ELECTRONIC RESEARCH ARCHIVE
Volume **29**, Number **6**, December **2021**

doi:10.3934/era.2021081
pp. **4229–4241**

PLANAR VORTICES IN A BOUNDED DOMAIN WITH A HOLE

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Abstract. In this paper, we consider the inviscid, incompressible planar flows in a bounded domain with a hole and construct stationary classical solutions with single vortex core, which is closed to the hole. This is carried out by constructing solutions to the following semilinear elliptic problem

$$\begin{cases} -\Delta\psi = \lambda(\psi - \frac{\kappa}{4\pi}\ln\lambda)_+^p, & \text{in } \Omega, \\ \psi = \rho_\lambda, & \text{on } \partial O_0, \\ \psi = 0, & \text{on } \partial\Omega_0, \end{cases} \tag{1}$$

where $p > 1$, κ is a positive constant, ρ_λ is a constant, depending on λ, $\Omega = \Omega_0 \setminus \bar{O}_0$ and Ω_0, O_0 are two planar bounded simply-connected domains. We show that under the assumption $(\ln\lambda)^\sigma \le \rho_\lambda \le (\ln\lambda)^{1-\sigma}$ for some $\sigma > 0$ small, (1) has a solution ψ_λ, whose vorticity set $\{y \in \Omega : \psi(y) - \kappa + \rho_\lambda\eta(y) > 0\}$ shrinks to the boundary of the hole as $\lambda \to +\infty$.

1. **Introduction.** In this paper, we consider a planar incompressible flow in a bounded smooth domain

$$\Omega = \Omega_0 \setminus \bar{O}_0,$$

where O_0, Ω_0 are two bounded simply-connected open subsets of $\mathbb{R}^2$, such that $\bar{O}_0 \subset \Omega_0$. A simple model describing this flow is

$$\begin{cases} \nabla^\perp\psi \cdot \nabla\omega = 0, & \text{in } \Omega, \\ -\Delta\psi = \omega, & \text{in } \Omega, \\ \psi = constant, & \text{on } \partial O_0, \\ \psi = 0, & \text{on } \partial\Omega_0, \end{cases} \tag{2}$$

where ψ and ω are the stream function and the vorticity of this flow, respectively, and $\nabla^\perp\psi := (\partial_2\psi, -\partial_1\psi)$. For a detailed presentation of this model, we refer the readers to [9].

2020 *Mathematics Subject Classification.* Primary: 58F15, 58F17; Secondary: 53C35.

Key words and phrases. The Euler flow, semilinear elliptic equation, variational method, free boundary problem, reduction.

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An existence result obtained by Smets and Schaftingen [15] via a variational method shows that (2) has a solution $(\psi_\lambda, \omega_\lambda)$, such that $\psi_\lambda = \rho_\lambda$ on ∂O_0,

$$\int_{\partial O_0} \frac{\partial \psi_\lambda}{\partial \nu} = 0,$$

and $\omega_\lambda = \lambda(\psi_\lambda - q_\lambda)_+^p$, where $1 < p < +\infty$, $q_\lambda = q + \frac{\kappa}{4\pi} \ln \lambda$ with $\kappa > 0$ and q is a harmonic function in Ω. Moreover, as $\lambda \to +\infty$, the total vorticity

$$\int_\Omega \omega_\lambda \to \kappa,$$

and the vorticity set $\{y \in \Omega | \, \omega_\lambda(y) > 0\}$ shrinks to a point in Ω, which is a critical point of the Kirchhoff-Routh function corresponding to κ and q. For flows past obstacles, we refer the readers to [11, 12, 13, 14] for other results.

Note that in [15], the value ρ_λ of ψ_λ on ∂O_0 is as a lagrangian multiplier, which is unknown. In this paper, we assume that ρ_λ is a prescribed constant, and we remove the condition

$$\int_{\partial O_0} \frac{\partial \psi}{\partial \nu} = 0.$$

The first equation in (2) suggests that ψ and ω are functionally dependent. So, for simplicity, we consider the following elliptic equation

$$\begin{cases} -\Delta \psi = \lambda(\psi - \frac{\kappa}{4\pi} \ln \lambda)_+^p, & \text{in } \Omega, \\ \psi = \rho_\lambda, & \text{on } \partial O_0, \\ \psi = 0, & \text{on } \partial \Omega_0, \end{cases} \tag{3}$$

where $1 < p < +\infty$, $\kappa > 0$ is a constant, and $\rho_\lambda > 0$ is a constant, depending on λ.

In this paper, we mainly focus on the solvability of (3), and the effect from the constant ρ_λ on the location of the vorticity set $\Omega_\lambda := \{y \in \Omega | \, \psi(y) > \frac{\kappa}{4\pi} \ln \lambda\}$. We expect that for large $\rho_\lambda > 0$, the vorticity set Ω_λ concentrates near the boundary of O_0, as $\lambda \to +\infty$.

Let η be the unique solution of the following problem

$$\Delta \eta = 0 \ \text{ in } \Omega, \quad \eta = 1 \ \text{ on } \partial O_0, \quad \eta = 0 \ \text{ on } \partial \Omega_0. \tag{4}$$

Making the change of $\psi = \frac{\ln \lambda}{4\pi} u + \rho_\lambda \eta$, $\varepsilon = \lambda^{-\frac{1}{2}} \left(\frac{\ln \lambda}{4\pi}\right)^{\frac{1-p}{2}}$ and $\lambda_\varepsilon = \frac{4\pi}{\ln \lambda} \rho_\lambda$, (3) can be changed into

$$\begin{cases} -\varepsilon^2 \Delta u = (u - \kappa + \lambda_\varepsilon \eta)_+^p, & \text{in } \Omega, \\ u = 0, & \text{on } \partial \Omega. \end{cases} \tag{5}$$

Now we want to find a solution to (3) by constructing a solution for (5), whose vorticity set is close to the boundary of O_0. For this purpose, the following assumption can be imposed on ρ_λ:

(H_λ) There is a small constant $\sigma > 0$, such that

$$(\ln \lambda)^\sigma \le \rho_\lambda \le (\ln \lambda)^{1-\sigma}.$$

It is easy to see that if ρ_λ satisfies the condition (H_λ), then λ_ε satisfies the condition:

(H_ε) There are constants γ_1 and γ_2 with $0 < \gamma_1 < \gamma_2 < 1$, such that

$$\frac{1}{|\ln \varepsilon|^{\gamma_2}} \le \lambda_\varepsilon \le \frac{1}{|\ln \varepsilon|^{\gamma_1}}.$$

Before we state the main result, we give the following definition.

Definition 1.1. Let $f:\mathbb{R}\mapsto\mathbb{R}$ be a continuous function. We call $I=[a.b]$, $a\le b$ a minimum interval of f, if $f(t_1)=f(t_2)$ for any $t_1,t_2\in I$, and there is $\sigma_0>0$ such that for $0<\sigma<\sigma_0$, $f(a-\sigma)>f(a)$ and $f(b+\sigma)>f(b)$.

Our main result of this paper can be stated as follows.

Theorem 1.2. *Let κ be a given positive number. Suppose that ρ_λ satisfies (H_λ). Then there is a constant $C_0>0$, such that for any $\lambda>C_0$, (3) has a solution ψ_λ, such that*

$$\Big\{y\in\Omega|\,\psi_\lambda(y)>\frac{\kappa}{4\pi}\ln\lambda\Big\}\subset B_{\frac{L}{\sqrt{\lambda}}}(x_\lambda),$$

where $x_\lambda\in\Omega$ and $L>0$ is a constant independent of λ.

Moreover, as $\lambda\to+\infty$,

$$dist(x_\lambda,\partial\Omega)\to 0,$$

$$\lambda\int_\Omega(\psi_\lambda-\frac{\kappa}{4\pi}\ln\lambda)_+^p\to\kappa.$$

In particular, if I is a minimum interval of $\frac{\partial\eta(x(s))}{\partial\nu}$ defined for $x(s)\in\partial\Omega$, then as $\lambda\to+\infty$,

$$dist(x_\lambda,x(I))\to 0.$$

As a result of Theorem 1.2, we obtain a flow in Ω with single non-vanishing anti-clockwise vortex, which concentrates on the boundary of O_0. Let us point out that we can also construct a planar Euler flow with single clockwise vortex, which nears the boundary of Ω_0. This is carried out by considering the problem (3) with nonlinearity $\lambda(\psi-\frac{\kappa}{4\pi}\ln\lambda)_+^p$ replaced by $-\lambda(-\psi-\frac{\kappa}{4\pi}\ln\lambda)_+^p$.

Theorem 1.2 is proved via the following theorem.

Theorem 1.3. *Let κ be a given positive number. Suppose that λ_ε satisfies (H_ε). Then there is a constant $\varepsilon_0>0$, such that for any $0<\varepsilon<\varepsilon_0$, (5) has a solution u_ε, such that*

$$\Big\{y\in\Omega|\,u_\varepsilon(y)-\kappa+\lambda_\varepsilon\eta(y)>0\Big\}\subset B_{L\varepsilon|\ln\varepsilon|^{\frac{p-1}{2}}}(x_\varepsilon),$$

where $x_\varepsilon\in\Omega$ and $L>0$ is a constant independent of ε. Moreover, as $\varepsilon\to 0$,

$$dist(x_\varepsilon,\partial\Omega)\to 0.$$

In particular, if there is a minimum interval I of $\frac{\partial\eta(x(s))}{\partial\nu}$ defined for $x(x)\in\partial\Omega$, then as $\varepsilon\to 0$,

$$dist(x_\varepsilon,x(I))\to 0.$$

To prove Theorem 1.3, we will use a finite reduction argument as in [6, 7, 8, 9]. Since we consider the vortex nears the boundary, it turns out that more delicate estimates are needed in the proof of Theorem 1.3 than those estimates in [6, 7, 8, 9]. In particular, we need the estimates of the functions η and G near the boundary, where G is the Green's function for $-\Delta$ in Ω with zero boundary condition, written as

$$G(y,x)=\frac{1}{2\pi}\ln\frac{1}{|y-x|}-H(y,x),\quad x,y\in\Omega, \tag{6}$$

and $H(y,x)$ is the regular part of the Green's function. Recall that the Robin's function is defined by $\varphi(x)=H(x,x)$.

The stationary incompressible Euler equations have been studied by many authors, see for instance [1, 4, 5, 6, 7, 8, 9, 10, 3, 15, 16, 17] and references therein. Roughly speaking, there are two commonly used methods to study the existence of the stationary incompressible Euler equations: the vorticity method and the stream-function method. The vorticity method was first established by Arnold [2] and further developed by Burton [4, 5] and Tukington [16]. This argument roughly consists in maximizing the kinetic energy under a constrained sublevel set of ω. In this paper, we will use the stream-function method.

This paper is organized as follows. In section 2, we construct approximate solutions for (5). We will carry out a reduction procedure in section 3 and the results of existence will be proved in section 4. In appendix A, we give the estimates for the radius of vortex core.

2. **Approximate solutions.** In this section, following [10], we will construct an approximate solution for (5).

For $p > 1$, there is a unique solution ϕ for the following problem:

$$-\Delta\phi = \phi^p, \quad \phi > 0,\ \phi \in H_0^1\big(B_1(0)\big). \tag{7}$$

Moreover, ϕ is a radial function and satisfies

$$\int_{B_1(0)} \phi^{p+1} = \frac{\pi(p+1)}{2}|\phi'(1)|^2, \quad \int_{B_1(0)} \phi^p = 2\pi|\phi'(1)|.$$

Let $R > 0$ be a large constant, such that for any $x \in \Omega$, $\Omega \subset\subset B_R(x)$. Now we consider

$$\begin{cases} -\varepsilon^2\Delta u = (u-a)_+^p, & y \in B_R(0),\\ u = 0, & y \in \partial B_R(0), \end{cases} \tag{8}$$

where $a > 0$ is a constant. Then, (8) has a solution $U_{\varepsilon,a}$, which can be represented by

$$U_{\varepsilon,a}(y) = \begin{cases} a + \left(\frac{\varepsilon}{s_\varepsilon}\right)^{\frac{2}{p-1}}\phi\big(\frac{|y|}{s_\varepsilon}\big), & |y| \le s_\varepsilon,\\ a\ln\frac{|y|}{R}\big/\ln\frac{s_\varepsilon}{R}, & s_\varepsilon \le |y| \le R, \end{cases} \tag{9}$$

where and $\phi(y)$ is a radial solution of (7), and s_ε is a constant, such that $U_{\varepsilon,a} \in C^1(B_R(0))$. So, s_ε is determined by

$$\left(\frac{\varepsilon}{s_\varepsilon}\right)^{\frac{2}{p-1}}\phi'(1) = \frac{a}{\ln\frac{s_\varepsilon}{R}}, \tag{10}$$

which gives an expansion for s_ε as follows

$$s_\varepsilon = \left(\frac{|\phi'(1)|}{a}\right)^{\frac{p-1}{2}}\varepsilon|\ln\varepsilon|^{\frac{p-1}{2}}\left(1 + O(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|})\right). \tag{11}$$

For any $x \in \Omega$, define $U_{\varepsilon,x,a}(y) = U_{\varepsilon,a}(y-x)$. Because $U_{\varepsilon,x,a}$ does not vanish on $\partial\Omega$, we need to make a projection. Let $PU_{\varepsilon,x,a}$ be the solution of

$$\begin{cases} -\varepsilon^2\Delta u = (U_{\varepsilon,x,a} - a)_+^p, & y \in \Omega,\\ u = 0, & y \in \partial\Omega. \end{cases}$$

Then

$$PU_{\varepsilon,x,a}(y) = U_{\varepsilon,x,a}(y) - \frac{a}{\ln\frac{R}{s_\varepsilon}}g(y,x), \tag{12}$$

where $g(y,x) = \ln R + 2\pi H(y,x)$ and $H(y,x)$ is the regular part of the Green's function G defined by (6).

Take

$$d_1 = \frac{r_1}{|\ln\varepsilon|\lambda_\varepsilon}, \quad d_2 = \frac{r_2}{|\ln\varepsilon|\lambda_\varepsilon}, \tag{13}$$

where $0 < r_1 < r_2$ are two fixed constants, which will be determined later, and set

$$S =: (\partial O_0)_{d_2} \setminus \overline{(\partial O_0)_{d_1}}, \tag{14}$$

where $(\partial O_0)_{d_i} = \{y \in \Omega | \, dist(y, \partial O_0) < d_i\}$ is the neighborhood of ∂O_0. Denote $d(x) = dist(x, \partial O_0)$ and take $\hat{x} \in \partial O_0$ such that $d(x) = |x - \hat{x}|$. Define

$$r(x) = |\ln \varepsilon| \lambda_\varepsilon d(x). \tag{15}$$

Then we can rewrite S as follows

$$S = \{x \in \Omega : \, r_1 < r(x) < r_2\}. \tag{16}$$

In the rest of this paper, we assume that $x \in S$. We want to construct solutions for (5) of the form

$$u = PU_{\varepsilon,x,a_\varepsilon} + \omega_\varepsilon, \tag{17}$$

where ω_ε is a perturbation term. To make $PU_{\varepsilon,x,a_\varepsilon}$ a good approximate solution, we need to require that a_ε and s_ε satisfy

$$\left(\frac{\varepsilon}{s_\varepsilon}\right)^{\frac{2}{p-1}} \phi'(1) = \frac{a_\varepsilon}{\ln \frac{s_\varepsilon}{R}}, \quad a_\varepsilon = \kappa - \lambda_\varepsilon \eta(x) + \frac{a_\varepsilon}{\ln \frac{R}{s_\varepsilon}} g(x,x). \tag{18}$$

Let us now show that (18) is solvable for $\varepsilon > 0$ small. From the Taylor expansion we have

$$\eta(x) = \eta(\hat{x}) - \frac{\partial \eta(\hat{x})}{\partial \nu} d(x) + O(d^2(x)) = 1 - \frac{\partial \eta(\hat{x})}{\partial \nu} d(x) + O(d^2(x)),$$

where ν is the outward unit normal of $\partial\Omega$. On the other hand, the following expansion for Robin's function is proved in [15] Appendix B:

$$\varphi(x) = \frac{1}{2\pi} \ln \frac{1}{2d(x)} + O(d(x)).$$

Then by our assumption on λ_ε, as in Lemma 2.1 in [6], we can solve (18) to obtain $a_\varepsilon(x)$ and $s_\varepsilon(x)$. For simplicity, we use a_ε and s_ε instead of $a_\varepsilon(x)$ and $s_\varepsilon(x)$, respectively. Then for $y \in B_{Ls_\varepsilon}(x)$, where $L > 0$ is any fixed constant, we have

$$\begin{aligned} PU_{\varepsilon,x,a_\varepsilon}(y) - \kappa + \lambda_\varepsilon \eta(y) =& U_{\varepsilon,x,a_\varepsilon}(y) - \kappa + \lambda_\varepsilon \eta(y) - \frac{a_\varepsilon}{\ln \frac{R}{s_\varepsilon}} g(y,x) \\ =& U_{\varepsilon,x,a_\varepsilon}(y) - a_\varepsilon + \lambda_\varepsilon \langle \nabla \eta(x), y - x \rangle \\ &- \frac{a_\varepsilon}{\ln \frac{R}{s_\varepsilon}} \langle \nabla g(x,x), y - x \rangle + O(s_\varepsilon^{1+\sigma}), \end{aligned} \tag{19}$$

where $0 < \sigma < 1$ is a small constant.

Remark 1. It is not difficult to get the following expansions:

$$\begin{aligned} \frac{1}{\ln \frac{R}{s_\varepsilon}} =& \frac{1}{A_\varepsilon} + O\left(\frac{\lambda_\varepsilon}{|\ln \varepsilon|^2}\right), \\ a_\varepsilon =& \left(1 + \frac{\ln R}{|\ln \varepsilon|}\right) \kappa - \lambda_\varepsilon + \lambda_\varepsilon \frac{\partial \eta(\hat{x})}{\partial \nu} d(x) + \frac{\kappa}{|\ln \varepsilon|} \ln \frac{1}{2d(x)} + O\left(\frac{1}{|\ln \varepsilon|^{1+\sigma}}\right), \end{aligned} \tag{20}$$

where $\sigma > 0$ is a small constant and

$$A_\varepsilon = |\ln \varepsilon| - \frac{p-1}{2} \ln |\ln \varepsilon| + \ln R + \frac{p-1}{2} \ln \frac{\kappa}{|\phi'(1)|}.$$

Moreover, in view of $\frac{\partial \varphi(x)}{\partial x_i} = O(\frac{1}{d(x)})$, we can prove that

$$\frac{\partial a_\varepsilon}{\partial x_i} = O(\lambda_\varepsilon), \quad \frac{\partial s_\varepsilon}{\partial x_i} = O(\lambda_\varepsilon s_\varepsilon), \quad i = 1, 2. \tag{21}$$

By (9) and (21), we have the following expansion, which will be used in the rest of this paper,

$$\frac{\partial U_{\varepsilon,x,a_\varepsilon}(y)}{\partial x_i} = \begin{cases} \dfrac{a_\varepsilon}{|\phi'(1)|}\dfrac{1}{\ln\frac{R}{s_\varepsilon}}\phi'(\dfrac{|y-x|}{s_\varepsilon})\dfrac{x_i-y_i}{|y-x|}\dfrac{1}{s_\varepsilon} & +O\left(\frac{\lambda_\varepsilon}{|\ln\varepsilon|}\right), \\ & y \in B_{s_\varepsilon}(x), \\ \dfrac{a_\varepsilon}{\ln\frac{R}{s_\varepsilon}}\dfrac{x_i-y_i}{|y-x|^2} + O\left(\dfrac{\lambda_\varepsilon \ln|y-x|}{|\ln\varepsilon|}\right), & y \in \Omega\setminus B_{s_\varepsilon}(x). \end{cases} \tag{22}$$

3. **The reduction.** In this section, we reduce the problem of finding a solution for (5) of the form (17) to a finite dimension problem.

Define

$$E_{\varepsilon,x} = \Big\{u : u \in W^{2,p}(\Omega)\cap H_0^1(\Omega),\ \int_\Omega u\,\Delta\frac{\partial PU_{\varepsilon,x,a_\varepsilon}}{\partial x_i}\,dy = 0,\ \ i = 1,2\Big\},$$

and

$$F_{\varepsilon,x} = \Big\{u : u \in L^p(\Omega),\ \int_\Omega u\,\frac{\partial PU_{\varepsilon,x,a_\varepsilon}}{\partial x_i}\,dy = 0,\ \ i = 1,2\Big\}.$$

For any $u \in L^p(\Omega)$, define the following projection

$$\mathbb{Q}_\varepsilon u =: u + \sum_{j=1}^2 b_j \varepsilon^2 \Delta \frac{\partial PU_{\varepsilon,x,a_\varepsilon}}{\partial x_j},$$

where b_1 and b_2 are the constants such that $Q_\varepsilon u \in F_{\varepsilon,x}$. Thus b_1 and b_2 should satisfy

$$\sum_{j=1}^2 b_j\varepsilon^2 \int_\Omega \nabla\frac{\partial PU_{\varepsilon,x,a_\varepsilon}}{\partial x_j}\nabla\frac{\partial PU_{\varepsilon,x,a_\varepsilon}}{\partial x_i} = \int_\Omega u\,\frac{\partial PU_{\varepsilon,x,a_\varepsilon}}{\partial x_i}. \tag{23}$$

The existence of b_1 and b_2 can be obtained by the following estimate

$$\begin{aligned} &\varepsilon^2 \int_\Omega \nabla\frac{\partial PU_{\varepsilon,x,a_\varepsilon}}{\partial x_j}\nabla\frac{\partial PU_{\varepsilon,x,a_\varepsilon}}{\partial x_i} \\ =&p\int_\Omega (U_{\varepsilon,x,a_\varepsilon} - a_\varepsilon)_+^{p-1}\left(\frac{\partial U_{\varepsilon,x,a_\varepsilon}}{\partial x_j} - \frac{\partial a_\varepsilon}{\partial x_j}\right)\frac{\partial PU_{\varepsilon,x,a_\varepsilon}}{\partial x_i} \\ =&c(\delta_{ij}+o(1))\frac{1}{|\ln\varepsilon|^{p+1}}, \end{aligned} \tag{24}$$

where $c > 0$ is a constant, $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$, if $i \neq j$.

Set

$$\mathbb{L}_\varepsilon u = -\varepsilon^2\Delta u - p\,(PU_{\varepsilon,x,a_\varepsilon} - \kappa + \lambda_\varepsilon\eta)_+^{p-1}\,u. \tag{25}$$

We have the following result for the operator $\mathbb{Q}_\varepsilon\mathbb{L}_\varepsilon$.

Proposition 1. *There are constants $\varepsilon_0 > 0$ and $\sigma_0 > 0$, such that for any $\varepsilon \in (0,\varepsilon_0)$, $x \in S$, $u \in E_{\varepsilon,x}$ with $\mathbb{Q}_\varepsilon\mathbb{L}_\varepsilon u = 0$ in $\Omega\setminus B_{Ls_\varepsilon}(x)$ for some large $L > 0$, then*

$$\frac{s_\varepsilon^{2-\frac{2}{p}}}{\varepsilon^2}\|\mathbb{Q}_\varepsilon\mathbb{L}_\varepsilon u\|_{L^p(B_{Ls_\varepsilon}(x))} \geq \sigma_0\|u\|_{L^\infty(\Omega)}.$$

As a consequence, $\mathbb{Q}_\varepsilon\mathbb{L}_\varepsilon$ is one to one and onto from $E_{\varepsilon,x}$ to $F_{\varepsilon,x}$.

Proof. Suppose to the contrary that there are $\varepsilon_n \to 0$, $x_n \in S_n$, $u_n \in E_{\varepsilon_n,x_n}$ with $\mathbb{Q}_{\varepsilon_n}\mathbb{L}_{\varepsilon_n}u_n = 0$ in $\Omega \setminus B_{Ls_{\varepsilon_n}}(x_n)$, and $\|u_n\|_{L^\infty(\Omega)} = 1$, such that

$$\frac{s_{\varepsilon_n}^{2-\frac{2}{p}}}{\varepsilon_n^2}\|\mathbb{Q}_{\varepsilon_n}\mathbb{L}_{\varepsilon_n}u_n\|_{L^p(B_{Ls_{\varepsilon_n}}(x_n))} \le \frac{1}{n}. \tag{26}$$

First of all, we estimate $b_{1,n}$ and $b_{2,n}$ in the following formula:

$$\mathbb{Q}_{\varepsilon_n}\mathbb{L}_{\varepsilon_n}u_n = \mathbb{L}_{\varepsilon_n}u_n + \sum_{j=1}^{2} b_{j,n}\varepsilon_n^2\Delta\frac{\partial PU_{\varepsilon_n,x_n,a_{\varepsilon_n}}}{\partial x_j}, \tag{27}$$

where $b_{1,n}$ and $b_{2,n}$ satisfy

$$\sum_{j=1}^{2} b_{j,n}\varepsilon_n^2\int_\Omega \nabla\frac{\partial PU_{\varepsilon_n,x_n,a_{\varepsilon_n}}}{\partial x_j}\nabla\frac{\partial PU_{\varepsilon_n,x_n,a_{\varepsilon_n}}}{\partial x_i} = \int_\Omega \frac{\partial PU_{\varepsilon_n,x_n,a_{\varepsilon_n}}}{\partial x_i}\mathbb{L}_{\varepsilon_n}u_n. \tag{28}$$

From (19), (21) and Lemma A.1, we have

$$\begin{aligned}
&\int_\Omega \frac{\partial PU_{\varepsilon_n,x_n,a_{\varepsilon_n}}}{\partial x_i}\mathbb{L}_{\varepsilon_n}u_n = \int_\Omega u_n\mathbb{L}_{\varepsilon_n}\frac{\partial PU_{\varepsilon_n,x_n,a_{\varepsilon_n}}}{\partial x_i}\\
=&p\int_\Omega \Big[\big(U_{\varepsilon_n,x_n,a_{\varepsilon_n}} - a_{\varepsilon_n}\big)_+^{p-1}\left(\frac{\partial U_{\varepsilon_n,x_n,a_{\varepsilon_n}}}{\partial x_i} - \frac{\partial a_{\varepsilon_n}}{\partial x_i}\right)\\
&-\big(PU_{\varepsilon_n,x_n,a_{\varepsilon_n}} - \kappa + \lambda_{\varepsilon_n}\eta\big)_+^{p-1}\frac{\partial PU_{\varepsilon_n,x_n,a_{\varepsilon_n}}}{\partial x_i}\Big]u_n\\
=&p\int_\Omega \Big[\big(U_{\varepsilon_n,x_n,a_{\varepsilon_n}} - a_{\varepsilon_n}\big)_+^{p-1}\frac{\partial U_{\varepsilon_n,x_n,a_{\varepsilon_n}}}{\partial x_i}\\
&-\big(U_{\varepsilon_n,x_n,a_{\varepsilon_n}} - a_{\varepsilon_n} + O(\lambda_{\varepsilon_n}s_{\varepsilon_n})\big)_+^{p-1}\frac{\partial PU_{\varepsilon_n,x_n,a_{\varepsilon_n}}}{\partial x_i}\Big]u_n + O\left(\frac{\lambda_{\varepsilon_n}s_{\varepsilon_n}^2}{|\ln\varepsilon_n|^{p-1}}\right)\\
=&O\left(\frac{\lambda_{\varepsilon_n}s_{\varepsilon_n}^2}{|\ln\varepsilon_n|^{p-1}}\right).
\end{aligned} \tag{29}$$

Then by (24), (28) and (29), we obtain

$$b_{j,n} = O\left(\lambda_{\varepsilon_n}s_{\varepsilon_n}^2|\ln\varepsilon_n|^2\right). \tag{30}$$

Write

$$-\varepsilon_n^2\Delta u_n = p\big(PU_{\varepsilon_n,x_n,a_{\varepsilon_n}} - \kappa + \lambda_{\varepsilon_n}\eta\big)_+^{p-1}u_n + f_n, \tag{31}$$

where

$$f_n = \mathbb{Q}_{\varepsilon_n}\mathbb{L}_{\varepsilon_n}u_n - \sum_{j=1}^{2} b_{j,n}\varepsilon_n^2\Delta\frac{\partial PU_{\varepsilon_n,x_n,a_{\varepsilon_n}}}{\partial x_j},$$

Define

$$\tilde{u}_n(y) = u_n(s_{\varepsilon_n}y + x_n), \quad \tilde{f}_n(y) = f_n(s_{\varepsilon_n}y + x_n).$$

Then we have

$$-\Delta\tilde{u}_n = p\frac{s_{\varepsilon_n}^2}{\varepsilon_n^2}\big(\tilde{PU}_{\varepsilon_n,x_n,a_{\varepsilon_n}} - \kappa + \lambda_{\varepsilon_n}\tilde{\eta}\big)_+^{p-1}\tilde{u}_n + \frac{s_{\varepsilon_n}^2}{\varepsilon_n^2}\tilde{f}_{2,n}. \tag{32}$$

From (26), (30) and Lemma A.1, we find

$$\frac{s_{\varepsilon_n}^2}{\varepsilon_n^2}\|\tilde{f}_n\|_{L^p_{loc}(\mathbb{R}^2)} = o_n(1) + O\Big(s_{\varepsilon_n}^{2-\frac{2}{p}}\Big\|\sum_{j=1}^{2} b_{j,n}\Delta\frac{\partial PU_{\varepsilon_n,x_n,a_{\varepsilon_n}}}{\partial x_j}\Big\|_{L^p(\Omega)}\Big) = o_n(1). \tag{33}$$

Since the right hand side of (32) is bounded in $L^p_{loc}(\mathbb{R}^2)$, $\tilde{u}_n$ is bounded in $W^{2,p}_{loc}(\mathbb{R}^2)$. By the Sobolev embedding, $\tilde{u}_n$ is bounded in $C^{\alpha}_{loc}(\mathbb{R}^2)$ for some $\alpha > 0$. So, we can assume that $\tilde{u}_n$ converges uniformly in any compact set of $\mathbb{R}^2$ to $\tilde{u} \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$. It is easy to check that $\tilde{u}$ satisfies

$$-\Delta\tilde{u} - pw_+^{p-1}\tilde{u} = 0, \quad \text{in } \mathbb{R}^2, \tag{34}$$

where

$$w(y) = \begin{cases} \phi(|y|), & |y| \le 1, \\ \phi'(1)\ln|y|, & |y| > 1. \end{cases}$$

So there exist constants b_1 and b_2 (see [10]), such that

$$\tilde{u} = b_1 \frac{\partial w}{\partial y_1} + b_2 \frac{\partial w}{\partial y_2}. \tag{35}$$

From $u_n \in E_{\varepsilon_n, x_n}$, we see that

$$\int_{B_1(0)} \phi^{p-1} \frac{\partial \phi}{\partial y_j}\tilde{u} = 0, \;\; j = 1, 2.$$

So we get $b_1 = b_2 = 0$. That is, $\tilde{u} \equiv 0$. Then we find

$$u_n = o_n(1), \quad \text{in } C(B_{Ls_{\varepsilon_n}}(x_n)). \tag{36}$$

By our assumption,

$$\mathbb{Q}_{\varepsilon_n}\mathbb{L}_{\varepsilon_n} u_n = 0, \quad \text{in } \Omega \setminus B_{Ls_{\varepsilon_n}}(x_n).$$

We find that

$$\Delta u_n = 0, \quad \text{in } \Omega \setminus B_{Ls_{\varepsilon_n}}(x_n).$$

However, $u_n = 0$ on $\partial\Omega$ and $u_n = o_n(1)$ on $\partial B_{Ls_{\varepsilon_n}}(x_n)$. By the maximum principle,

$$u_n = o_n(1), \quad \text{in } \Omega \setminus B_{Ls_{\varepsilon_n}}(x_n).$$

So, we have proved that

$$\|u_n\|_{L^\infty(\Omega)} = o_n(1),$$

which contradicts $\|u_n\|_{L^\infty(\Omega)} = 1$.

Using the same argument as in Proposition 3.2 in [6], it is not difficult to prove that $\mathbb{Q}_\varepsilon\mathbb{L}_\varepsilon$ is one to one and onto from $E_{\varepsilon,x}$ to $F_{\varepsilon,x}$. Therefore, we complete the proof. □

We now want to find a solution for (5) of the form $PU_{\varepsilon,x,a_\varepsilon} + \omega$. Then ω should satisfy

$$\mathbb{L}_\varepsilon \omega = l_\varepsilon + R_\varepsilon(\omega), \tag{37}$$

where

$$l_\varepsilon = (PU_{\varepsilon,x,a_\varepsilon} - \kappa + \lambda_\varepsilon\eta)_+^p - (U_{\varepsilon,x,a_\varepsilon} - a_\varepsilon)_+^p, \tag{38}$$

and

$$R_\varepsilon(\omega) = (PU_{\varepsilon,x,a_\varepsilon} + \omega - \kappa + \lambda_\varepsilon\eta)_+^p - (PU_{\varepsilon,x,a_\varepsilon} - \kappa + \lambda_\varepsilon\eta)_+^p - p(PU_{\varepsilon,x,a_\varepsilon} - \kappa + \lambda_\varepsilon\eta)_+^{p-1}\omega. \tag{39}$$

From (37), we see

$$\mathbb{Q}_\varepsilon\mathbb{L}_\varepsilon\omega = \mathbb{Q}_\varepsilon l_\varepsilon + \mathbb{Q}_\varepsilon R_\varepsilon(\omega). \tag{40}$$

Fot $\omega \in E_{\varepsilon,x}$, using Proposition 1, (40) is equivalent to

$$\omega = \mathbb{G}_\varepsilon\omega =: (\mathbb{Q}_\varepsilon\mathbb{L}_\varepsilon)^{-1}\mathbb{Q}_\varepsilon(l_\varepsilon + R_\varepsilon(\omega)). \tag{41}$$

We have the following estimates for l_ε and $R_\varepsilon(\omega)$.

Lemma 3.1. *It holds*

$$\|l_\varepsilon\|_{L^p(B_{Ls_\varepsilon}(x))} = O\left(\frac{\lambda_\varepsilon s_\varepsilon^{1+\frac{2}{p}}}{|\ln\varepsilon|^{p-1}}\right),$$

and if $\|\omega\|_{L^\infty(\Omega)} = O(s_\varepsilon)$*, then*

$$\|R_\varepsilon(\omega)\|_{L^p(B_{Ls_\varepsilon}(x))} = O\left(\frac{s_\varepsilon^{\frac{2}{p}}}{|\ln\varepsilon|^{p-2}}\|\omega\|^2_{L^\infty(\Omega)}\right).$$

Proof. For any $y\in B_{Ls_\varepsilon}(x)$, from (19), we have

$$\begin{aligned}|l_\varepsilon| =&|(PU_{\varepsilon,x,a_\varepsilon}-\kappa+\lambda_\varepsilon\eta)^p_+ - (U_{\varepsilon,x,a_\varepsilon}-a_\varepsilon)^p_+|\\ =&|(U_{\varepsilon,x,a_\varepsilon}-a_\varepsilon+O(\lambda_\varepsilon s_\varepsilon))^p_+ - (U_{\varepsilon,x,a_\varepsilon}-a_\varepsilon)^p_+|\\ \le& C\left(\lambda_\varepsilon s_\varepsilon(U_{\varepsilon,x,a_\varepsilon}-a_\varepsilon)^{p-1}_+ + (\lambda_\varepsilon s_\varepsilon)^p\right)\\ \le& C\frac{\lambda_\varepsilon s_\varepsilon}{|\ln\varepsilon|^{p-1}}.\end{aligned}$$

So we get

$$\|l_\varepsilon\|_{L^p(B_{Ls_\varepsilon}(x))} \le s_\varepsilon^{\frac{2}{p}}\|l_\varepsilon\|_{L^\infty(B_{Ls_\varepsilon}(x))} \le C\frac{\lambda_\varepsilon s_\varepsilon^{1+\frac{2}{p}}}{|\ln\varepsilon|^{p-1}}.$$

Similarly, using (19) and Lemma A.1, it ie east to see that

$$\|R_\varepsilon(\omega)\|_{L^p(B_{Ls_\varepsilon}(x))} \le C\|\omega\|^2_{L^\infty(\Omega)}\|(PU_{\varepsilon,x,a_\varepsilon}-\kappa+\lambda_\varepsilon\eta)^{p-2}_+\|_{L^p(B_{Ls_\varepsilon}(x))} \le C\frac{s_\varepsilon^{\frac{2}{p}}}{|\ln\varepsilon|^{p-2}}\|\omega\|^2_{L^\infty(\Omega)}.$$

So we complete the proof. □

Using Lemma 3.1, we can solve (41) in a standard way and obtain the following proposition.

Proposition 2. *There is* $\varepsilon_0>0$*, such that for any* $\varepsilon\in(0,\varepsilon_0)$ *and* $x\in S$*, (40) has a unique solution* $\omega_{\varepsilon,x}\in E_{\varepsilon,x}$*, with*

$$\|\omega_{\varepsilon,x}\|_{L^\infty(\Omega)} = O(\lambda_\varepsilon s_\varepsilon).$$

Furthermore, $\omega_{\varepsilon,x}$ *is a* C^1 *map from* $x\in S$ *to* $E_{\varepsilon,x}$*.*

4. Proof of main results.

In this section, we will prove our main results.

Define

$$I(u) = \frac{\varepsilon^2}{2}\int_\Omega|\nabla u|^2 - \frac{1}{p+1}\int_\Omega(u-\kappa+\lambda_\varepsilon\eta)^{p+1}_+,$$

and set

$$F(x) = I(PU_{\varepsilon,x,a_\varepsilon}+\omega_{\varepsilon,x}),$$

where $x\in S$ and $\omega_{\varepsilon,x}$ is found in Proposition 2. It is well-known that if x is a critical point of F, then $PU_{\varepsilon,x,a_\varepsilon}+\omega_{\varepsilon,x}$ is a solution of (5).

Lemma 4.1. *There holds:*

$$F(x) = I(PU_{\varepsilon,x,a_\varepsilon}) + O\left(\frac{\lambda_\varepsilon^2 s_\varepsilon^4}{|\ln\varepsilon|^{p-1}}\right).$$

Proof. Since $\omega_{\varepsilon,x} \in E_{\varepsilon,x}$, then we have the following energy expansion

$$\begin{aligned}
&I(PU_{\varepsilon,x,a_\varepsilon}+\omega_{\varepsilon,x})\\
=&I(PU_{\varepsilon,x,a_\varepsilon})+\langle I'(PU_{\varepsilon,x,a_\varepsilon}),\omega_{\varepsilon,x}\rangle+\frac{1}{2}\langle I''(PU_{\varepsilon,x,a_\varepsilon})\omega_{\varepsilon,x},\omega_{\varepsilon,x}\rangle+\tilde{R}_\varepsilon(\omega_{\varepsilon,x})\\
=&I(PU_{\varepsilon,x,a_\varepsilon})-\int_\Omega l_\varepsilon\omega_{\varepsilon,x}+\frac{1}{2}\int_\Omega \omega_{\varepsilon,x}\mathbb{L}_\varepsilon\omega_{\varepsilon,x}+\tilde{R}_\varepsilon(\omega_{\varepsilon,x})\\
=&I(PU_{\varepsilon,x,a_\varepsilon})-\int_\Omega l_\varepsilon\omega_{\varepsilon,x}+\frac{1}{2}\int_\Omega\left(l_\varepsilon+R_\varepsilon(\omega_{\varepsilon,x})-\sum_{j=1}^{2}b_j\varepsilon^2\Delta\frac{\partial PU_{\varepsilon,x,a_\varepsilon}}{\partial x_j}\right)\omega_{\varepsilon,x}+\tilde{R}_\varepsilon(\omega_{\varepsilon,x})\\
=&I(PU_{\varepsilon,x,a_\varepsilon})-\frac{1}{2}\int_\Omega l_\varepsilon\omega_{\varepsilon,x}+\frac{1}{2}\int_\Omega R_\varepsilon(\omega_{\varepsilon,x})\omega_{\varepsilon,x}+\tilde{R}_\varepsilon(\omega_{\varepsilon,x}),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{R}_\varepsilon(\omega_{\varepsilon,x})=&-\frac{1}{p+1}\int_\Omega\Big[(PU_{\varepsilon,x,a_\varepsilon}+\omega_{\varepsilon,x}-\kappa+\lambda_\varepsilon\eta)_+^{p+1}-(PU_{\varepsilon,x,a_\varepsilon}-\kappa+\lambda_\varepsilon\eta)_+^{p+1}\\
&-(p+1)(PU_{\varepsilon,x,a_\varepsilon}-\kappa+\lambda_\varepsilon\eta)_+^{p}\omega_{\varepsilon,x}-\frac{p(p+1)}{2}(PU_{\varepsilon,x,a_\varepsilon}-\kappa+\lambda_\varepsilon\eta)_+^{p-1}\omega_{\varepsilon,x}^2\Big].
\end{aligned}$$

One sees

$$|\tilde{R}_\varepsilon(\omega_{\varepsilon,x})|\le C\int_\Omega(PU_{\varepsilon,x,a_\varepsilon}-\kappa+\lambda_\varepsilon\eta)_+^{p-2}\omega_{\varepsilon,x}^3\le C\frac{\lambda_\varepsilon^3 s_\varepsilon^5}{|\ln\varepsilon|^{p-2}}.$$

Then by Lemma 3.1, we can easily check that

$$F(x)=I(PU_{\varepsilon,x,a_\varepsilon})+O\left(\frac{\lambda_\varepsilon^2 s_\varepsilon^4}{|\ln\varepsilon|^{p-1}}\right).$$

□

Lemma 4.2. *We have*

$$I(PU_{\varepsilon,x,a_\varepsilon})=C_\varepsilon+\frac{\pi\kappa\varepsilon^2}{A_\varepsilon}\left(2\lambda_\varepsilon\frac{\partial\eta(\hat{x})}{\partial\nu}d(x)+\frac{\kappa}{|\ln\varepsilon|}\ln\frac{1}{2d(x)}\right)+O\left(\frac{\varepsilon^2}{|\ln\varepsilon|^{2+\sigma}}\right),$$

where $\sigma>0$ is a small constant, A_ε is given in Remark 1, and C_ε is a constant, depending on ε.

Proof. We have

$$\begin{aligned}
&\varepsilon^2\int_\Omega|\nabla PU_{\varepsilon,x,a_\varepsilon}|^2\\
=&\int_\Omega\big(-\varepsilon^2\Delta PU_{\varepsilon,x,a_\varepsilon}\big)PU_{\varepsilon,x,a_\varepsilon}\\
=&\int_\Omega\big(U_{\varepsilon,x,a_\varepsilon}-a_\varepsilon\big)_+^pPU_{\varepsilon,x,a_\varepsilon}\\
=&\int_\Omega\big(U_{\varepsilon,x,a_\varepsilon}-a_\varepsilon\big)_+^{p+1}+a_\varepsilon\int_\Omega\big(U_{\varepsilon,x,a_\varepsilon}-a_\varepsilon\big)_+^p-\frac{a_\varepsilon}{\ln\frac{R}{s_\varepsilon}}\int_\Omega g(y,x)\big(U_{\varepsilon,x,a_\varepsilon}-a_\varepsilon\big)_+^p\\
=&\frac{\pi(p+1)}{2}\varepsilon^2\left(\frac{a_\varepsilon}{\ln\frac{R}{s_\varepsilon}}\right)^2+2\pi\varepsilon^2\frac{a_\varepsilon}{\ln\frac{R}{s_\varepsilon}}(\kappa-\lambda_\varepsilon\eta(x))+O\left(\frac{\lambda_\varepsilon s_\varepsilon^3}{|\ln\varepsilon|^p}\right).
\end{aligned}$$

On the other hand, by (19)

$$\begin{aligned}\int_\Omega \left(PU_{\varepsilon,x,a_\varepsilon}-\kappa+\lambda_\varepsilon\eta\right)_+^{p+1} &= \int_\Omega \left(U_{\varepsilon,x,a_\varepsilon}-a_\varepsilon+O(\lambda_\varepsilon s_\varepsilon)\right)_+^{p+1}\\ &= \int_\Omega \left(U_{\varepsilon,x,a_\varepsilon}-a_\varepsilon\right)_+^{p+1}+O\left(\frac{\lambda_\varepsilon s_\varepsilon^3}{|\ln\varepsilon|^p}\right)\\ &= \frac{\pi(p+1)}{2}\varepsilon^2\left(\frac{a_\varepsilon}{\ln\frac{R}{s_\varepsilon}}\right)^2+O\left(\frac{\lambda_\varepsilon s_\varepsilon^3}{|\ln\varepsilon|^p}\right).\end{aligned}$$

Then we get that

$$I(PU_{\varepsilon,x,a_\varepsilon})=\frac{\pi(p-1)}{4}\varepsilon^2\left(\frac{a_\varepsilon}{\ln\frac{R}{s_\varepsilon}}\right)^2+\pi\varepsilon^2\frac{a_\varepsilon}{\ln\frac{R}{s_\varepsilon}}(\kappa-\lambda_\varepsilon\eta(x))+O\left(\frac{\lambda_\varepsilon s_\varepsilon^3}{|\ln\varepsilon|^p}\right).$$

Therefore, since λ_ε satisfies (H_ε), it is not difficult from Remark 1 to see that

$$I(PU_{\varepsilon,x,a_\varepsilon})=C_\varepsilon+\frac{\pi\kappa\varepsilon^2}{A_\varepsilon}\left(2\lambda_\varepsilon\frac{\partial\eta(\hat{x})}{\partial\nu}d(x)+\frac{\kappa}{|\ln\varepsilon|}\ln\frac{1}{2d(x)}\right)+O\left(\frac{\varepsilon^2}{|\ln\varepsilon|^{2+\sigma}}\right),$$

where $\sigma>0$ is a small constant, A_ε is given in Remark 1, and

$$C_\varepsilon=\frac{\pi\varepsilon^2}{A_\varepsilon}\left(\frac{(p-1)\kappa^2}{4A_\varepsilon}+(\kappa-\lambda_\varepsilon)\Big((1+\frac{\ln R}{|\ln\varepsilon|})\kappa-\lambda_\varepsilon\Big)\right).$$

□

Proof of Theorem 1.3. By Lemma 4.1 and Lemma 4.2, we have that for $x\in S$,

$$F(x)=C_\varepsilon+\frac{\pi\kappa\varepsilon^2}{A_\varepsilon}\left(2\lambda_\varepsilon\frac{\partial\eta(\hat{x})}{\partial\nu}d(x)+\frac{\kappa}{|\ln\varepsilon|}\ln\frac{1}{2d(x)}\right)+O\left(\frac{\varepsilon^2}{|\ln\varepsilon|^{2+\sigma}}\right). \tag{42}$$

Consider the following minimization problem

$$\min\{F(x):\, x\in\bar{S}\}. \tag{43}$$

Let

$$\rho(x)=2\lambda_\varepsilon\frac{\partial\eta(\hat{x})}{\partial\nu}d(x)+\frac{\kappa}{|\ln\varepsilon|}\ln\frac{1}{2d(x)}. \tag{44}$$

The Hopf lemma shows that for any $y\in\partial O_0$, $\frac{\partial\eta(y)}{\partial\nu}>0$. Define

$$m_1=\frac{\kappa}{2}\min_{y\in\partial\Omega}\{(\frac{\partial\eta(y)}{\partial\nu})^{-1}\},\quad m_2=\frac{\kappa}{2}\max_{y\in\partial\Omega}\{(\frac{\partial\eta(y)}{\partial\nu})^{-1}\},$$

and choose $r_1,r_2\in\mathbb{R}$, such that $0<r_1<m_1\le m_2<r_2$. Then there exists $x_0\in S$, such that

$$r(x_0)=\frac{\kappa}{2}\left(\frac{\partial\eta(\hat{x}_0)}{\partial\nu}\right)^{-1},$$

where $r(x)$ is given in (15). So

$$\begin{aligned}\rho(x_0)=&2\lambda_\varepsilon\frac{\partial\eta(\hat{x}_0)}{\partial\nu}d(x_0)+\frac{\kappa}{|\ln\varepsilon|}\ln\frac{1}{2d(x_0)}\\ =&\frac{\kappa}{|\ln\varepsilon|}\left(1+\ln\frac{1}{2r(x_0)}+\ln(\lambda_\varepsilon|\ln\varepsilon|)\right).\end{aligned} \tag{45}$$

For $x \in \partial S$, $r(x) = r_1$, we have

$$\begin{aligned}\rho(x) =& 2\lambda_\varepsilon \frac{\partial \eta(\hat{x})}{\partial \nu} d(x) + \frac{\kappa}{|\ln \varepsilon|} \ln \frac{1}{2d(x)} \\ =& \frac{2}{|\ln \varepsilon|} \frac{\partial \eta(\hat{x})}{\partial \nu} r_1 + \frac{\kappa}{|\ln \varepsilon|} \ln \frac{1}{2r_1} + \frac{\kappa}{|\ln \varepsilon|} \ln(\lambda_\varepsilon |\ln \varepsilon|) \\ \geq& \frac{\kappa}{|\ln \varepsilon|} \left(\frac{r_1}{m_2} + \ln \frac{1}{2r_1} + \ln(\lambda_\varepsilon |\ln \varepsilon|) \right) \\ >& h(x_0),\end{aligned} \tag{46}$$

if $0 < r_1 < m_1$ small enough.

Similarly, for $x \in \partial S$, $r(x) = r$, we have

$$\begin{aligned}\rho(x) \geq& \frac{\kappa}{|\ln \varepsilon|} \left(\frac{r_2}{m_2} + \ln \frac{1}{2r_2} + \ln(\lambda_\varepsilon |\ln \varepsilon|) \right) \\ >& h(x_0),\end{aligned} \tag{47}$$

if $r_2 \geq m_2$ large enough.

Therefore, from (42), (46) and (47), for any $x \in \partial S$, we have

$$F(x_0) < F(x).$$

Thus there is a minimum point $x_\varepsilon \in S$ of $F(x)$ in S, which is a critical point. As a result, $u_\varepsilon = PU_{\varepsilon,x_\varepsilon,a_\varepsilon} + \omega_{\varepsilon,x_\varepsilon}$ is a solution of (5).

By our construction, we find

$$B_{s_\varepsilon(1-Ls_\varepsilon^\sigma)}(x_\varepsilon) \subset \{y \in \Omega : u_\varepsilon(y) - \kappa + \lambda_\varepsilon \eta(y) \geq 0\} \subset B_{s_\varepsilon(1+Ls_\varepsilon^\sigma)}(x_\varepsilon),$$

and as $\varepsilon \to 0$, $dist(x_\varepsilon, \partial O_0) \to 0$.

We remark that if there is a minimum interval I of $\frac{\partial \eta(x(s))}{\partial \nu}$ for $x(s) \in \partial O_0$, then we can find a critical point x_ε of F, which nears $x(I)$. □

Appendix A. **The estimate for the radius of vortex core.** In this appendix, we give the estimates for the radius of vortex core. The proofs of such results can be found in [6] Lemma A.1.

In the following, we assume that $x \in S$, where S is given in (14).

Lemma A.1. *Let $0 < \sigma < 1$ be a constant. Then there are $\varepsilon_0 > 0$ and $L > 0$ large enough, such that for any $\varepsilon \in (0, \varepsilon_0)$,*

$$PU_{\varepsilon,x,a_\varepsilon}(y) - \kappa + \lambda_\varepsilon \eta(y) > 0, \quad y \in B_{s_\varepsilon(1-Ls_\varepsilon^\sigma)}(x),$$

while

$$PU_{\varepsilon,x,a_\varepsilon}(y) - \kappa + \lambda_\varepsilon \eta(y) < 0, \quad y \in \Omega \setminus B_{s_\varepsilon(1+Ls_\varepsilon^\sigma)}(x).$$

Lemma A.2. *Suppose that ω satisfies*

$$\|\omega\|_{L^\infty(\Omega)} = O(s_\varepsilon).$$

Then there is $L > 0$ large enough, such that

$$PU_{\varepsilon,x,a_\varepsilon}(y) + \omega(y) - \kappa + \lambda_\varepsilon \eta(y) > 0, \quad y \in B_{s_\varepsilon(1-Ls_\varepsilon^\sigma)}(x),$$

while

$$PU_{\varepsilon,x,a_\varepsilon}(y) + \omega(y) - \kappa + \lambda_\varepsilon \eta(y) < 0, \quad y \in \Omega \setminus B_{s_\varepsilon(1+Ls_\varepsilon^\sigma)}(x).$$

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Received for publication August 2021; early access October 2021.

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