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CONGRUENCES FOR SIXTH ORDER MOCK THETA FUNCTIONS $\lambda(q)$ AND $\rho(q)$

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Abstract. Ramanujan introduced sixth order mock theta functions $\lambda(q)$ and $\rho(q)$ defined as:

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n},$$
$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q;q)_n}{(q;q^2)_{n+1}},$$

listed in the Lost Notebook. In this paper, we present some Ramanujan-like congruences and also find their infinite families modulo 12 for the coefficients of mock theta functions mentioned above.

1. **Introduction.** In 1920, Ramanujan introduced 17 mock theta functions of odd order in his last letter to Hardy. In addition Ramanujan also gave the mock theta functions of order 6 that are listed in the Lost Notebook. In the study of the arithmetic properties of mock theta functions, many authors have found some congruence properties for their coefficients. For instance, Andrews et al. [1] found several congruences for the partition functions $p_\omega(n)$ and $p_\nu(n)$ corresponding to the mock theta functions $\omega(q)$ and $\nu(q)$ respectively, defined as:

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2(n^2+n)}}{(q;q^2)_{n+1}^2},$$
$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}},$$

where

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), \tag{1}$$

for some positive integer n and

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j) \tag{2}$$

with $|q| < 1$. They proved congruences for modulo 2 and some infinite families of congruences for $p_\omega(n)$ and $p_\nu(n)$. In 2017, Fathima and Pore [4] obtained a

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number of congruences for $p_\omega(n)$ and $p_\nu(n)$ modulo 20 and some infinite families of congruences modulo 2. In the sequel, Baruah and Begum [2] in 2019 established many congruences for the same partition functions modulo powers of 5.

Zhang in 2018 proved some congruences for the sixth order mock theta function $\beta(q)$ shown below and also gave some conjectures in [8].

$$\beta(q) = \sum_{n=0}^{\infty} \frac{q^{3n^2+3n+1}}{(q;q^3)_{n+1}(q^2;q^3)_{n+1}}.$$

Brietzke, Silva, and Sellers [3] in 2019 found many arithmetic properties satisfied by the coefficients of the eighth order mock theta function $V_0(q)$ given as:

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q;q^2)_n}.$$

Silva and Sellers in [7] proved some congruence relations for the third order mock theta function $\xi(q)$ given by Gordon and McIntosh given below:

$$\xi(q) = 1 + 2\sum_{n=1}^{\infty} \frac{q^{6n^2-6n+1}}{(q;q^6)_n(q^5;q^6)_n}.$$

The main purpose of this paper is to study the arithmetic properties of the sixth order mock theta functions $\lambda(q)$ and $\rho(q)$ given by Ramanujan where the two mock theta functions are defined as:

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n} = \sum_{n=0}^{\infty} p_\lambda(n) q^n, \tag{3}$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q;q)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} p_\rho(n) q^n. \tag{4}$$

Ramanujan also listed linear relations connecting the sixth order mock theta functions with each other as:

$$2q^{-1}\psi_6(q^2) + \lambda(-q) = (-q;q^2)_\infty^2 f(q,q^5), \tag{5}$$

$$q^{-1}\psi_6(q^2) + \rho(q) = (-q;q^2)_\infty^2 f(q,q^5), \tag{6}$$

where $\psi_6(q)$ is the sixth order mock theta function

$$\psi_6(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}}.$$

The proof technique of all the congruences for mock theta functions involves applying identities on the coefficients in arithmetic progressions, we use the same idea to prove infinite family of congruences modulo certain numbers of the form $2^\alpha \cdot 3^\beta$ for $p_\lambda(n)$ and $p_\rho(n)$. The main results are found in Theorem 3.2–3.7 given in Section 3. Before proceeding towards the main theorems, we need some preliminary results given in Section 2 for proving the results in Section 3.

2. Preliminaries.

To shorten the notations, we define

$$f_l = (q^l;q^l)_\infty,$$

for some positive integer l.

Now, we define Ramanujan's general theta function

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2},\ \text{for } |ab|<1, \tag{7}$$

Jacobi's triple product identity is defined as:

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

The special cases for $f(a,b)$ are:

$$\varphi(q)=f(q,q)=\sum_{n=-\infty}^{\infty} q^{n^2}=\frac{f_2^5}{f_1^2f_4^2}, \tag{8}$$

$$\psi(q)=f(q,q^3)=\sum_{n=0}^{\infty} q^{n(n+1)/2}=\frac{f_2^2}{f_1}, \tag{9}$$

$$\varphi(-q)=\frac{f_1^2}{f_2}, \tag{10}$$

$$\psi(-q)=\frac{f_1f_4}{f_2}. \tag{11}$$

In some of the proofs, we make use of the following identities:

$$f_1^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}, \text{(Jacobi's identity)}$$

$$f_1=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}, \text{(Euler's Pentagonal number theorem)}.$$

The following lemma exhibits the 3-dissection of $\psi(q)$ and $1/\varphi(-q)$.

Lemma 2.1. *We have*

$$\psi(q)=\frac{f_6f_9^2}{f_3f_{18}}+q\frac{f_{18}^2}{f_9}, \tag{12}$$

$$\frac{1}{\varphi(-q)}=\frac{f_6^4f_9^6}{f_3^8f_{18}^3}+2q\frac{f_6^3f_9^3}{f_3^7}+4q^2\frac{f_6^2f_{18}^3}{f_3^6}. \tag{13}$$

Proof. Identity (12) is equation (14.3.3) of [5]. Identity (13) comes from [6] shown in equation (2). □

Lemma 2.2. *We have*

$$\frac{f_1^3}{f_3}=\frac{f_4^3}{f_{12}}-3q\frac{f_2^2f_{12}^3}{f_4f_6^2}, \tag{14}$$

$$\frac{f_3}{f_1^3}=\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}. \tag{15}$$

Proof. The above identity (14) is (22.1.13) in [5]. And (15) follows from (14) by replacing q by $-q$ and using

$$(-q;-q)_\infty=\frac{f_2^3}{f_1f_4}.$$

□

From the binomial theorem, for any positive integer l and prime p, we have

$$(q;q)_\infty^{p^l}\equiv(q^p;q^p)_\infty^{p^{l-1}} \pmod{p^l}.$$

3. **Congruence relations for $\lambda(q)$ and $\rho(q)$.** We first prove the 2-dissection of $\lambda(q)$.

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty} p_\lambda(2n)q^n = \frac{f_2^3 f_3^2}{f_1^3 f_6}, \tag{16}$$

$$\sum_{n=0}^{\infty} p_\lambda(2n+1)q^n = -3\frac{f_6^3}{f_1 f_2} + 2q^{-1}\psi_6(q). \tag{17}$$

Proof. To prove the above dissections, consider (5) and replacing q by $-q$ in (5), we have

$$\begin{aligned}\lambda(q) &= (q;q^2)_\infty^2 f(-q,-q^5) + 2q^{-1}\psi_6(q^2),\\ &= \frac{f_1^3 f_6^2}{f_2^3 f_3} + 2q^{-1}\psi_6(q^2),\\ &= \frac{f_6^2}{f_2^3}\left(\frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}\right) + 2q^{-1}\psi_6(q^2),\end{aligned}$$

where the last equality follows from (14). Extracting even and odd terms from above equation, we get

$$\sum_{n=0}^{\infty} p_\lambda(2n)q^{2n} = \frac{f_6^2 f_4^3}{f_2^3 f_{12}}, \tag{18}$$

$$\sum_{n=0}^{\infty} p_\lambda(2n+1)q^{2n+1} = -3q\frac{f_{12}^3}{f_2 f_4} + 2q^{-1}\psi_6(q^2). \tag{19}$$

Replace q^2 by q in (18) to arrive at (16). Divide (19) by q and replace q^2 by q to obtain (17). ☐

Theorem 3.2. *We have*

$$p_\lambda(2n) \equiv \begin{cases} 1 \pmod 2 & \text{if } n \text{ is triangular number},\\ 0 \pmod 2 & \text{otherwise.}\end{cases}$$

Proof. According to (16), we have

$$p_\lambda(2n) = \frac{f_3^2 f_2^3}{f_1^3 f_6} \equiv \frac{f_6 f_2^3}{f_1 f_2 f_6} = \frac{f_2^2}{f_1} = \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \pmod 2. \tag{20}$$

Therefore, we complete the proof. ☐

Theorem 3.3. *We have*

$$p_\lambda(2n) \equiv \begin{cases} (-1)^k \pmod 3 & \text{if } n = \frac{3k(3k-1)}{2},\\ 0 \pmod 3 & \text{otherwise.}\end{cases}$$

Proof. According to (16), we have

$$\sum_{n=0}^{\infty} p_\lambda(2n)q^n = \frac{f_3^2 f_2^3}{f_1^3 f_6} \equiv \frac{f_3^2 f_6}{f_3 f_6} = f_3 = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(3n-1)/2} \pmod 3. \tag{21}$$

Therefore, we complete the proof. ☐

Corollary 1. *Let $p > 3$ be a prime and r an integer such that $8r+1$ is a quadratic non-residue modulo p. Then for all $n \geq 0$,*

$$p_\lambda(2(pn+r)) \equiv 0 \pmod{6}.$$

Proof. As

$$pn + r = \frac{k(k+1)}{2} \Rightarrow r \equiv \frac{k(k+1)}{2} \pmod{p}.$$

Thus, $2r \equiv k^2 + k \pmod{p}$ or $8r+1 \equiv (2k+1)^2 \pmod{p}$ which contradicts the fact that $8r+1$ is a quadratic non-residue modulo p. Therefore, from Theorem 3.2,

$$p_\lambda(2(pn+r)) \equiv 0 \pmod{2}. \tag{22}$$

Similarly,

$$pn + r = \frac{3k(3k-1)}{2} \Rightarrow r \equiv \frac{3k(3k-1)}{2} \pmod{p}.$$

then $2r \equiv 9k^2 - 3k \pmod{p}$ or $8r+1 \equiv (6k-1)^2 \pmod{p}$ which contradicts the fact that $8r+1$ is a quadratic non-residue modulo p. Therefore, from Theorem 3.3,

$$p_\lambda(2(pn+r)) \equiv 0 \pmod{3}. \tag{23}$$

From (22) and (23), we readily arrive at the main result. □

Since gcd(6, p)=1, among the $p-1$ residues modulo p, there are $(p-1)/2$ residues r for which $8r+1$ is a quadratic non-residue modulo p. So the above result leads us to a number of congruences for different primes $p > 3$ as shown below:

$$p_\lambda(10n+i) \equiv 0 \pmod{6},\ i \in \{4, 8\},$$
$$p_\lambda(14n+i) \equiv 0 \pmod{6},\ i \in \{4, 8, 10\}.$$

Theorem 3.4. *We have*

$$p_\lambda(6n+2) \equiv 0 \pmod{3}, \tag{24}$$
$$p_\lambda(6n+4) \equiv 0 \pmod{6}, \tag{25}$$
$$p_\lambda(18n+8) \equiv 0 \pmod{18}, \tag{26}$$
$$p_\lambda(18n+14) \equiv 0 \pmod{72}. \tag{27}$$

Proof. From (16), we get

$$\sum_{n=0}^{\infty} p_\lambda(2n)q^n = \frac{f_3^2}{f_6} \cdot \frac{f_2^2}{f_1} \cdot \frac{f_2}{f_1^2}. \tag{28}$$

Using (9) and (10), we have

$$\sum_{n=0}^{\infty} p_\lambda(2n)q^n = \frac{\varphi(-q^3)\psi(q)}{\varphi(-q)}.$$

Using Lemma 2.1, we obtain

$$\sum_{n=0}^{\infty} p_\lambda(2n)q^n = \varphi(-q^3)\left(\frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}\right)\left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right).$$

Extracting the terms involving $q^{3n}, q^{3n+1}, q^{3n+2}$ from above equation, we have

$$\sum_{n=0}^{\infty} p_\lambda(6n)q^{3n} = \varphi(-q^3)\left(\frac{f_6^5 f_9^8}{f_3^9 f_{18}^4} + 4q^3\frac{f_6^2 f_{18}^5}{f_3^6 f_9}\right), \tag{29}$$

$$\sum_{n=0}^{\infty} p_\lambda(6n+2)q^{3n+1} = \varphi(-q^3)\left(2q\frac{f_6^4 f_9^5}{f_3^8 f_{18}} + q\frac{f_6^4 f_9^5}{f_3^8 f_{18}}\right), \tag{30}$$

$$\sum_{n=0}^{\infty} p_\lambda(6n+4)q^{3n+2} = \varphi(-q^3)\left(4q^2\frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} + 2q^2\frac{f_6^3 f_9^2 f_{18}^2}{f_3^7}\right). \tag{31}$$

To prove (25), dividing (31) by q^2 and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} p_\lambda(6n+4)q^n = 6\varphi(-q)\frac{f_2^3 f_3^2 f_6^2}{f_1^7}$$

which can also be written as

$$\sum_{n=0}^{\infty} p_\lambda(6n+4)q^n = 6\frac{f_2^2 f_3^2 f_6^2}{f_1^5}.$$

The above equation readily implies (25). Consider (30), dividing by q and replacing q^3 by q, we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_\lambda(6n+2)q^n &= \varphi(-q)\left(2\frac{f_2^4 f_3^5}{f_1^8 f_6} + \frac{f_2^4 f_3^5}{f_1^8 f_6}\right),\\ &= 3\frac{f_3^5}{f_6}\left(\frac{f_2}{f_1^2}\right)^3,\end{aligned} \tag{32}$$

which proves (24). Now using (13) in above equation,

$$\sum_{n=0}^{\infty} p_\lambda(6n+2)q^n = 3\frac{f_3^5}{f_6}\left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right)^3. \tag{33}$$

Extracting the terms involving $q^{3n}, q^{3n+1}, q^{3n+2}$ from (33), we have

$$\sum_{n=0}^{\infty} p_\lambda(18n+2)q^{3n} = 3\frac{f_3^5}{f_6}\left(\frac{f_6^{12} f_9^{18}}{f_3^{24} f_{18}^9} + 56q^3\frac{f_6^9 f_9^9}{f_3^{21}} + 64q^6\frac{f_6^6 f_{18}^9}{f_3^{18}}\right), \tag{34}$$

$$\sum_{n=0}^{\infty} p_\lambda(18n+8)q^{3n+1} = 3\frac{f_3^5}{f_6}\left(6q\frac{f_6^{11} f_9^{15}}{f_3^{23} f_{18}^6} + 96q^4\frac{f_6^8 f_9^6 f_{18}^3}{f_3^{20}}\right), \tag{35}$$

$$\sum_{n=0}^{\infty} p_\lambda(18n+14)q^{3n+2} = 3\frac{f_3^5}{f_6}\left(24q^2\frac{f_6^{10} f_9^{12}}{f_3^{15} f_{18}^3} + 96q^5\frac{f_6^7 f_9^3 f_{18}^6}{f_3^{19}}\right). \tag{36}$$

Dividing (35) and (36) by q and q^2 respectively, replacing q^3 by q, we arrive at

$$\sum_{n=0}^{\infty} p_\lambda(18n+8)q^n = 3\frac{f_1^5}{f_2}\left(6\frac{f_2^{11} f_3^{15}}{f_1^{23} f_6^6} + 96q\frac{f_2^8 f_3^6 f_6^3}{f_1^{20}}\right),$$

$$\sum_{n=0}^{\infty} p_\lambda(18n+14)q^n = 3\frac{f_1^5}{f_2}\left(24\frac{f_2^{10} f_3^{12}}{f_1^{22} f_6^3} + 96q\frac{f_2^7 f_3^3 f_6^6}{f_1^{19}}\right).$$

From the above equations, we get (26) and (27). □

Corollary 2. *We have*

$$\sum_{n=0}^{\infty} p_\lambda(18n+8)q^n \equiv 18 f_1^2 f_3^3 \pmod{72}. \tag{37}$$

Now we present the infinite families of congruences modulo 12.

Theorem 3.5. *For prime $p \geq 5$, we have*

$$p_\lambda\left(6p^2n + 6pi + \frac{p^2-1}{4}\right) \equiv 0 \pmod{12} \tag{38}$$

where $i = 1, 2, \cdots, p-1$.

Proof. From (29), replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} p_\lambda(6n)q^n = \varphi(-q)\left(\frac{f_2^5 f_3^8}{f_1^9 f_6^4} + 4q\frac{f_2^2 f_6^5}{f_1^6 f_3}\right)$$

Reducing modulo 4,

$$\begin{aligned}\sum_{n=0}^{\infty} p_\lambda(6n)q^n &\equiv \frac{f_1^2}{f_2}\frac{f_2^5 f_3^8}{f_1^9 f_6^4} \pmod 4\\ &\equiv \frac{f_2^4 f_1}{f_1^8} \pmod 4\\ &\equiv f_1 \pmod 4. \end{aligned} \tag{39}$$

By (21), we have

$$\sum_{n=0}^{\infty} p_\lambda(2n)q^n \equiv f_3 \pmod 3.$$

Extracting the terms involving q^{3n} and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} p_\lambda(6n)q^n \equiv f_1 \pmod 3. \tag{40}$$

From (39) and (40), we have

$$\sum_{n=0}^{\infty} p_\lambda(6n)q^n \equiv f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} \pmod{12}.$$

This implies that $p_\lambda(6k) \equiv 0 \pmod{12}$ unless k is a pentagonal number, or equivalently, unless $24k+1$ is a square. Now letting $k = p^2n + pi + (p^2-1)/24$ where $i = 1, 2, \cdots, (p-1)$ and p is a prime, we have that $24k+1 = 24p^2n + 24pi + p^2$, and this is evidently not a square since p^2 divides the first and third terms but not the middle term. Thus $p_\lambda(6k) = p_\lambda(6p^2n + 6pi + (p^2-1)/4) \equiv 0 \pmod{12}$. □

Theorem 3.6. *For $m \geq 1$, we have*

$$\sum_{n=0}^{\infty} p_\lambda\left(3^{4m-3}\cdot 2n + 2\sum_{j=0}^{2m-2} 3^{2j}\right) q^n \equiv 3\frac{f_2 f_3 f_6}{f_1^2} \pmod{12}, \tag{41}$$

$$\sum_{n=0}^{\infty} p_\lambda\left(3^{4m-1}\cdot 2n + 2\sum_{j=0}^{2m-1} 3^{2j}\right) q^n \equiv 9\frac{f_2 f_3 f_6}{f_1^2} \pmod{12}, \tag{42}$$

$$\sum_{n=0}^{\infty} p_\lambda\left(3^{4m-2}\cdot 2n+2\sum_{j=0}^{2m-2}3^{2j}\right)q^n \equiv 3\frac{f_2^3f_3^2}{f_1^3f_6} \pmod{12}, \tag{43}$$

$$\sum_{n=0}^{\infty} p_\lambda\left(3^{4m}\cdot 2n+2\sum_{j=0}^{2m-1}3^{2j}\right)q^n \equiv 9\frac{f_2^3f_3^2}{f_1^3f_6} \pmod{12}, \tag{44}$$

$$\sum_{n=0}^{\infty} p_\lambda\left(3^{2m+1}\cdot 2n+3^{2m}\cdot 2^2+2\sum_{j=0}^{m-1}3^{2j}\right)q^n \equiv 6\frac{f_6^3}{f_1} \pmod{12}, \tag{45}$$

$$\sum_{n=0}^{\infty} p_\lambda\left(3^{2m}\cdot 2n+3^{2m-1}\cdot 2+2\sum_{j=0}^{m-1}3^{2j}\right)q^n \equiv 6f_2f_3^3 \pmod{12}, \tag{46}$$

$$\sum_{n=0}^{\infty} p_\lambda\left(3^{4m-1}\cdot 2n+2\sum_{j=0}^{2m-2}3^{2j}\right)q^n \equiv 3f_1 \pmod{12}, \tag{47}$$

$$\sum_{n=0}^{\infty} p_\lambda\left(3^{4m+1}\cdot 2n+2\sum_{j=0}^{2m-1}3^{2j}\right)q^n \equiv 9f_1 \pmod{12}. \tag{48}$$

Theorem 3.7. *For $m \geq 1$, we have*

$$p_\lambda\left(3^{2m}\cdot 2n+3^{2m-1}\cdot 2^2+2\sum_{j=0}^{m-1}3^{2j}\right) \equiv 0 \pmod{12}. \tag{49}$$

Proof of Theorems 3.6 and 3.7. The proof for the above theorems follows by induction. Let us first prove the first step of induction, for $m = 1$.

From (32), we have

$$\sum_{n=0}^{\infty} p_\lambda(6n+2)q^n = 3\frac{f_2^3f_3^5}{f_1^6f_6}.$$

Reducing modulo 12, we have

$$\sum_{n=0}^{\infty} p_\lambda(6n+2)q^n \equiv 3f_3f_6\frac{f_2}{f_1^2} \pmod{12} \tag{50}$$

which proves (41) for $m = 1$. Using (13), we get

$$\sum_{n=0}^{\infty} p_\lambda(6n+2)q^n \equiv 3f_3f_6\left(\frac{f_6^4f_9^6}{f_3^8f_{18}^3}+2q\frac{f_6^3f_9^3}{f_3^7}+4q^2\frac{f_6^2f_{18}^3}{f_3^6}\right) \pmod{12}$$

or

$$\sum_{n=0}^{\infty} p_\lambda(6n+2)q^n \equiv 3\frac{f_6^5f_9^6}{f_3^7f_{18}^3}+6q\frac{f_6^4f_9^3}{f_3^6} \pmod{12}.$$

Extracting the terms involving $q^{3n}, q^{3n+1}, q^{3n+2}$ from above, we have

$$\sum_{n=0}^{\infty} p_\lambda(18n+2)q^{3n} \equiv 3\frac{f_6^5f_9^6}{f_3^7f_{18}^3} \pmod{12}, \tag{51}$$

$$\sum_{n=0}^{\infty} p_\lambda(18n+8)q^{3n+1} \equiv 6q\frac{f_6^4f_9^3}{f_3^6} \pmod{12}, \tag{52}$$

$$\sum_{n=0}^{\infty} p_\lambda(18n+14)q^{3n+2} \equiv 0 \pmod{12}. \tag{53}$$

Dividing (52) and (53) by q and q^2 respectively then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} p_\lambda(18n+8)q^n \equiv 6\frac{f_2^4 f_3^3}{f_1^6} \equiv 6f_2 f_3^3 \pmod{12}, \tag{54}$$

$$\sum_{n=0}^{\infty} p_\lambda(18n+14)q^n \equiv 0 \pmod{12}. \tag{55}$$

Here (54) and (55) proves (46) and (49), respectively for $m=1$. Replacing q^3 by q in (51), we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_\lambda(18n+2)q^n &\equiv 3\frac{f_2^5 f_3^6}{f_1^7 f_6^3} \pmod{12}\\ &\equiv 3\frac{f_2^3 f_3^2}{f_1^3 f_6} \pmod{12}\\ &= 3\frac{f_2^2}{f_1}\cdot\frac{f_2}{f_1^2}\cdot\frac{f_3^2}{f_6} \pmod{12}.\end{aligned}$$

The above equation proves (43) for $m=1$ and it can also be written as:

$$\sum_{n=0}^{\infty} p_\lambda(18n+2)q^n \equiv 3\frac{\varphi(-q^3)\psi(q)}{\varphi(-q)} \pmod{12}. \tag{56}$$

Using Lemma 2.1, we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_\lambda(18n+2)q^n \quad &\equiv \quad 3\varphi(-q^3)\left(\frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}\right)\\ &\qquad \left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right) \pmod{12}.\end{aligned}$$

Extracting the terms involving $q^{3n}, q^{3n+1}, q^{3n+2}$ from above, we get

$$\sum_{n=0}^{\infty} p_\lambda(54n+2)q^{3n} \equiv 3\varphi(-q^3)\frac{f_6^5 f_9^8}{f_3^9 f_{18}^4} \pmod{12}, \tag{57}$$

$$\sum_{n=0}^{\infty} p_\lambda(54n+20)q^{3n+1} \equiv 9q\varphi(-q^3)\frac{f_6^4 f_9^5}{f_3^8 f_{18}} \pmod{12}, \tag{58}$$

$$\sum_{n=0}^{\infty} p_\lambda(54n+38)q^{3n+2} \equiv 6q^2\varphi(-q^3)\frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} \pmod{12}. \tag{59}$$

Replacing q^3 by q in (57), we have

$$\sum_{n=0}^{\infty} p_\lambda(54n+2)q^n \equiv 3\frac{f_2^4 f_3^8}{f_1^7 f_6^4} \equiv 3f_1 \pmod{12}$$

which proves (47) for $m=1$. Now, dividing (58) and (59) by q and q^2 respectively then replacing q^3 by q,

$$\sum_{n=0}^{\infty} p_\lambda(54n+20)q^n \equiv 9\frac{f_2^3 f_3^5}{f_1^6 f_6} \equiv 9\frac{f_2 f_3 f_6}{f_1^2} \pmod{12},$$

$$\sum_{n=0}^{\infty} p_\lambda(54n+38)q^n \equiv 6\frac{f_2^2 f_3^2 f_6^2}{f_1^5} \equiv 6\frac{f_6^3}{f_1} \pmod{12}.$$

The above congruences imply (42) and (45) for $m=1$. Consider

$$\sum_{n=0}^{\infty} p_\lambda(54n+20)q^n \equiv 9f_3f_6\left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right) \pmod{12}$$

or

$$\sum_{n=0}^{\infty} p_\lambda(54n+20)q^n \equiv 9f_3f_6\left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7}\right) \pmod{12}.$$

Extracting the terms involving $q^{3n}, q^{3n+1}, q^{3n+2}$ from above, we arrive at

$$\sum_{n=0}^{\infty} p_\lambda(162n+20)q^{3n} \equiv 9\frac{f_6^5 f_9^6}{f_3^7 f_{18}^3} \pmod{12}, \tag{60}$$

$$\sum_{n=0}^{\infty} p_\lambda(162n+74)q^{3n+1} \equiv 6q\frac{f_6^4 f_9^3}{f_3^6} \pmod{12}, \tag{61}$$

$$\sum_{n=0}^{\infty} p_\lambda(162n+128)q^{3n+2} \equiv 0 \pmod{12}. \tag{62}$$

Replacing q^3 by q in (60), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p_\lambda(162n+20)q^n &\equiv 9\frac{f_2^5 f_3^6}{f_1^7 f_6^3} \pmod{12}\\ &\equiv 9\frac{f_2^3 f_3^2}{f_1^3 f_6} \pmod{12}\end{aligned}$$

which proves (44) for $m=1$.

$$\sum_{n=0}^{\infty} p_\lambda(162n+20)q^n \equiv 9\frac{\varphi(-q^3)\psi(q)}{\varphi(-q)} \pmod{12}$$

Similar to (56), extracting the terms involving $q^{3n}, q^{3n+1}, q^{3n+2}$, we ultimately get

$$\sum_{n=0}^{\infty} p_\lambda(162\cdot 3n+20)q^n \equiv 9f_1 \pmod{12}, \tag{63}$$

$$\sum_{n=0}^{\infty} p_\lambda(162\cdot(3n+1)+20)q^n \equiv 3\frac{f_2f_3f_6}{f_1^2} \pmod{12}, \tag{64}$$

$$\sum_{n=0}^{\infty} p_\lambda(162\cdot(3n+2)+20)q^n \equiv 6\frac{f_6^3}{f_1} \pmod{12}. \tag{65}$$

Here (63) is the case when $m=1$ in (48). For the next step of induction, let us suppose that (41)-(49) holds true for $m=k$. Then, for $m=(k+1)$, we prove the relations in similar manner mentioned above starting from (50) (taking $m=k$) and obtain (64) $(m=k+1)$. Same process follows for other parts. □

Corollary 3. *For $m \geq 1$, we have*

$$p_\lambda\left(3^{4m-1}\cdot 2n + 2\sum_{j=0}^{2m-1} 3^{2j}\right) \equiv 3p_\lambda\left(3^{4m-3}\cdot 2n + 2\sum_{j=0}^{2m-2} 3^{2j}\right) \pmod{12}, \quad (66)$$

$$p_\lambda\left(3^{4m}\cdot 2n + 2\sum_{j=0}^{2m-1} 3^{2j}\right) \equiv 3p_\lambda\left(3^{4m-2}\cdot 2n + 2\sum_{j=0}^{2m-2} 3^{2j}\right) \pmod{12}, \quad (67)$$

$$p_\lambda\left(3^{4m+1}\cdot 2n + 2\sum_{j=0}^{2m-1} 3^{2j}\right) \equiv 3p_\lambda\left(3^{4m-1}\cdot 2n + 2\sum_{j=0}^{2m-2} 3^{2j}\right) \pmod{12}. \quad (68)$$

Now, we prove the 2-dissection of $\rho(q)$.

Theorem 3.8. *We have*

$$\sum_{n=0}^{\infty} p_\rho(2n)q^n = \frac{f_2^3 f_3^2}{f_1^3 f_6}, \quad (69)$$

$$\sum_{n=0}^{\infty} p_\rho(2n+1)q^n = 3\frac{f_6^3}{f_1 f_2} - q^{-1}\psi_6(q^2). \quad (70)$$

Proof. From (6), we have

$$\rho(q) = \frac{f_2^6 f_{12}}{f_4^3 f_6}\cdot\frac{f_3}{f_1^3} - q^{-1}\psi_6(q^2).$$

Substituting the value from (15), we have

$$\rho(q) = \frac{f_2^6 f_{12}}{f_4^3 f_6}\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right) - q^{-1}\psi_6(q^2).$$

Extracting even and odd terms from above equation, we easily arrive at (69) and (70). ∎

Finally on comparing (16) and (69), we arrive at the following theorem.

Theorem 3.9. *We have*

$$p_\lambda(2n) = p_\rho(2n). \quad (71)$$

The above theorem yields that all the results shown above for $\lambda(q)$ also hold for $\rho(q)$.

4. Conclusion and discussions.

We have provided elementary proofs of numerous infinite family of congruences satisfied by $p_\lambda(n)$ and $p_\rho(n)$. We did not carry out a computer search for congruences, and so we are unaware whether other congruences hold beyond the ones we prove in this paper, but certainly there is a possibility to explore more in this direction.

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