}\bar{n})\lambda dxdt \\
&- \int_0^T \int_\Omega (\tilde{c}_t - \Delta\tilde{c} + \bar{u}\cdot\nabla\tilde{c} + \tilde{u}\cdot\nabla\bar{c} + \tilde{c} - \tilde{n})\,\eta dxdt + \beta_7 \int_0^T \int_{\Omega_d} \tilde{f}\bar{f}dxdt \\
&- \int_0^T \int_\Omega (\tilde{u}_t - \Delta\tilde{u} + \bar{u}\cdot\nabla\tilde{u} + \tilde{u}\cdot\nabla\bar{u} - \tilde{n}\nabla\varphi)\,\rho dxdt + \int_\Omega \tilde{n}(0)\xi dx + \int_\Omega \tilde{c}(0)\varphi dx \\
&+ \int_\Omega \tilde{u}(0)\omega dx + \beta_6 \int_{\Omega_d} (\bar{u}(T) - u_\Omega)\tilde{u}(T)dx + \int_0^T \int_\Omega \tilde{f}\eta dxdt \geq 0, \qquad (5.15)
\end{aligned}
$$

where $\mathcal{C}(\bar{f}) = \{\theta(f - \bar{f}) : \theta \geq 0, f \in \mathcal{U}\}$.

Proof. With the Lemma 5.3 in hand, we get that $\bar{s} \in \mathcal{S}_{ad}$ is a regular point. Then, togather with Theorem 3.1 in [20], it follows that there exist Lagrange multipliers $(\lambda, \eta, \rho, \xi, \varphi, \omega) \in L^2(Q) \times (L^2(0,T;H^1(\Omega)))' \times L^2(Q) \times (H^1(\Omega))' \times (H^2(\Omega))' \times (H^1(\Omega))'$ such that

$$
\begin{aligned}
&J'(\bar{s})[r] - \langle G_1'(\bar{s})[r], \lambda\rangle - \langle G_2'(\bar{s})[r], \eta\rangle - \langle G_3'(\bar{s})[r], \rho\rangle - \langle G_4'(\bar{s})[r], \xi\rangle \\
&- \langle G_5'(\bar{s})[r], \varphi\rangle - \langle G_6'(\bar{s})[r], \omega\rangle \geq 0,
\end{aligned}
$$

for all $r = (\tilde{n}, \tilde{c}, \tilde{u}, \tilde{f}) \in V_n \times V_c \times V_u \times \mathcal{C}(\bar{f})$. Hence, the proof follows from Lemmas 5.1 and 5.2. □

Corollary 5.1. *Assume that* $\bar{s} = (\bar{n}, \bar{c}, \bar{u}, \bar{f}) \in \mathcal{S}_{ad}$ *be an optimal solution for the control problem (5.3). Then, there exist Lagrange multipliers* $(\lambda, \eta, \rho) \in L^2(Q) \times (L^2(0,T;H^1(\Omega)))' \times L^2(Q)$*, satisfying*

$$
\begin{aligned}
&\int_0^T \int_\Omega (\tilde{n}_t - \Delta\tilde{n} + \bar{u}\cdot\nabla\tilde{n} + \nabla(\tilde{n}\nabla\bar{c}) - \gamma\tilde{n} + 2\mu\tilde{n}\bar{n})\lambda dxdt - \int_0^T \int_\Omega \tilde{n}\eta dxdt \\
&- \int_0^T \int_\Omega \tilde{n}\nabla\varphi\rho dxdt = \beta_1 \int_0^T \int_{\Omega_d} (\bar{n} - n_d)\tilde{n}dxdt, \qquad (5.16)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T \int_\Omega (\tilde{c}_t - \Delta\tilde{c} + \bar{u}\cdot\nabla\tilde{c} + \tilde{c})\,\eta dxdt + \int_0^T \int_\Omega \nabla\cdot(\bar{n}\nabla\tilde{c})\lambda dxdt \\
=&\beta_2 \int_0^T \int_{\Omega_d} (\bar{c} - c_d)\tilde{c}dxdt, \qquad (5.17)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T \int_\Omega (\tilde{u}_t - \Delta\tilde{u} + \bar{u}\cdot\nabla\tilde{u} + \tilde{u}\cdot\nabla\bar{u})\,\rho dxdt + \int_0^T \int_\Omega \tilde{u}\nabla\bar{n}\lambda dxdt \\
&+ \int_0^T \int_\Omega \tilde{u}\cdot\nabla\bar{c}\eta dxdt = \beta_3 \int_0^T \int_{\Omega_d} (\bar{u} - u_d)\tilde{u}dxdt, \qquad (5.18)
\end{aligned}
$$

which corresponds to the linear system

$$
\begin{cases}
-\lambda_t - \Delta\lambda + \bar{u}\cdot\nabla\lambda - \nabla\lambda\nabla\bar{c} - \gamma\lambda + 2\mu\lambda\bar{n} - \eta - \nabla\varphi\rho \\
= \beta_1(\bar{n} - n_d), \\
-\eta_t - \Delta\eta + \bar{u}\cdot\nabla\eta + \eta + \nabla(\bar{n}\nabla\lambda) = \beta_2(\bar{c} - c_d), \\
-\rho_t - \Delta\rho + (\bar{u}\cdot\nabla)\rho + (\rho\cdot\nabla^T)\bar{u} + \lambda\nabla\bar{n} + \eta\nabla\bar{c} = \beta_3(\bar{u} - u_d),
\end{cases} \qquad (5.19)
$$

subject to the following boundary and final conditions

$$\begin{cases} \nabla\cdot\rho=0, & \text{in } Q,\\ \dfrac{\partial\lambda}{\partial\nu}=\dfrac{\partial\eta}{\partial\nu},\rho=0, & \text{on } (0,T)\times\partial\Omega,\\ \lambda(T)=\beta_4(\bar n(T)-n_\Omega),\eta(T)=\beta_5(\bar c(T)-c_\Omega), & \\ \rho(T)=\beta_5(\bar c(T)-c_\Omega), & \text{in } \Omega,\end{cases}$$

and the following identities hold

$$\int_0^T\int_{\Omega_d}(\beta_7\bar f+\eta)(f-\bar f)dxdt\ge 0,\ \forall f\in\mathcal{U}. \tag{5.20}$$

Proof. By taking $(\tilde c,\tilde u,\tilde f)=(0,0,0)$ in (5.15), then it follows that the equation (5.16) holds. In light of an analogous argument, and in light of the (5.15), it guarantees that (5.17) and (5.18) hold. On the other hand, let $(\tilde n,\tilde c,\tilde u)=(0,0,0)$, as an immediate consequence we obtain

$$\beta_7\int_0^T\tilde f\bar f dxdt+\int_0^T\tilde f\eta dxdt\ge 0,\quad \forall\tilde f\in\mathcal{C}(\bar f).$$

By choosing $\tilde f=f-\bar f\in\mathcal{C}(\bar f)$ for all $\bar f\in\mathcal{U}$, thus we achieve (5.20). □

Theorem 5.2. *Under the assumptions of Theorem 5.1, system (5.19) has a unique weak solution such that*

$$\|\lambda\|_{H^1}^2+\|\eta\|_{L^2}^2+\|\rho\|_{L^2}^2+\int_0^t\|\lambda\|_{H^2}^2d\tau+\int_0^t\|\eta\|_{H^1}^2d\tau+\int_0^t\|\rho\|_{H^1}^2d\tau\le C.$$

Proof. For convenience, we set $\tilde\lambda=\lambda(T-t)$, $\tilde\eta=\eta(T-t)$, $\tilde\rho=\rho(T-t)$, in order to simplify notations, we still write λ, η, ρ instead of $\tilde\lambda$, $\tilde\eta$, $\tilde\rho$, then the adjoint system (5.19) can be written as follow

$$\begin{cases} \lambda_t-\Delta\lambda+\bar u\cdot\nabla\lambda-\nabla\lambda\nabla\bar c-\gamma\lambda+2\mu\lambda\bar n-\eta-\nabla\varphi\rho & \\ =\beta_1(\bar n-n_d), & \text{in } Q,\\ \eta_t-\Delta\eta+\bar u\cdot\nabla\eta+\eta+\nabla(\bar n\nabla\lambda)=\beta_2(\bar c-c_d), & \text{in } Q,\\ \rho_t-\Delta\rho+(\bar u\cdot\nabla)\rho+(\rho\cdot\nabla^T)\bar u+\lambda\nabla\bar n+\eta\nabla\bar c=\beta_3(\bar u-u_d), & \text{in } Q,\end{cases} \tag{5.21}$$

subject to the following boundary and final conditions

$$\begin{cases} \nabla\cdot\rho=0, & \text{in } Q,\\ \dfrac{\partial\lambda}{\partial\nu}=\dfrac{\partial\eta}{\partial\nu},\rho=0, & \text{on } (0,T)\times\partial\Omega,\\ \lambda(0)=\beta_4(\bar n(T)-n_\Omega),\eta(0)=\beta_5(\bar c(T)-c_\Omega), & \\ \rho(0)=\beta_5(\bar c(T)-c_\Omega), & \text{in } \Omega.\end{cases}$$

Following an analogous reasoning as in the proof of Lemma 5.3, we omit the process and just give a number of a priori estimates as follows.

Taking the L^2-inner product with λ for the first equation of (5.21) implies

$$\begin{aligned}&\frac12\frac{d}{dt}\int_\Omega\lambda^2dx+\int_\Omega|\nabla\lambda|^2dx+2\mu\int_\Omega\lambda^2\bar n dx\\ =&\int_\Omega\nabla\lambda\nabla\bar c dx+\gamma\int_\Omega\lambda^2dx+\int_\Omega\lambda\eta dx+\int_\Omega\lambda\nabla\varphi\rho dx+\beta_1\int_\Omega(\bar n-n_d)\lambda dx\\ \le&\|\nabla\lambda\|_{L^2}\|\nabla\bar c\|_{L^2}+\gamma\|\lambda\|_{L^2}^2+\|\lambda\|_{L^2}(\|\eta\|_{L^2}+\|\rho\|_{L^2})+\beta_1\|\bar n-n_d\|_{L^2}\|\lambda\|_{L^2}\end{aligned}$$

$$\leq \frac{1}{2}\|\nabla\lambda\|_{L^2}^2 + C(\|\lambda\|_{L^2}^2 + \|\eta\|_{L^2}^2 + \|\rho\|_{L^2}^2) + C\|\bar{n} - n_d\|_{L^2}^2.$$

Then, we have

$$\frac{d}{dt}\|\lambda\|_{L^2}^2 + \|\lambda\|_{H^1}^2 \leq C(\|\lambda\|_{L^2}^2 + \|\eta\|_{L^2}^2 + \|\rho\|_{L^2}^2) + C\|\bar{n} - n_d\|_{L^2}^2. \tag{5.22}$$

Taking the L^2-inner product with $-\Delta\eta$ for the first equation of (5.21) implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\lambda|^2 dx + \int_\Omega |\Delta\lambda|^2 dx \\
=&\int_\Omega \bar{u}\cdot\nabla\lambda\Delta\lambda dx - \int_\Omega \nabla\lambda\nabla\bar{c}\Delta\lambda dx - \gamma\int_\Omega \lambda\Delta\lambda dx + 2\mu\int_\Omega \lambda\bar{n}\Delta\lambda dx \\
&- \int_\Omega \eta\Delta\lambda dx - \int_\Omega \nabla\varphi\rho\Delta\lambda dx + \beta_1\int_\Omega (\bar{n} - n_d)\Delta\lambda dx \\
\leq&\|\bar{u}\|_{L^4}\|\nabla\lambda\|_{L^4}\|\Delta\lambda\|_{L^2} + \|\nabla\lambda\|_{L^4}\|\nabla\bar{c}\|_{L^4}\|\Delta\lambda\|_{L^2} + \gamma\|\nabla\lambda\|_{L^2}^2 \\
&+ \|\lambda\|_{L^4}\|\bar{n}\|_{L^4}\|\Delta\lambda\|_{L^2} + \|\eta\|_{L^2}\|\Delta\lambda\|_{L^2} + \|\rho\|_{L^2}\|\Delta\lambda\|_{L^2} \\
&+ \beta_1\|\Delta\lambda\|_{L^2}\|\bar{n} - n_d\|_{L^2}^2 \\
\leq&\|\bar{u}\|_{H^1}(\|\nabla\lambda\|_{L^2}^{\frac{1}{2}}\|\Delta\lambda\|_{L^2}^{\frac{1}{2}} + \|\nabla\lambda\|_{L^2})\|\Delta\lambda\|_{L^2} + \gamma\|\nabla\lambda\|_{L^2}^2 \\
&+ (\|\nabla\lambda\|_{L^2}^{\frac{1}{2}}\|\Delta\lambda\|_{L^2}^{\frac{1}{2}} + \|\nabla\lambda\|_{L^2})\|\nabla\bar{c}\|_{H^1}\|\Delta\lambda\|_{L^2} + \|\eta\|_{L^2}\|\Delta\lambda\|_{L^2} \\
&+ \|\rho\|_{L^2}\|\Delta\lambda\|_{L^2} + \beta_1\|\Delta\lambda\|_{L^2}\|\bar{n} - n_d\|_{L^2}^2 \\
\leq&\frac{1}{2}\|\Delta\lambda\|_{L^2}^2 + C(\|\nabla\lambda\|_{L^2}^2 + \|\eta\|_{L^2}^2 + \|\rho\|_{L^2}^2).
\end{aligned}$$

Thus, we get

$$\frac{d}{dt}\|\nabla\lambda\|_{L^2}^2 + \|\nabla\lambda\|_{H^1}^2 \leq C(\|\nabla\lambda\|_{L^2}^2 + \|\eta\|_{L^2}^2 + \|\rho\|_{L^2}^2) + C\|\bar{n} - n_d\|_{L^2}^2. \tag{5.23}$$

Taking the L^2-inner product with η for the second equation of (5.21) implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \eta^2 dx + \int_\Omega |\nabla\eta|^2 dx + \int_\Omega \eta^2 dx \\
=&\int_\Omega \bar{n}\nabla\lambda\nabla\eta dx + \beta_2\int_\Omega \eta(\bar{c} - c_d)dx \\
\leq&\|\bar{n}\|_{L^4}\|\nabla\lambda\|_{L^4}\|\nabla\eta\|_{L^2} + \beta_2\|\eta\|_{L^2}\|\bar{c} - c_d\|_{L^2} \\
\leq&\|\bar{n}\|_{H^1}(\|\nabla\lambda\|_{L^2}^{\frac{1}{2}}\|\Delta\lambda\|_{L^2}^{\frac{1}{2}} + \|\nabla\lambda\|_{L^2})\|\nabla\eta\|_{L^2} + \beta_2\|\eta\|_{L^2}\|\bar{c} - c_d\|_{L^2} \\
\leq&\frac{1}{2}\|\nabla\eta\|_{L^2}^2 + \delta\|\Delta\lambda\|_{L^2}^2 + C\|\nabla\lambda\|_{L^2} + C\|\eta\|_{L^2}^2 + C\|\bar{c} - c_d\|_{L^2}^2.
\end{aligned}$$

As an immediate consequence, we obtain

$$\frac{d}{dt}\|\eta\|_{L^2}^2 + \|\eta\|_{H^1}^2 \leq \delta\|\Delta\lambda\|_{L^2}^2 + C\|\nabla\lambda\|_{L^2} + C\|\eta\|_{L^2}^2 + C\|\bar{c} - c_d\|_{L^2}^2. \tag{5.24}$$

Taking the L^2-inner product with ρ for the third equation of (5.21) implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^2 dx + \int_\Omega |\nabla\rho|^2 dx \\
=&-\int_\Omega (\rho\cdot\nabla^T)\bar{u}\rho dx - \lambda\int_\Omega \nabla\bar{n}\rho dx - \int_\Omega \eta\nabla\bar{c}\rho dx + \beta_3\int_\Omega (\bar{u} - u_d)\rho dx \\
\leq&\|\rho\|_{L^2}\|\nabla\bar{u}\|_{L^4}\|\rho\|_{L^4} + \lambda\|\nabla\bar{n}\|_{L^2}\|\rho\|_{L^2} + \|\eta\|_{L^2}\|\nabla\bar{c}\|_{L^4}\|\rho\|_{L^4}
\end{aligned}$$

$$
\begin{aligned}
&+\beta_3\|\rho\|_{L^2}\|\bar{u}-u_d\|_{L^2}\\
\leq&\|\rho\|_{L^2}\|\nabla\bar{u}\|_{H^1}(\|\rho\|_{L^2}^{\frac{1}{2}}\|\nabla\rho\|_{L^2}^{\frac{1}{2}}+\|\rho\|_{L^2})+\lambda\|\nabla\bar{n}\|_{L^2}\|\rho\|_{L^2}\\
&+\|\eta\|_{L^2}\|\nabla\bar{c}\|_{H^1}(\|\rho\|_{L^2}^{\frac{1}{2}}\|\nabla\rho\|_{L^2}^{\frac{1}{2}}+\|\rho\|_{L^2})+\beta_3\|\rho\|_{L^2}\|\bar{u}-u_d\|_{L^2}\\
\leq&\frac{1}{2}\|\nabla\rho\|_{L^2}^2+C\|\rho\|_{L^2}^2(\|\nabla\bar{u}\|_{H^1}^2+1)+C\|\eta\|_{L^2}^2+C\|\bar{u}-u_d\|_{L^2}^2.
\end{aligned}
$$

Therefore, we see that

$$
\frac{d}{dt}\|\rho\|_{L^2}^2+\|\rho\|_{H^1}^2\leq C\|\rho\|_{L^2}^2(\|\nabla\bar{u}\|_{H^1}^2+1)+C\|\eta\|_{L^2}^2+C\|\bar{u}-u_d\|_{L^2}^2. \tag{5.25}
$$

Combining (5.22)-(5.25) and taking δ small enough, we have

$$
\begin{aligned}
&\frac{d}{dt}(\|\lambda\|_{H^1}^2+\|\eta\|_{L^2}^2+\|\rho\|_{L^2}^2)+\|\lambda\|_{H^2}^2+\|\eta\|_{H^1}^2+\|\rho\|_{H^1}^2\\
\leq&C(\|\nabla\bar{u}\|_{H^1}^2+1)(\|\lambda\|_{H^1}^2+\|\eta\|_{L^2}^2+\|\rho\|_{L^2}^2)+C\|\bar{n}-n_d\|_{L^2}^2\\
&+C\|\bar{c}-c_d\|_{L^2}^2+C\|\bar{u}-u_d\|_{L^2}^2.
\end{aligned}
$$

Applying Gronwall's lemma to the resulting differential inequality, we know

$$
\|\lambda\|_{H^1}^2+\|\eta\|_{L^2}^2+\|\rho\|_{L^2}^2+\int_0^t\|\lambda\|_{H^2}^2d\tau+\int_0^t\|\eta\|_{H^1}^2d\tau+\int_0^t\|\rho\|_{H^1}^2d\tau\leq C.
$$

The proof is complete. □

Acknowledgment. The authors would like to express their deep thanks to the referee's valuable suggestions for the revision and improvement of the manuscript.

REFERENCES

[1] P. Biler, W. Hebisch and T. Nadzieja, The Debye system: Existence and large time behavior of solutions, *Nonlinear Anal.*, **23** (1994), 1189–1209.

[2] B. Chen and C. Liu, Optimal distributed control of a Allen-Cahn/Cahn-Hilliard system with temperature, *Applied Mathematics and Optimization*, 2021.

[3] B. Chen, H. Li and C. Liu, Optimal distributed control for a coupled phase-field system, *Discrete and Continuous Dynamical Systems Series B*.

[4] P. Colli, G. Gilardi, G. Marinoschi and E. Rocca, Optimal control for a conserved phase field system with a possibly singular potential, *Evol. Equ. Control Theory*, **7** (2018), 95–116.

[5] P. Colli, G. Gilardi, E. Rocca and J. Sprekels, Optimal distributed control of a diffuse interface model of tumor growth, *Nonlinearity*, **30** (2017), 2518–2546.

[6] E. Espejo and T. Suzuki, Reaction terms avoiding aggregation in slow fluids, *Nonlinear Anal. Real World Appl.*, **21** (2015), 110–126.

[7] A. Friedman, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.

[8] S. Frigeri, M. Grasselli and J. Sprekels, Optimal distributed control of two-dimensional nonlocal Cahn-Hilliard-Navier-Stokes systems with degenerate mobility and singular potential, *Appl. Math. Optim.*, **81** (2020), 899–931.

[9] F. Guillén-González, E. Mallea-Zepeda and M. Rodríguez-Bellido, Optimal bilinear control problem related to a chemo-repulsion system in 2D domains, *ESAIM Control Optim. Calc. Var.*, **26** (2020), 21pp.

[10] A. Helmut and T. Yutaka, On Stokes operators with variable viscosity in bounded and unbounded domains, *Math. Ann.*, **344** (2009), 381–429.

[11] C. Jin, Large time periodic solutions to coupled chemotaxis-fluid models, *Z. Angew. Math. Phys.*, **68** (2017), 24pp.

[12] C. Jin, Large time periodic solution to the coupled chemotaxis-Stokes model, *Math. Nachr.*, **290** (2017), 1701–1715.

[13] C. Liu and X. Zhang, Optimal distributed control for a new mechanochemical model in biological patterns, *J. Math. Anal. Appl.*, **478** (2019), 825–863.

[14] C. Liu and X. Zhang, Optimal control of a new mechanochemical model with state constraint, *Math. Methods Appl. Sci.*, **44** (2021), 9237–9263.
[15] S. U. Ryu and A. Yagi, Optimal control of Keller-Segel equations, *J. Math. Anal. Appl.*, **256** (2001), 45–66.
[16] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.*, **146** (1987), 65–96.
[17] Y. Tao and M. Winkler, Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system, *Z. Angew. Math. Phys.*, **66** (2015), 2555–2573.
[18] Y. Tao and M. Winkler, Blow-up prevention by quadratic degradation in a two-dimensional Keller-Segel-Navier-Stokes system, *Z. Angew. Math. Phys.*, **67** (2016), 1–23.
[19] X. Zhang, H. Li and C. Liu, Optimal control problem for the Cahn-Hilliard/Allen-Cahn equation with state constraint, *Appl. Math. Optim.*, **82** (2020), 721–754.
[20] J. Zowe and S. Kurcyusz, Regularity and stability for the mathematical programming problem in Banach spaces, *Appl. Math. Optim.*, **5** (1979), 49–62.

Received August 2021; revised September 2021; early access October 2021.

E-mail address: yuanyf19@mails.jlu.edu.cn
E-mail address: liucc@jlu.edu.cn