ELECTRONIC RESEARCH ARCHIVE
Volume **29**, Number **6**, December **2021**

doi:10.3934/era.2021073
pp. **4087–4098**

NON-GLOBAL SOLUTION FOR VISCO-ELASTIC DYNAMICAL SYSTEM WITH NONLINEAR SOURCE TERM IN CONTROL PROBLEM

Xiaoqiang Dai
Department of Electronic Information, Jiangsu University of Science and Technology
Zhenjiang 212003, China

Wenke Li*
College of Power and Energy Engineering, Harbin Engineering University
Harbin 150001, China

(Communicated by Runzhang Xu)

Abstract. In this paper, we study the initial boundary value problem of the visco-elastic dynamical system with the nonlinear source term in control system. By variational arguments and an improved convexity method, we prove the global nonexistence of solution, and we also give a sharp condition for global existence and nonexistence.

1. **Introduction.** This paper considers the initial-boundary value problem of visco-elastic dynamical system (viscoelasticity equation) with nonlinear source term

$$u_{tt} - \Delta u_t - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{m-1}\frac{\partial u}{\partial x_i}\right) = |u|^{p-1}u,\ x\in\Omega, t>0, \tag{1}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x\in\Omega, \tag{2}$$

$$u(x,t) = 0,\ x\in\partial\Omega,\ t\geq 0, \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, m and p satisfy the assumption

$$\text{(H)}\ \begin{cases} (i)\ \ 2\leq m+1<p+1<\frac{N(m+1)}{N-m-1} & \text{for } m+1<N,\\ (ii)\ 2\leq m+1<p+1<\infty & \text{for } m+1\geq N.\end{cases}$$

The viscoelasticity equation

$$u_{tt} - u_{xxt} = \sigma(u_x)_x, \tag{4}$$

was suggested and studied by Greenberg et al in [13] from viscoelasticity mechanics to describe the visco-elastic dynamical system and control problems. Under the condition $\sigma'(s) > 0$ and higher smooth conditions on $\sigma(s)$ and initial data they obtained the global existence of classical solutions for the initial boundary value problem of Eq. (4). After that many authors in [13], [1], [2], [6] and [3]

2020 *Mathematics Subject Classification.* Primary: 35L05.

Key words and phrases. Visco-elastic dynamical system, initial boundary value problem, global nonexistence, convexity, potential well.

The first author is supported by Jiangsu key R & D plan(BE2018007).

* Corresponding author: liwenke@hrbeu.edu.cn.

considered the global well-posedness of the IBVP for Eq.(4). And it is worth noting that special mentions, Dafermos [7], Greenberg [12], Davis [9], Andrems [1], [2] and Engler [10] obtained the global existence, the uniqueness and the stability of solution respectively. In [5], [6] and [3] Clements and Ang suggested some multidimensional viscoelasticity equation and gave the global existence of solution. In [24] Yang studied the blow up of solution for the initial-boundary value problem of multidimensional viscoelasticity equation. Then in [20] Yacheng Liu considered the initial-boundary value problem of multidimensional viscoelasticity equation $u_{tt} - \Delta u_t - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) + f(u_t) = g(u)$. By using the potential well theory they proved the global existence of weak solution under some conditions on the nonlinear effects $\sigma_i(s)$, $f(s)$, $g(s)$ and initial values. And the existence of the non-global solution is still unsolved. As we know, for the linear dynamical system, if the initial data are smooth enough, we can always expect the global existence of the solutions. However, we cannot expect the same for the nonlinear dynamical systems see [23], [22] and [21] as examples. Hence a lot of interest has been paid to the finite-time blow-up phenomena [26], [11], [4], not only for the parabolic type model [26], [21], [22], but also for the hyperbolic type model, and also the Schrödinger model [25]. Furthermore, combing the results from both sides of global existence and finite-time blow-up, we are also interested in the so-called sharp conditions [21], [23], [22].

The main aim of the present paper is to deal with the global nonexistence of solution to problem (1)-(3). Due to the presence of the term $-\Delta u_t$, the classical convexity method cannot be applied directly, hence some new skills are needed.

In Section 2 we give some definitions and prove some lemmas. In Section 3 we discuss the invariant sets of solution. In Section 4 we prove the global existence and nonexistence of solution and obtain a sharp condition for global existence and nonexistence of solution for problem (1)-(3).

In this paper, we denote $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ and $(u, v) = \int_\Omega uv\mathrm{d}x$.

2. **Some notations and lemmas.** We shall introduce some necessary notations and definitions, and prove some preliminary lemmas for our further analysis.

For problem (1)-(3) we define

$$J(u) = \frac{1}{m+1}\|\nabla u\|_{m+1}^{m+1} - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$
$$I(u) = \|\nabla u\|_{m+1}^{m+1} - \|u\|_{p+1}^{p+1},$$
$$d = \inf_{u\in\mathcal{N}} J(u),$$

where

$$\mathcal{N} = \{u \in W_0^{1,m+1}(\Omega) \mid I(u) = 0 \ u \neq 0\},$$

and

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{m+1}\|\nabla u\|_{m+1}^{m+1} - \frac{1}{p+1}\|u\|_{p+1}^{p+1} = \frac{1}{2}\|u_t\|^2 + J(u).$$

Lemma 2.1. *Let m and p satisfy (H), $u \in W_0^{1,m+1}(\Omega), u \neq 0$.*

(i) In the interval $0 < \lambda < \infty$ there exists a unique $\lambda^* = \lambda^*(u)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u) \mid_{\lambda=\lambda^*} = 0;$$

(ii) $J(\lambda u)$ is increasing on $0 < \lambda \le \lambda^*$, decreasing on $\lambda^* \le \lambda < \infty$ and takes the maximum at $\lambda = \lambda^*$;

(iii) $I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < \infty$ and $I(\lambda^* u) = 0$.

Proof. (i) From

$$J(\lambda u) = \frac{\lambda^{m+1}}{m+1}\|\nabla u\|_{m+1}^{m+1} - \frac{\lambda^{p+1}}{p+1}\|u\|_{p+1}^{p+1},$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u) = \lambda^m\|\nabla u\|_{m+1}^{m+1} - \lambda^p\|u\|_{p+1}^{p+1} = \lambda^m\left(\|\nabla u\|_{m+1}^{m+1} - \lambda^{p-m}\|u\|_{p+1}^{p+1}\right). \tag{5}$$

Hence there exists a unique

$$\lambda^* = \left(\frac{\|\nabla u\|_{m+1}^{m+1}}{\|u\|_{p+1}^{p+1}}\right)^{\frac{1}{p-m}}$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u)\mid_{\lambda=\lambda^*} = 0.$$

(ii) From (5) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u) > 0$$

for $0 < \lambda < \lambda^*$, and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u) < 0$$

for $\lambda^* < \lambda < \infty$, which gives the conclusion of (ii).

(iii) The conclusion of (iii) follows from

$$I(\lambda u) = \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u)$$

and the proof of part (iii).

□

Lemma 2.2. *Let m and p satisfy (H), $u \in W_0^{1,m+1}(\Omega), u \neq 0$.*

(i) When $0 < \|\nabla u\|_{m+1} < r_0$, one has $I(u) > 0$;
(ii) When $I(u) < 0$, one has $\|\nabla u\|_{m+1} > r_0$;
(iii) When $I(u) = 0$ and $u \neq 0$, one has $\|\nabla u\|_{m+1} \geq r_0$, where

$$r_0 = C_*^{-\frac{p+1}{p-m}},\ C_* = \sup_{u\in W_0^{1,m+1}(\Omega)/0}\frac{\|u\|_{p+1}}{\|\nabla u\|_{m+1}}.$$

Proof. (i) For the case $0 < \|\nabla u\|_{m+1} < r_0$, one has

$$\|u\|_{p+1}^{p+1} \leq C_*^{p+1}\|\nabla u\|_{m+1}^{p+1} = C_*^{p+1}\|\nabla u\|_{m+1}^{p-m}\|\nabla u\|_{m+1}^{m+1} < \|\nabla u\|_{m+1}^{m+1}$$

which says $I(u) > 0$.

(ii) For the case $I(u) < 0$, by

$$\|\nabla u\|_{m+1}^{m+1} < \|u\|_{p+1}^{p+1} \leq C_*^{p+1}\|\nabla u\|_{m+1}^{p-m}\|\nabla u\|_{m+1}^{m+1} < \|\nabla u\|_{m+1}^{m+1},$$

one has $\|\nabla u\|_{m+1} > r_0$.

(iii) For the case $I(u) = 0$ and $u \neq 0$, from

$$\|\nabla u\|_{m+1}^{m+1} < \|u\|_{p+1}^{p+1} \leq C_*^{p+1}\|\nabla u\|_{m+1}^{p-m}\|\nabla u\|_{m+1}^{m+1},$$

one knows $\|\nabla u\|_{m+1} \geq r_0$.

□

Lemma 2.3. *Let m and p satisfy (H). Then*

$$d \geq d_0 = \frac{1}{\alpha} C_*^{-\alpha}, \tag{6}$$

where

$$\alpha = \frac{(m+1)(p+1)}{p-m}.$$

Proof. For any $u \in \mathcal{N}$, considering Lemma 2.2 we know that $\|\nabla u\|_{m+1} \geq r_0$, where r_0 is defined in Lemma 2.2. Hence we have

$$\begin{aligned} J(u) &= \frac{1}{m+1}\|\nabla u\|_{m+1}^{m+1} + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &= \left(\frac{1}{m+1} - \frac{1}{p+1}\right)\|\nabla u\|_{m+1}^{m+1} + \frac{1}{p+1} I(u) \\ &= \frac{p-m}{(m+1)(p+1)}\|\nabla u\|_{m+1}^{m+1} \\ &\geq \frac{p-m}{(m+1)(p+1)} r_0^{m+1} \\ &= \frac{p-m}{(m+1)(p+1)} \left(C_*^{-\frac{p+1}{p-m}}\right)^{m+1}, \end{aligned}$$

which gives (6) □

Lemma 2.4. *Let m and p satisfy (H). Assume that $u \in W_0^{1,m+1}(\Omega)$ and $I(u) < 0$. Then there holds*

$$I(u) < (p+1)(J(u) - d). \tag{7}$$

Proof. From $I(u) < 0$ and Lemma 2.1 it follows that there exists a $\lambda^* \in (0,1)$ such that $I(\lambda^* u) = 0$.

Set

$$g(\lambda) = (p+1)J(\lambda u) - I(\lambda u), \quad \lambda > 0.$$

Then from

$$\begin{aligned} g(\lambda) =& (p+1)\left(\frac{\lambda^{m+1}}{m+1}\|\nabla u\|_{m+1}^{m+1} - \frac{\lambda^{p+1}}{p+1}\|\nabla u\|_{p+1}^{p+1}\right) \\ &- \left(\lambda^{m+1}\|\nabla u\|_{m+1}^{m+1} - \lambda^{p+1}\|\nabla u\|_{p+1}^{p+1}\right) \\ =& \frac{p-m}{m+1}\lambda^{m+1}\|\nabla u\|_{p+1}^{p+1}, \end{aligned}$$

we get

$$g'(\lambda) = (p-m)\lambda^m \|\nabla u\|_{m+1}^{m+1} > 0, \quad \lambda > 0.$$

Hence $g(\lambda)$ is strictly increasing for $\lambda > 0$, which gives that $g(1) > g(\lambda^*)$, i.e.

$$(p+1)J(u) - I(u) > (p+1)J(\lambda^* u) - I(\lambda^* u) = (p+1)J(\lambda^* u) \geq (p+1)d,$$

which gives (7). □

Now for problem (1)-(3) we define

$$\begin{aligned} W &= \{u \in W_0^{1,m+1}(\Omega) \mid I(u) > 0,\ J(u) < d\} \cup \{0\}, \\ V &= \{u \in W_0^{1,m+1}(\Omega) \mid I(u) < 0,\ J(u) < d\}, \\ W' &= \{u \in W_0^{1,m+1}(\Omega) \mid I(u) > 0, \} \cup \{0\}, \\ V' &= \{u \in W_0^{1,m+1}(\Omega) \mid I(u) < 0\}. \end{aligned}$$

3. **Invariant sets of solution.** In this section we discuss the invariant sets of the solution to problem (1)-(3). First we give the definition of the weak solution to the problem (1)-(3).

Definition 3.1. Function $u = u(x,t)$ is called a weak solution of problem (1)-(3) on $\Omega \times [0,T)$ provided

$$u \in L^\infty(0,T; W_0^{1,m+1}(\Omega))$$

and

$$u_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$$

satisfying

(i)

$$\begin{aligned}&(u_t, v) + (\nabla u, \nabla v) + \sum_{i=1}^{N} \int_0^t \left(|\frac{\partial u}{\partial x_i}|^{m-1} \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right) \mathrm{d}\tau \\ &= \int_0^t \left(|u|^{p-1} u, v \right) \mathrm{d}\tau + (u_1, v) + (\nabla u_0, v),\end{aligned} \tag{8}$$

$$\text{for all } v \in W_0^{1,m+1}(\Omega), \quad t \in [0,T)$$

(ii)

$$u(x,0) = u_0(x) \text{ in } W_0^{1,m+1}(\Omega), \quad u_t(x,0) = u_1(x,0) = u_1(x) \text{ in } L^2(\Omega). \tag{9}$$

(iii)

$$E(t) + \int_0^t \|\nabla u_\tau\|^2 \mathrm{d}\tau \le E(t), \quad 0 \le t < T. \tag{10}$$

Theorem 3.2. *Let m and p satisfy (H), $u_0(x) \in W_0^{1,m+1}(\Omega)$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) < d$. Then both sets W' and V' are invariant under the flow of (1)-(3) respectively.*

Proof. (i) Let $u(x)$ be any weak solution of problem (1)-(3) with $E(0) < d$, $u_0(x) \in W'$, for $0 < t < T$. If it is false, then there exists a $t_0 \in (0,T)$ such that $u(t_0) \in \partial W'$, i.e. $I((u(t_0)) = 0$, $u(t_0) \neq 0$, which implies $u(t_0) \in \mathcal{N}$. Hence by the definition of d we have $J(u(t_0)) \ge d$, which contradicts (10), i.e.

$$\frac{1}{2}\|u_t\|^2 + J(u) + \int_2^t \|\nabla u_\tau\|^2 \mathrm{d}\tau \le E(0) < d. \tag{11}$$

(ii) Let $u(x)$ be any weak solution of problem (1)-(3) with $E(0) < d$, $I(u_0) < 0$, T be the existence time of u. Let us prove that $u(t) \in V'$ for $0 < t < T$. By contradiction, suppose it is not true, then there exists a $t_0 \in (0,T)$ such that $I(u(t_0)) < 0$ and $I(u) < 0$ for $0 \le t < t_0$. Hence we have $\|\nabla u(t_0)\| \ge r_0$. Again by the definition of d we get $J(u(t_0) \ge r_0$, which contradicts (11).

□

Corollary 1. *Let m and p satisfy (H), $u_0(x) \in W_0^{1,m+1}(\Omega)$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) < d$.*

(i) A weak solutions of problem (1)-(3) belong to W, provided $u_0(x) \in W'$.
(ii) A weak solutions of problem (1)-(3) belong to V, provided $u_0(x) \in V'$.

Corollary 2. *Let $u(x)$ be any weak solution of problem (1)-(3) with $E(0) < d$, $I(u_0) < 0$ or $E(0) = 0$, $u_0(x) \neq 0$. Then all weak solution of problem (1)-(3) belong to V.*

4. **Sharp condition: Global and non-global solutions.** We shall deal with the global and non-global solutions in this section in order to finally give a sharp condition.

Lemma 4.1. *Let m and p satisfy (H). Then the embedding $W^{1,m+1}(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact.*

Lemma 4.2. [14], [15] *Assume that there exists a $\beta > 0$ such that*

$$\phi(t)\ddot{\phi}(t) - (\beta+1)(\dot{\phi}(t))^2 \geq 0, \quad t > 0$$

and $\phi(0) > 0$, $\dot{\phi}(0) > 0$. Then there exists a T satisfying

$$T \leq \frac{\phi(0)}{\beta\dot{\phi}(0)}$$

such that

$$\lim_{t\to T} \phi(t) = +\infty.$$

Next we first consider the global existence of solution of problem (1)-(3). Note that for $\sigma_i(s) = |s|^{m-1}s$ and $g(s) = |s|^{p-1}s$, where m and p satisfy (H), the assumption (H_1) and (H_1) and (H_3) in [20] hold. However the definition of weak solution for the IBVP of the multi-dimensional model does not allow the energy inequality (10). Hence the result of Theorem 3.1 in [20] should be seriously treated, and for the energy inequality (10) we need to prove the global existence of solution for problem (1)-(3).

Theorem 4.3. *Let m and p satisfy (H), $u_0(x) \in W_0^{1,m+1}(\Omega)$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) < d$, $u_0(x) \in W'$. Then problem (1)-(3) admits a global weak solution*

$$u \in L^\infty(0,T; W_0^{1,m+1}(\Omega)),$$
$$u_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$$

and

$$u \in W \text{ for } 0 \leq t < \infty.$$

Proof. Let $\{w_j(x)\}_{j=1}^\infty$ be the basis in $W_0^{1,m+1}(\Omega)$. We take the approximate solutions of problem (1)-(3) as

$$u_n(x,t) = \sum_{j=1}^n g_{jn}(t) w_j(x), \quad n = 1, 2\cdots$$

satisfying

$$(u_t, w_s) + (\nabla u, \nabla w_s) + \sum_{i=1}^N \left(|\frac{\partial u}{\partial x_i}|^{m-1} \frac{\partial u}{\partial x_i}, \frac{\partial w_s}{\partial x_i} \right) = \left(|u|^{p-1}u, w_s \right), \tag{12}$$

$$u_n(x,0) = \sum_{j=1}^n g_{jn}(0) w_j(x) \to u_0(x) \text{ in } W_0^{1,m+1}(\Omega), \tag{13}$$

$$u_{nt}(x,0) = \sum_{j=1}^n g'_{jn}(0) w_j(0) \to u_1(x) \text{ in } L^2(\Omega). \tag{14}$$

Multiplying (12) by $g'_{sn}(t)$ and summing for s we get

$$\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} + \|\nabla u_{nt}\|^2 = 0$$

and

$$E_n(t) + \int_0^t \|\nabla u_{n\tau}\|^2 \mathrm{d}\tau = E_n(0), \quad 0 \le t < \infty. \tag{15}$$

Hence we have

$$E_n(t) = \frac{1}{2}\|u_{nt}\|^2 + J(u_n).$$

From $E(0) < d$ (13) and (14) we can get $E_n(0) < 0$ for sufficiently large n, which together with (15) gives

$$\frac{1}{2}\|u_{nt}\|^2 + J(u_n) + \int_0^t \|\nabla u_{n\tau}\|^2 \mathrm{d}\tau < d, \quad 0 \le t < \infty, \tag{16}$$

for sufficiently large n. Since W' is an open set in $W_0^{m+1}(\Omega)$, from $u_0(x) \in W'$ and (13), we get $u_n(0) \in W'$, for sufficiently large n. Next we prove that $u_n(t) \in W'$,for sufficiently large n. In fact, if it is false, then there exists a $t_0 > 0$ such that $u_n(t_0) \in \partial W'$, i.e., $I(u_n(t_0)) = 0$ and $u_n(t_0) \neq 0$, which means $u_n(t_0) \in \mathcal{N}$. Hence we have $J(u_n(t_0) \ge d$, which contradicts (16).

From (16) we get

$$\frac{1}{2}\|u_{n\tau}\|^2 + \frac{p-m}{(m+1)(p+1)}\|\nabla u_n\|_{m+1}^{m+1} + \frac{1}{p+1}I(u_n) + \int_0^t \|\nabla u_{n\tau}\|^2 \mathrm{d}\tau < d, \tag{17}$$

$$0 \le t < \infty,$$

which together with $u_n(t) \in W'$ gives

$$\|\nabla u_n\|_{m+1}^{m+1} < \frac{(m+1)(p+1)}{p-m} d, \ 0 \le t < \infty, \tag{18}$$

$$\|u_{nt}\|^2 < 2d, \ 0 \le t < \infty, \tag{19}$$

$$\int_0^t \|\nabla u_{n\tau}\|^2 \mathrm{d}\tau < d, \ 0 \le t < \infty. \tag{20}$$

From (18)-(20) it follows that there exist a subsequence $\{u_v\}$ of $\{u_n\}$ such that as $v \to \infty$,

$u_v \to u$ in $L^\infty(0,\infty; W_0^{1,m+1}(\Omega))$ weakly star, and a.e. in $Q = \Omega \times [0,\infty)$,

$u_v \to u$ in $L^{p+1}(\Omega)$ strongly for any $t > 0$,

$u_{vt} \to u_t$ in $L^\infty(0,\infty; L^2(\Omega))$ weakly star and in $L^2(0,\infty; H_0^1(\Omega))$ weakly, and

$$|\frac{\partial u_v}{\partial x_i}|^{m-1}\frac{\partial u_v}{\partial x_i} \to \chi_i = |\frac{\partial u}{\partial x_i}|^{m-1}\frac{\partial u}{\partial x_i} \text{ in } L^\infty(0,\infty; L^r(\Omega)) \text{ weakly star,}$$

where

$$r = \frac{m+1}{m}.$$

Then in the following distribution,

$$\begin{aligned}&(u_{nt}, w_s) + (\nabla u_n, \nabla w_s) + \sum_{i=1}^N \int_0^t \left(|\frac{\partial u_n}{\partial x_i}|^{m-1}\frac{\partial u_n}{\partial x_i}, \frac{\partial w_s}{\partial x_i}\right) \mathrm{d}\tau \\ =& \int_0^t \left(|u_n|^{p-1}u_n, w_s\right) \mathrm{d}\tau + (u_{nt}(0), w_s) + (\nabla u_n(0), \nabla w_s),\end{aligned} \tag{21}$$

by taking $m = v \to \infty$ we get

$$(u_t, w_s) + (\nabla u, \nabla w_s) + \sum_{i=1}^{N} \int_0^t \left(|\frac{\partial u}{\partial x_i}|^{m-1} \frac{\partial u}{\partial x_i}, \frac{\partial w_s}{\partial x_i} \right) \mathrm{d}\tau$$
$$= \int_0^t \left(|u|^{p-1} u, w_s \right) \mathrm{d}\tau + (u_1, w_s) + (\nabla u_0, \nabla w_s),$$
$$\forall\ s\ ,\ 0 \le t < \infty,$$

and

$$(u_t, v) + (\nabla u, \nabla v) + \sum_{i=1}^{N} \int_0^t \left(|\frac{\partial u}{\partial x_i}|^{m-1} \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right) \mathrm{d}\tau$$
$$= \int_0^t \left(|u|^{p-1} u, v \right) \mathrm{d}\tau + (u_1, v) + (\nabla u_0, \nabla v),$$
$$\forall\ v \in W_0^{1,m+1}(\Omega), \quad \forall\ t \in [0, \infty).$$

Also, from (13) and (14) we have $u(x,0) = u_0(x)$ in $W_0^{1,m+1}(\Omega)$, $u_t(x,0) = u_1(x)$ in $L^2(\Omega)$. Next we prove that above equation satisfies (10) for $0 \le t < \infty$. Let $\{u_v\}$ be the subsequence of $\{u_n\}$ liked above. Then we have $E_v(0) \to E(0)$ and $\|u_v\|_{p+1} \to \|u\|^{p+1}$ as $v \to \infty$. Hence from (15) we get

$$\frac{1}{2}\|u_t\|^2 + \frac{1}{m+1}\|\nabla u\|_{m+1}^{m+1} + \int_0^t \|\nabla u_\tau\|^2 \mathrm{d}\tau$$
$$\le \liminf_{v\to\infty} \frac{1}{2}\|u_{vt}\|^2 + \liminf_{v\to\infty} \frac{1}{m+1}\|\nabla u_v\|_{m+1}^{m+1} + \liminf_{v\to\infty} \int_0^t \|\nabla u_{v\tau}\|^2 \mathrm{d}\tau$$
$$= \liminf_{v\to\infty} \left(\frac{1}{2}\|u_{vt}\|^2 + \frac{1}{m+1}\|\nabla u_v\|_{m+1}^{m+1} + \int_0^t \|\nabla u_{v\tau}\|^2 \mathrm{d}\tau \right)$$
$$= \liminf_{v\to\infty} \left(E_v(0) + \frac{1}{m+1}\|\nabla u_v\|_{m+1}^{m+1} \right)$$
$$= \lim_{v\to\infty} \left(E_v(0) + \frac{1}{m+1}\|\nabla u_v\|_{m+1}^{m+1} \right)$$
$$= E(0) + \frac{1}{m+1}\|\nabla u\|_{m+1}^{m+1}, \quad 0 \le t < \infty,$$

which gives

$$E(t) + \int_0^t \|\nabla u_\tau\|^2 \mathrm{d}\tau < E(0), \quad 0 \le t < \infty.$$

Finally from Corollary 10 we get $u \in W$ for $0 \le t < \infty$. □

Theorem 4.4. *Let m and p satisfy (H), $u_0(x) \in W_0^{1,m+1}(\Omega)$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) < d$, $I(u_0) < 0$. Then problem (1)-(3) does not admits any global weak solution.*

Proof. Let $u \in L^\infty(0,T; W_0^{1,m+1}(\Omega))$, $u_t \in L^\infty(0,T; L^2(\Omega) \cap L^2(0,T; H_0^1(\Omega))$ be any weak solution of problem (1)-(3) with $E(0) < 0$, $I(u_0) < 0$, T be the existence time of u. Next we prove $T < \infty$. If it is false, then we have $T = +\infty$, and $u \in L^\infty(0,T; W_0^{1,m+1}(\Omega))$, $u_t \in L^\infty(0,T; L^2(\Omega) \cap L^2(0,T; H_0^1(\Omega))$. Set

$$\phi(t) = \|u\|^2 + \int_0^t \|\nabla u_\tau\|^2 \mathrm{d}\tau + (T_0 - t)\|\nabla u_0\|^2, \quad 0 \le t < \infty,$$

Then

$$\dot{\phi}(t) = 2(u_t, u) + \|\nabla u\|^2 - \nabla u_0\|^2 = 2(u_t, u) + 2\int_0^t (\nabla u_\tau, \nabla u)\mathrm{d}\tau, \qquad (22)$$

$$0 \le t < \infty,$$

and

$$\ddot{\phi}(t) = 2\|u_t\|^2 + 2(u_{tt}, u) + 2(\nabla u_t, \nabla u). \qquad (23)$$

Note that from Definition 3.1 and Eq. (1) we get $u_{tt} \in L^\infty(0, \infty; W^{-1,r}(\Omega))$, $r = \frac{m+1}{m}$. Hence the inner product (u_{tt}, u) makes sense. So by Eq. (1) we get

$$2(u_{tt}, u) = -2(\nabla u_t, \nabla u) - 2I(u)$$

and

$$\ddot{\phi}(t) = 2\|u_t\|^2 - 2I(u), \quad 0 \le t < \infty. \qquad (24)$$

Moreover from (21) we get

$$\begin{aligned}\ddot{\phi}(t) &= \left((u_t, u) + \int_0^t (\nabla u_\tau, \nabla u)\mathrm{d}\tau\right)^2 \\ &= 4\left((u_t, u)^2 + 2(u_t, u)\int_0^t (\nabla u_\tau, \nabla u)^2\mathrm{d}\tau + \left(\int_0^t (\nabla u_\tau, \nabla u)^2\mathrm{d}\tau\right)^2\right).\end{aligned} \qquad (25)$$

By using Schwartz inequality we have

$$(u_t, u)^2 \le \|u_t\|^2\|u\|^2, \qquad (26)$$

$$\begin{aligned}\left(\int_0^t (\nabla u_\tau, \nabla u)^2\mathrm{d}\tau\right)^2 &\le \int_0^t \|\nabla u_\tau\|^2\mathrm{d}\tau \int_0^t \|\nabla u\|^2\mathrm{d}\tau \\ &\le 2(u_t, u)\int_0^t (\nabla u_\tau, \nabla u)^2\mathrm{d}\tau \\ &\le 2\|u_t\|^2\|u\|^2\left(\int_0^t \|\nabla u_\tau\|^2\mathrm{d}\tau\right),\end{aligned} \qquad (27)$$

and

$$\left(\int_0^t \|\nabla u\|^2\mathrm{d}\tau\right)^2 \le \|u\|^2\int_0^t \|\nabla u_\tau\|^2\mathrm{d}\tau + \|u_\tau\|^2\int_0^t \|\nabla u\|^2\mathrm{d}\tau. \qquad (28)$$

Substituting (26)-(28) into (25) we obtain

$$\begin{aligned}\dot{\phi}(t) &\le 4\left(\|u\|^2 + \int_0^t \|\nabla u\|^2\mathrm{d}\tau\right)\left(\|u_t\|^2 + \int_0^t \|\nabla u_\tau\|^2\mathrm{d}\tau\right) \\ &\le 4\phi(t)\left(\|u_t\|^2 + \int_0^t \|\nabla u_\tau\|^2\mathrm{d}\tau\right), \quad 0 \le t < T_0.\end{aligned}$$

Hence we have

$$\begin{aligned}&\phi(t)\ddot{\phi}(t) - \frac{p+4}{4}\dot{\phi}^2(t) \\ \ge&\phi(t)\left(2\|u_t\|^2 - 2I(u) - (p+3)\|u_t\|^2 - (p+3)\int_0^t \|\nabla u_\tau\|^2\mathrm{d}\tau\right) \\ =&-(p+1)\|u_t\|^t - 2I(u) - (p+3)\int_0^t \|\nabla u_\tau\|^2\mathrm{d}\tau.\end{aligned} \qquad (29)$$

From the energy inequality

$$\frac{1}{2}\|u_t\|^2 + J(u) + \int_0^t \|\nabla u_\tau\|^2 \mathrm{d}\tau \leq E(0)$$

we get

$$-(p+1)\|u_t\|^t \geq 2(p+1)(J(u) - E(0)) + 2(p+1)\int_0^t \|\nabla u_\tau\|^2 \mathrm{d}\tau.$$

Hence we have

$$\begin{aligned}
&\phi(t)\ddot{\phi}(t) - \frac{p+4}{4}\dot{\phi}^2(t) \\
\geq& 2\phi(t)\left((p+1)(J(u) - E(0)) - I(u) + (p-1)\int_0^t \|\nabla u_\tau\|^2 \mathrm{d}\tau\right) \\
\geq& 2\phi(t)\left((p+1)(J(u) - E(0)) - I(u)\right) \\
\geq& 2\phi(t)\left((p+1)(J(u) - d) - I(u)\right), \quad 0 \leq t \leq T_0.
\end{aligned} \tag{30}$$

From $I(u_0) < 0$ and Theorem 3.2 we have $I(u) < 0$ for $0 \leq t < \infty$. Hence by Lemma 2.4 we get

$$(p+1)(J(u) - d) - I(u) > 0.$$

In addition from $I(u) < 0$ and Lemma 6 we have $\|\nabla u\|_{m+1} > r_0 > 0$, which gives $\phi(t) > 0$ for $0 \leq t \leq T_0$. Thus we have

$$\phi(t)\ddot{\phi}(t) - \frac{p+4}{4}\dot{\phi}^2(t) > 0, \;\; 0 \leq t \leq T_0. \tag{31}$$

Additionally, from (24) and (7) we get

$$\ddot{\phi}(t) \geq -2I(u) > 2(p+1)(d - J(u)) \geq 2(p+1)(d - E(0)) = C_0 > 0, \;\; 0 \leq t \leq \infty,$$

and

$$\dot{\phi}(t) \geq C_0 t + \dot{\phi}(0), \quad 0 \leq t < \infty.$$

Hence there exists a $t_0 \geq 0$ such that

$$\dot{\phi}(t_0) > \frac{4}{p-1}\|\nabla u_0\|^2.$$

Define

$$T_0 = \frac{\|u(t_0)\|^2 + \int_0^{t_0} \|\nabla u\|^2 \mathrm{d}\tau}{\frac{p-1}{4}\dot{\phi}(t_0) - \|\nabla u_0\|^2} + t_0.$$

Then $T_0 > t_0$ and

$$\frac{\|u(t_0)\|^2 + \int_0^{t_0} \|\nabla u\|^2 \mathrm{d}\tau + (T_0 - t_0)\|\nabla u_0\|^2}{\frac{p-1}{4}\dot{\phi}(t_0)} = T_0 - t_0.$$

Hence from Lemma 13 (consider $t = t_0$ as initial time) it follows that there exists a T_1 satisfying

$$T_1 - t_0 \leq \frac{\phi(t_0)}{\frac{p-1}{4}\dot{\phi}(t_0)} = T_0 - t,$$

such that

$$\lim_{t \to T_1} \phi(t) = +\infty,$$

which implies

$$\lim_{t \to T_1}\left(\|u\|^2 + \int_0^t \|\nabla u\|^2\right) = +\infty,$$

and contradicts $T = +\infty$. □

From above two established theorems for the global and non-global solution respectively, we can now arrive at the sharp condition as follows.

Theorem 4.5. *Let m and p satisfy (H), $u_0(x) \in W_0^{1,m+1}(\Omega)$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) < d$, Then when $I(u_0) > 0$ problem (1)-(3) admits a global weak solution, and when $I(u_0) < 0$ problem (1)-(3) dose not admit any global weak solution.*

Acknowledgments. We would like to thank the reviewers for their careful and constructive suggestions that have led to improvements in this manuscript.

REFERENCES

[1] G. Andrews, On the existence of solutions to the equation $u_{tt} - u_{xxt} = \sigma(u_x)_x$, *J. Differential Equations*, **35** (1980), 200–231.
[2] G. Andrews and J. M. Ball, Asymptotic behavior and changes in phase in one-dimensional nonlinear viscoelasticity, *J. Differential Equations*, **44** (1982), 306–341.
[3] D. D. Áng and A. Pham Ngoc Dinh, Strong solutions of a quasilinear wave equation with nonlinear damping, *SIAM J. Math. Anal.*, **19** (1988), 337–347.
[4] Y. Cao, Blow-up criterion for the 3D viscous polytropic fluids with degenerate viscosities, *Elec. Res. Arch.*, **28** (2020), 27–46.
[5] J. Clements, Existence theorems for a quasilinear evolution equation, *SIAM. J. Appl. Math.*, **226** (1974), 745–752.
[6] J. C. Clements, On the existence and uniqueness of solutions of the equation $u_{tt} - \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) - \Delta_N u_t = f$, *Canad. Math. Bull.*, **18** (1975), 181–187.
[7] C. M. Dafermos, The mixed initial-boundary value problem for the equations of nonlinear one-dimensional visco-elasticity, *J. Differential Equations*, **6** (1969), 71–86.
[8] X. Dai, C. Yang, S. Huang, Y. Tao and Y. Zhu, Finite time blow-up for a wave equation with dynamic boundary condition at critical and high energy levels in control systems, *Elec. Res. Arch.*, **28** (2020), 91–102.
[9] P. L. Davis, A quasi-linear hyperbolic and related third-order equation, *J. Math. Anal. Appl.*, **51** (1975), 596–606.
[10] H. Engler, Strong solutions for strongly damped quasilinear wave equations, *Contemp. Math.*, **64** (1987), 219–237.
[11] J. A. Esquivel-Avila, Blow-up in damped abstract nonlinear equations, *Elec. Res. Arch.*, **28** (2020), 347–367.
[12] J. M. Greenberg and R. C. MacCamy, On the exponential stability of solutions of $E(u_x)u_{xx} + \lambda u_{xtx} = \rho u_{tt}$, *J. Math. Anal. Appl.*, **31** (1970), 406–417.
[13] J. M. Greenberg, R. C. MacCamy and V. J. Mizel, On the existence, uniqueness and stability of the equation $\sigma(u_x)u_{xx} + \lambda u_{xxt} = \rho_0 u_{tt}$, *J. Math. Mech.*, **17** (1968), 707–728.
[14] H. A. Levine, Instablity and nonexistence of global solutions to nonlinear wave equations of the form $Pu = Au + F(u)$, *Trans. Amer. Math. Soc.*, **192** (1974), 1–21.
[15] H. A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, *SIAM J. Math. Anal.*, **5** (1974), 138–146.
[16] W. Lian, M. S. Ahmed and R. Xu, Global existence and blow up of solution for semi-linear hyperbolic equation with the product of logarithmic and power-type nonlinearity, *Opuscula Math.*, **40** (2020), 111–130.
[17] W. Lian and R. Xu, Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term, *Adv. Nonlinear Anal.*, **9** (2020), 613–632.
[18] G. Liu, The existence, general decay and blow-up for a plate equation with nonlinear damping and a logarithmic source term, *Elec. Res. Arch.*, **28** (2020), 263–289.
[19] Y. Liu and W. Li, A family of potential wells for a wave equation, *Elec. Res. Arch.*, **28** (2020), 807–820.
[20] Y. Liu and J. Zhao, Multidimensional viscoelasticity equations with nonlinear damping and source terms, *Nonlinear Anal.*, **56** (2004), 851–865.

[21] X. Wang and R. Xu, Global existence and finite time blowup for a nonlocal semilinear pseudo-parabolic equation, *Adv. Nonlinear Anal.*, **10** (2021), 261–288.

[22] R. Xu, W. Lian and Y. Niu, Global well-posedness of coupled parabolic systems, *Sci. China Math.*, **63** (2020), 321–356.

[23] R. Xu and J. Su, Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations, *J. Funct. Anal.*, **264** (2013), 2732–2763.

[24] Z. Yang, Initial-boundary value problem and Cauchy problem for a quasilinear evolution equation, *Acta Math. Sci.*, **19** (1999), 487–496.

[25] M. Zhang and M. S. Ahmed, Sharp conditions of global existence for nonlinear Schrödinger equation with a harmonic potential, *Adv. Nonlinear Anal.*, **9** (2020), 882-894.

[26] M. Zhang, Q. Zhao, Y. Liu and W. Li, Finite time blow-up and global existence of solutions for semilinear parabolic equations with nonlinear dynamical boundary condition, *Elec. Res. Arch.*, **28** (2020), 369–381.

Received July 2021; revised August 2021; early access September 2021.

E-mail address: daixiaoqiang@just.edu.cn
E-mail address: liwenke@hrbeu.edu.cn