ON THE NUMBER OF CRITICAL POINTS OF SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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Dedicated to Norman Dancer, a gentleman of mathematical analysis.

ABSTRACT. In this survey we discuss old and new results on the number of critical points of solutions of the problem

$$\begin{cases} -\Delta u = f(u) & in \ \Omega \\ u = 0 & on \ \partial\Omega \end{cases} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ is a smooth bounded domain. Both cases where u is a positive or nodal solution will be considered.

1. **Introduction.** This paper is a survey of old and new results about the number of critical points of solutions of the following problem,

$$\begin{cases} -\Delta u = f(u) & in \ \Omega \\ u = 0 & on \ \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ is a smooth bounded domain and f is a smooth (say C^1) nonlinearity. In all the paper we consider classical solutions $u \in C^2(\bar{\Omega})$.

This is a classical topic in PDEs, where many techniques and important results were developed in the literature (Morse theory, degree theory, etc.). Despite the great interest aroused by the problem, many questions are still unanswered and we are far from a complete understanding of the phenomenon.

We do not care about the condition on f for which (1.1) admits solutions, we just assume that such a solution u exists.

Observe that if $f \geq 0$, by the maximum principle we get that minimum points for u cannot occur and it turns out that u is positive.

The first part of this survey (Section 2) is devoted to the classification of critical points. As we will see, without some restriction on the class of solutions or on the nonlinearity f, no complete classification is available. Indeed, although (1.1) is an important constraint, "pathological" situations for the set of critical points of u cannot be excluded.

In next sections (3-5) we discuss some situations where it is possible to compute the number of critical points of the solution u.

Probably the only "easy" case appears when we consider *radial* solutions to (1.1) and Ω is a ball. In this case the condition $f \geq 0$ gives immediately (up to an

integration) that the solution is positive and strictly radially decreasing and so the uniqueness of the critical point of u follows (of course its maximum).

For a general domain Ω the situation becomes much more delicate. It is well known that this problem is strictly related to topological and geometrical properties of the domain.

A first relationship is clearly highlighted in the following beautiful *Poincaré-Hopf Theorem* which we state in the particular case where Ω is a bounded smooth domain of $\mathbb{R}^N$. Here and in the rest of the paper $B(y,r)$ denote the ball centered at y and radius r and $deg\big(v, B(x,\delta), 0\big)$ the classical Brouwer degree of a vector field v.

Theorem A (Poincaré-Hopf Theorem). *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain. Let v be a vector field on Ω with isolated zeroes $x_1, .., x_k$ and such that $v(x) \cdot \nu(x) < 0$ for any $x \in \partial\Omega$ (here ν is the outward normal vector to $\partial\Omega$). Then we have the formula*

$$\sum_{i=1}^{k} index_{x_i}(v) = (-1)^N \chi(\Omega), \tag{1.2}$$

where $index_x(v) = deg\big(v, B(x,\delta), 0\big)$ with small fixed $\delta > 0$ and $\chi(\Omega)$ is the Euler characteristic of Ω.

Choosing $v = \nabla u$ in Theorem A we get a beautiful link between an *analytic* problem (to compute the number of critical points of u) and a *topological* invariant (the Euler characteristic of Ω).

If u is a solution to (1.1) we can try to get by (1.2) more precise information about the number (and the type) of its critical point. For example it is interesting to understand when the number k in (1.2) is the lowest possible compatible with the Euler characteristic of Ω. This leads us to consider different situations depending on the topology of Ω. In Section 3 we start to consider *contractible* domains Ω.

Historically this is the first case that has been studied. In particular, if Ω is convex then $\chi(\Omega) = 1$ in (1.2) and the uniqueness of the critical point ($k = 1$ in (1.2)) of u is expected. If we add the symmetry hypothesis to Ω, an optimal answer to this question is given by the fundamental result by Gidas, Ni and Nirenberg [18].

Theorem 1.1 (**Gidas, Ni, Nirenberg**). *Let $\Omega \subset \mathbb{R}^n$ a bounded, smooth domain which is symmetric with respect to the plane $x_i = 0$ for any $i = 1, .., n$ and convex in the x_i direction for $i = 1, .., n$. Suppose that u is a positive solution to (1.1) where f is a locally Lipschitz nonlinearity. Then*

- *u is symmetric with respect to $x_1, .., x_n$. (Symmetry)*
- *$\frac{\partial u}{\partial x_i} < 0$ for $x_i > 0$ and $i = 1, \ldots, n$. (Monotonicity)*

Let us recall that Ω is convex in the direction x_1 (say) if whenever $P = (p_1, x') \in \Omega$ and $Q = (q_1, x') \in \Omega$ then the line segment $\overline{PQ}$ is contained in Ω.

An easy consequence of the symmetry and monotonicity properties in the previous theorem is that

$$\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} < 0 \ \ \forall x \neq 0$$

that is all the superlevel sets are *starshaped* with respect to the origin. Of course this last property implies that the origin is the unique critical point of u. Let us note that the symmetry of the domain is a crucial assumption in the proof of the monotonicity property. Although it is expected that it is not necessary and that

the uniqueness of the critical point (as well as the starlikeness of superlevel sets) holds without this assumption, this is a very difficult hypothesis to remove.

In Section 3 we assume that the solution u is positive and state some results where the symmetry assumption is removed and the solution u admits a unique critical point. The majority of the results appear in the planar case $\Omega \subset \mathbb{R}^2$, the case of higher dimensions $N \geq 3$ is much more complicated.

In Section 4 we consider domains Ω with rich topology, mainly with many *holes* (we again consider the case $u > 0$). In the last few decades there has been a great deal of interest in problems like (1.1) in domains with nontrivial topology, for example where f is superlinear or with exponential growth. However, the vast majority of these papers is related to the existence of solutions. Much less is known about the shape of the solution and its number of critical points.

If Ω has a "rich" topology then $\chi(\Omega)$ in (1.2) can be different of 1. For example if $D = B(0,1) \setminus B(x_0, r)$ with $\overline{B(x_0, r)} \subset B(0,1)$ then we have that

$$\chi(D) = \begin{cases} 0 & \text{if } N \text{ is even} \\ 2 & \text{if } N \text{ is odd} \end{cases} \tag{1.3}$$

Since we always have a *maximum* point then by (1.2) we deduce the existence of at least an *additional* critical point for u (note that this result could be also deduced by using the Lusternik-Schnirelmann category). As in Section 3 it is natural to ask when we have a *minimal* number k, i.e. in which cases the solution u has exactly two critical points. It will be discussed in Section 4.

In Section 5 we consider sign changing solutions. In this case there are few results in the literature (for example Gidas-Ni-Nirenberg Theorem is not applicable). For this reason we focus with a basic but classical problems, namely the eigenfunctions of $-\Delta$ with Dirichlet boundary conditions. Here the number of critical points is strongly affected by the set of *zeros* of the eigenfunctions. The structure of this set has been of great interest in the literature and is a problem far to be completely understood. For this reason we discuss the particular case of the "second" eigenfunction where $\Omega \subset \mathbb{R}^2$.

Section 5 starts with a discussion on the shape of the nodal line of the second eigenfunction, including the history of the problem and some conjectures.

Next we focus on the case where $\Omega \subset \mathbb{R}^2$ is a convex domain with large *eccentricity* (see (5.2) for the definition of *eccentricity* of a domain). About this domains there are some important results by Jerison and Grieser-Jerison ([26] and [21]) where it is proved that the nodal line of the second eigenfunction approaches a *segment*. These results will allow the computation of the critical points.

2. **Classification of critical points.** A complete classification of the critical points of a smooth function is not an easy task. Indeed we can have very "bad" behaviors, as for example in a celebrate example by Whitney [41] where a non-constant solution of class C^1 with a connected set of critical values is constructed.

If we assume that u is a solution to (1.1) we can remove some "pathological" situations, but even in this case we do not have a clear scenario.

Major difficulties in any possible classification arise because the critical points do not need to be isolated. In this setting, up to some restrictions, some results were obtained in [4]. Note that $non-isolated$ critical points naturally appear in simple situations, it is enough to think of positive radial solutions to (1.1) in annuli.

For this reason we focus on the more manageable case of *isolated* critical points. A great vantage is that in this case it is possible to define the *index* of a critical point, namely

$$index_P(\nabla u) = deg\big(\nabla u, B(P,\delta), 0\big) \tag{2.1}$$

where $deg\big(\nabla u, B(P,\delta), 0\big)$ is the classical Brouwer degree of ∇u in a small ball centered at P. This is the case where Theorem A in the Introduction holds.

The simplest case, i.e. where the index can be easily computed, is that of $non-$ $degenerate$ critical points. This means that the Hessian matrix of u at P, denoted by $H(u)(P)$, satisfies $\det H(u)(P) \neq 0$. So we have that,

- If the quadratic form associated to $H(u)(P)$ is definite *positive* we have that P is a *minimum* point to u and its index is 1.
- If the quadratic form associated to $H(u)(P)$ is definite *negative* we have that P is a *maximum* point to u and its index is $(-1)^N$.
- If the quadratic form associated to $H(u)(P)$ is indefinite we have that P is a *saddle* point to u and its index is ± 1 according to the shape of u and the dimension N.

See [33] for other properties of nondegenerate critical points. On the other hand if we allow to the critical point to be *degenerate* (for example the case on non-isolated critical points), then it is impossible to have a classification as the previous one. Even in the case where u is a polynomial there are delicate problems to solve ([37]).

Adding some additional conditions to u there are some results in [39] concerning the computation of the index but we would like to mention the results in [3] where a classification of critical points of suitable functions in the plane is provided.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and let its boundary $\partial\Omega$ be composed of k closed simple smooth curves. Let u be a non-negative solution of* (1.1) *such that $f(t) > 0$ if $t > 0$. If P is an isolated critical point of u then the following alternative holds,*

- *P is a maximum point with $index_P(\nabla u) = 1$.*
- *There exist a positive integer $L \leq 2$ and a neighborhood U of P such that the level set $\{(x, y \in U : u(x,y) = u(P)\}$ consists of L curves which cross at P only. We have $index_P(\nabla u) = 1 - L$.*

Moreover if $L = 1$ we call P as a trivial point, if $L = 1$ as a simple saddle point. Finally,

$$\sharp\{maxima\ of\ u\} - \sharp\{saddles\ of\ u\} = 2 - k \tag{2.2}$$

Proof. See Section 3 in [3]. □

The *trivial* critical points are the most unpleasant. They are difficult to manage, both because they have index 0 and do not induce a change of topology in the superlevel set of u. Unfortunately they can appear in simple cases, as the following example shows,

Example 2.2. Let us consider, for $z = x + iy$,

$$u(x,y) = c - \frac{y^2}{2} - \big(x^3 - 3xy^2\big) - 12\big(x^4 - 6x^2y^2 + y^4\big) = c - \frac{y^2}{2} - Re(z^3) - 12Re(z^4)$$

Some tedious but not difficult computations show that $D = \big\{(x,y) \in \mathbb{R}^2 : u(x,y) > 0\big\}$ is a smooth bounded domain for c small enough (see fig.1). A straightforward

computation shows that u verifies

$$\begin{cases} -\Delta u = 1 & in\ D \\ u = 0 & on\ \partial D. \end{cases} \tag{2.3}$$

Finally we have that u admits two critical points P_1 and P_1 in D; $P_1 = \left(-\frac{1}{16}, 0\right)$ is a maximum point and $P_2 = (0,0)$ is a trivial point.
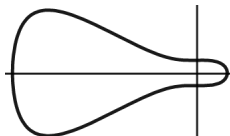

FIGURE 1. A picture of D with $c = \frac{1}{300000}$

We end this section discussing the conditions ensuring that all critical points of u are isolated. Usually analyticity of f (and so of u) is a good assumption but, as remarked before, the geometry of the domain is crucial. A sufficient condition is stated below,

Theorem 2.3. *Let u be a positive solution to* (1.1) *and suppose that Ω is simply connected, f is analytic and satisfies $f(t) > 0$ if $t > 0$. Then the critical points of u are isolated, and we have*

$$\sharp\{maxima\ of\ u\} - \sharp\{saddles\ of\ u\} = 1 \tag{2.4}$$

Proof. See Corollary 3.4 in [3]. □

3. **The case where Ω is a contractible domain.** This is actually the first case studied in the literature. Here we have that $\chi(\Omega) = 1$ and so (1.2) becomes

$$\sum_{i=1}^{k} index_{x_i}(\nabla u) = (-1)^N. \tag{3.1}$$

Of course since u is a solution to (1.1), we always have a maximum point for u whose index is $(-1)^N$. Analogously as in the previous section we address the "minimality" of the set of critical points of u. So our question is now

when does the sum in (3.1) *reduce to a singleton?* (3.2)

The results of this section are related with the quasi-concavity property of a function. We recall that a function is called quasiconcave if its superlevel sets are all convex. A beautiful result on the quasiconcavity of solutions to (1.1) concern the classical case of the torsion problem,

Theorem 3.1 (Makar-Limanov [31])**.** *Let u a positive solution to*

$$\begin{cases} -\Delta u = 1 & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{3.3}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth convex domain. Then the level set of u are strictly convex (in particular u has only one critical point).

The proof of the previous theorem is carried out introducing the function

$$I(x) = -u_{xx}u_y^2 + 2u_{xy}u_xu_y^2 - u_{yy}u_x^2 + 2u\underbrace{(u_{xx}u_{xx} - u_{xy}^2)}_{=\det Hess(u)} \tag{3.4}$$

$$= 8u^2 \det(Hess(\sqrt{u})).$$

A straightforward computation and the convexity assumption on Ω show that

$$\begin{cases} -\Delta I = 8\,u\,\underbrace{\left(u_{xxx}^2 + u_{yyy}^2\right)}_{\geq 0} & \text{in } \Omega \\ I \geq 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

From (3.5) the claim follows (see [31] for the details). In particular $\sqrt{u}$ is strictly concave. A natural question is the following,

Question. *Is it possible to extend Theorem 3.1 to higher dimensions?*

Next result prove the sharpness of Theorem 3.1. It will be proved that the same result is not true if we replace the RHS by more general nonlinearities $f(u)$.

Theorem 3.2 (**Hamel, Nadirashvili, Sire** [24]). *In dimension $N = 2$ there are some smooth bounded convex domains Ω and some C^∞ functions $f : [0, +\infty) \to \mathbb{R}$ for which problem (1.1) admits a solution u which is not quasiconcave.*

The domain in the previous theorem is a "like-stadium " domain, where in Fig. 2 the number a will be chosen large. Note that Ω_a verifies the assumptions in Gidas-
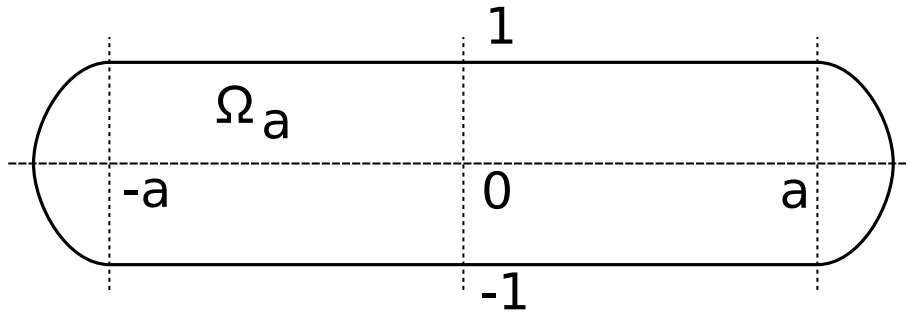

FIGURE 2. The domain Ω in Theorem 3.2

Ni-Nirenberg Theorem and then any solution u of (1.1) in Ω_a has only one critical point. In order to construct the example in Theorem 3.2, the authors consider a suitable nonlinearity $f \geq 1$ such that (1.1) admits *two* solutions in Ω_a for a large. One of this solution is *quasiconcave* and the other one is that of the claim of Theorem 3.2.

On the other hand, for other nonlinearities f the counter-example in Theorem 3.2 does not hols.

Indeed the convexity of the level set was proved for the first eigenfunction of the Laplacian in strictly convex bounded domain by Brascamp and Lieb [7] (see also Acker, Payne and Philippin [1] and Korevaar [27]). As a consequence of these results the following question naturally arises,

Question. *Find sufficient conditions on Ω and f such that the positive solution of (1.1) to be semi-concave.*

Next we investigate how to generalize the Makar-Limanov result in order to get the uniqueness of the critical point. Of course, by Gidas-Ni-Nirenberg Theorem we are interested in domains which are **not** symmetric.

A good class of solutions to extend our result is that of the *semi−stable* solutions. We recall that a solution u to (1.1) is semi-stable if the linearized operator at u admits a nonnegative first eigenvalue. Minima of the functional

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega F(u)$$

where F is the primitive of f are typical examples of semi-stable solutions. An important result concerning this class of solution is the following,

Theorem 3.3 (**Cabré, Chanillo** [8]). *Assume Ω is a smooth, bounded and convex domain of $\mathbb{R}^2$ whose boundary has positive curvature. Suppose $f \geq 0$ and u is a semi-stable positive solution to (1.1). Then u has a unique critical point, which is non-degenerate.*

This result was extended to the case of *non − negative* curvature in [15].

The proof of Theorem 3.3 relies on a careful study of the nodal lines of the derivatives $\frac{\partial u}{\partial \theta} = \cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y}$, $\theta \in [0, 2\pi)$. In particular, the semi-stability conditions implies that $\nabla\left(\frac{\partial u}{\partial \theta}\right) \neq 0$. Finally a topological argument gives the claim of the theorem.

There are very interesting (and popular!) questions about the extension of the last result. We mention some of them, they seem quite hard to solve.

Question. *Is it possible to extend Cabré-Chanillo's Theorem to solutions of Morse index 1 (i.e. the first eigenvalue of the linearized operator at u is negative and the second non-negative)? Is it true if $f(s) = s^p$ with $p > 1$?*

Question. *What about Cabré-Chanillo's Theorem in higher dimensions $N \geq 3$?*

A partial contribution to this question was given in [15] where it was constructed a solution to (1.1) of the torsion problem (3.5) where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, its boundary $\partial\Omega$ has *positive mean curvature* and the solution u as k maximum points. Our feeling is that the correct request to extend the Cabré-Chanillo's Theorem is that all the *principal curvatures* are positive.

Regarding again the higher dimensions $N \geq 3$, the only known results concern special nonlinearities as perturbations of critical powers (see [23] or [20]). In these cases the uniqueness of the critical point is proved as well as some information about the geometry of the level sets.

Our next investigation concerns what happens if the curvature of $\partial\Omega \subset \mathbb{R}^2$ is *negative somewhere* (recall that we are considering *contractible domains*). Next result shows that it is substantially impossible to have some similar result as in Theorem 3.1.

Theorem 3.4. *(see [19]) For any integer $k \geq 2$ there exists a family of smooth bounded domains $\Omega_\varepsilon \subset \mathbb{R}^2$ and smooth functions $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}^+$ which solves the torsion problem*

$$\begin{cases} -\Delta u_\varepsilon = 1 & in \ \Omega_\varepsilon \\ u_\varepsilon = 0 & on \ \partial\Omega_\varepsilon \end{cases} \tag{3.6}$$

such that for ε small enough,

- Ω_ε is starshaped with respect to an interior point.
- For suitable $c > 0$ the set $\{u_\varepsilon > c\}$ is non-empty and has at least k connected components; in particular u_ε has at least k maximum points.
- If S is the strip $S = \{(x, y) \in \mathbb{R}^2 \text{ such that } |y| < 1\}$ and Q is any compact set of $\mathbb{R}^2$ then $\Omega_\varepsilon \cap Q \xrightarrow[\varepsilon \to 0]{} S \cap Q$.
- The curvature of $\partial\Omega_\varepsilon$ changes sign and vanishes exactly at two points. Moreover $\min \left(Curv_{\partial\Omega_\varepsilon} \right) \xrightarrow[\varepsilon \to 0]{} 0$.

A picture of Ω_ε for $k = 2$ and ε small is given in Fig.1. Hence the last results

FIGURE 3. Domain Ω_ε with $k = 2$ and level set $\{u_\varepsilon = c\}$

says that the convexity assumption in Makar-Limanov result cannot be relaxed. The domain Ω_ε turns out to be a *bounded perturbation of a strip* and the solution u_ε has the following representation,

$$u_\varepsilon(x, y) = \underbrace{\frac{1}{2}(1 - y^2)}_{\text{solution in the strip } s} + \underbrace{\varepsilon h(x, y)}_{\text{homogeneous harmonic polynomial}}$$

Some generalization of this result to higher dimension as well as to more general nonlinearities f are considered in [14].

On the other hand it is interesting to investigate domains with boundary with *negative* curvature and try to deduce sufficient conditions which imply the uniqueness of the critical point.

We end this section raising some questions concerning solutions of the *Poisson* problem,

$$\begin{cases} -\Delta u = f & in \ \Omega \subset \mathbb{R}^N, \quad N \geq 3 \\ u = 0 & on \ \partial\Omega \end{cases} \tag{3.7}$$

Question. *What conditions on f and Ω imply the uniqueness of the critical point of the solution u?*

Strangely this question has an affirmative answer in very few cases:

- $f \equiv 1$ with $\Omega \subset \mathbb{R}^2$ convex (Makar-Limanov)
- f radially decreasing in the ball (Gidas-Ni-Nirenberg Theorem)
- f positive and radial in the ball (a simple integration gives the claim)

4. **The case of domains with holes.** In this section we assume $f \geq 0$. It is known that the presence of holes in the domain increases the number of critical points of the function u. A classic result by Lusternik and Schnirelmann is the following,

Theorem 4.1. *Let $u : \Omega \to \mathbb{R}$ be a smooth function such that $u = 0$ on $\partial\Omega$. Then we have*

$$\sharp\{\text{critical points of } u\} \geq cat(\Omega) \tag{4.1}$$

where $cat(\Omega)$ is the Lusternik-Schnirelmann category of Ω.

Let us recall that the Lusternik-Schnirelmann category of Ω is the smallest number k such that there is an open covering U_k of Ω with each U_i is contractible in Ω. Of course if Ω is a domain with at least one hole we have that $cat(\Omega) \geq 2$. We stress that no non-degeneracy assumption is required to the critical points of u.

As will be proven shortly, the estimate (4.1) is sharp if Ω has *one* hole. On the other hand it is not satisfactory if there are many holes in Ω.

In the last few decades, there has been a lot of work to prove the existence of multi-peak solutions to (1.1) for various nonlinearities f where Ω is a domain with holes. It is impossible to provide en exhaustive literature on this subject, we just recall a result concerning the critical Sobolev exponent,

Theorem 4.2. *[(see [38] or [28])] Let us assume that $f(s) = s^{\frac{N+2}{N-2}}$ and Ω a domain with k holes. Then there exists k positive solutions which blow-up like a volcano at the center of the hole.*

Similar results were obtained in a great variety of different problems; it is impossible to provide an exhaustive list of references, we just mention the papers [5], [9], [10], [13], [16], [17], [34] and the references therein. One interesting question is the following,

Question. *Let us consider the multi-peak solutions in [38] and [28] (or other analogous problems). What about the exact number of critical points?*

A partial answer to the previous question can be done in the simpler case when Ω has a small hole. In some suitable situations we will show that the corresponding solution admits exactly *two* critical points. Analogously as in the previous section we get some kind of minimality of k in the Poincaré-Hopf Theorem.

Denote by u_ε a solution to

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega_\varepsilon, \\ u > 0 & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{4.2}$$

where

$$\Omega_\varepsilon = \Omega \backslash B(P, \varepsilon) \text{ with } P \in \Omega \text{ and } \varepsilon \text{ small.}$$

Additionally we require that the solution u_ε is uniformly bounded in Ω_ε, i.e.,

$$0 < u_\varepsilon \leq C \text{ in } \Omega_\varepsilon \text{ with } C \text{ independent of } \varepsilon. \tag{4.3}$$

By the standard regularity theory, extending u_ε to 0 in $B(P, \varepsilon)$, we get that $u_\varepsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. In this setting we observe that, if P is not a critical point of u_0, then (1.2) becomes

$$\sum_{i=1}^k index_{x \in \left(B(P,d) \backslash B(P,\varepsilon)\right)} u_\varepsilon(x) = -1. \tag{4.4}$$

where d is sufficiently small such that no critical point lies in $B(P, d) \backslash B(P, \varepsilon)$. Next result (see [22]) claims that the sum in (4.4) reduces to a singleton.

Theorem 4.3. *Suppose that u_ε is a solution to (4.2) which verifies (4.3) and u_0 its weak limit. We have that if*

$$P \text{ is not a critical point of } u_0, \tag{4.5}$$

then for ε small enough there is exactly one critical point for u_ε in $B(P,d) \setminus B(P,\varepsilon)$ (here $B(P,d) \subset \Omega$ is chosen not containing any critical point of u_0). Moreover the critical point $x_\varepsilon \in B(P,d)$ of u_ε is a saddle point of index -1 which verifies

$$u_\varepsilon(x_\varepsilon) \to u_0(P). \qquad (4.6)$$

Let us make some comments to the previous result.

- The crucial step in the previous result relies in a careful analysis of the solution u_ε near the boundary of $B(P,\varepsilon)$. In this region the standard regularity theory cannot be applied.
- The condition that P is not a critical point of u_0 cannot be removed. An easy counterexample can be constructed when $\Omega = B(0,1)$ and u_0 is the first eigenfunction of $-\Delta$ with zero Dirichlet boundary condition. If $P = 0$ we have that $\Omega_\varepsilon = B(0,1) \setminus B(0,\varepsilon)$ and u_ε is the first *radial* eigenfunction in the annulus Ω_ε. Of course u_ε has infinitely many critical points in $B(P,d) \setminus B(P,\varepsilon)$ for any $\varepsilon > 0$ small and $d \in (0,1)$.
- If u_0 has k *nondegenerate* critical points then Theorem 4.3 says that any solution u_ε to (4.2) has *exactly* $k+1$ critical points. A corollary is that if Ω is a Gidas-Ni-Nirenberg domain then any solution u_ε to (4.2) (satisfying (4.3)) has *exactly* two critical points in $\Omega \setminus B(P,\varepsilon)$, for any $P \neq 0$. The same happens for the first eigenfunction of the Laplacian.

Next we discuss what happens if in the previous result we have that P is a critical point of u_0. In this case we need to impose some additional assumption. They are contained in the next

Theorem 4.4. *Assume that Ω is a Gidas-Ni-Nirenberg domain and u_ε, u_0 as in Theorem 4.3.*

If O is a nondegenerate critical point of u_0, for small ε we have following results.

- *If all the eigenvalues of the hessian matrix $H(O)$ are simple then for ε small enough we have that*

$$\sharp\{\text{critical points of } u_\varepsilon \text{ in } \Omega_\varepsilon\} = 2N$$

and $P_{1,\varepsilon}^+, P_{1,\varepsilon}^-, .., P_{N,\varepsilon}^+, P_{N,\varepsilon}^- \to O$ (the precise asymptotic behavior can be found in [22])

- *If at least one negative eigenvalue of $\boldsymbol{H}(P)$ is multiple, then*

$$\sharp\{\text{critical points of } u_\varepsilon\} \geq 2N.$$

In Theorems 4.3 and 4.4 the smallness of the hole is an assumption that cannot be removed. Known results in the literature (see [29] for example) claim the existence with a lot of critical points in domains with *large* holes.

The assumption (4.3) prevents the blow-up of the solution u_ε. This is typically verified in the *subcritical* regime, i.e. for nonlinearity which behaves like $f(s) = s^p$, $1 < p < \frac{N+2}{N-2}$. In the *critical or supercritical* regime like Theorem 4.2 the previous theorems cannot be applied. On the other hand we suspect that similar results hold.

5. **Sign changing solutions to** (1.1)**.** If one considers solutions to (1.1) that change sign, the literature on the number of critical points is much poorer. We just restrict our attention to the eigenfunctions of the Laplace operator in planar

domains, namely,

$$\begin{cases} -\Delta u = \lambda_k u & in \ \Omega \subset \mathbb{R}^2 \\ u = 0 & on \ \partial\Omega \end{cases} \tag{5.1}$$

where λ_k is the $k-th$ eigenvalue.

It is known that if $k \geq 2$ then u must change sign and the geometry and location of its nodal line $\Lambda = \overline{\{(x,y) \in \Omega : u(x,y) = 0\}}$ has addressed a lot of interest. We pointed out that, even in "very symmetric" cases, the nodal line of an eigenfunction can be very complicated. In Fig. 4 it is showed an example in the case of the rectangle (see Stern [40] and [6])
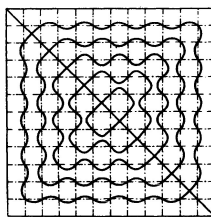

FIGURE 4. Nodal line of an eigenfuncion in the rectangle (Stern's PhD thesis)

So we focus our interest in the simplest case, the *second* eigenfunction, which we denote by u_2. By the Courant principle it is known that u_2 change sign once and so its nodal line Λ splits Ω it *two* parts. A longstanding conjecture is the following (see [35, 36, 42])

(C) *For which domains $\Omega \subset \mathbb{R}^2$ does the nodal line Λ touch $\partial\Omega$ at exactly two points?*

In other words one requires if Λ intersects $\partial\Omega$ or not. There was a lot work about this conjecture (see for instance [30, 12]) until in [32] where it was proved in convex domains (see [2] for non-smooth domains). The conjecture is not true in any domain: in [25] it was given an example of a domain whose boundary has a large number of connected components and the nodal line of the second eigenfunction does not touch the boundary. In the same paper it was conjectured that (C) holds in planar simply-connected domains. An interesting question raised in [25] is the following

Question. *What is the maximal connectivity of a domain for which the nodal line must hit the boundary?*

In [11] it was provided, using a computer assisted proof, an example of a planar domain with 6 holes such that the nodal line does not touch the boundary. As pointed out by the authors, it is likely that this is not the optimal number.

Of course the computation of the critical points of eigenfunctions to (1.1) is strongly influenced by the geometry of the nodal lines. If Λ is a closed curve contained in Ω we expect at least 3 critical points, otherwise 2 is the minimum number. For this reason we restrict our interest to the case of *convex* domains where the second alternative occurs.

However, even in this case, without no other restrictions, in the literature there are very few results about the qualitative properties of the nodal line and the corresponding eigenfunction. A good subclass of the convex domains are those with *large eccentricity.* Let us recall that the eccentricity of a planar domain is defined as

$$\mathrm{ecc}(\Omega) = \frac{\mathrm{diameter}(\Omega)}{\mathrm{inradius}(\Omega)} \tag{5.2}$$

where inradius(Ω) is the radius of the largest circle contained in Ω. These domains were considered by Jerison ([26]) and Grieser-Jerison ([21]) where the location of the nodal line Λ was characterized. In order to state their result we need to normalize the domain Ω in an appropriate way. First let us rotate Ω so that its projection on the y-axis has the shortest possible length, and then dilate so that this projection has length 1. Denote by M the length of the projection of Ω on the x-axis. Then $M \geq 1$, and M is essentially the diameter of Ω. From now we denote by $\Omega = \Omega_M$ a domain satisfying the previous properties.

Accordingly we denote by $u = u_M$ a solution to (1.1) in Ω_M and Λ_M its nodal line.

Note that in this setting the domain Ω_M is close to the strip (in a suitable way) $\Omega_\infty = \{ (x,y) \in \mathbb{R}^2 : 0 < y < 1 \}$. We have the following result.

Theorem 5.1 ([21, Theorem 1])**.** *There is an absolute constat C_0 such that the width of the nodal line Λ_M is at most C_0/M. In other words, up to translate Ω_M, one has*

$$(x,y) \in \Lambda_M \implies |x| < \frac{C_0}{M}.$$

This result is the starting point to compute the number of critical points of u_M in Ω_M. We have the following theorem (see [14]),

Theorem 5.2. *For M large enough, u_M has exactly two critical points $P_M, Q_M \in \Omega_M$. Moreover P_M (say) is a nondegenerate maximum point while Q_M is a nondegenerate minimum. Finally $|P_M|, |Q_M| \to +\infty$ as $M \to +\infty$.*

The proof of the previous theorem relies on the asymptotic behavior of the eigenfunction u_M. Indeed it is proved that, for some $c \in \mathbb{R}$, we have that $u_M(x,y) \to Cx \sin y$ uniformly on the compact set of Ω_∞. Note that this results imply that necessarily $|P_M|, |Q_M| \to +\infty$ as $M \to +\infty$. Finally a topological argument gives the uniqueness and nondegeneracy of these points.

The last question concerns the generalization of the previous result.

Question. *In which domains of the plane does the second Laplacian eigenfunction have exactly 2 critical points?*

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