

PULLBACK DYNAMICS OF A 3D MODIFIED NAVIER-STOKES EQUATIONS WITH DOUBLE DELAYS

PAN ZHANG AND LAN HUANG*

College of Mathematics and Statistics
North China University of Water Resources and Electric Power
Zhengzhou 450011, China

RUI LU

Department of Mathematics
China University of Mining and Technology
Beijing 100083, China

XIN-GUANG YANG

College of Mathematics and Information Science
Henan Normal University
Xinxiang 453007, China

(Communicated by Alain Miranville)

ABSTRACT. This paper is concerned with the tempered pullback dynamics for a 3D modified Navier-Stokes equations with double time-delays, which includes delays on external force and convective terms respectively. Based on the property of monotone operator and some suitable hypotheses on the external forces, the existence and uniqueness of weak solutions can be shown in an appropriate functional Banach space. By using the energy equation technique and weak convergence method to achieve asymptotic compactness for the process, the existence of minimal family of pullback attractors has also been derived.

1. Introduction. The three dimensional incompressible Navier-Stokes equations is expressed as

$$\begin{cases} \frac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0, \end{cases} \quad (1)$$

which was proposed by Navier and Stokes respectively for the motion of incompressible viscous fluid with very small velocity gradient. Since last century, many mathematicians and physicists have studied the existence, uniqueness, regularity and long time behavior of Navier-Stokes equations deeply, and have obtained a series of significant results (see [1, 3, 19, 21, 24, 29, 33]). However, for the 3D Navier-Stokes model, the uniqueness of weak solution and the existence of strong solutions have not been solved as our best knowledge.

2020 *Mathematics Subject Classification.* 35Q30, 35B40, 35B41, 76D03, 76D05.

Key words and phrases. 3D modified Navier-Stokes equations, double delays, tempered pullback attractors.

Research partly supported by the NSFC (No. 11501199 and No. 11871212) and the Young Key Teachers Project in Higher Vocational Colleges of Henan Province (No. 2020GZGG109).

* Corresponding author: Lan Huang.

In the 1960s, Ladyzhenskaya [22, 23] relaxed the limitation of small fluid velocity gradient, and derive several modified Navier-Stokes equations, one of which reads as

$$\begin{cases} u_t - \operatorname{div}[(\mu_0 + \mu_1 \|\nabla u\|_{L^2(\Omega)}^2) \mathcal{D}u] + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0, \\ \mathcal{D}u = \nabla u + \nabla u^T, \end{cases} \quad (2)$$

and reduced to

$$\begin{cases} \frac{\partial u}{\partial t} - (\mu_0 + \mu_1 \|\nabla u\|_{L^2(\Omega)}^2) \Delta u + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0 \end{cases} \quad (3)$$

by Lions, see [27]. Note that it reduces into the classical Navier-Stokes equations (1) when $\mu_1 = 0$. Lions [27] proved the existence and uniqueness of global weak solutions for the initial boundary value problem of the model (3) by using the Faedo-Galerkin method. In 2008, Dong and Jiang [10] studied the optimal upper and lower bounds of the decay for higher order derivatives of the solution of (3). Under some assumptions on the external force and initial data, the existence and structure of uniform attractors of the system (3) were proved in [9]. Recently, Yang and Feng et al [37] considered the pullback dynamics of (3), and proved the existence of minimal and unique family of pullback attractors, and also presented the finite fractal dimension of pullback attractors. In addition, the upper semi-continuity of pullback attractor was also studied in [37] when the perturbed external force disappears as parameter tends to zero.

The Navier-Stokes equations with delays were firstly considered by Caraballo and Real in [5], then there are a lot of works concerning asymptotic behavior, stability, the existence of pullback attractors and the fractal dimensional of pullback attractors for time-delayed Navier-Stokes equations (see, e.g., [6, 7, 15, 30, 38]). It is worth to be pointed out that García-Luengo, Marín-Rubio and Real [15] obtained that the existence of pullback attractors for the 2D Navier-Stokes model with finite delay. Furthermore, the bounded fractal and Hausdorff dimension of the pullback attractors for 2D non-autonomous incompressible Navier-Stokes equations with delay was studied in [38]. The above work is to study the time delay which only exists in the external force. Later on, Planas et al [17, 32] considered Navier-Stokes equations with double delays and proved the exponential stability of stationary solutions. For the double time-delayed 2D Navier-Stokes model, the existence of pullback attractors was proved in [12]. As far as we know, there are less results on the research for 3D modified Navier-Stokes equations with double time delays till we know now. Motivated by the results [12, 15, 37, 39], this paper is concerned with the pullback dynamics for a three dimensional modified Navier-Stokes equation with double time delays defined on $(\tau, \infty) \times \Omega$, which can be described as

$$\begin{cases} \frac{\partial u}{\partial t} - (\mu_0 + \mu_1 \|\nabla u\|_{L^2(\Omega)}^2) \Delta u + (u(t - \rho(t)) \cdot \nabla)u + \nabla p = f(x, t) + g(t, u_t), \\ \nabla \cdot u = 0, & (t, x) \in (\tau, \infty) \times \Omega, \\ u(t, x)|_{\partial\Omega} = 0, & t \in (\tau, \infty), \\ u(t, x)|_{t=\tau} = u^\tau(x), & x \in \Omega, \\ u_\tau(s, x) = u(\tau + s, x) = \phi(s, x), & s \in (-h, 0), x \in \Omega, \end{cases} \quad (4)$$

here $\Omega \in \mathbb{R}^3$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $u = (u_1, u_2, u_3)$ is the velocity field of the fluid, p is the pressure, $\mu_0 > 0$ and $\mu_1 > 0$ are

the kinematic viscosities of the fluid, $f(x, t)$ is a generic external force, $g(t, u_t)$ is an external force with some hereditary characteristics and $h > 0$ is a fixed positive constant. The function u_t appeared in the delay term $g(t, u_t)$ is defined on $(-h, 0)$ by the relation $u_t(s) = u(t + s)$, $s \in (-h, 0)$. Assume that the delay function ρ in the convective term satisfies $\rho(t) \in C^1(\mathbb{R}; [0, h])$ and $\rho'(t) \leq \rho^* < 1$ for all $t \in \mathbb{R}$, where ρ^* is a constant.

The main features and results of this paper can be summarized as follows:

(I) By using the Galerkin approximated technique and compact argument, we can prove the existence of global weak solution. Using the energy equation approach similar as in Ball [2], the pullback asymptotic compactness of the process can be shown, which leads to the existence of pullback attractors together with pullback dissipation of our problem. In addition, based on different universes, we can present the minimal family of pullback attractors on functional Banach space.

(II) Since the topic in [16] only contains the delay on external force, the result in this paper is a further extension of our former result [16]. Moreover, the problem (4) has a delay function ρ in the convection term, hence, there are more difficulties to achieve well-posedness and pullback asymptotic compactness, which also require appropriate restriction on the delay function ρ .

(III) The upper semi-continuity of pullback attractors as perturbed delay can not be shown easily similar as in [16] since the lack of regular estimate in appropriate phase spaces.

The structure of this paper is arranged as follows. In Section 2, we give the definitions of some usual functional spaces and operators. Moreover, Some lemmas used later are also given in the end of the section. In Section 3, we first give the needed assumptions on the external forces, then show the abstract equivalent form of the system (4), and establish the well-posedness of the global weak solutions for system (4). Finally, the existence of the minimal pullback attractors for the abstract non-autonomous system is showed in Section 4.

2. Mathematical setting.

2.1. Some usual functional spaces and operators. Denote

$$E := \{u | u \in (C_0^\infty(\Omega))^3, \operatorname{div} u = 0\}$$

and H is the closure of E in $(L^2(\Omega))^3$ topology. The inner product and norm in H are represented by (\cdot, \cdot) and $|\cdot|$ respectively, which are defined as

$$(u, v) = \sum_{i=1}^3 \int_{\Omega} u_i(x) v_i(x) dx, \quad |u|^2 = (u, u), \quad \forall u, v \in (L^2(\Omega))^3.$$

V is the closure of E in $(H_0^1(\Omega))^3$ topology. The inner product and norm in V are represented by $((\cdot, \cdot))$ and $\|\cdot\|$ respectively, which are defined as

$$((u, v)) = \sum_{i=1}^3 \int_{\Omega} \nabla u_i(x) \nabla v_i(x) dx, \quad \|u\|^2 = ((u, u)), \quad \forall u, v \in (H_0^1(\Omega))^3.$$

Then we have $\|u\| = \|\nabla u\|_{(L^2(\Omega))^3} = |\nabla u|$ for all $u \in V$, and it is easy to verify that H and V are Hilbert spaces. Let H' and V' be dual spaces of H and V respectively. Then there is $V \hookrightarrow H \equiv H' \hookrightarrow V'$, where the injections are dense and continuous.

Let $\|\cdot\|_*$ be the norm in V' , $\langle \cdot, \cdot \rangle$ be the dual product in V and V' , where $\|\cdot\|_*$ is defined as

$$\|f\|_* = \sup_{v \in V, \|v\|=1} |\langle f, v \rangle|, \quad \forall f \in V'. \quad (5)$$

P denotes the Helmholtz-Leray orthogonal projection from $(L^2(\Omega))^3$ onto the space H (see [11, 34]). We define $A_1 := -P\Delta$ as the Stokes operator on $D(A_1) = (H^2(\Omega))^3 \cap V$, then $A_1 : V \rightarrow V'$ satisfy $\langle A_1 u, v \rangle = ((u, v))$, and A_1 is an isomorphism from V into V' . There is $\|A_1 u\|_* = \sup_{v \in V, \|v\|=1} |\langle A_1 u, v \rangle| = \sup_{v \in V, \|v\|=1} |((u, v))| \leq \|u\|$, i.e., $\|A_1\| \leq 1$. Let $\{\lambda_i\}_{i=1}^\infty$ be the eigenvalues of the operator A_1 with Dirichlet boundary condition, which satisfies $0 < \lambda_1 \leq \lambda_2 \leq \dots$. By the property of the Stokes operator, the corresponding eigenfunctions $\{\omega_i\}_{i=1}^\infty$ form an orthonormal complete basis in H . In addition, at this time we have the following Poincaré inequality

$$|u|^2 \leq \frac{1}{\lambda_1} \|u\|^2, \quad \forall u \in V. \quad (6)$$

In order to deal with the nonlinear term $\mu_1 \|\nabla u\|_{L^2(\Omega)}^2$, we define the operator $A_2 : V \rightarrow V'$ as $A_2 u := -\mu_1 \|\nabla u\|_{L^2(\Omega)}^2 \Delta u$, which satisfies

$$\langle A_2 u, v \rangle = \mu_1 \|\nabla u\|_{L^2(\Omega)}^2 \langle -\Delta u, v \rangle = \mu_1 \|u\|^2 ((u, v)), \quad \forall u, v \in V. \quad (7)$$

It is easy to verify that $\langle A_2 u - A_2 v, u - v \rangle \geq 0$ for any $u, v \in V$, that is, A_2 is a monotone operator. We can obtain from (5) and (7) that

$$\|A_2 u\|_* = \sup_{v \in V, \|v\|=1} |\langle A_2 u, v \rangle| = \sup_{v \in V, \|v\|=1} \mu_1 \|u\|^2 ((u, v)) \leq \mu_1 \|u\|^3, \quad \forall u \in V. \quad (8)$$

We also introduce the bilinear operator

$$B(u, v) = P((u \cdot \nabla)v), \quad \forall u, v \in V$$

and trilinear operator

$$b(u, v, \omega) = (B(u, v), \omega) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} \omega_j dx, \quad \forall u, v, \omega \in V.$$

2.2. Some functional spaces with delay. We define some functional Banach space as

$$C_H = C([-h, 0]; H) \text{ with the norm } \|\phi\|_{C_H} = \sup_{s \in [-h, 0]} |\phi|$$

and some Lebesgue spaces on delayed interval as $L_H^p = L^p(-h, 0; H)$ and $L_V^p = L^p(-h, 0; V)$. The inner product and norm in L_H^2 are defined as

$$(u, v)_{L_H^2 \times L_H^2} = \int_{-h}^0 (u(s), v(s)) ds, \quad \|u\|_{L_H^2}^2 = \int_{-h}^0 |u(s)|^2 ds, \quad \forall u, v \in L_H^2.$$

The inner product and norm in L_V^2 are defined as

$$(u, v)_{L_V^2 \times L_V^2} = \int_{-h}^0 ((u(s), v(s))) ds, \quad \|u\|_{L_V^2}^2 = \int_{-h}^0 \|u(s)\|^2 ds, \quad \forall u, v \in L_V^2.$$

2.3. Some Lemmas. The following lemmas are used to prove the existence of weak solutions.

Lemma 2.1. ([24, 34]) *The bilinear operator $B(u, v)$ and trilinear operator $b(u, v, \omega)$ satisfy the properties*

$$\begin{cases} \|B(u, v)\|_* \leq C\|u\|\|v\|, & \forall u, v \in V, \\ b(u, v, v) = 0, & \forall u, v \in V, \\ b(u, v, \omega) = -b(u, \omega, v), & \forall u, v, \omega \in V, \\ |b(u, v, \omega)| \leq C\|u\|\|v\|\|\omega\|, & \forall u, v, \omega \in V, \\ |b(u, v, \omega)| \leq C|u|^{\frac{1}{4}}\|u\|^{\frac{3}{4}}\|v\|\|\omega\|^{\frac{1}{4}}\|\omega\|^{\frac{3}{4}}, & \forall u, v, \omega \in V. \end{cases}$$

Lemma 2.2. ([41]) *Suppose that A be a nonlinear monotone operator from a separable Banach space V to V' satisfying conditions*

(i) *For all $v \in V$,*

$$\|A(v)\|_* \leq C\|v\|^{p-1},$$

where $1 < p < \infty$ and C is a positive constant independent of v .

(ii) *(semi-continuous) for all $u, v, \omega \in V$ and $\lambda \in \mathbb{R}$, $\langle A(u + \lambda v), \omega \rangle$ is a continuous function of λ .*

If $u_n \in L^p([0, T], V)$ with $1 < p < \infty$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } L^p([\tau, T], V),$$

$$A(u_n) \rightharpoonup \psi \quad \text{weakly in } L^{p'}([\tau, T], V')$$

with $\frac{1}{p} + \frac{1}{p'} = 1$, and

$$\overline{\lim}_{n \rightarrow \infty} \int_{\tau}^T \langle A(u_n), u_n \rangle dt \leq \int_{\tau}^T \langle \psi, u \rangle dt,$$

then

$$\psi = A(u).$$

3. Global well-posedness. The following assumptions on the external forces are imposed for our results.

(H_g) Let the function $g : \mathbb{R} \times C_H \rightarrow H$ satisfies the following properties:

- (a) The function $g(\cdot, \xi)$ is measurable for any $\xi \in C_H$ and $g(\cdot, 0) \equiv 0$.
- (b) There exists a constant $L_g > 0$ such that for all $\xi, \eta \in C_H$,

$$|g(t, \xi) - g(t, \eta)| \leq L_g \|\xi - \eta\|_{C_H}, \quad \forall \tau \leq t.$$

- (c) There exists a constant $C_g > 0$ such that for all $u, v \in C([\tau - h, t]; H)$,

$$\int_{\tau}^t |g(s, u_s) - g(s, v_s)|^2 ds \leq C_g^2 \int_{\tau-h}^t |u(s) - v(s)|^2 ds, \quad \forall \tau \leq t.$$

(H_f) The function $f \in L_{loc}^{\frac{4}{3}}(\mathbb{R}, V')$ satisfies that there exists some $\sigma \in (0, \mu_0 \lambda_1)$ such that

$$\int_{-\infty}^t e^{\sigma s} \|f(s, \cdot)\|_{V'}^{\frac{4}{3}} ds < \infty, \quad \forall \tau \leq t.$$

Based on the previous definitions of operators P , A_1 and A_2 , the system (4) can be written as the following abstract equivalent form

$$\begin{cases} \frac{\partial u}{\partial t} + \mu_0 A_1 u + P(A_2 u + B(u(t - \rho(t)), u)) = Pf(x, t) + Pg(t, u_t), \\ \quad (t, x) \in (\tau, \infty) \times \Omega, \\ \nabla \cdot u = 0, \quad (t, x) \in (\tau, \infty) \times \Omega, \\ u(t, x)|_{\partial\Omega} = 0, \quad t \in (\tau, \infty), \\ u(t, x)|_{t=\tau} = u^\tau(x), \quad x \in \Omega, \\ u_\tau(s, x) = u(\tau + s, x) = \phi(s, x), \quad s \in (-h, 0), x \in \Omega. \end{cases}$$

Definition 3.1. Let $T > \tau$, assume that the initial data $(u^\tau, \phi) \in H \times (C_H \cap L_V^2) \triangleq M_H$, $f \in L_{loc}^{\frac{4}{3}}(\mathbb{R}, V')$ and g satisfies the assumption (H_g) , a function $u = u(t, x) \in C([\tau - h, T]; H) \cap L^4(\tau, T; V)$ is called a weak solution to problem (4) if

$$\begin{cases} \frac{\partial}{\partial t}(u, v) + \mu_0(A_1 u, v) + (A_2 u, v) + b(u(t - \rho(t)), u, v) \\ \quad = \langle f(t), v \rangle + (g(t, u_t), v), \\ u(\tau, x) = u^\tau(x), \\ u_\tau(s, x) = u(\tau + s, x) = \phi(s, x), \quad s \in (-h, 0) \end{cases}$$

holds for all $v \in V$ in the sense of $D'(\tau, T)$.

Theorem 3.2. For any $T > \tau$, if the initial data $(u^\tau, \phi) \in M_H = H \times (C_H \cap L_V^2)$, $f \in L_{loc}^{\frac{4}{3}}(\mathbb{R}, V')$ and g satisfies the assumption (H_g) , then problem (4) possesses a unique weak solution $u(t, x) \in C([\tau - h, T]; H) \cap L^4(\tau, T; V)$.

Proof. **Step 1.** Local approximating sequence.

From the property of the Stokes operator A_1 , we can see that the sequence of characteristic functions $\{\omega_i\}_{i=1}^\infty \subset D(A_1)$, satisfying $A_1 \omega_i = \lambda_i \omega_i$, constitutes a complete orthogonal basis in H . Let $H_m = \text{span}\{\omega_1, \dots, \omega_m\}$, and projection $P_m : H \rightarrow H_m$ is defined as

$$P_m u = \sum_{i=1}^m (u, \omega_i) \omega_i, \quad \forall u \in H.$$

Let $u_m(t) = \sum_{i=1}^m h_{im}(t) \omega_i$ be the approximated solutions satisfying the following Cauchy problem

$$\begin{cases} (\frac{\partial u_m}{\partial t}, \omega_i) + \mu_0(A_1 u_m, \omega_i) + (A_2 u_m, \omega_i) + b(u_m(t - \rho(t)), u_m, \omega_i) \\ \quad = \langle f(t), \omega_i \rangle + (g(t, u_{mt}), \omega_i), \\ u_m(\tau, x) = u_m^\tau(x) = P_m u^\tau, \\ u_{m\tau}(s, x) = \phi_m(s, x) = P_m \phi(s, x) \text{ for } s \in [-h, 0]. \end{cases} \quad (9)$$

The problem (9) is equivalent to a system of functional differential equation with respect to the unknown variables $\{h_{1m}(t), h_{2m}(t), \dots, h_{mm}(t)\}$. According to Theorem A1 in the appendix of [5] for the Cauchy problem of functional differential equation, the system (9) possesses a local solution on the interval $[\tau, t_m]$, that is, there exists a solution $u_m(t) = \sum_{i=1}^m h_{im}(t) \omega_i$ satisfying the approximation problem (9) on $[\tau, t_m]$.

Step 2. The priori estimates for $\{u_m\}$ and $\{\frac{\partial u_m}{\partial t}\}$.

Multiplying (9) by $h_{im}(t)$, and then summing from $i = 1$ to $i = m$, in view of Young's inequality and the embedding $V \hookrightarrow H$, we obtain

$$\begin{aligned} & \frac{d}{dt} |u_m(t)|^2 + 2\mu_0 \|u_m(t)\|^2 + 2\mu_1 \|u_m(t)\|^4 \\ &= 2\langle f(t), u_m(t) \rangle + 2(g(t, u_{mt}), u_m(t)) \\ &\leq 2\|f(t)\|_* \|u_m(t)\| + 2\mu_0 \lambda_1 |u_m(t)|^2 + \frac{1}{2\mu_0 \lambda_1} |g(t, u_{mt})|^2 \\ &\leq \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \|f(t)\|_*^{\frac{4}{3}} + \mu_1 \|u_m(t)\|^4 + 2\mu_0 \|u_m(t)\|^2 + \frac{1}{2\mu_0 \lambda_1} |g(t, u_{mt})|^2. \end{aligned} \quad (10)$$

Integrating (10) with respect to the variable t from τ to t , using the definition of operator P_m , $f \in L_{loc}^{\frac{4}{3}}(\mathbb{R}, V')$ and the assumption (H_g) , we have

$$\begin{aligned} & |u_m(t)|^2 + \mu_1 \int_{\tau}^t \|u_m(s)\|^4 ds \\ &\leq |u_m^{\tau}|^2 + \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \int_{\tau}^t \|f(s)\|_*^{\frac{4}{3}} ds + \frac{1}{2\mu_0 \lambda_1} \int_{\tau}^t |g(s, u_{ms})|^2 ds \\ &\leq |u^{\tau}|^2 + \frac{C_g^2}{2\mu_0 \lambda_1} \left(\int_{\tau-h}^{\tau} |u_m(s)|^2 ds + \int_{\tau}^t |u_m(s)|^2 ds \right) \\ &\quad + \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \int_{\tau}^t \|f(s)\|_*^{\frac{4}{3}} ds \\ &\leq C + \frac{C_g^2}{2\mu_0 \lambda_1} \int_{\tau}^t |u_m(s)|^2 ds. \end{aligned}$$

Thus, it follows from Gronwall's inequality that

$$|u_m(t)|^2 + \int_{\tau}^t \|u_m(s)\|^4 ds \leq C, \quad \tau \leq t \leq T,$$

which implies $t_m = T$ and

$$\{u_m(t)\} \text{ is uniformly bounded in } L^{\infty}(\tau, T; H) \cap L^4(\tau, T; V). \quad (11)$$

In particular, the sequence of functions $\{u_m(t - \rho(t))\}$ is bounded in $L^2(\tau, T; V)$. Furthermore, approximate equations (9) can be rewritten as

$$\begin{aligned} \frac{\partial u_m}{\partial t} &= P_m f(t, x) + P_m g(t, u_{mt}) - P_m (A_2 u_m) \\ &\quad - P_m (\mu_0 A_1 u_m) - P_m B(u_m(t - \rho(t)), u_m). \end{aligned} \quad (12)$$

We notice that $\|P_m\|_{\mathcal{L}(V, V)} \leq 1$ and $P_m^* = P_m$, then $\|P_m\|_{\mathcal{L}(V', V')} \leq 1$. By the assumption (H_g) , in particular, we get

$$\int_{\tau}^t |g(s, u_{ms})|^2 ds \leq C_g^2 \int_{\tau-h}^t |u_m(s)|^2 ds \leq C, \quad (13)$$

which implies that $g(s, u_{ms})$ is bounded in $L^{\frac{4}{3}}(\tau, T; V')$.

For the convective term with delay, the desired estimation can be obtained by using Lemma 2.1 and Young's inequality. Indeed, it holds that

$$\begin{aligned} \int_{\tau}^t \|B(u_m(t - \rho(t)), u_m)\|_*^{\frac{4}{3}} ds &\leq \int_{\tau}^t (C\|u_m(t - \rho(t))\|)^{\frac{4}{3}} \|u_m\|^{\frac{4}{3}} ds \\ &\leq \frac{2C^2}{3} \int_{\tau}^t \|u_m(t - \rho(t))\|^2 ds + \frac{1}{3} \int_{\tau}^t \|u_m\|^4 ds \\ &\leq C, \end{aligned} \quad (14)$$

i.e., $B(u_m(t - \rho(t)), u_m)$ is bounded in $L^{\frac{4}{3}}(\tau, T; V')$. By virtue of $\|A_1\| \leq 1$, it is easy to verify from (8) and (11) that

$$A_1 u_m \text{ and } A_2 u_m \text{ are bounded in } L^{\frac{4}{3}}(\tau, T; V'). \quad (15)$$

Combining (13)-(15), $f \in L_{loc}^{\frac{4}{3}}(\mathbb{R}, V')$ and (12), we can get

$$\left\{ \frac{\partial u_m}{\partial t} \right\} \text{ is uniformly bounded in } L^{\frac{4}{3}}(\tau, T; V'). \quad (16)$$

Step 3. Compact results and strong convergence.

Let

$$W = \{u | u \in L^4(\tau, T; V); \frac{\partial u}{\partial t} \in L^{\frac{4}{3}}(\tau, T; V')\}.$$

Now, applying the Aubin-Lions Lemma, we can derive

$$W \hookrightarrow L^4(\tau, T; H). \quad (17)$$

By using (11) and (15)-(17), we deduce that there exist a subsequence (still denote it by $\{u_m\}$) and $u \in L^\infty(\tau - h, T; H) \cap L^4(\tau, T; V)$ such that

$$\begin{cases} u_m \rightarrow u \text{ strongly in } L^4(\tau, T; H), \\ g(t, u_{mt}) \rightarrow g(t, u_t) \text{ strongly in } L^2(\tau, T; H), \\ u_m(t - \rho(t)) \rightarrow u(t - \rho(t)) \text{ strongly in } L^2(\tau, T; H), \\ u_m \rightharpoonup u \text{ weakly }^* \text{ in } L^\infty(\tau, T; H), \\ u_m \rightharpoonup u \text{ weakly in } L^4(\tau, T; V), \\ \frac{\partial u_m}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ weakly in } L^{\frac{4}{3}}(\tau, T; V'), \\ A_2(u_m) \rightharpoonup \psi \text{ weakly in } L^{\frac{4}{3}}(\tau, T; V'), \end{cases} \quad (18)$$

and

$$u_m(\tau) = P_m u^\tau \rightarrow u(\tau) = u^\tau \text{ in } H, \quad \phi_m(s) = P_m \phi(s) \rightarrow \phi(s) \text{ in } C_H \cap L_V^2.$$

Next, we prove that

$$u_m \rightarrow u \text{ in } C([\tau, T]; H). \quad (19)$$

Let u_m and u_n be two solutions to the approximated system corresponding to initial value (u_m^τ, ϕ_m) and (u_n^τ, ϕ_n) respectively, and set $\omega(t) = u_m(t) - u_n(t)$, then we can get

$$\begin{aligned} \frac{\partial \omega}{\partial t} + \mu_0 A_1 \omega + A_2 u_m - A_2 u_n + B(u_m(t - \rho(t)), u_m) - B(u_n(t - \rho(t)), u_n) \\ = g(t, u_{mt}) - g(t, u_{nt}). \end{aligned} \quad (20)$$

We observe the fact that

$$\begin{aligned} & B(u_m(t - \rho(t)), u_m) - B(u_n(t - \rho(t)), u_n) \\ &= B(\omega(t - \rho(t)), u_m) + B(u_n(t - \rho(t)), \omega). \end{aligned} \quad (21)$$

Multiplying (20) by $\omega(t)$, then integrating the resultant over Ω and applying (21), we obtain

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} |\omega|^2 + \mu_0 \|\omega\|^2 + (A_2 u_m - A_2 u_n, u_m - u_n) + b(\omega(t - \rho(t)), u_m, \omega) \\ = (g(t, u_{mt}) - g(t, u_{nt}), \omega). \end{aligned}$$

Thanks to the assumption (H_g) and the monotonicity of operator A_2 , we can deduce from Lemma 2.1 and Young's inequality that

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} |\omega|^2 + \mu_0 \|\omega\|^2 &\leq |b(\omega(t - \rho(t)), u_m, \omega)| + (g(t, u_{mt}) - g(t, u_{nt}), \omega) \\ &\leq C |\omega(t - \rho(t))|^{\frac{1}{4}} \|\omega(t - \rho(t))\|^{\frac{3}{4}} \|u_m\| |\omega|^{\frac{1}{4}} \|\omega\|^{\frac{3}{4}} \\ &\quad + |g(t, u_{mt}) - g(t, u_{nt})| |\omega| \\ &\leq C \|\omega_t\|_{C_H}^{\frac{1}{2}} \|u_m\| \|\omega(t - \rho(t))\|^{\frac{3}{4}} \|\omega\|^{\frac{3}{4}} + L_g \|\omega_t\|_{C_H} |\omega| \\ &\leq \frac{C^4}{4\varepsilon^3} \|\omega_t\|_{C_H}^2 \|u_m\|^4 + \frac{3\varepsilon}{4} \|\omega(t - \rho(t))\| \|\omega\| + L_g \|\omega_t\|_{C_H} |\omega| \\ &\leq \frac{C^4}{4\varepsilon^3} \|\omega_t\|_{C_H}^2 \|u_m\|^4 + \frac{9\varepsilon^2}{32\mu_0} \|\omega(t - \rho(t))\|^2 \\ &\quad + \frac{\mu_0}{2} \|\omega\|^2 + L_g \|\omega_t\|_{C_H} |\omega|. \end{aligned} \quad (22)$$

Here ε is a positive number, which is determined later. Note the fact that $|\omega(s)| \leq \|\omega_s\|_{C_H}$. Integrating (22) with respect to time s on $[\tau, t]$, it holds that

$$\begin{aligned} & |\omega(t)|^2 + \mu_0 \int_{\tau}^t \|\omega(s)\|^2 ds \\ &\leq \frac{C^4}{2\varepsilon^3} \int_{\tau}^t \|\omega_s\|_{C_H}^2 \|u_m(s)\|^4 ds + \frac{9\varepsilon^2}{16\mu_0} \int_{\tau}^t \|\omega(s - \rho(s))\|^2 ds \\ &\quad + 2L_g \int_{\tau}^t \|\omega_s\|_{C_H}^2 ds + |\omega(\tau)|^2. \end{aligned}$$

With the help of a change of variable in the integral of $\omega(s - \rho(s))$ and $\rho'(t) \leq \rho^* < 1$, we can deduce by choosing $\varepsilon^2 = \frac{16(1-\rho^*)}{9} \mu_0^2$ that

$$\begin{aligned} |\omega(t)|^2 &\leq |\omega(\tau)|^2 + \frac{C^4}{2\varepsilon^3} \int_{\tau}^t \|\omega_s\|_{C_H}^2 \|u_m(s)\|^4 ds \\ &\quad + \mu_0 \int_{\tau-h}^{\tau} \|\omega(s)\|^2 ds + 2L_g \int_{\tau}^t \|\omega_s\|_{C_H}^2 ds \\ &\leq \|\omega_{\tau}\|_{C_H}^2 + \mu_0 \int_{\tau-h}^{\tau} \|\omega(s)\|^2 ds \\ &\quad + \left(\frac{C^4}{2\varepsilon^3} + 2L_g\right) \int_{\tau}^t (\|u_m(s)\|^4 + 1) \|\omega_s\|_{C_H}^2 ds. \end{aligned}$$

For simplicity, we set that $C' = (\frac{C^4}{2\varepsilon^3} + 2L_g)$. If $t \geq \tau + h$, it holds that $t + \theta \geq \tau$ for all $\theta \in [-h, 0]$ and

$$\begin{aligned} |\omega(t + \theta)|^2 &\leq \|\omega_\tau\|_{C_H}^2 + \mu_0 \int_{\tau-h}^\tau \|\omega(s)\|^2 ds \\ &\quad + C' \int_\tau^{t+\theta} (\|u_m(s)\|^4 + 1) \|\omega_s\|_{C_H}^2 ds. \end{aligned}$$

Consequently, it follows that for all $t \geq \tau + h$,

$$\begin{aligned} \|u_{mt} - u_{nt}\|_{C_H}^2 &\leq \|\phi_m - \phi_n\|_{C_H}^2 + \mu_0 \|\phi_m - \phi_n\|_{L_V^2}^2 \\ &\quad + C' \int_\tau^t (\|u_m(s)\|^4 + 1) \|u_{ms} - u_{ns}\|_{C_H}^2 ds, \end{aligned} \quad (23)$$

which, by the Gronwall inequality, implies

$$\begin{aligned} \|u_{mt} - u_{nt}\|_{C_H}^2 &\leq (\|\phi_m - \phi_n\|_{C_H}^2 + \mu_0 \|\phi_m - \phi_n\|_{L_V^2}^2) \\ &\quad \times e^{\int_\tau^t C'(\|u_m(s)\|^4 + 1) ds}. \end{aligned} \quad (24)$$

Since $\phi_m \rightarrow \phi$ in $C_H \cap L_V^2$, (24) indicates that $\{u_m\}$ is a Cauchy sequence in $C([\tau - h, T]; H)$. Thus we complete the proof of (19).

Step 4. Passing the limit and uniqueness.

In order to pass the limit of (9), we need to discuss the convergence of nonlinear terms $b(u_m, u_m, \omega_i)$ and $\langle A_2 u_m, \omega_i \rangle$. Using the properties of trilinear operator $b(u, v, \omega)$, Lemma 2.1, and the fact that ω_i is an eigenfunction of the Stokes operator, we obtain

$$\begin{aligned} &\int_\tau^T |b(u_m(t - \rho(t)), u_m, \omega_i) - b(u(t - \rho(t)), u, \omega_i)| ds \\ &\leq \int_\tau^T |b(u_m(t - \rho(t)) - u(t - \rho(t)), \omega_i, u_m)| \\ &\quad + |b(u(t - \rho(t)), \omega_i, u_m - u)| ds \\ &\leq C \int_\tau^T |u_m(t - \rho(t)) - u(t - \rho(t))|^{\frac{1}{4}} \|u_m(t - \rho(t)) \\ &\quad - u(t - \rho(t))\|^{\frac{3}{4}} |u_m|^{\frac{1}{4}} \|u_m\|^{\frac{3}{4}} ds \\ &\quad + C \int_\tau^T |u(t - \rho(t))|^{\frac{1}{4}} \|u(t - \rho(t))\|^{\frac{3}{4}} |u_m - u|^{\frac{1}{4}} \|u_m - u\|^{\frac{3}{4}} ds \\ &\triangleq I_1 + I_2. \end{aligned} \quad (25)$$

We use the Hölder inequality to give

$$\begin{cases} I_1 \leq C \|u_m(t - \rho(t)) - u(t - \rho(t))\|_{L^2(\tau, T; H)}^{\frac{1}{4}} \|u_m(t - \rho(t)) \\ \quad - u(t - \rho(t))\|_{L^2(\tau, T; V)}^{\frac{3}{4}} \|u_m\|_{L^2(\tau, T; H)}^{\frac{1}{4}} \|u_m\|_{L^2(\tau, T; V)}^{\frac{3}{4}}, \\ I_2 \leq \|u(t - \rho(t))\|_{L^2(\tau, T; H)}^{\frac{1}{4}} \|u(t - \rho(t))\|_{L^2(\tau, T; V)}^{\frac{3}{4}} \\ \quad \times \|u_m - u\|_{L^2(\tau, T; H)}^{\frac{1}{4}} \|u_m - u\|_{L^2(\tau, T; V)}^{\frac{3}{4}}, \end{cases}$$

which, together with (18)₁, (18)₃ and (25), yields that

$$\int_\tau^T |b(u_m, u_m, \omega_i) - b(u, u, \omega_i)| ds \rightarrow 0, \quad \text{as } m \rightarrow \infty. \quad (26)$$

By a similar technique to the proof of Lemma 3.2 in [35], we can deduce

$$\overline{\lim}_{n \rightarrow \infty} \int_{\tau}^T \langle A_2(u_m), u_m \rangle dt \leq \int_{\tau}^T \langle \psi, u \rangle dt. \quad (27)$$

Moreover, it is easy to verify that the operator A_2 satisfies the conditions (i) and (ii) of Lemma 2.2. Combining (18)₅, (18)₇, (27) and Lemma 2.2, we get

$$\psi = A_2(u). \quad (28)$$

Now, passing to the limit of (9), by combining (18)-(19), (26) and (28), we can infer that u is indeed a weak solution to problem (4).

Last, we consider the uniqueness of solutions for our problem. Let u, v be two weak solutions with the same initial (u^τ, ϕ) and set $\omega = u - v$. By a similar technique as proving (23), we can obtain that for $t \in [\tau, T]$,

$$\|u_t - v_t\|_{C_H}^2 \leq C' \int_{\tau}^t (\|u_m(s)\|^4 + 1) \|u_s - v_s\|_{C_H}^2 ds.$$

By Gronwall's inequality, it yields $\omega(t) \equiv 0$, i.e., the solution is unique. Similarly, we can also verify that the solution is continuously dependent on the initial value. \square

4. Pullback dynamics for (4). In this section, we shall obtain the existence of pullback attractors for the process associated to (4).

4.1. Preliminaries for pullback attractors. Let (X, d_X) be a given metric space, and we denote $\mathbb{R}_d^2 = \{(t, \tau) \in \mathbb{R}^2 : \tau \leq t\}$.

A process on X is a mapping $U(\cdot, \cdot)$ such that $\mathbb{R}_d^2 \times X \ni (t, \tau, x) \mapsto U(t, \tau)x \in X$ with $U(\tau, \tau)x = x$ for any $(\tau, x) \in \mathbb{R} \times X$, and $U(t, r)U(r, \tau)x = U(t, \tau)x$ for any $\tau \leq r \leq t$ and all $x \in X$. A process $U(\cdot, \cdot)$ is said to be continuous if for any pair $\tau \leq t$, $U(t, \tau) : X \rightarrow X$ is continuous. A process $U(\cdot, \cdot)$ is said to be closed if for any $\tau \leq t$, and any sequence $\{x_n\} \subset X$, if $x_n \rightarrow x \in X$ and $U(t, \tau)x_n \rightarrow y \in X$, then $U(t, \tau)x = y$. Clearly, every continuous process is closed.

Let $\mathcal{P}(X)$ be the family of all nonempty subsets of X , and consider a family of nonempty sets parameterized in time $\hat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. Let \mathcal{D} be a nonempty class of families parameterized in time $\hat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class \mathcal{D} will be called a universe in $\mathcal{P}(X)$.

Definition 4.1. (1) A process $U(\cdot, \cdot)$ on X is said to be pullback \hat{D}_0 -asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \rightarrow -\infty$ and $x_n \in D_0(\tau_n)$ for all n , the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X .

(2) Further, we say that a process $U(\cdot, \cdot)$ on X is pullback \mathcal{D} -asymptotically compact if it is pullback \hat{D} -asymptotically compact for all $\hat{D} \in \mathcal{D}$.

Definition 4.2. $\hat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback \mathcal{D} -absorbing for the process $U(\cdot, \cdot)$ on X if for any $t \in \mathbb{R}$ and any $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, there exists a $\tau_0(t, \hat{D}) \leq t$ such that

$$U(t, \tau)D(\tau) \subset D_0(t), \quad \forall \tau \leq \tau_0(t, \hat{D}).$$

Observe from the above definition that \hat{D}_0 does not belong necessarily to the class \mathcal{D} .

Denote

$$\Lambda(\widehat{D}_0, t) := \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau) D_0(\tau)}^X, \quad \forall t \in \mathbb{R},$$

where $\overline{\{\cdot\cdot\}}^X$ is the closure in X . We denote by $\text{dist}_X(X_1, X_2)$ the Hausdorff semi-distance in X between two sets X_1 and X_2 , defined as

$$\text{dist}_X(X_1, X_2) = \sup_{x \in X_1} \inf_{y \in X_2} d_X(x, y), \quad \forall X_1, X_2 \subset X.$$

In order to get our result, we need to use a classical theorem in [14, 15].

Theorem 4.3. *Consider a closed process $U : \mathbb{R}_d^2 \times X \mapsto X$, a universe $\mathcal{D} \in \mathcal{P}(X)$, and a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ which is pullback \mathcal{D} -absorbing for U , and assume also that U is pullback \widehat{D}_0 -asymptotically compact. Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ is a family of pullback \mathcal{D} -attractors which is defined by*

$$\mathcal{A}_{\mathcal{D}}(t) = \bigcup_{\widehat{D} \in \mathcal{D}} \overline{\Lambda(\widehat{D}, t)}^X, \quad t \in \mathbb{R},$$

and has the following properties:

(a) for any $t \in \mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X , and $\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda(\widehat{D}_0, t)$.

(b) $\mathcal{A}_{\mathcal{D}}$ is pullback \mathcal{D} -attracting, i.e.,

$$\lim_{\tau \rightarrow -\infty} \text{dist}_X(U(t, \tau) D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0, \quad \forall \widehat{D} \in \mathcal{D}, t \in \mathbb{R}.$$

(c) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e., $U(t, \tau) \mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$, $\forall \tau \leq t$.

(d) if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$, $\forall t \in \mathbb{R}$.

Moreover, family $\mathcal{A}_{\mathcal{D}}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$,

$$\lim_{\tau \rightarrow -\infty} \text{dist}_X(U(t, \tau) D(\tau), C(t)) = 0,$$

then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Remark 1. If $\widehat{D}_0 \in \mathcal{D}$, $D_0(t)$ is closed for any $t \in \mathbb{R}$ and \mathcal{D} is inclusion-closed, then $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ and $\mathcal{A}_{\mathcal{D}}$ is unique family in \mathcal{D} that satisfies (a), (b) and (c) above.

Let \mathcal{D}_F^X be the universe of fixed nonempty bounded subsets of X , i.e., the class of all families \widehat{D} of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X .

Lemma 4.4. ([14]) *Under the assumptions of Theorem 4.3, if the universe \mathcal{D} contains the universe \mathcal{D}_F^X , then both attractors, $\mathcal{A}_{\mathcal{D}_F^X}$ and $\mathcal{A}_{\mathcal{D}}$, exist, and the following relation holds:*

$$\mathcal{A}_{\mathcal{D}_F^X} \subset \mathcal{A}_{\mathcal{D}}, \quad \forall t \in \mathbb{R}.$$

4.2. The process and pullback dissipation. In view of Theorem 3.2, here we take the phase space $M_H = H \times (C_H \cap L_V^2)$ equipped with the norm $\|(\xi, \phi)\|_{M_H} = \|\xi\|_H + \|\phi\|_{C_H} + \|\phi\|_{L_V^2}$ for $(\xi, \phi) \in M_H$. It is easy to verify the following proposition.

Proposition 1. Consider given $g : \mathbb{R} \times C_H \rightarrow H$ and $f \in L_{loc}^{\frac{4}{3}}(\mathbb{R}; V')$ satisfying assumptions (H_g) and (H_f) . Then the solution of problem (1.1) generates a bipara-metric family of mappings $U(t, \tau) : M_H \rightarrow M_H$ by $U(t, \tau)(u^\tau, \phi) = (u(t; \tau, u^\tau, \phi), u_t(\cdot; \tau, u^\tau, \phi))$, which is a continuous process.

In order to prove the existence of pullback attractors for the process U , we need the following assumption:

(H_1) For every $u \in L^2(\tau - h, T; V)$, there exists a value $\sigma \in (0, \mu_0 \lambda_1)$ which is independent of u such that

$$\int_{\tau}^t e^{\sigma s} |g(s, u_s)|^2 ds < C_g^2 \int_{\tau-h}^t e^{\sigma s} |u(s)|^2 ds, \quad \forall t \leq T.$$

Lemma 4.5. $g : \mathbb{R} \times C_H \rightarrow H$ and $f \in L_{loc}^{\frac{4}{3}}(\mathbb{R}; V')$ satisfy assumptions (H_g) , (H_1) and (H_f) . Then, for any $(u^\tau, \phi) \in M_H$, there exists a value $\sigma \in (0, \mu_0 \lambda_1)$ such that the solution u of (4) holds the estimates

$$\begin{aligned} |u(t)|^2 &\leq C e^{-\sigma(t-\tau)} (|u^\tau|^2 + \frac{C_g^2}{\mu_0 \lambda_1} \|\phi\|_{L_H^2}) \\ &\quad + C \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} e^{-\sigma t} \int_{\tau}^t e^{\sigma s} \|f(s)\|_*^{\frac{4}{3}} ds, \end{aligned} \quad (29)$$

$$\begin{aligned} \mu_0 \int_s^t \|u(r)\|^2 dr &\leq |u(s)|^2 + \frac{C_g^2}{\mu_0 \lambda_1} \|u_s\|_{L_H^2} \\ &\quad + \frac{C_g^2}{\mu_0 \lambda_1} \int_s^t |u(r)|^2 dr + \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \int_s^t \|f(r)\|_*^{\frac{4}{3}} dr, \end{aligned} \quad (30)$$

and

$$\begin{aligned} \mu_1 \int_s^t \|u(r)\|^4 dr &\leq |u(s)|^2 + \frac{C_g^2}{\mu_0 \lambda_1} \|u_s\|_{L_H^2} \\ &\quad + \frac{C_g^2}{\mu_0 \lambda_1} \int_s^t |u(r)|^2 dr + \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \int_s^t \|f(r)\|_*^{\frac{4}{3}} dr. \end{aligned} \quad (31)$$

Proof. Multiplying (4) by u and integrating the resultant over Ω , using integration by parts and Young's inequality, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u(t)|^2 + \mu_0 \|u(t)\|^2 + \mu_1 \|u(t)\|^4 \\ &= \langle f(t), u(t) \rangle + (g(t, u_t), u(t)) \\ &\leq \|f(t)\|_* \|u(t)\| + |g(t, u_t)| |u(t)| \\ &\leq \frac{1}{2} \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \|f(t)\|_*^{\frac{4}{3}} + \frac{\mu_1}{2} \|u(t)\|^4 \\ &\quad + \frac{1}{2\mu_0 \lambda_1} |g(t, u_t(t))|^2 + \frac{\mu_0 \lambda_1}{2} |u(t)|^2. \end{aligned} \quad (32)$$

Now multiplying (32) by $2e^{\sigma t}$ and applying Poincaré's inequality (6), one has

$$\begin{aligned} \frac{d}{dt} (e^{\sigma t} |u(t)|^2) &= \sigma e^{\sigma t} |u(t)|^2 + e^{\sigma t} \frac{d}{dt} |u(t)|^2 \\ &\leq (\sigma + \mu_0 \lambda_1) e^{\sigma t} |u(t)|^2 - 2\mu_0 \|u(t)\|^2 e^{\sigma t} \\ &\quad + \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} e^{\sigma t} \|f(t)\|_*^{\frac{4}{3}} + \frac{e^{\sigma t}}{\mu_0 \lambda_1} |g(t, u_t)|^2 \end{aligned}$$

$$\begin{aligned} &\leq \left(\frac{\sigma}{\lambda_1} - \mu_0\right) \|u(t)\|^2 e^{\sigma t} \\ &\quad + \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} e^{\sigma t} \|f(t)\|_*^{\frac{4}{3}} + \frac{e^{\sigma t}}{\mu_0 \lambda_1} |g(t, u_t)|^2. \end{aligned} \quad (33)$$

We integrate (33) over the interval $[\tau, t]$ with respect to t , use $\sigma \in (0, \mu_0 \lambda_1)$ and assumption (H_1) , to conclude

$$\begin{aligned} e^{\sigma t} |u(t)|^2 &\leq e^{\sigma \tau} |u^\tau|^2 + \frac{1}{\mu_0 \lambda_1} \int_\tau^t e^{\sigma s} |g(s, u_s)|^2 ds \\ &\quad + \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \int_\tau^t e^{\sigma s} \|f(s)\|_*^{\frac{4}{3}} ds \\ &\leq e^{\sigma \tau} |u^\tau|^2 + \frac{C_g^2}{\mu_0 \lambda_1} \int_{\tau-t}^t e^{\sigma s} |u(s)|^2 ds \\ &\quad + \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \int_\tau^t e^{\sigma s} \|f(s)\|_*^{\frac{4}{3}} ds \\ &\leq e^{\sigma \tau} |u^\tau|^2 + \frac{C_g^2}{\mu_0 \lambda_1} (e^{\sigma \tau} \int_{-h}^0 |\phi(s)|^2 ds + \int_\tau^t e^{\sigma s} |u(s)|^2 ds) \\ &\quad + \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \int_\tau^t e^{\sigma s} \|f(s)\|_*^{\frac{4}{3}} ds \\ &\leq e^{\sigma \tau} (|u^\tau|^2 + \frac{C_g^2}{\mu_0 \lambda_1} \int_{-h}^0 |\phi(s)|^2 ds) + \frac{C_g^2}{\mu_0 \lambda_1} \int_\tau^t e^{\sigma s} |u(s)|^2 ds \\ &\quad + \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \int_\tau^t e^{\sigma s} \|f(s)\|_*^{\frac{4}{3}} ds. \end{aligned} \quad (34)$$

On account of the assumption (H_f) , we apply the Gronwall inequality to (34) to derive

$$e^{\sigma t} |u(t)|^2 \leq C e^{\sigma \tau} (|u^\tau|^2 + \frac{C_g^2}{\mu_0 \lambda_1} \int_{-h}^0 |\phi(s)|^2 ds) + C \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \int_\tau^t e^{\sigma s} \|f(s)\|_*^{\frac{4}{3}} ds,$$

which means

$$\begin{aligned} |u(t)|^2 &\leq C e^{-\sigma(t-\tau)} (|u^\tau|^2 + \frac{C_g^2}{\mu_0 \lambda_1} \|\phi\|_{L_H^2}) \\ &\quad + C \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} e^{-\sigma t} \int_\tau^t e^{\sigma s} \|f(s)\|_*^{\frac{4}{3}} ds, \quad \forall t \geq \tau. \end{aligned}$$

Consequently, the estimation (29) is proved. Thanks to (32) and (6), we obtain

$$\frac{d}{dt} |u(t)|^2 + \mu_0 \|u(t)\|^2 + \mu_1 \|u(t)\|^4 \leq \frac{1}{\mu_0 \lambda_1} |g(t, u_t)|^2 + \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \|f(t)\|_*^{\frac{4}{3}}. \quad (35)$$

Integrating (35) over the interval (s, t) , and using the assumptions on g , we can get

$$\begin{aligned} &|u(r)|^2 + \mu_0 \int_s^t \|u(r)\|^2 dr + \mu_1 \int_s^t \|u(r)\|^4 dr \\ &\leq |u(s)|^2 + \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \int_s^t \|f(r)\|_*^{\frac{4}{3}} dr \\ &\quad + \frac{C_g^2}{\mu_0 \lambda_1} \left(\int_{-h}^0 |u_s(r)|^2 dr + \int_s^t |u(r)|^2 dr \right). \end{aligned} \quad (36)$$

Thus, (30) and (31) are obtained immediately from (36). Now we complete the proof. \square

Definition 4.6. (Universe) We will denote by $\mathcal{D}_\sigma^{M_H}$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(M_H)$ such that

$$\lim_{\tau \rightarrow -\infty} (e^{\sigma\tau} \sup_{(\xi, \phi) \in D(\tau)} \|(\xi, \phi)\|_{M_H}^2) = 0.$$

Remark 2. According to the above definition and the notation \mathcal{D}_F^X , it is obvious that $\mathcal{D}_F^{M_H} \subset \mathcal{D}_\sigma^{M_H}$ and that both are inclusion-closed.

Based on the above universe and Lemma 4.5, we can present the pullback dissipation in M_H . Let $\overline{B}_X(0, R)$ be the closed ball with zero as the center and R as the radius in X .

Proposition 2. Suppose that $g : \mathbb{R} \times C_H \rightarrow H$ and $f \in L_{loc}^{\frac{4}{3}}(\mathbb{R}; V')$ satisfy assumptions (H_g) , (H_f) and (H_1) . Then, the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset M_H$ is defined by

$$D_0(t) = \overline{B}_H(0, R_H) \times (\overline{B}_{C_H}(0, R_{C_H}) \cap \overline{B}_{L_V^2}(0, R_{L_V^2}))$$

is pullback $\mathcal{D}_\sigma^{M_H}$ -absorbing for the process $U(t, \tau)$ on M_H and $\widehat{D}_0 \in \mathcal{D}_\sigma^{M_H}$, where

$$\begin{aligned} R_H^2(t) &= 1 + C \cdot \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} e^{-\sigma(t-2h)} \int_{-\infty}^t e^{\sigma s} \|f(s)\|_{*}^{\frac{4}{3}} ds, \\ R_{L_V^2}^2(t) &= \frac{1}{\mu_0} \left[\left(1 + \frac{2C_g^2 h}{\mu_0 \lambda_1}\right) R_H^2 + \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} \|f(r)\|_{L^{\frac{4}{3}}(t-h, t; V')} \right]. \end{aligned}$$

Proof. Fix $t \in \mathbb{R}$, we derive from (29) that there exists a pullback time $\tau(\widehat{D}, t) \leq t - 2h$ such that

$$\begin{aligned} |u(t; \tau, u^\tau, \phi)|^2 &\leq 1 + C \cdot \left(\frac{27}{16\mu_1}\right)^{\frac{1}{3}} e^{-\sigma t} \int_{-\infty}^t e^{\sigma s} \|f(s)\|_{*}^{\frac{4}{3}} ds \\ &\leq R_H^2(t), \quad \forall \widehat{D} \in \mathcal{D}_\sigma^{M_H}(t) \end{aligned}$$

holds for all $t \geq \tau$ with $\tau \leq \tau(\widehat{D}, t)$ and $(u^\tau, \phi) \in D(\tau)$.

In particular, we observe that $\|u_t\|_{C_H} \leq R_H^2(t)$. Now, putting $s = t - h$ in the estimate (30) and using (29), we deduce immediately that $\|u_t\|_{L_V^2}^2 \leq R_{L_V^2}^2(t)$. Moreover, in view of the above estimates, (H_f) and the definition of universe, we can see clearly the fact that \widehat{D}_0 belongs to $\mathcal{D}_\sigma^{M_H}$. Therefore, the proof is done. \square

4.3. Pullback asymptotic compactness. To use Theorem 4.3, we also need to establish the asymptotically compact of the process. We give the following result.

Theorem 4.7. Assume that $g : \mathbb{R} \times C_H \rightarrow H$ and $f \in L_{loc}^{\frac{4}{3}}(\mathbb{R}; V')$ satisfy assumptions (H_g) and (H_1) . Then, the process $U(t, \tau)$ defined in proposition 1 is pullback $\mathcal{D}_\sigma^{M_H}$ -asymptotically compact.

Proof. Fix a value $t \in \mathbb{R}$ and consider a family $\widehat{D} \in \mathcal{D}_\sigma^{M_H}$, let $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \rightarrow -\infty$ and $\{(u^{\tau_n}, \phi^n)\}$ with $\{(u^{\tau_n}, \phi^n)\} \in D(\tau_n)$ be two sequences for all n , then we denote $\{(u^n, u_t^n)\} \in \widehat{D}$ as a sequence with $u^n(\cdot) = u(\cdot; \tau_n, u^{\tau_n}, \phi^n)$.

In the same way as Proposition 2, by using the estimations in Lemma 4.5, we obtain that there exists a pullback time $\tau_1(\widehat{D}, t) \leq t - 4h - 1$ such that the subsequence $\{u^n : \tau_n \leq \tau_1(\widehat{D}, t)\}$ is bounded in $L^\infty(t - 4h - 1, t; H) \cap L^4(t - 3h - 1, t; V)$. By virtue of the equation (12) and the embedding theorem, we have that

$$\begin{aligned} & \| (u^n)' \|_{L^{\frac{4}{3}}(t-2h-1, t; V')} \\ & \leq \mu_0 \| u^n \|_{L^{\frac{4}{3}}(t-2h-1, t; V)} + \mu_1 \| u^n \|_{L^4(t-2h-1, t; V)}^3 \\ & \quad + \| f \|_{L^{\frac{4}{3}}(t-2h-1, t; V')} + \| (u^n(t - \rho(t)) \cdot \nabla) u^n \|_{L^{\frac{4}{3}}(t-2h-1, t; V')} \\ & \quad + C \| g(t, u_t^n) \|_{L^2(t-2h-1, t; H)}. \end{aligned}$$

By applying a similar technique as proving (16) in Theorem 3.2, we derive from the assumption (H_g) that $\{(u^n)'\}$ is uniformly bounded in $L^{\frac{4}{3}}(t - 2h - 1, t; V')$. Thanks to the Aubin-Lions Lemma, the assumptions on g and the diagonal procedure, there exists a subsequence (still denote it by $\{u^n\}$) and a function $u \in L^\infty(t - 4h - 1, t; H) \cap L^4(t - 3h - 1, t; V)$ such that

$$\begin{cases} u^n \rightharpoonup u \text{ weakly }^* \text{ in } L^\infty(t - 4h - 1, t; H), \\ u^n \rightharpoonup u \text{ weakly in } L^4(t - 2h - 1, t; V), \\ (u^n)' \rightharpoonup u' \text{ weakly in } L^{\frac{4}{3}}(t - 2h - 1, t; V'), \\ u^n \rightarrow u \text{ strongly in } L^4(t - 2h - 1, t; H), \\ u^n(s) \rightarrow u(s) \text{ strongly in } H, \quad a.e. \ s \in (t - 2h - 1, t), \\ g(\cdot, u_t^n) \rightarrow g(\cdot, u_t) \text{ strongly in } L^2(t - h - 1, t; H). \end{cases} \quad (37)$$

From the above convergences, we derive that $u \in C([t - 2h - 1, t]; H)$ is a weak solution for problem (1.1) in the interval $(t - h - 1, t)$ with u_{t-h-1} as initial data.

By using the same technique as proving (19) in Theorem 3.2, we can get

$$u^n \rightarrow u \text{ strongly in } C([t - h - 1, t]; H).$$

Consequently, we obtain that for any sequence $\{s_n\} \subset [t - h - 1, t]$ with $s_n \rightarrow s_*$,

$$u^n(s_n) \rightharpoonup u(s_*) \text{ weakly in } H. \quad (38)$$

Now, our goal is to obtain that

$$|u^n(s_n) - u(s_*)| \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

First of all, we can conclude from the weak convergence (38) that

$$|u(s_*)| \leq \liminf_{n \rightarrow \infty} |u^n(s_n)|. \quad (39)$$

Furthermore, in view of the energy equality (32), we infer that for all $t - h - 1 \leq s_1 \leq s_2 \leq t$,

$$\begin{aligned} & \frac{1}{2} |y(s_2)|^2 + \mu_0 \int_{s_1}^{s_2} \|y(r)\|^2 dr + \mu_1 \int_{s_1}^{s_2} \|y(r)\|^4 dr \\ & = \frac{1}{2} |y(s_1)|^2 + \int_{s_1}^{s_2} \langle f(r), y(r) \rangle dr + \int_{s_1}^{s_2} (g(r, y_r), y(r)) dr, \end{aligned} \quad (40)$$

where y can be u and all u^n . Hence we can define the continuous functions on the interval $[t - h - 1, t]$ as

$$\mathcal{J}(s) = \frac{1}{2} |u(s)|^2 - \int_{t-h-1}^s \langle f(r), u(r) \rangle dr - \int_{t-h-1}^s (g(r, u_r), u(r)) dr$$

and

$$\mathcal{J}_n(s) = \frac{1}{2}|u^n(s)|^2 - \int_{t-h-1}^s \langle f(r), u^n(r) \rangle dr - \int_{t-h-1}^s (g(r, u_r^n), u^n(r)) dr.$$

It is clear that \mathcal{J} and \mathcal{J}_n are non-increasing functions. In addition, the convergences (37) indicates that

$$\mathcal{J}_n(s) \rightarrow \mathcal{J}(s) \quad a.e. \quad s \in (t-h-1, t).$$

Therefore, it is possible to choose a sequence $\{s_k\} \subset (t-h-1, t)$ satisfying that $\lim_{k \rightarrow \infty} s_k = s_*$ and

$$\lim_{n \rightarrow \infty} \mathcal{J}_n(s_k) = \mathcal{J}(s_k), \quad \forall k.$$

Since $\mathcal{J}(s)$ is continuous, for $\forall \varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that

$$|\mathcal{J}(s_k) - \mathcal{J}(s_*)| < \frac{\varepsilon}{2}, \quad \forall k \geq k_\varepsilon. \quad (41)$$

Because $\mathcal{J}_n(s)$ is uniformly continuous with respect to time s , there exists $n(k_\varepsilon)$ such that

$$s_{k_\varepsilon} \leq s_n, \quad |\mathcal{J}_n(s_{k_\varepsilon}) - \mathcal{J}(s_{k_\varepsilon})| < \frac{\varepsilon}{2}, \quad \forall n \geq n(k_\varepsilon). \quad (42)$$

According to the non-increasing property of all \mathcal{J}_n and (41)-(42), we derive that for all $n \geq n(k_\varepsilon)$,

$$\begin{aligned} \mathcal{J}_n(s_n) - \mathcal{J}(s_*) &\leq \mathcal{J}_n(s_{k_\varepsilon}) - \mathcal{J}(s_*) \\ &\leq |\mathcal{J}_n(s_{k_\varepsilon}) - \mathcal{J}(s_*)| \\ &\leq |\mathcal{J}_n(s_{k_\varepsilon}) - \mathcal{J}(s_{k_\varepsilon})| + |\mathcal{J}(s_{k_\varepsilon}) - \mathcal{J}(s_*)| < \varepsilon. \end{aligned}$$

Because of the arbitrariness of ε , we have that $\limsup_{n \rightarrow \infty} \mathcal{J}_n(s_n) \leq \mathcal{J}(s_*)$, which, by virtue of (37) again, implies

$$\limsup_{n \rightarrow \infty} |u^n(s_n)| \leq |u(s_*)|. \quad (43)$$

Therefore, combining (38), (39) and (43), we conclude that

$$u^n(s_n) \rightarrow u(s_*) \text{ strongly in } C([t-h, t]; H). \quad (44)$$

By using again the energy equality (40) satisfied by u and u^n , the convergences in (37) and (44), we can deduce that

$$\|u^n\|_{L^2(t-h, t; V)} \rightarrow \|u\|_{L^2(t-h, t; V)},$$

which, together with the weak convergence already proved in (37), gives

$$u^n(s_n) \rightarrow u(s_*) \text{ strongly in } L^2(t-h, t; V). \quad (45)$$

Combining (44) with (45), we derive that the process is pullback $\mathcal{D}_\sigma^{M_H}$ -asymptotically compact. Thus we finish the proof. \square

4.4. Existence of pullback attractors. In this subsection, by using the results obtained in subsection 4.2 and subsection 4.3, we shall establish the main result of the paper as follows.

Theorem 4.8. *Assume that $g : \mathbb{R} \times C_H \rightarrow H$ satisfies (H_g) and $f \in L_{loc}^{\frac{4}{3}}(\mathbb{R}; V')$ fulfills conditions (H_f) and (H_1) . Then, there exist the minimal pullback $\mathcal{D}_F^{M_H}$ -attractor*

$$\mathcal{A}_{\mathcal{D}_F^{M_H}} = \{\mathcal{A}_{\mathcal{D}_F^{M_H}}(t) : t \in \mathbb{R}\}$$

and the minimal pullback $\mathcal{D}_\sigma^{M_H}$ -attractor

$$\mathcal{A}_{\mathcal{D}_\sigma^{M_H}} = \{\mathcal{A}_{\mathcal{D}_\sigma^{M_H}}(t) : t \in \mathbb{R}\},$$

for the process defined in Proposition 1. The family $\mathcal{A}_{\mathcal{D}_\sigma^{M_H}} \in \mathcal{D}_\sigma^{M_H}$ and the following relation holds:

$$\mathcal{A}_{\mathcal{D}_F^{M_H}}(t) \subset \mathcal{A}_{\mathcal{D}_\sigma^{M_H}}(t) \subset D_0(t), \quad \forall t \in \mathbb{R}. \quad (46)$$

Moreover, the pullback attractor $\mathcal{A}_{\mathcal{D}_\sigma^{M_H}}$ is unique (in the sense of Remark 1).

Proof. From Proposition 1, we observe that the process U is continuous in M_H . Furthermore, we can also obtain that there exists a pullback absorbing family $\widehat{D}_0 \in \mathcal{D}_\sigma^{M_H}$ from Proposition 2 and the process U is pullback $\mathcal{D}_\sigma^{M_H}$ -asymptotically compact from Theorem 4.7. Consequently, by using Theorem 4.3 and Lemma 4.4, we derive that the pullback attractors $\mathcal{A}_{\mathcal{D}_\sigma^{M_H}}$ and $\mathcal{A}_{\mathcal{D}_F^{M_H}}$ exist and

$$\mathcal{A}_{\mathcal{D}_F^{M_H}}(t) \subset \mathcal{A}_{\mathcal{D}_\sigma^{M_H}}(t), \quad \forall t \in \mathbb{R}. \quad (47)$$

Since $\widehat{D}_0 \in \mathcal{D}_\sigma^{M_H}$, $D_0(t)$ is closed for any $t \in \mathbb{R}$ and $\mathcal{D}_\sigma^{M_H}$ is inclusion-closed, we conclude from Remark 1 that $\mathcal{A}_{\mathcal{D}_\sigma^{M_H}}$ belongs to $\mathcal{D}_\sigma^{M_H}$ and $\mathcal{A}_{\mathcal{D}_\sigma^{M_H}}$ is unique. Moreover, in view of the property (d) in Theorem 4.3, we can get

$$\mathcal{A}_{\mathcal{D}_\sigma^{M_H}}(t) \subset D_0(t), \quad \forall t \in \mathbb{R},$$

which, along with (47), gives (46). The proof is complete. \square

Appendix: Background of the modified fluid flow models. In the Appendix of Ladyzhenskaya [23], the classical incompressible Navier-Stokes equations is approximated by using a class of regular Navier-Stokes systems which are described as

$$\begin{cases} u_t - \nu_0 \operatorname{div} \left[(1 + \varepsilon \hat{u}^2) \mathcal{D}u \right] + (u \cdot \nabla)u + \nabla p = f(t, x), \\ \nabla \cdot u = 0, \\ \mathcal{D}u = \nabla u + \nabla u^T, \quad \hat{u}^2 = \|\mathcal{D}u\|_{L^2}^2, \end{cases} \quad (48)$$

and its special case (2), which reflects the physical phenomena that $\|\nabla u(x, t)\|_{L^2(\Omega)}$ should not be too large or infinite. In this line of work, Smagorinsky in 1960s proposed a similar approximating equation, known as Ladyzhenskaya-Smagorinsky model

$$\begin{cases} u_t - \operatorname{div} \left[(\nu_0 + \nu_1 \|\mathcal{D}u\|_{L^2(\Omega)}^{p-2}) \mathcal{D}u \right] + (u \cdot \nabla)u + \nabla p = f(t, x), \\ \nabla \cdot u = 0, \quad \mathcal{D}u = \nabla u + \nabla u^T. \end{cases} \quad (49)$$

However, even for these systems, the uniqueness and stability are still open questions when Reynold number is large. To overcome this difficulty and simplify Ladyzhenskaya models, Lions [27] replaced $\mathcal{D}u$ by ∇u , and thus deduced another two systems as

$$\begin{cases} u_t - \nu_0 \Delta u - \nu_1 \sum_{i=1}^n \frac{\partial}{\partial x_i} (|\nabla u|^{p-1} \frac{\partial u}{\partial x_i}) + (u \cdot \nabla)u + \nabla p = f(t, x), \\ \nabla \cdot u = 0 \end{cases} \quad (50)$$

and

$$\begin{cases} u_t - \nu \sum_{i=1}^n \frac{\partial}{\partial x_i} (|\nabla u|^{p-1} \frac{\partial u}{\partial x_i}) + (u \cdot \nabla)u + \nabla p = f(t, x), \\ \nabla \cdot u = 0. \end{cases} \quad (51)$$

Lions [27] proved the existence of weak solutions to systems (50) and (51) for $p \geq 1 + 2n/(n+2)$ and the uniqueness of solutions to system (50) for $p \geq (n+2)/2$. However, the uniqueness of the system (51) is still an open problem.

Acknowledgments. The authors are grateful to the referees for their helpful suggestions which improved the presentation of this paper. Lan Huang was partially supported by the NSFC (No. 11501199 and No. 11871212) and the Young Key Teachers Project in Higher Vocational Colleges of Henan Province (No. 2020GZGG109). Xin-Guang Yang was partially supported by the Fund of Young Backbone Teachers in Henan Province (No. 2018GGJS039), Incubation Fund Project of Henan Normal University (No. 2020PL17) and Henan Overseas Expertise Introduction Center for Discipline Innovation (No. CXJD2020003).

REFERENCES

- [1] H. Bae, [Existence and analyticity of Lei-Lin solution to the Navier-Stokes equations](#), *Proc. Amer. Math. Soc.*, **143** (2015), 2887–2892.
- [2] J. M. Ball, [Global attractors for damped semi-linear wave equations](#), *Disc. Cont. Dyn. Syst.*, **10** (2004), 31–52.
- [3] T. Buckmaster and V. Vicol, [Nonuniqueness of weak solutions to the Navier-Stokes equation](#), *Ann. Math.*, **189** (2019), 101–144.
- [4] T. Caraballo and X. Han, [A survey on Navier-Stokes models with delays: Existence, uniqueness and asymptotic behavior of solutions](#), *Discrete Contin. Dyn. Syst. Ser. S*, **8** (2015), 1079–1101.
- [5] T. Caraballo and J. Real, [Navier-Stokes equations with delays](#), *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, **457** (2001), 2441–2453.
- [6] T. Caraballo and J. Real, [Asymptotic behaviour of Navier-Stokes equations with delays](#), *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, **459** (2003), 3181–3194.
- [7] T. Caraballo and J. Real, [Attractors for 2D Navier-Stokes models with delays](#), *J. Differential Equations*, **205** (2004), 271–297.
- [8] T. Caraballo, J. Real and A. M. Márquez, [Three-dimensional system of globally modified Navier-Stokes equations with delay](#), *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **20** (2010), 2869–2883.
- [9] Y. Chen, X. Yang and M. Si, [The long-time dynamics of 3D non-autonomous Navier-Stokes equations with variable viscosity](#), *ScienceAsia*, **44** (2018), 18–26.
- [10] B. Dong and W. Jiang, [On the decay of higher order derivatives of solutions to Ladyzhenskaya model for incompressible viscous flows](#), *Sci. China Ser. A*, **51** (2008), 925–934.
- [11] C. Foias, O. Manley, R. Rosa and R. Temam, [Navier-Stokes Equations and Turbulence](#), Cambridge University Press, Cambridge, 2001.
- [12] J. García-Luengo, P. Marín-Rubio and G. Planas, [Attractors for a double time-delayed 2D-Navier-Stokes model](#), *Disc. Contin. Dyn. Syst.*, **34** (2014), 4085–4105.

- [13] J. García-Luengo, P. Marín-Rubio and J. Real, [Pullback attractors for three-dimensional non-autonomous Navier-Stokes-Voigt equations](#), *Nonlinearity*, **25** (2012), 905–930.
- [14] J. García-Luengo, P. Marín-Rubio and J. Real, [Pullback attractors in \$V\$ for nonautonomous 2D-Navier-Stokes equations and their tempered behavior](#), *J. Differential Equations*, **252** (2012), 4333–4356.
- [15] J. García-Luengo, P. Marín-Rubio and J. Real, [Pullback attractors for 2D Navier-Stokes equations with delays and their regularity](#), *Adv. Nonlinear Stud.*, **13** (2013), 331–357.
- [16] C. Guo, R. Lu, X. Yang and P. Zhang, Dynamics for three dimensional generalized Navier-Stokes equations with delay, Preprint, (2021).
- [17] S. M. Guzzo and G. Planas, [On a class of three dimensional Navier-Stokes equations with bounded delay](#), *Discrete Contin. Dyn. Syst. Ser. B*, **16** (2011), 225–238.
- [18] X. Han, P. E. Kloeden and B. Usman, [Upper semi-continuous convergence of attractors for a Hopfield-type lattice model](#), *Nonlinearity*, **33** (2020), 1881–1906.
- [19] E. Hopf, [Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen](#), *Math. Nachr.*, **4** (1951), 213–231.
- [20] A. V. Kapustyan and J. Valero, [Weak and strong attractors for the 3D Navier-Stokes system](#), *J. Differential Equations*, **240** (2007), 249–278.
- [21] H. Koch and D. Tataru, [Well-posedness for the Navier-Stokes equations](#), *Adv. Math.*, **157** (2001), 22–35.
- [22] O. A. Ladyzhenskaya, On some nonlinear problems in the theory of continuous media, *Am. Math. Soc. Transl.*, **70** (1968), 73–89.
- [23] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, New York: Gordon and Breach, 1969.
- [24] J. Leray, Essai sur les mouvements plans d'un liquide visqueux que limitent des parois, *J. Math. Pure Appl.*, **13** (1934), 331–418.
- [25] D. Li, Q. Liu and X. Ju, [Uniform decay estimates for solutions of a class of retarded integral inequalities](#), *J. Differential Equations*, **271** (2021), 1–38.
- [26] L. Li, X.-G. Yang, X. Li, X. Yan and Y. Lu, [Dynamics and stability of the 3D Brinkman-Forchheimer equation with variable delay \(I\)](#), *Asymptot. Anal.*, **113** (2019), 167–194.
- [27] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Gauthier-Villars, Paris, 1969.
- [28] J.-L. Lions and G. Prodi, Une théorème d'existence et unicité dans les équations de Navier-Stokes en dimension 2, *C. R. Acad. Sci. Paris*, **248** (1959), 3519–3521.
- [29] P. L. Lions, *Mathematical Topics in Fluid Dynamics, Vol. 1, Incompressible Models*, Oxford Science Publication, Oxford, 1996.
- [30] L. Liu, T. Caraballo and P. Marín-Rubio, [Stability results for 2D Navier-Stokes equations with unbounded delay](#), *J. Differential Equations*, **265** (2018), 5685–5708.
- [31] P. Marín-Rubio, A. Márquez-Durán and J. Real, [Pullback attractors for globally modified Navier-Stokes equations with infinite delays](#), *Disc. Contin. Dyn. Syst.*, **31** (2011), 779–796.
- [32] G. Planas and E. Hernández, [Asymptotic behaviour of two-dimensional time-delayed Navier-Stokes equations](#), *Disc. Contin. Dyn. Syst.*, **21** (2008), 1245–1258.
- [33] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, Revised edition, North Holland Publishing Company-Amsterdam, New York, 1979.
- [34] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, 2nd edition, Applied Mathematical Sciences, **68**. Springer-Verlag, New York, 1997.
- [35] B. Wang and B. Guo, Asymptotic behavior of non-autonomous stochastic parabolic equations with nonlinear Laplacian principal part, *Electron. J. Differential Equations*, (2013), No. 191, 25 pp.
- [36] J. Wang, C. Zhao and T. Caraballo, [Invariant measures for the 3D globally modified Navier-Stokes equations with unbounded variable delays](#), *Comm. Nonl. Sci. Numer. Simul.*, **91** (2020), 105459, 14 pp.
- [37] X.-G. Yang, B. Feng, S. Wang, Y. Lu and T. F. Ma, [Pullback dynamics of 3D Navier-Stokes equations with nonlinear viscosity](#), *Nonlinear Anal. RWA*, **48** (2019), 337–361.
- [38] X.-G. Yang, B. Guo, C. Guo and D. Li, [The fractal dimension of pullback attractors for the 2D Navier-Stokes equations with delay](#), *Math. Meth. Appl. Sci.*, **43** (2020), 9637–9653.
- [39] X.-G. Yang, L. Li, X. Yan and L. Ding, [The structure and stability of pullback attractors for 3D Brinkman-Forchheimer equation with delay](#), *Electron. Res. Arch.*, **28** (2020), 1395–1418.

- [40] X.-G. Yang, R.-N. Wang, X. Yan and A. Miranville, [Dynamics of the 2D Navier-Stokes equations with sublinear operators in Lipschitz-like domain](#), *Disc. Contin. Dyn. Syst.*, **41** (2021), 3343–3366.
- [41] S. Zheng, [Nonlinear Evolution Equations](#), Monographs and Surveys in Pure and Applied Mathematics, 2004.

Received May 2021; revised July 2021; early access October 2021.

E-mail address: zhangpan95@hotmail.com

E-mail address: huanglan82@hotmail.com

E-mail address: luruilr1995@126.com

E-mail address: yangxinguang@hotmail.com