ELECTRONIC RESEARCH ARCHIVE
Volume **29**, Number **6**, December **2021**
doi:10.3934/era.2021082
pp. **4243–4255**

WELL-POSEDNESS IN SOBOLEV SPACES OF THE TWO-DIMENSIONAL MHD BOUNDARY LAYER EQUATIONS WITHOUT VISCOSITY

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Abstract. We consider the two-dimensional MHD Boundary layer system without hydrodynamic viscosity, and establish the existence and uniqueness of solutions in Sobolev spaces under the assumption that the tangential component of magnetic fields dominates. This gives a complement to the previous works of Liu-Xie-Yang [Comm. Pure Appl. Math. 72 (2019)] and Liu-Wang-Xie-Yang [J. Funct. Anal. 279 (2020)], where the well-posedness theory was established for the MHD boundary layer systems with both viscosity and resistivity and with viscosity only, respectively. We use the pseudo-differential calculation, to overcome a new difficulty arising from the treatment of boundary integrals due to the absence of the diffusion property for the velocity.

1. **Introduction.** In this work we study the existence and uniqueness of solution to the two-dimensional magnetohydrodynamic (MHD) boundary layer system without viscosity which reads, letting $\Omega := \mathbb{T}\times\mathbb{R}_+ = \big\{(x,y)\in\mathbb{R}^2;\ x\in\mathbb{T}, y>0\big\}$ be the fluid domain,

$$\begin{cases}\partial_t u+(u\partial_x+v\partial_y)u-(f\partial_x+g\partial_y)f+\partial_x p=0,\\ \partial_t f+(u\partial_x f+v\partial_y)f-(f\partial_x+g\partial_y)u-\mu\partial_y^2 f=0,\\ \partial_t g+(u\partial_x+v\partial_y)g-\mu\partial_y^2 g=f\partial_x v-g\partial_x u,\\ \partial_x u+\partial_y v=0,\quad \partial_x f+\partial_y g=0,\\ \big(v,\partial_y f,g\big)|_{y=0}=(0,0,0),\quad \lim\limits_{y\to+\infty}(u,f)=(U,B),\\ u|_{t=0}=u_0,\quad f|_{t=0}=f_0\end{cases}\tag{1}$$

where (u,v) and (f,g) stand for velocity and magnetic fields, respectively, and μ is resistivity coefficients, and U,B and p are the values on the boundary of the

2020 *Mathematics Subject Classification.* Primary: 35Q35, 76D10; Secondary: 76W05.
Key words and phrases. MHD boundary layer, well-posedness, Sobolev space.
The work was supported by NSF of China(Nos. 11961160716, 11871054, 11771342), the Natural Science Foundation of Hubei Province (No. 2019CFA007) and the Fundamental Research Funds for the Central Universities(No. 2042020kf0210).
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tangential velocity, magnetic fields and pressure, respectively, in the ideal MHD system satisfying the Bernoulli's law:

$$\begin{cases} \partial_t U + U\partial_x U + \partial_x p = B\partial_x B, \\ \partial_t B + U\partial_x B = B\partial_x U. \end{cases} \tag{2}$$

Note the MHD boundary system with a nonzero hydrodynamic viscosity will reduce to the classical Prandtl equations in the absence of a magnetic field, and the main difficulty for investigating Prandtl equation lies in the nonlocal property coupled with the loss of one order tangential derivative when dealing with the terms $v\partial_y u$. The mathematical study on the Prandtl boundary layer has a long history, and there have been extensive works concerning its well/ill-posedness theories. So far the two-dimensional (2D) Prandtl equation is well-explored in various function spaces, see, e.g., [1, 2, 4, 5, 6, 7, 9, 10, 11, 13, 14, 22, 23, 24] and the references therein. Compared with the Prandtl equation the treatment is more complicated since we have a new difficulty caused by the additional loss of tangential derivative in the magnetic field. So far the MHD boundary layer system is mainly explored in the two settings.

- Without any structural assumption on initial data the well-posedness for 2D and 3D MHD boundary systems was established in Gevrey space by the first author and T. Yang [15] with Gevrey index up to $3/2$, and it remains interesting to relax the Gevrey index therein to 2 inspired the previous works of [4, 12] on the well-posedness for the Prandtl equations in Gevrey space with optimal index 2.
- Under the structural assumption that the tangential magnetic field dominates, i.e., $f \neq 0$, the well-posedness in weighted Sobolev space was established by Liu-Xie-Yang [18] and Liu-Wang-Xie-Yang [16] without Oleinik's monotonicity assumption, where the two cases that with both viscosity and resistivity and with only viscosity are considered, respectively; see also the work of Gérard-Varet and Prestipino [8] for the stability analysis of the MHD boundary layer system with insulating boundary conditions (i.e. $f|_{y=0} = 0$). These works, together with the essential role of the Oleinik's monotonicity for well-posedness theory of the Prandtl equations (see, e.g., [1, 19, 20]), justify the stabilizing effect of the magnetic field on MHD boundary layer, no matter whether or not there is resistivity in the magnetic boundary layer equation.

The aforementioned works [18, 16] investigated the well-posedness for MHD boundary layer system with the nonzero viscosity coefficient. This work aims to consider the case without viscosity coefficient, giving a complement to the previous works [18, 16]. To simply the argument we will assume without loss of generality that $\mu = 1$ and $(U, B) \equiv 0$ in the system (1) since the result will hold true in the general case if we use some kind of the nontrivial weighted functions similar to those used in for the Prandtl equation. Hence we consider the following 2D MHD boundary layer system in the region $\Omega = \mathbb{T} \times \mathbb{R}_+$

$$\begin{cases} \big(\partial_t + u\partial_x + v\partial_y\big)u - (f\partial_x + g\partial_y)f = 0, \\ \big(\partial_t + u\partial_x + v\partial_y - \partial_y^2\big)f - (f\partial_x + g\partial_y)u = 0, \\ \big(\partial_t + u\partial_x + v\partial_y - \partial_y^2\big)g = f\partial_x v - g\partial_x u, \\ \partial_x u + \partial_y v = \partial_x f + \partial_y g = 0, \\ (v, \partial_y f, g)|_{y=0} = (0,0,0), \quad (u, f)|_{y\to+\infty} = (0,0), \\ (u, f)|_{t=0} = (u_0, f_0). \end{cases} \tag{3}$$

By the boundary condition and divergence-free condition above, we have

$$v(t,x,y) = -\int_0^y \partial_x u(t,x,\tilde{y})d\tilde{y}, \quad g(t,x,y) = -\int_0^y \partial_x f(t,x,\tilde{y})d\tilde{y}.$$

We remark that the equation for g in (3) can be derived from the one for f and the main difficulty in analysis is the loss of x-derivatives in the two terms v and g. As to be seen in the next Section 2, The system (3) can be derived from the MHD system

$$\begin{cases} \partial_t \boldsymbol{u}^\varepsilon + (\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{u}^\varepsilon - (\boldsymbol{H}^\varepsilon \cdot \nabla)\boldsymbol{H}^\varepsilon + \nabla P^\varepsilon = 0, \\ \partial_t \boldsymbol{H}^\varepsilon + (\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{H}^\varepsilon = (\boldsymbol{H}^\varepsilon \cdot \nabla)\boldsymbol{u}^\varepsilon + \varepsilon \Delta \boldsymbol{H}^\varepsilon, \\ \nabla \cdot \boldsymbol{u}^\varepsilon = \nabla \cdot \boldsymbol{H}^\varepsilon = 0, \end{cases} \tag{4}$$

where $\boldsymbol{u}^\varepsilon = (u^\varepsilon, v^\varepsilon), \boldsymbol{H}^\varepsilon = (f^\varepsilon, g^\varepsilon)$ denote velocity and magnetic field, respectively. The MHD system (4) is complemented with the boundary condition that

$$v^\varepsilon|_{y=0} = 0, \quad (\partial_y f^\varepsilon,\ g^\varepsilon)|_{y=0} = (0,0).$$

It is an important issue in both mathematics and physics to ask the high Reynolds number limit for MHD systems, and so far it is justified mathematically by Liu-Xie-Yang [17] with the presence of viscosity and the other cases remain unclear.

Notation. Before stating the main result we first list some notation used frequently in this paper. Given the domain $\Omega = \mathbb{T}\times\mathbb{R}_+$, we will use $\|\cdot\|_{L^2}$ and $(\cdot,\cdot)_{L^2}$ to denote the norm and inner product of $L^2 = L^2(\Omega)$ and use the notation $\|\cdot\|_{L^2_x}$ and $(\cdot,\cdot)_{L^2_x}$ when the variable x is specified. Similar notation will be used for L^∞. Moreover, we use $L^p_x(L^q_y) = L^p(\mathbb{T}; L^q(\mathbb{R}_+))$ for the classical Sobolev space. Let $H^m = H^m(\Omega)$ be the standard Sobolev space and define the weighted Sobolev space H^m_ℓ by setting, for $\ell \in \mathbb{R}$,

$$H^m_\ell = \Big\{ f(x,y) : \Omega \to \mathbb{R}; \quad \|f\|^2_{H^m_\ell} := \sum_{i+j\le m} \|\left\langle y\right\rangle^{\ell+j} \partial_x^i \partial_y^j f(x,y)\|^2_{L^2} < +\infty \Big\},$$

where here and below $\langle y\rangle = \big(1+|y|^2\big)^{1/2}$. With the above notation the well-posedness theory of (1) in weighted Sobolev space can be stated as below. Here the main assumption is that the tangential magnetic field in (1) dominates, that is, $f \neq 0$.

Theorem 1.1. *Let $\ell > \frac{1}{2}$ and $\delta > \ell + \frac{1}{2}$ be two given numbers. Suppose the initial data u_0, f_0 of (3) lie in $H^4_\ell(\Omega)$ satisfying that there exists a constant $c_0 > 0$ such that for any $(x,y) \in \Omega$,*

$$f_0(x,y) \ge c_0 \left\langle y\right\rangle^{-\delta} \ \text{ and } \sum_{j\le 2} \left|\partial_y^j f_0(x,y)\right| \le c_0^{-1} \left\langle y\right\rangle^{-\delta-j}. \tag{5}$$

Then the MHD boundary layer system (3) admits a unique local-in-time solution

$$u, f \in L^\infty([0,T]; H^4_\ell)$$

for some $T > 0$. Moreover a constant $c > 0$ exists such that for any $(t,x,y) \in [0,T]\times\Omega$,

$$f(t,x,y) \ge c \left\langle y\right\rangle^{-\delta} \ \text{ and } \sum_{j\le 2} \left|\partial_y^j f(t,x,y)\right| \le c^{-1} \left\langle y\right\rangle^{-\delta-j}.$$

Remark 1. The solution in Theorem 1.1 lies in the same Sobolev space as that for initial data, different from the previous works [15, 16, 18] where the loss of regularity occurs at positive time. Here the tangential magnetic field is allowed to decay polynomially at infinity, and this relaxes the condition in [16, 18] where the infimum of the tangential component is strictly positive.

Remark 2. The result above confirms that the magnetic field may act as a stabilizing factor on MHD boundary layer. The stabilizing effect was justified by [18] for the case with both viscosity and resistivity, and by [16] for the case without resistivity.

2. **Derivation of the boundary layer system.**

This section is devoted to deriving the boundary layer system (1). We consider the MHD system in Ω

$$\begin{cases} \partial_t \boldsymbol{u}^\varepsilon + (\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{u}^\varepsilon - (\boldsymbol{H}^\varepsilon \cdot \nabla)\boldsymbol{H}^\varepsilon + \nabla p^\varepsilon = 0, \\ \partial_t \boldsymbol{H}^\varepsilon + (\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{H}^\varepsilon - (\boldsymbol{H}^\varepsilon \cdot \nabla)\boldsymbol{u}^\varepsilon - \mu\varepsilon\Delta \boldsymbol{H}^\varepsilon = 0, \\ \nabla \cdot \boldsymbol{u}^\varepsilon = \nabla \cdot \boldsymbol{H}^\varepsilon = 0, \\ \boldsymbol{u}^\varepsilon|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{H}^\varepsilon|_{t=0} = \boldsymbol{b}_0, \end{cases} \tag{6}$$

where $\boldsymbol{u}^\varepsilon = (u^\varepsilon, v^\varepsilon), \boldsymbol{H}^\varepsilon = (f^\varepsilon, g^\varepsilon)$ denote velocity and magnetic fields, respectively. The above system is complemented with the no-slip boundary condition on the normal component of velocity field and perfectly conducting boundary condition on the magnetic field, that is,

$$v^\varepsilon|_{y=0} = (0), \quad (\partial_y f^\varepsilon, g^\varepsilon)|_{y=0} = (0,0). \tag{7}$$

A boundary layer will appear in order to overcome a mismatch on the boundary $y = 0$ for the tangential magnetic fields between (6) and the limiting equations by letting $\varepsilon \to 0$. To derive the governing equations for boundary layers we consider the ansatz

$$\begin{cases} u^\varepsilon(t,x,y) = u^0(t,x,y) + u^b(t,x,\tilde{y}) + O(\sqrt{\varepsilon}), \\ v^\varepsilon(t,x,y) = v^0(t,x,y) + \sqrt{\varepsilon} v^b(t,x,\tilde{y}) + O(\varepsilon), \\ f^\varepsilon(t,x,y) = f^0(t,x,y) + f^b(t,x,\tilde{y}) + O(\sqrt{\varepsilon}), \\ g^\varepsilon(t,x,y) = g^0(t,x,y) + \sqrt{\varepsilon} g^b(t,x,\tilde{y}) + O(\varepsilon), \\ p^\varepsilon(t,x,y) = p^0(t,x,y) + p^b(t,x,\tilde{y}) + O(\sqrt{\varepsilon}), \end{cases} \tag{8}$$

where we used the notation $\tilde{y} = y/\sqrt{\varepsilon}$. We suppose u^b, f^b, and p^b in the expansion (8) polynomially trend to zero as $\tilde{y} \to +\infty$, that is, as $\varepsilon \to 0$. Similarly for the expansion of the initial data.

Boundary conditions. Taking trace on $y = 0$ for the second and the fourth expansions in (8) and recalling the boundary condition (7), we derive that

$$v^0|_{y=0} = g^0|_{y=0} = 0, \tag{9}$$

and using again the second and the fourth equations in (8) and letting $\varepsilon \to 0$, we get that

$$v^b|_{y=0} = g^b|_{y=0} = 0. \tag{10}$$

Moreover observe

$$0 = \partial_y f^\varepsilon|_{y=0} = \partial_y f^0|_{y=0} + \frac{1}{\sqrt{\varepsilon}} \partial_{\tilde{y}} f^b|_{\tilde{y}=0} + o(1).$$

This gives

$$\partial_{\tilde{y}} f^b|_{\tilde{y}=0} = 0. \tag{11}$$

The governing equations of the fluid behavior near and far from the boundary. We substitute the ansatz (8) into (6) and consider the order of ε. At the order $\varepsilon^{-1/2}$ we get

$$\partial_{\tilde{y}} p^b \equiv 0.$$

This with the assumption that p^b goes to 0 as $\tilde{y} \to +\infty$ implies

$$p^b \equiv 0. \tag{12}$$

At the order ε^0, letting $\tilde{y} \to +\infty$ ($\varepsilon \to 0$) and taking into account (9) and fact that u^b, f^b, and p^b polynomially trend to zero as $\varepsilon \to 0$, we see the limiting system is the ideal incompressible MHD system:

$$\begin{cases} \partial_t \boldsymbol{u}^0 + (\boldsymbol{u}^0 \cdot \nabla)\boldsymbol{u}^0 - (\boldsymbol{H}^0 \cdot \nabla)\boldsymbol{H}^0 + \nabla p^0 = 0, \\ \partial_t \boldsymbol{H}^0 + (\boldsymbol{u}^0 \cdot \nabla)\boldsymbol{H}^0 - (\boldsymbol{H}^0 \cdot \nabla)\boldsymbol{u}^0 = 0, \\ \nabla \cdot \boldsymbol{u}^0 = \nabla \cdot \boldsymbol{H}^0 = 0, \end{cases} \tag{13}$$

complemented with the boundary condition (9) and initial data $\boldsymbol{u}^0_{in}$ and $\boldsymbol{H}^0_{in}$, where $\boldsymbol{u}^0 = (u^0, v^0), \boldsymbol{H}^0 = (f^0, g^0)$.

Next we will derive the boundary layer equations. Let $\boldsymbol{u}^0 = (u^0, v^0), \boldsymbol{H}^0 = (f^0, g^0)$ be the solution to the ideal MHD system (13). By Taylor expansion we write $u^0(t, x, y)$ as

$$\begin{aligned} u^0(t,x,y) &= u^0(t,x,0) + y\partial_y u^0(t,x,0) + \frac{y^2}{2}\partial_y^2 u^0(t,x,0) + \cdots \\ &= \overline{u^0} + \sqrt{\varepsilon}\tilde{y}\overline{\partial_y u^0} + O(\varepsilon), \end{aligned}$$

where here and below we use the notation $\overline{h}$ to stand for the trace of a function h on the boundary $y = 0$. Similarly,

$$\begin{aligned} v^0(t,x,y) &= \sqrt{\varepsilon}\tilde{y}\overline{\partial_y v^0} + O(\varepsilon), \quad f^0(t,x,y) = \overline{f^0} + \sqrt{\varepsilon}\tilde{y}\overline{\partial_y f^0} + O(\varepsilon), \\ g^0(t,x,y) &= \sqrt{\varepsilon}\tilde{y}\overline{\partial_y g^0} + O(\varepsilon), \quad p^0(t,x,y) = \overline{p^0} + \sqrt{\varepsilon}\tilde{y}\overline{\partial_y g^0} + O(\varepsilon). \end{aligned}$$

Now we compare the order ε^0 for the resulting equation by substituting the ansatz (8) as well as the above Taylor expansion of $\boldsymbol{u}^0 = (u^0, v^0), \boldsymbol{H}^0 = (f^0, g^0)$ into (6); this gives, by virtue of (12) and (13) ,

$$\begin{cases} \partial_t(\overline{u^0} + u^b) + (\overline{u^0} + u^b)\partial_x(\overline{u^0} + u^b) + (\tilde{y} \cdot \overline{\partial_y v^0} + v^b) \cdot \partial_{\tilde{y}} u^b \\ \qquad - (\overline{f^0} + f^b)\partial_x(\overline{f^0} + f^b) - (\tilde{y} \cdot \overline{\partial_y g^0} + g^b) \cdot \partial_{\tilde{y}} f^b + \partial_x \overline{p^0} = 0, \\ \partial_t(\overline{f^0} + f^b) + (\overline{u^0} + u^b)\partial_x(\overline{f^0} + f^b) + (\tilde{y} \cdot \overline{\partial_y v^0} + v^b) \cdot \partial_{\tilde{y}} f^b \\ \qquad - (\overline{f^0} + f^b)\partial_x(\overline{u^0} + u^b) - (\tilde{y} \cdot \overline{\partial_y g^0} + g^b) \cdot \partial_{\tilde{y}} u^b - \mu\partial_{\tilde{y}}^2 f^b = 0, \\ \partial_x(\overline{u^0} + u^b) + \partial_y(\tilde{y} \cdot \overline{\partial_y v^0} + v^b) = \partial_x(\overline{f^0} + f^b) + \partial_y(\tilde{y} \cdot \overline{\partial_y g^0} + g^b) = 0. \end{cases} \tag{14}$$

Denoting

$$\begin{aligned} u(t,x,\tilde{y}) &= \overline{u^0} + u^b(t,x,\tilde{y}), \quad v(t,x,\tilde{y}) = \tilde{y}\partial_y\overline{v^0} + v^b(t,x,\tilde{y}), \\ f(t,x,\tilde{y}) &= \overline{f^0} + f^b(t,x,\tilde{y}), \quad g(t,x,\tilde{y}) = \tilde{y}\partial_y\overline{g^0} + g^b(t,x,\tilde{y}), \end{aligned}$$

and recalling u^b, f^b polynomially trend to 0 as $\tilde{y} \to +\infty$, we combine (14) and the boundary conditions (9)–(11) to conclude that all equations except the third one in

(1) are fulfilled by u, v, f, and g. For simplicity of notation, we have replaced $\tilde{y}$ by y. Note that the third equation in (1) can be derived from the second one and the boundary condition $\partial_y f|_{y=0} = 0$ by observing that

$$g(t,x,y) = -\int_0^y \partial_x f(t,x,z)dz.$$

Finally we remark the Bernoulli's law (2) follows by taking trace on $y = 0$ for the ideal MHD system (13).

3. **A priori energy estimates.** The general strategy for constructing solutions to (3) involves mainly two ingredients. One is to construct appropriate approximate solutions, which reserve a similar properties as (5) for initial data by applying the standard maximum principle for parabolic equations in the domain Ω (see [19, Lemmas E.1 and E.2] for instance). Next we need to deduce the uniform estimate for these approximate solutions. For sake of simplicity we only present the following a priori estimate for regular solutions, which is a key part to prove the main result Theorem 1.1.

Theorem 3.1. *Let $\ell > \frac{1}{2}$ and $\delta > \ell + \frac{1}{2}$ be two given numbers, and let $u, f \in L^\infty([0,T]; H_\ell^4)$ solve the MHD boundary layer system (3) satisfying that a constant $c > 0$ exists such that for any $(t,x,y) \in [0,T] \times \Omega$,*

$$f(t,x,y) \geq c\langle y\rangle^{-\delta} \text{ and } \sum_{j\leq 2} \left|\partial_y^j f(t,x,y)\right| \leq c^{-1}\langle y\rangle^{-\delta-j}.$$

Then there exists a constant $C > 0$ such that

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(s)ds \leq C\Big(\mathcal{E}(0) + \int_0^t \left(\mathcal{E}(s) + \mathcal{E}(s)^2\right) ds\Big),$$

where here and below

$$\mathcal{E}(t) := \|u(t)\|_{H_\ell^4}^2 + \|f(t)\|_{H_\ell^4}^2, \quad \mathcal{D}(t) := \|\partial_y f(t)\|_{H_\ell^4}^2. \tag{15}$$

We will present the proof of Theorem 3.1 in the next two subsections, one of which is devoted to the estimates on tangential and another to the normal derivatives. To simplify the notation we will use the capital letter C in the following argument to denote some generic constants that may vary from line to line, and moreover use C_ε to denote some generic constants depending on a given number $0 < \varepsilon \ll 1$.

3.1. **Energy estimates: Tangential derivatives.** In this part, we will derive the estimate on tangential derivatives, following the cancellation mechanism observed in the previous work of Liu-Xie-Yang [18].

Lemma 3.2. *Under the same assumption as in Theorem 3.1 we have, for any $t \in [0,T]$,*

$$\sum_{i\leq 4}\Big(\|\langle y\rangle^\ell \partial_x^i u(t)\|_{L^2}^2 + \|\langle y\rangle^\ell \partial_x^i f(t)\|_{L^2}^2\Big) + \sum_{i\leq 4}\int_0^t \|\langle y\rangle^\ell \partial_x^i \partial_y f(s)\|_{L^2}^2 ds \leq C\Big(\mathcal{E}(0) + \int_0^t \left(\mathcal{E}(s) + \mathcal{E}(s)^2\right) ds\Big).$$

Recall $\mathcal{E}$ is defined in (15).

Proof. Without loss of generality we may consider $i = 4$, apply ∂_x^4 to the first second equations and ∂_x^3 to the third equation in (3), respectively, this gives

$$\begin{cases} \big(\partial_t + u\partial_x + v\partial_y\big)\partial_x^4 u - \big(f\partial_x + g\partial_y\big)\partial_x^4 f = -(\partial_y u)\partial_x^4 v + (\partial_y f)\partial_x^4 g + F_4, \\ \big(\partial_t + u\partial_x + v\partial_y - \partial_y^2\big)\partial_x^4 f - \big(f\partial_x + g\partial_y\big)\partial_x^4 u = -(\partial_y f)\partial_x^4 v + (\partial_y u)\partial_x^4 g + P_4, \\ \big(\partial_t + u\partial_x + v\partial_y - \partial_y^2\big)\partial_x^3 g = f\partial_x^4 v - g\partial_x^4 u + Q_4, \end{cases} \tag{16}$$

where

$$F_4 = \sum_{j=1}^{4}\binom{4}{j}\big[(\partial_x^j f)\partial_x^{5-j} f - (\partial_x^j u)\partial_x^{5-j} u\big] + \sum_{j=1}^{3}\binom{4}{j}\big[(\partial_x^j g)\partial_x^{4-j}\partial_y f - (\partial_x^j v)\partial_x^{4-j}\partial_y u\big],$$

$$\begin{aligned} P_4 = &\sum_{j=1}^{4}\binom{4}{j}\big[(\partial_x^j f)\partial_x^{5-j} u - (\partial_x^j u)\partial_x^{5-j} f\big] \\ &\quad + \sum_{j=1}^{3}\binom{4}{j}\big[(\partial_x^j g)\partial_x^{4-j}\partial_y u - (\partial_x^j v)\partial_x^{4-j}\partial_y f\big] \end{aligned}$$

and

$$\begin{aligned} Q_4 &= \sum_{j=1}^{3}\binom{3}{j}\big[(\partial_x^j f)\partial_x^{4-j} v - (\partial_x^j v)\partial_x^{3-j}\partial_y g\big] - \sum_{j=1}^{3}\binom{3}{j}\big[(\partial_x^j u)\partial_x^{4-j} g + (\partial_x^j g)\partial_x^{4-j} u\big] \\ &= \sum_{j=1}^{3}\binom{3}{j}\big[(\partial_x^j f)\partial_x^{4-j} v + (\partial_x^j v)\partial_x^{4-j} f\big] - \sum_{j=1}^{3}\binom{3}{j}\big[(\partial_x^j u)\partial_x^{4-j} g + (\partial_x^j g)\partial_x^{4-j} u\big]. \end{aligned}$$

In order to eliminate the terms $\partial_x^4 v$ and $\partial_x^4 g$ where the fifth order tangential derivatives are involved, we introduce, observing $f > 0$ by assumption,

$$\psi \overset{\text{def}}{=} \partial_x^4 f + \frac{\partial_y f}{f}\partial_x^3 g = -f\partial_y\Big(\frac{\partial_x^3 g}{f}\Big), \quad \varphi \overset{\text{def}}{=} \partial_x^4 u + \frac{\partial_y u}{f}\partial_x^3 g. \tag{17}$$

Multiplying the third equation in (16) by $(\partial_y f)/f$ and then taking summation with the second one in (16), we obtain the equation solved by ψ, that is,

$$\big(\partial_t + u\partial_x + v\partial_y - \partial_y^2\big)\psi - \big(f\partial_x + g\partial_y\big)\varphi = L_4, \tag{18}$$

where

$$\begin{aligned} L_4 = P_4 + \frac{\partial_y f}{f}Q_4 - \frac{g(\partial_y f)}{f}\partial_x^4 u + \Big[\frac{g(\partial_y u)}{f} + 2\partial_y\big((\partial_y f)/f\big)\Big]\partial_x^4 f \\ + \Big[\frac{2(\partial_y f)\partial_y^2 f}{f^2} + \frac{(\partial_x u)\partial_y f}{f} - \frac{2(\partial_y f)^3}{f^3} - \frac{(\partial_y u)\partial_x f}{f}\Big]\partial_x^3 g \end{aligned}$$

with P_4 and Q_4 given in (16). Similarly we multiply the third equation in (16) by $(\partial_y u)/f$ and then add the resulting equation by the first one in (16), to obtain

$$\big(\partial_t + u\partial_x + v\partial_y\big)\varphi - \big(f\partial_x + g\partial_y\big)\psi = -\frac{\partial_y u}{f}\partial_y\psi + M_4, \tag{19}$$

where

$$M_4 = F_4 + \frac{\partial_y u}{f} Q_4 - \frac{g(\partial_y u)}{f} \partial_x^4 u + \Big[\frac{g(\partial_y f)}{f} + \frac{(\partial_y u)\partial_y f}{f})\Big]\partial_x^4 f + \Big[\frac{(\partial_y f)\partial_x f}{f} + \frac{g(\partial_y f)^2}{f^2} - \frac{(\partial_y f)^2 \partial_y u}{f^3} - \frac{(\partial_x u)\partial_y u}{f} - \frac{g(\partial_y u)^2}{f^2}\Big]\partial_x^3 g$$

with F_4 and Q_4 given in (16). Note that (18) is complemented with the boundary condition that

$$\partial_y \psi|_{y=0} = 0.$$

Thus we perform the weighted energy estimate for (18) and (19) and use the fact that

$$\big((f\partial_x + g\partial_y)\langle y\rangle^\ell \varphi,\ \langle y\rangle^\ell \psi\big)_{L^2} + \big((f\partial_x + g\partial_y)\langle y\rangle^\ell \psi,\ \langle y\rangle^\ell \varphi\big)_{L^2} = 0$$

to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\langle y\rangle^\ell \psi\|_{L^2}^2 + \|\langle y\rangle^\ell \varphi\|_{L^2}^2\Big) + \|\partial_y(\langle y\rangle^\ell \psi)\|_{L^2}^2 \\
&= \big(\langle y\rangle^\ell L_4,\ \langle y\rangle^\ell \psi\big)_{L^2} + \big(\langle y\rangle^\ell M_4,\ \langle y\rangle^\ell \varphi\big)_{L^2} - \big(\langle y\rangle^\ell \big((\partial_y u)/f\big)\partial_y \psi,\ \langle y\rangle^\ell \varphi\big)_{L^2} \\
&\quad + \big([v\partial_y,\ \langle y\rangle^\ell]\psi,\ \langle y\rangle^\ell \psi\big)_{L^2} - \big([\partial_y^2,\ \langle y\rangle^\ell]\psi,\ \langle y\rangle^\ell \psi\big)_{L^2} - \big([g\partial_y,\ \langle y\rangle^\ell]\varphi,\ \langle y\rangle^\ell \psi\big)_{L^2} \\
&\quad + \big([v\partial_y,\ \langle y\rangle^\ell]\varphi,\ \langle y\rangle^\ell \varphi\big)_{L^2} - \big([g\partial_y,\ \langle y\rangle^\ell]\psi,\ \langle y\rangle^\ell \varphi\big)_{L^2},
\end{aligned} \tag{20}$$

where here and below we use $[\mathcal{T}_1,\ \mathcal{T}_2]$ to denote the commutator between two operators $\mathcal{T}_1, \mathcal{T}_2$, that is

$$[\mathcal{T}_1,\ \mathcal{T}_2] = \mathcal{T}_1\mathcal{T}_2 - \mathcal{T}_2\mathcal{T}_1. \tag{21}$$

Observe the derivatives are at most up to the fourth order for the terms on the right of (20). Then by direct compute we have

$$\begin{aligned}
&\big(\langle y\rangle^\ell L_4,\ \langle y\rangle^\ell \psi\big)_{L^2} + \big(\langle y\rangle^\ell M_4,\ \langle y\rangle^\ell \varphi\big)_{L^2} - \big(\langle y\rangle^\ell \big((\partial_y u)/f\big)\partial_y \psi,\ \langle y\rangle^\ell \varphi\big)_{L^2} \\
&\quad + \big([v\partial_y,\ \langle y\rangle^\ell]\psi,\ \langle y\rangle^\ell \psi\big)_{L^2} - \big([\partial_y^2,\ \langle y\rangle^\ell]\psi,\ \langle y\rangle^\ell \psi\big)_{L^2} - \big([g\partial_y,\ \langle y\rangle^\ell]\varphi,\ \langle y\rangle^\ell \psi\big)_{L^2} \\
&\quad\quad + \big([v\partial_y,\ \langle y\rangle^\ell]\varphi,\ \langle y\rangle^\ell \varphi\big)_{L^2} - \big([g\partial_y,\ \langle y\rangle^\ell]\psi,\ \langle y\rangle^\ell \varphi\big)_{L^2} \\
&\le \frac{1}{2}\|\partial_y(\langle y\rangle^\ell \psi)\|_{L^2}^2 + C\left(\mathcal{E} + \mathcal{E}^2\right).
\end{aligned}$$

Substituting the above estimate into (20) and then integrating over $[0,t]$ for any $0 < t < T$ gives

$$\begin{aligned}
&\|\langle y\rangle^\ell \psi(t)\|_{L^2}^2 + \|\langle y\rangle^\ell \varphi(t)\|_{L^2}^2 + \int_0^t \|\partial_y(\langle y\rangle^\ell \psi)\|_{L^2}^2 ds \\
&\qquad \le C\Big(\mathcal{E}(0) + \int_0^t \left(\mathcal{E}(s) + \mathcal{E}(s)^2\right) ds\Big).
\end{aligned} \tag{22}$$

Next we will derive the estimates for f, u from the ones of ψ, φ. In fact in view of the representation of ψ given in (17), we use Hardy-type inequality (cf. [19, Lemma B.1] for instance) to conclude

$$\|\langle y\rangle^{\ell-1} \partial_x^3 g\|_{L^2} \le C\|\langle y\rangle^\ell \psi\|_{L^2}.$$

As a result, using the representation of ψ and φ given in (17) and the fact that $|(\partial_y f)/f| \lesssim \langle y\rangle^{-1}$ gives

$$\|\langle y\rangle^\ell \partial_x^4 f\|_{L^2} \le C\|\langle y\rangle^{\ell-1} \partial_x^3 g\|_{L^2} + \|\langle y\rangle^\ell \psi\|_{L^2} \le C\|\langle y\rangle^\ell \psi\|_{L^2}$$

and

$$\| \langle y\rangle^{\ell} \partial_x^4 u\|_{L^2} \leq C\| \langle y\rangle^{\ell-1} \partial_x^3 g\|_{L^2} + \| \langle y\rangle^{\ell} \varphi\|_{L^2} \leq C\| \langle y\rangle^{\ell} \psi\|_{L^2} + \| \langle y\rangle^{\ell} \varphi\|_{L^2}.$$

Moreover, using again (17),

$$\begin{aligned}\| \langle y\rangle^{\ell} \partial_y\partial_x^4 f\|_{L^2} &\leq C\| \langle y\rangle^{\ell} \partial_y\big[((\partial_y f)/f)\partial_x^3 g]\|_{L^2} + \| \langle y\rangle^{\ell} \partial_y\psi\|_{L^2}\\ &\leq C\|\partial_y(\langle y\rangle^{\ell} \psi)\|_{L^2} + C\mathcal{E}^{1/2}.\end{aligned}$$

Combining these inequality with (22) we conclude

$$\begin{aligned}\| \langle y\rangle^{\ell} \partial_x^4 u(t)\|_{L^2}^2 + \| \langle y\rangle^{\ell} \partial_x^4 f(t)\|_{L^2}^2 + \int_0^t \| \langle y\rangle^{\ell} \partial_x^4\partial_y f\|_{L^2}^2 ds\\ \leq C\Big(\mathcal{E}(0) + \int_0^t \big(\mathcal{E}(s) + \mathcal{E}(s)^2\big)\, ds\Big).\end{aligned}$$

Note the above estimate still holds true if we replace ∂_x^4 by ∂_x^i with $i \leq 4$. This gives the desired estimate in Lemma 3.2, completing the proof. □

3.2. **Estimate for normal derivatives.** In this part, we perform the estimate for normal derivatives. Compared with [18] a new difficulty arises when dealing the boundary integrals because of the absence of the hydrodynamic viscosity.

Lemma 3.3. *Under the same assumption as in Theorem 3.1 we have, for any $t \in [0,T]$,*

$$\begin{aligned}&\sum_{\substack{i+j\leq 4\\ j\geq 1}} \Big(\| \langle y\rangle^{\ell+j} \partial_x^i\partial_y^j u(t)\|_{L^2}^2 + \| \langle y\rangle^{\ell+j} \partial_x^i\partial_y^j f(t)\|_{L^2}^2\Big)\\ &+\int_0^t \sum_{\substack{i+j\leq 4\\ j\geq 1}} \| \langle y\rangle^{\ell+j} \partial_x^i\partial_y^{j+1} f(s)\|_{L^2}^2 ds \leq \varepsilon \int_0^t \mathcal{D}(s)ds + C_\varepsilon\Big(\mathcal{E}(0) + \int_0^t (\mathcal{E}(s) + \mathcal{E}^2(s))ds\Big).\end{aligned}$$

Recall $\mathcal{E}, \mathcal{D}$ are defined by (15).

Proof. **Step 1).** We first consider the case of $i = 0$ and $j = 4$. In this step we will prove that, for any $\varepsilon > 0$,

$$\begin{aligned}&\Big(\| \langle y\rangle^{\ell+4} \partial_y^4 u(t)\|_{L^2}^2 + \| \langle y\rangle^{\ell+4} \partial_y^4 f(t)\|_{L^2}^2\Big) + \int_0^t \| \langle y\rangle^{\ell+4} \partial_y^5 f(s)\|_{L^2}^2 ds\\ &\leq \int_0^t \Big(\int_{\mathbb{T}} (\partial_y^4 f)(f\partial_y^3\partial_x u)|_{y=0}\, dx\Big) ds + \varepsilon \int_0^t \mathcal{D}(s)ds + C_\varepsilon\Big(\mathcal{E}(0) + \int_0^t (\mathcal{E}(s) + \mathcal{E}^2(s))ds\Big).\end{aligned} \tag{23}$$

Applying $\langle y\rangle^{\ell+4} \partial_y^4$ to (3) yields that

$$\begin{aligned}\big(\partial_t + u\partial_x + v\partial_y\big) \langle y\rangle^{\ell+4} \partial_y^4 u - \big(f\partial_x + g\partial_y\big) \langle y\rangle^{\ell+4} \partial_y^4 f\\ = [u\partial_x + v\partial_y,\ \langle y\rangle^{\ell+4} \partial_y^4]u - [f\partial_x + g\partial_y,\ \langle y\rangle^{\ell+4} \partial_y^4]f\end{aligned}$$

and

$$\begin{aligned}\big(\partial_t + u\partial_x + v\partial_y - \partial_y^2\big) \langle y\rangle^{\ell+4} \partial_y^4 f - \big(f\partial_x + g\partial_y\big) \langle y\rangle^{\ell+4} \partial_y^4 u\\ = [u\partial_x + v\partial_y,\ \langle y\rangle^{\ell+4} \partial_y^4]f - [f\partial_x + g\partial_y,\ \langle y\rangle^{\ell+4} \partial_y^4]u - [\partial_y^2,\ \langle y\rangle^{\ell+4} \partial_y^4]f.\end{aligned}$$

Recall $[\cdot,\cdot]$ is given by (21), standing for the commutator between two operators. Taking inner product with $\langle y\rangle^{\ell+4}\partial_y^4 u$ to the first equation above, and with $\langle y\rangle^{\ell+4}\partial_y^4 f$ to the second one, and then taking summation and observing

$$\Big(\big(f\partial_x+g\partial_y\big)\langle y\rangle^{\ell+4}\partial_y^4 f,\ \langle y\rangle^{\ell+4}\partial_y^4 u\Big)_{L^2} + \Big(\big(f\partial_x+g\partial_y\big)\langle y\rangle^{\ell+4}\partial_y^4 u,\ \langle y\rangle^{\ell+4}\partial_y^4 f\Big)_{L^2}=0,$$

we obtain

$$\frac12\frac{d}{dt}\Big(\|\langle y\rangle^{\ell+4}\partial_y^4 u\|_{L^2}^2+\|\langle y\rangle^{\ell+4}\partial_y^4 f\|_{L^2}^2\Big)+\|\partial_y\big(\langle y\rangle^{\ell+4}\partial_y^4 f\big)\|_{L^2}^2 = -\int_{\mathbb{T}}(\partial_y^4 f)\partial_y^5 f|_{y=0}dx+R_4 \tag{24}$$

with

$$R_4=\Big([u\partial_x+v\partial_y,\ \langle y\rangle^{\ell+4}\partial_y^4]u-[f\partial_x+g\partial_y,\ \langle y\rangle^{\ell+4}\partial_y^4]f,\ \langle y\rangle^{\ell+4}\partial_y^4 u\Big)_{L^2}$$
$$+\Big([u\partial_x+v\partial_y,\ \langle y\rangle^{\ell+4}\partial_y^4]f-[f\partial_x+g\partial_y,\ \langle y\rangle^{\ell+4}\partial_y^4]u-[\partial_y^2,\ \langle y\rangle^{\ell+4}\partial_y^4]f,\ \langle y\rangle^{\ell+4}\partial_y^4 f\Big)_{L^2}.$$

Direct computation shows

$$R_4\le\frac12\|\partial_y\big(\langle y\rangle^{\ell+4}\partial_y^4 f\big)\|_{L^2}^2+C\left(\mathcal{E}+\mathcal{E}^2\right). \tag{25}$$

It remains to deal with the boundary integeral on the right of (24). We first apply ∂_y to the second equation in (3) and then take trace on $y=0$; this together with the boundary condition in (3) gives

$$\partial_y^3 f|_{y=0}=2(\partial_y u)\partial_x f|_{y=0}-f\partial_x\partial_y u|_{y=0}. \tag{26}$$

By virtue of the above representation of $\partial_y^3 f|_{y=0}$, we compute directly that

$$\begin{aligned}\partial_y^5 f|_{y=0}&=\partial_y^3\big(\partial_t f+u\partial_x f+v\partial_y f-f\partial_x u-g\partial_y u\big)\big|_{y=0}\\&=\Big\{u\partial_x\partial_y^3 f-f\partial_x\partial_y^3 u-4(\partial_x u)\partial_y^3 f-7(\partial_x\partial_y u)\partial_y^2 f+4(\partial_x f)\partial_y^3 u\\&\quad+8(\partial_y u)\partial_x\partial_y^2 f-u(\partial_x f)\partial_x\partial_y u+uf\partial_x^2\partial_y u-2u(\partial_y u)\partial_x^2 f+2f(\partial_y u)\partial_x^2 u\Big\}\Big|_{y=0}.\end{aligned} \tag{27}$$

As a result we combine (27) with Sobolev's inequality to conclude

$$-\int_{\mathbb{T}}(\partial_y^4 f)\partial_y^5 f|_{y=0}dx\le\int_{\mathbb{T}}f(\partial_y^4 f)(\partial_x\partial_y^3 u)|_{y=0}dx+\varepsilon\mathcal{D}+C_\varepsilon\left(\mathcal{E}+\mathcal{E}^2\right).$$

Substituting the above inequality and (25) into (24) and then integrating over $[0,t]$ for any $t\in[0,T]$ we obtain the desired estimate (23).

Step 2). In this step we will treat the first term on the right of (23) and prove that

$$\int_0^t\Big(\int_{\mathbb{T}}(\partial_y^4 f)(f\partial_y^3\partial_x u)|_{y=0}\,dx\Big)ds\le\varepsilon\int_0^t\mathcal{D}(s)ds+C_\varepsilon\int_0^t\mathcal{E}^2(s)ds \tag{28}$$

holds true for any $\varepsilon>0$. To do so we recall some facts on the Fourier multiplier. Let $k\in\mathbb{Z}$ be the partial Fourier dual variable of $x\in\mathbb{T}$ and let $\Lambda_x^\sigma,\sigma\in\mathbb{R}$, be the Fourier multiplier with symbol $\left(1+k^2\right)^{\sigma/2}$, that is,

$$\mathcal{F}_x(\Lambda_x^\sigma f)(k)=(1+k^2)^{\sigma/2}\mathcal{F}_x(f)(k),$$

where $\mathcal{F}_x$ stands for the Fourier transform with respect to x variable:

$$(\mathcal{F}_x f)(k) := \int_{\mathbb{T}} f(x)e^{-ikx}dx, \quad k \in \mathbb{Z}.$$

Similarly we define $|D_x|^\sigma, \sigma > 0$, by setting

$$\mathcal{F}_x(|D_x|^\sigma f)(k) = |k|^\sigma \mathcal{F}_x(f)(k).$$

Given a C^1 function ρ of $x \in \mathbb{T}$ with bounded derivatives, we have, for $0 < \sigma < 1$,

$$\forall\ w \in L^2_x, \quad \|[|D_x|^\sigma\ ,\ \rho]w\|_{L^2_x} \leq C_\sigma \left(\|\rho\|_{L^\infty} + \|\partial_x\rho\|_{L^\infty}\right)\|w\|_{L^2_x} \tag{29}$$

with C_σ a constant depending only on σ, recalling the commutator $[|D_x|^\sigma\ ,\ \rho]$ is defined by (21). Note the counterpart for $x \in \mathbb{R}$ of (29) is clear, see, e.g., [3, Pages 702–704], the estimate (29) can be proven in a similar inspirit and we omit it for brevity and and refer to [21] and references therein for the comprehensive argument on the extension of the classical pseudo-differential calculus in $\mathbb{R}$ to the torus case $x \in \mathbb{T}$.

With the Fourier multipliers introduced above we use (29) to compute

$$\begin{aligned} &\Big|\int_{\mathbb{T}} (\partial_y^4 f)(f\partial_y^3\partial_x u)|_{y=0}\,dx\Big| \leq \|\Lambda_x^{1/2}\partial_y^3 u|_{y=0}\|_{L^2_x}\|\Lambda_x^{1/2}(f\partial_y^4 f)|_{y=0}\|_{L^2_x} \\ &\leq C\|\Lambda_x^{1/2}\partial_y^3 u|_{y=0}\|_{L^2_x}\Big(\|\,|D_x|^{1/2}(f\partial_y^4 f)|_{y=0}\|_{L^2_x} + \|(f\partial_y^4 f)|_{y=0}\|_{L^2_x}\Big) \\ &\leq C\|\Lambda_x^{1/2}\partial_y^3 u|_{y=0}\|_{L^2_x}\left(\|\partial_x f\|_{L^\infty} + \|f\|_{L^\infty}\right)\|\Lambda_x^{1/2}\partial_y^4 f|_{y=0}\|_{L^2_x} \\ &\leq \varepsilon\|\Lambda_x^{1/2}\partial_y^4 f|_{y=0}\|^2_{L^2_x} + C_\varepsilon\mathcal{E}\|\Lambda_x^{1/2}\partial_y^3 u|_{y=0}\|^2_{L^2_x}. \end{aligned} \tag{30}$$

On the other hand, using the fact that

$$\Big(\Lambda_x^{1/2}\partial_y^3 u(x,0)\Big)^2 = -2\int_0^{+\infty}\left(\Lambda_x^{1/2}\partial_y^3 u(x,y)\right)\Lambda_x^{1/2}\partial_y^4 u(x,y)dy,$$

we compute

$$\begin{aligned} \|\Lambda_x^{1/2}\partial_y^3 u(x,0)\|^2_{L^2_x} &= -2\int_{\mathbb{R}}\Big(\int_0^{+\infty}\left(\Lambda_x^{1/2}\partial_y^3 u(x,y)\right)\Lambda_x^{1/2}\partial_y^4 u(x,y)dy\Big)dx \\ &= -2\int_0^{+\infty}\Big(\int_{\mathbb{R}}\left(\Lambda_x^{1/2}\partial_y^3 u(x,y)\right)\Lambda_x^{1/2}\partial_y^4 u(x,y)dx\Big)dy \\ &\leq 2\int_0^{+\infty}\|\Lambda_x\partial_y^3 u(\cdot,y)\|_{L^2_x}\|\partial_y^4 u(\cdot,y)\|_{L^2_x}dy \leq \mathcal{E}. \end{aligned}$$

A similar argument gives

$$\|\Lambda_x^{1/2}\partial_y^4 f(x,0)\|^2_{L^2_x} \leq 2\int_0^{+\infty}\|\Lambda_x\partial_y^4 f(\cdot,y)\|_{L^2_x}\|\partial_y^5 f(\cdot,y)\|_{L^2_x}dy \leq \mathcal{D}.$$

Substituting the two inequalities above into (30) yields

$$\Big|\int_{\mathbb{T}}(\partial_y^4 f)(f\partial_y^3\partial_x u)|_{y=0}\,dx\Big| \leq \|\Lambda_x^{1/2}\partial_y^3 u\|_{L^2_x L^\infty_y}\|\Lambda_x^{1/2}(f\partial_y^4 f)\|_{L^2_x L^\infty_y} \leq \varepsilon\mathcal{D} + C_\varepsilon\mathcal{E}^2.$$

This gives the desired estimate (28).

Step 3). We combine (23) and (28) to obtain

$$\Big(\| \langle y\rangle^{\ell+4}\,\partial_y^4 u(t)\|_{L^2}^2 + \| \langle y\rangle^{\ell+4}\,\partial_y^4 f(t)\|_{L^2}^2\Big) + \int_0^t \| \langle y\rangle^{\ell+4}\,\partial_y^5 f(s)\|_{L^2}^2 ds$$
$$\leq \varepsilon \int_0^t \mathcal{D}(s)ds + C_\varepsilon\Big(\mathcal{E}(0) + \int_0^t \big(\mathcal{E}(s)+\mathcal{E}^2(s)\big)\,ds\Big).$$

Observe the above inequality still holds true if we replace $\langle y\rangle^{\ell+4}\,\partial_y^4$ by $\langle y\rangle^{\ell+j}\,\partial_x^i\partial_y^j$ with $i+j\leq 4$ and use the boundary conditions (26) and $\partial_y f|_{y=0}=0$. Since the argument is straightforward we omit it for brevity. The proof of Lemma 3.3 is completed. □

3.3. **Completing the proof of the energy estimate.** Combining the estimates in Lemmas 3.2–3.3 and letting ε be small enough we obtain the desired energy estimate, completing the proof of Theorem 3.1.

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Received for publication June 2021; early access October 2021.

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