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NONEXISTENCE OF ENTIRE POSITIVE SOLUTIONS FOR CONFORMAL HESSIAN QUOTIENT INEQUALITIES

Feida Jiang*
School of Mathematics and Shing-Tung Yau Center of Southeast University
Southeast University
Nanjing 211189, China
College of Mathematics and Statistics
Nanjing University of Information Science and Technology
Nanjing 210044, China

Xi Chen
College of Mathematics and Statistics
Nanjing University of Information Science and Technology
Nanjing 210044, China

Juhua Shi
Faculty of Mathematics and Statistics
Hubei Key Laboratory of Applied Mathematics, Hubei University
Wuhan 430062, China

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Abstract. In this paper, we consider the nonexistence problem for conformal Hessian quotient inequalities in $\mathbb{R}^n$. We prove the nonexistence results of entire positive k-admissible solution to a conformal Hessian quotient inequality, and entire (k,k')-admissible solution pair to a system of Hessian quotient inequalities, respectively. We use the contradiction method combining with the integration by parts, suitable choices of test functions, Taylor's expansion and Maclaurin's inequality for Hessian quotient operators.

1. Introduction.

In this paper, we study a conformal Hessian quotient inequality:

$$\frac{S_k}{S_l}\left(A^g u\right) \geq e^{\alpha u}, \tag{1}$$

where $\frac{S_k}{S_l}\left(A^g u\right) := \frac{S_k}{S_l}\left(\lambda\left(A^g u\right)\right)$, $0 \leq l < k \leq n$, $k \in \mathbb{N}^+$, $l \in \mathbb{N}$, u is the unknown function, α is a non-negative constant. Note that in Euclidean space $\mathbb{R}^n$, the conformal Hessian matrix $A^g u$ in (1) has the form

$$A^g u = D^2 u - \left[a(x)|Du|^2 I - b(x) Du \otimes Du\right], \tag{2}$$

where $a(x)$ and $b(x)$ denote the functions of x, Du and D^2u denote the gradient vector and Hessian matrix of u respectively, I is the $n \times n$ identity matrix. In (1), the

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* Corresponding author: Feida Jiang.

operator S_k denotes the Hessian operator or the k-th order elementary symmetric polynomial given by

$$S_k(\lambda) = \sum_{i_1<\cdots<i_s} \prod_{s=1}^{k} \lambda_{i_s}, \tag{3}$$

where $k = 1, \ldots, n$, $i_1, \ldots, i_s \in \{1, \ldots, n\}$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of the matrix $A^g u$. $S_k(A^g u) := S_k(\lambda(A^g u))$ denotes the sum of the $k \times k$ principle minors of the matrix $A^g u$. As usual, we define $S_0(\lambda) \equiv 1$, see [17]. Then, naturally we can define the quotient operator in (1) to be

$$\frac{S_k}{S_l}(A^g u) = \frac{S_k}{S_l}(\lambda(A^g u)) = \frac{\sum\limits_{i_1<\cdots<i_s} \prod\limits_{s=1}^{k} \lambda_{i_s}}{\sum\limits_{i_1<\cdots<i_t} \prod\limits_{t=1}^{l} \lambda_{i_t}}. \tag{4}$$

When $l = 0$, the leading term $S_k(A^g u)$ in the inequality (1) is related to the k-Yamabe problem, see [15, 18]. Specially, when $k = 1$, it is related to the well-known Yamabe problem, see [1, 2, 14, 16, 19]. According to Caffarelli-Nirenberg-Spruck (see [4]), we say that u is a k-admissible function of (2) if

$$\lambda (A^g u) \in \Gamma^k := \{\lambda \in \mathbb{R}^n : S_i(\lambda) > 0,\ i = 1, \ldots, k\}. \tag{5}$$

We give the following nonexistence result of positive k-admissible solution of the inequality (1).

Theorem 1.1. *The inequality* (1) *with $A^g u$ in the form* (2) *has no entire positive k-admissible solution in $\mathbb{R}^n$ for either $\inf\limits_{x\in\mathbb{R}^n}(na(x)-b(x)) \geq 0,\ \alpha > 0$ or $\inf\limits_{x\in\mathbb{R}^n}(na(x)-b(x)) > 0,\ \alpha \geq 0$.*

Remark 1. Note that the functions $a(x)$ and $b(x)$ should satisfy the conditions in Theorem 1.1. When $a(x)$ is positive, $b(x)$ can be positive or negative. When $a(x)$ is negative, $b(x)$ can only be negative.

The above theorem is a nonexistence result of positive solutions to a single Hessian quotient inequality. Naturally, we can consider the system of Hessian quotient inequalities. We then study the following system of Hessian quotient inequalities:

$$\begin{cases} \frac{S_k}{S_l}(A^g u) \geq e^{\alpha v}, \\ \frac{S_{k'}}{S_{l'}}(A^g v) \geq e^{\beta u}, \end{cases} \tag{6}$$

where $\alpha \geq 0$, $\beta \geq 0$, $0 \leq l < k \leq n$, $0 \leq l' < k' \leq n$, $k' \in \mathbb{N}^+$, $l' \in \mathbb{N}$, $A^g u$ is defined in (2), and

$$A^g v = D^2 v - \left(a(x)|Dv|^2 I - b(x) Dv \otimes Dv\right). \tag{7}$$

If a k-admissible u and a k'-admissible v satisfy (6), we call (u, v) the (k, k')-admissible solution pair. In some situation, we need to assume that

$$\frac{\alpha}{k-l} = \frac{\beta}{k'-l'} = p \tag{8}$$

holds for some constant $p \geq 1$.

We formulate the nonexistence result of positive admissible solution of the coupled inequalities (6).

Theorem 1.2. *Assume that the system* (6) *of Hessian quotient inequalities satisfies* (2) *and* (7). *Assume also that either* $\inf_{x\in\mathbb{R}^n}(na(x)-b(x))\geq 0,\ \alpha>0,\ \beta>0,$ (8), *or* $\inf_{x\in\mathbb{R}^n}(na(x)-b(x))>0,\ \alpha\geq 0,\ \beta\geq 0$ *hold. Then the system* (6) *has no entire positive* (k,k')*-admissible solution pair* (u,v) *in* $\mathbb{R}^n$.

We recall some related studies on the entire solutions of Hessian equations and Hessian inequalities. The classification of the nonnegative entire solutions of the equation

$$-\Delta u=u^{\alpha}\quad \text{in } \mathbb{R}^n \tag{9}$$

had been deduced for $1\leq\alpha<\frac{n+2}{n-2}$ in [5] and $\alpha=\frac{n+2}{n-2}$ in [3], respectively. The similar classification results are extended in [8] to admissible positive solutions of the conformal k-Hessian equations

$$S_k\left(A^g u\right)=u^{\alpha}\quad \text{in } \mathcal{M},\quad \text{for } \alpha\in[0,\infty), \tag{10}$$

where $g=u^{-2}dx^2$ is a locally conformally flat matrix in $\mathcal{M}$, $A^g u$ is given by $A^g u=g^{-1}A^{g_0}u$, g_0 is a given metric on $\mathcal{M}$. In [11], the same classification result for special case of $\mathbb{R}^n$ $(n=2k+1)$ is also obtained by suitable choices of the text functions and the argument of integration. Using the method as in [5] and [11], a nonexistence result of the Hessian inequality

$$S_k\left(-D^2u\right)\geq u^{\alpha} \tag{11}$$

is proved in [12] for $2k<n$ and $\alpha\in(-\infty,nk/(n-2k)]$. The conformal k-Hessian inequality

$$S_k\left(A^g u\right)\geq u^{\alpha} \tag{12}$$

with $A^g u=u\left(D^2u\right)-\frac{1}{2}|Du|^2 I$ is considered in [13], where the nonexistence results for $2k<n$ and $\alpha\in[k,\infty)$ is formulated. The conformal k-Hessian inequality $S_k\left(A^g u\right)\geq u^{\alpha}$ with $A^g u=D^2u-\left(\frac{1}{2}|Du|^2 I-Du\otimes Du\right)$ is considered in [6]. Note that $A^g u$ in (2) has a similar structure to that in [6], but our $A^g u$ here is more general because of the existence of $a(x)$ and $b(x)$.

The organization of this paper is as follows. In Section 2, we first formulate a nonexistence result, Lemma 2.1, for an inequality involving the Laplace operator. Then we give its proof by appropriate choice of text functions and the method of integration by parts. The proof is divided into two cases according to α and the coefficient of the $|Du|^2$, where Schwarz's inequality and Young's inequality are properly used. In section 3, we establish the relations between Hessian quotient operator and the Laplace operator by using Maclaurin's inequality. Based on this relationship and Lemma 2.1, we prove the nonexistence result in Theorem 3.2. The proof of Theorem 1.1 is completed by using Taylor's expansion and Theorem 3.2. At the end, we give the proof of nonexistence result of the coupled inequalities (6), Theorem 1.2. The proof is divided into two cases, one of which needs to be proved under condition (8), and the other is a direct consequence of Theorem 1.1.

2. **A preliminary result - semilinear case.**

Before proving Theorem 1.1, we first introduce a nonexistence result for an inequality involving the Laplace operator in this section, which is a preliminary result for proving Theorem 1.1. Since there is a quadratic term $|Du|^2$ in the inequality, the preliminary nonexistence result is nontrivial and different from [7] and [10].

Lemma 2.1. *The inequality*

$$\Delta u - m(x)|Du|^2 \geq u^\alpha, \tag{13}$$

has no entire positive solution for either $\inf_{x\in\mathbb{R}^n} m(x) \geq 0,\ \alpha > 1$ *or* $\inf_{x\in\mathbb{R}^n} m(x) > 0,\ \alpha \geq 0$*, where* $m(x)$ *denotes the function of* x*.*

When $m(x) \equiv 0$, the inequality (13) becomes $\Delta u \geq u^\alpha$, whose existence and nonexistence of entire positive solutions are related to the Keller-Osserman condition, see [7], [10]. The Keller-Osserman condition of $\Delta u = f(u)$ is to check whether $\int^\infty \left(\int_0^\tau f(s)ds\right)^{-\frac{1}{2}} d\tau$ equals to $+\infty$ or not, where the lower limit is omitted in the integral to admit any positive constant. By taking $f(u) = u^\alpha$, we have

$$\int^\infty \left(\int_0^\tau f(s)ds\right)^{-\frac{1}{2}} d\tau = \sqrt{\alpha+1}\int_1^\infty \tau^{-\frac{1+\alpha}{2}} d\tau \begin{cases} = +\infty, & \text{if } 0 < \alpha \leq 1, \\ < +\infty, & \text{if } \alpha > 1. \end{cases} \tag{14}$$

Therefore, the equation $\Delta u = u^\alpha$ has an entire positive solution when $0 < \alpha \leq 1$, but has no entire positive solution when $\alpha > 1$, which corresponds to our Lemma 2.1 in the case of $\inf_{x\in\mathbb{R}^n} m(x) \geq 0$.

However, when $\inf_{x\in\mathbb{R}^n} m(x) > 0$, the nonlinear term $m(x)|Du|^2$ comes into play so that the nonexistence of entire positive solutions can hold in a wider range of α, namely $\alpha \in [0, \infty)$. Comparing with the classical results related to the Keller-Osserman condition, this is an interesting and different point. We will prove Lemma 2.1 by multiplying proper test functions to the inequality (13) and integrating by parts.

When $k = 1$ and $l = 0$ in (1), the operator in (1) is just $\Delta u - m(x)|Du|^2$ with $m(x) = na(x) - b(x)$, which is the same as the operator in (13). In Section 3, we will prove the main results, Theorems 1.1 and 1.2, based on the preliminary result in Lemma 2.1.

Proof of Lemma 2.1. It is enough to prove nonexistence of entire positive solution for either $\inf_{x\in\mathbb{R}^n} m(x) \geq 0,\ \alpha > 1$ or $\inf_{x\in\mathbb{R}^n} m(x) > 0,\ 0 \leq \alpha \leq 1$. In fact, the situation when $\inf_{x\in\mathbb{R}^n} m(x) > 0,\ \alpha > 1$ is already covered by the former case.

Suppose that the inequality (13) has an entire positive solution u. We will deduce the contradiction.

Multiplying both sides of (13) by $u^\delta \zeta^\theta$ and integrating over $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} u^{\alpha+\delta}\zeta^\theta dx \leq -\int_{\mathbb{R}^n} m(x) u^\delta \zeta^\theta |Du|^2 dx + \int_{\mathbb{R}^n} u^\delta \zeta^\theta \Delta u dx, \tag{15}$$

where δ, θ are constants to be determined, $\zeta \in C^2$ is a cut-off function satisfying:

$$\zeta \equiv 1 \quad \text{in } B_R, \qquad\quad 0 \leq \zeta \leq 1 \quad \text{in } B_{2R}, \tag{16}$$

$$\zeta \equiv 0 \quad \text{in } \mathbb{R}^n \backslash B_{2R}, \qquad |D\zeta| \leq \frac{C}{R} \quad \text{in } \mathbb{R}^n, \tag{17}$$

where B_R denotes a ball in $\mathbb{R}^n$ centered at 0 with radius R, and C is a positive constant. In order to deal with the last term of (15), we use the integration by parts to get

$$\int_{\mathbb{R}^n} u^\delta \zeta^\theta \Delta u dx = -\delta \int_{\mathbb{R}^n} |Du|^2 u^{\delta-1}\zeta^\theta dx - \theta \int_U u^\delta \zeta^{\theta-1} (D_i\zeta)(D_i u)\, dx, \tag{18}$$

where $U := \mathrm{supp}|D\zeta| = \{x \in \mathbb{R}^n : R < |x| < 2R\}$.

Inserting (18) into (15), we have

$$\begin{aligned}\int_{\mathbb{R}^n} u^{\alpha+\delta}\zeta^\theta dx \leq -\int_{\mathbb{R}^n} m(x) u^\delta \zeta^\theta |Du|^2 dx - \delta \int_{\mathbb{R}^n} |Du|^2 u^{\delta-1}\zeta^\theta dx \\ -\theta \int_U u^\delta \zeta^{\theta-1} (D_i\zeta)(D_i u)\, dx.\end{aligned} \tag{19}$$

We next split the proof into the following two cases of α and $m(x)$:

(i) $\inf_{x\in\mathbb{R}^n} m(x) = a_0 > 0$, $0 \leq \alpha \leq 1$,

(ii) $\inf_{x\in\mathbb{R}^n} m(x) = a_0 \geq 0$, $\alpha > 1$.

In case (i) ,we further consider the two subcases:

(a) $\inf_{x\in\mathbb{R}^n} m(x) = a_0 > 0$, $\alpha = 0$,

(b) $\inf_{x\in\mathbb{R}^n} m(x) = a_0 > 0$, $0 < \alpha \leq 1$.

In case $(i)(a)$, we fix the constants δ and θ such that $\delta = 0$, $\theta > n$. The last term in (19) becomes

$$\begin{aligned}& -\theta \int_U u^\delta \zeta^{\theta-1} (D_i\zeta)(D_i u)\, dx \\ \leq & \theta\varepsilon_1 \int_U |Du|^2 \zeta^\theta dx + \theta C_{\varepsilon_1} \int_U |D\zeta|^2 \zeta^{\theta-2} dx \\ \leq & \theta\varepsilon_1 \int_U |Du|^2 \zeta^\theta dx + \varepsilon_2 C_{\varepsilon_1}(\theta-2) \int_U \zeta^\theta dx + 2C_{\varepsilon_1}C_{\varepsilon_2} \int_U |D\zeta|^\theta dx,\end{aligned} \tag{20}$$

where Schwarz's inequality is used to obtain the first inequality, Young's inequality $ab \leq \varepsilon_2 a^p/p + C_{\varepsilon_2} b^q/q$ with the exponent pair $(p,q) = (\theta/(\theta-2), \theta/2)$ is used to obtain the second inequality. We emphasize that ε_1 and ε_2 are positive constants to be determined, C_{ε_1} and C_{ε_2} denote some positive constants depending on ε_1 and ε_2 respectively.

Inserting (20) into (19), we have

$$\begin{aligned}\int_{\mathbb{R}^n} \zeta^\theta dx \leq & (\theta\varepsilon_1 - a_0) \int_{\mathbb{R}^n} \zeta^\theta |Du|^2 dx + \varepsilon_2 C_{\varepsilon_1}(\theta-2) \int_U \zeta^\theta dx \\ & + 2C_{\varepsilon_1}C_{\varepsilon_2} \int_U |D\zeta|^\theta dx.\end{aligned} \tag{21}$$

By selecting appropriate ε_1 and ε_2 such that $\varepsilon_1 \leq \frac{a_0}{\theta}$, $\varepsilon_2 < \frac{1}{C_{\varepsilon_1}(\theta-2)}$, and using (17), we get from (21) that

$$\int_{\mathbb{R}^n} \zeta^\theta dx \leq \frac{2C_{\varepsilon_1}C_{\varepsilon_2}}{1-\varepsilon_2 C_{\varepsilon_1}(\theta-2)} \int_U |D\zeta|^\theta dx \leq \frac{2C_{\varepsilon_1}C_{\varepsilon_2} w_n C}{1-\varepsilon_2 C_{\varepsilon_1}(\theta-2)} R^{n-\theta}, \tag{22}$$

where w_n denotes the volume of the unit ball in $\mathbb{R}^n$. Letting $R \to \infty$ in (22), since $\theta > n$, we have

$$\int_{\mathbb{R}^n} \zeta^\theta dx \leq 0. \tag{23}$$

Since $\zeta \in C^2$ is a cut-off function satisfying (16)-(17), we get a contradiction in case $(i)(a)$.

In case $(i)(b)$, we fix the constants δ and θ such that

$$\delta > \max\left\{\alpha\left(\frac{n}{4}-1\right),1\right\}, \quad \theta > 4\left(1+\frac{\delta}{\alpha}\right). \tag{24}$$

For the last term in (19), we have

$$\begin{aligned}
&-\theta\int_U u^{\delta}\zeta^{\theta-1}\left(D_i\zeta\right)\left(D_iu\right)dx\\
\leq&\theta\varepsilon_3\int_U |Du|^2u^{\delta-\alpha/2}\zeta^{\theta}dx+\theta C_{\varepsilon_3}\int_U|D\zeta|^2u^{\delta+\alpha/2}\zeta^{\theta-2}dx\\
\leq&\frac{\theta\varepsilon_3\varepsilon_4(2-\alpha)}{2}\int_{\mathbb{R}^n}|Du|^2u^{\delta}\zeta^{\theta}dx+\frac{\theta\varepsilon_3C_{\varepsilon_4}\alpha}{2}\int_{\mathbb{R}^n}|Du|^2u^{\delta-1}\zeta^{\theta}dx\\
&+\theta C_{\varepsilon_3}\int_U|D\zeta|^2u^{\delta+\alpha/2}\zeta^{\theta-2}dx,
\end{aligned} \tag{25}$$

where Schwarz's inequality is used to obtain the first inequality, Young's inequality $ab \leq \varepsilon_4 a^p/p + C_{\varepsilon_4}b^q/q$ with the exponent pair $(p,q) = (2/(2-\alpha), 2/\alpha)$ is used to obtain the second inequality, ε_3 and ε_4 are positive constants to be determined, C_{ε_3} and C_{ε_4} denote some positive constants depending on ε_3 and ε_4 respectively.

Inserting (25) into (19), we have

$$\begin{aligned}
\int_{\mathbb{R}^n}u^{\alpha+\delta}\zeta^{\theta}dx \leq& \left[\frac{\theta\varepsilon_3\varepsilon_4(2-\alpha)}{2}-a_0\right]\int_{\mathbb{R}^n}|Du|^2u^{\delta}\zeta^{\theta}dx\\
&+\left(\frac{\theta\varepsilon_3C_{\varepsilon_4}\alpha}{2}-\delta\right)\int_{\mathbb{R}^n}|Du|^2u^{\delta-1}\zeta^{\theta}dx\\
&+\theta C_{\varepsilon_3}\int_U|D\zeta|^2u^{\delta+\alpha/2}\zeta^{\theta-2}dx.
\end{aligned} \tag{26}$$

By selecting appropriate ε_3 and ε_4 such that $\varepsilon_3 \leq \min\left\{\frac{2\delta}{\theta C_{\varepsilon_4}\alpha},1\right\}$, $\varepsilon_4 \leq \frac{2a_0}{\theta(2-\alpha)}$, we can discard the first two terms on the right hand side of (26). Hence, (26) becomes

$$\begin{aligned}
&\int_{\mathbb{R}^n}u^{\alpha+\delta}\zeta^{\theta}dx\\
\leq&\theta C_{\varepsilon_3}\int_U|D\zeta|^2u^{\delta+\alpha/2}\zeta^{\theta-2}dx\\
\leq&\frac{\theta\varepsilon_5C_{\varepsilon_3}(\alpha+2\delta)}{2\alpha+2\delta}\int_U u^{\alpha+\delta}\zeta^{\theta}dx+\frac{\theta C_{\varepsilon_3}C_{\varepsilon_5}\alpha}{2\alpha+2\delta}\int_U\zeta^{\theta-4(1+\delta/\alpha)}|D\zeta|^{4(1+\delta/\alpha)}dx,
\end{aligned} \tag{27}$$

where Young's inequality $ab \leq \varepsilon_5 a^p/p + C_{\varepsilon_5}b^q/q$ with the exponent pair $(p,q) = \left(\frac{\alpha+\delta}{\alpha/2+\delta}, \frac{2\alpha+2\delta}{\alpha}\right)$ is used to obtain the last inequality, ε_5 is a positive constant to be determined, and C_{ε_5} denote some positive constants depending on ε_5.

By choosing $\varepsilon_5 < \frac{2\alpha+2\delta}{\theta C_{\varepsilon_3}(\alpha+2\delta)}$ and using (17), we get from (27) that

$$\begin{aligned}
&\int_{\mathbb{R}^n}u^{\alpha+\delta}\zeta^{\theta}dx\\
\leq&\frac{\theta C_{\varepsilon_3}C_{\varepsilon_5}\alpha}{2\alpha+2\delta-\theta\varepsilon_5C_{\varepsilon_3}(\alpha+2\delta)}\int_U\zeta^{\theta-4(1+\delta/\alpha)}|D\zeta|^{4(1+\delta/\alpha)}dx\\
\leq&\frac{\theta C_{\varepsilon_3}C_{\varepsilon_5}\alpha w_nC}{2\alpha+2\delta-\theta\varepsilon_5C_{\varepsilon_3}(\alpha+2\delta)}R^{n-4(1+\delta/\alpha)},
\end{aligned} \tag{28}$$

where w_n denotes the volume of the unit ball in $\mathbb{R}^n$. By (24), the exponent $n-4(1+\delta/\alpha)$ of R is negative. Letting $R\to\infty$ in (28), we can obtain

$$\int_{\mathbb{R}^n} u^{\alpha+\delta}\zeta^\theta dx \le 0. \tag{29}$$

Since $\zeta\in C^2$ is a cut-off function satisfying (16)-(17) and u is positive, we get a contradiction in case $(i)(b)$.

In case (ii), we fix the constants δ and θ such that $\delta>\max\{(n-2)\alpha-n+1,0\}$, $\theta>\frac{2(\alpha+\delta)}{\alpha-1}$. Hence, we always have $\delta>0$ and $\theta>2$. For the last term in (18), using Schwarz's inequality, we have

$$\begin{aligned}&-\theta\int_U u^\delta\zeta^{\theta-1}\left(D_i\zeta\right)\left(D_iu\right)dx\\ \le&\varepsilon_6\theta\int_{\mathbb{R}^n}|Du|^2u^{\delta-1}\zeta^\theta dx+\theta C_{\varepsilon_6}\int_U|D\zeta|^2u^{\delta+1}\zeta^{\theta-2}dx,\end{aligned} \tag{30}$$

where ε_6 is any positive constant and C_{ε_6} denotes some positive constant depending on ε_6. Inserting (30) into (19), we have

$$\begin{aligned}\int_{\mathbb{R}^n}u^{\alpha+\delta}\zeta^\theta dx\le&-a_0\int_{\mathbb{R}^n}u^\delta\zeta^\theta|Du|^2dx\\ &+(\varepsilon_6\theta-\delta)\int_{\mathbb{R}^n}|Du|^2u^{\delta-1}\zeta^\theta dx+\theta C_{\varepsilon_6}\int_U|D\zeta|^2u^{\delta+1}\zeta^{\theta-2}dx.\end{aligned} \tag{31}$$

Hence, by taking $\varepsilon_6\le\delta/\theta$ in (31), we have

$$\int_{\mathbb{R}^n}u^{\alpha+\delta}\zeta^\theta dx\le\theta C_{\varepsilon_6}\int_U|D\zeta|^2u^{\delta+1}\zeta^{\theta-2}dx. \tag{32}$$

Then ε_6 is now fixed. Applying Young's inequality to the last term in (32), we have

$$\int_{\mathbb{R}^n}u^{\alpha+\delta}\zeta^\theta dx\le\theta\varepsilon_7C_{\varepsilon_6}\int_U\frac{\left(u^{\delta+1}\zeta^p\right)^s}{s}dx+\theta C_{\varepsilon_7}C_{\varepsilon_6}\int_U\frac{\left(\zeta^q|D\zeta|^2\right)^t}{t}dx, \tag{33}$$

where p,q are positive constants satisfying

$$p+q=\theta-2,\quad s>1,\quad t>1,\quad \text{and}\quad \tfrac{1}{s}+\tfrac{1}{t}=1,$$

ε_7 is a positive constant, and C_{ε_7} denotes some positive constant depending on ε_7.

Setting $s=\frac{\alpha+\delta}{\delta+1}>1$, we get

$$t=\tfrac{\alpha+\delta}{\alpha-1}>1,\quad p=\tfrac{\theta(\delta+1)}{\alpha+\delta}>0,\quad q=\theta-2-p>0,\quad\text{and}\quad qt=\theta-\tfrac{2(\alpha+\delta)}{\alpha-1}>0.$$

Inserting p,q,s and t into (33), we have

$$\begin{aligned}\int_{\mathbb{R}^n}u^{\alpha+\delta}\zeta^\theta dx\le&\frac{\theta\varepsilon_7C_{\varepsilon_6}(\delta+1)}{\alpha+\delta}\int_U u^{\alpha+\delta}\zeta^\theta dx\\ &+\frac{\theta C_{\varepsilon_7}C_{\varepsilon_6}(\alpha-1)}{\alpha+\delta}\int_U\zeta^{\theta-\frac{2(\alpha+\delta)}{\alpha-1}}|D\zeta|^{\frac{2(\alpha+\delta)}{\alpha-1}}dx.\end{aligned} \tag{34}$$

Taking $\varepsilon_7<\frac{\alpha+\delta}{\theta C_{\varepsilon_6}(\delta+1)}$, and using (17), we get from (34) that

$$\int_{\mathbb{R}^n}u^{\alpha+\delta}\zeta^\theta dx\le\frac{\theta C_{\varepsilon_7}C_{\varepsilon_6}(\alpha-1)w_nC}{\alpha+\delta-\theta\varepsilon_7C_{\varepsilon_6}(\delta+1)}R^{n-\frac{2(\alpha+\delta)}{\alpha-1}}, \tag{35}$$

where w_n denotes the volume of the unit ball in $\mathbb{R}^n$. Now the constant ε_7 is fixed. Recalling that $\delta > \max\{(n-2)\alpha - n+1, 0\}$. It is easy to check that $n - \frac{2(\alpha+\delta)}{\alpha-1} < 0$. Letting $R \to \infty$ in (35), we can obtain

$$\int_{\mathbb{R}^n} u^{\alpha+\delta}\zeta^\theta dx \le 0. \tag{36}$$

Since $\zeta \in C^2$ is a cut-off function satisfying (16)-(17) and u is positive, we get a contradiction in case (ii).

We have completed the proof of Lemma 2.1. □

3. **Proofs of the main results.**

In this section, we prove the nonexistence result of entire positive solutions of single conformal Hessian quotient inequality (1). Then we will consider the nonexistence result of entire positive solutions of the system (6) of Hessian quotient inequalities.

3.1. **Single conformal Hessian quotient inequality.**

In order to prove the Theorem 1.1, we first establish the relationship between Hessian quotient operator and the Laplace operator by using Maclaurin's inequality of Lemma 15.13 in [9].

Lemma 3.1. *For $\lambda \in \Gamma^k$, and $n \ge k > l \ge 0$, $r > s \ge 0$, $k \ge r$, $l \ge s$, we have*

$$\left[\frac{S_k(\lambda)/C_n^k}{S_l(\lambda)/C_n^l}\right]^{\frac{1}{k-l}} \le \left[\frac{S_r(\lambda)/C_n^r}{S_s(\lambda)/C_n^s}\right]^{\frac{1}{r-s}}, \tag{37}$$

and the inequality holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n > 0$.

The inequality (37) is a consequence of Newton's inequality, see [9].

Taking $r = 1,\ s = 0$ in (37), we can obtain the following relationship between Hessian quotient operator and Laplace operator,

$$\left[\frac{C_n^l}{C_n^k}\right]^{\frac{1}{k-l}} \left[\frac{S_k(\lambda)}{S_l(\lambda)}\right]^{\frac{1}{k-l}} \le \frac{1}{C_n^1} S_1(\lambda), \tag{38}$$

which is crucial in proving the nonexistence of (1). Since

$$\frac{C_n^k}{C_n^l} = \frac{l!(n-l)!}{k!(n-k)!} = \prod_{i=1}^{k-l} \frac{n-k+i}{l+i} \le n^{k-l},$$

when $\lambda(A^g u) \in \Gamma^k$, we have from (38) that

$$\left[\frac{S_k}{S_l}\left(A^g u\right)\right]^{\frac{1}{k-l}} \le S_1\left(A^g u\right). \tag{39}$$

Since $A^g u$ has the form (2), it can be readily calculated that

$$S_1\left(A^g u\right) = \Delta u - (na(x) - b(x))|Du|^2. \tag{40}$$

Hence, combining (39)-(40) with Lemma 2.1, we get the following theorem.

Theorem 3.2. *The inequality*

$$\frac{S_k}{S_l}\left(A^g u\right) \ge u^\alpha, \tag{41}$$

with $A^g u$ in the form (2) has no entire positive k-admissible solution in $\mathbb{R}^n$ for either $\inf_{x\in\mathbb{R}^n}(na(x) - b(x)) \ge 0$, $\alpha > k-l$ or $\inf_{x\in\mathbb{R}^n}(na(x) - b(x)) > 0$, $\alpha \ge 0$.

Proof. We replace α with $\frac{\alpha}{k-l}$ in (13) and take $m(x) = na(x) - b(x)$. Then we repeat the proof of the Lemma 2.1. Suppose that the inequality (41) has an entire positive solution u. We can get that

$$\Delta u - (na(x) - b(x))|Du|^2 \geq u^{\frac{\alpha}{k-l}} \tag{42}$$

has no entire positive k-admissible solution in $\mathbb{R}^n$ for either $\inf\limits_{x\in\mathbb{R}^n}(na(x)-b(x)) \geq 0$, $\alpha > k-l$ or $\inf\limits_{x\in\mathbb{R}^n}(na(x) - b(x)) > 0$, $\alpha \geq 0$. Combining (39), (40) and (42), we get that

$$\left[\frac{S_k}{S_l}(A^g u)\right]^{\frac{1}{k-l}} \geq u^{\frac{\alpha}{k-l}}, \tag{43}$$

which implies that the inequality (41) has no entire positive k-admissible solution in $\mathbb{R}^n$ for either $\inf\limits_{x\in\mathbb{R}^n}(na(x) - b(x)) \geq 0$, $\alpha > k-l$ or $\inf\limits_{x\in\mathbb{R}^n}(na(x) - b(x)) > 0$, $\alpha \geq 0$. □

Remark 2. In Theorem 3.2, when $\inf\limits_{x\in\mathbb{R}^n}(na(x)-b(x)) \geq 0$, the nonexistence of entire positive k-admissible solution is proved for $\alpha > k-l$. According to the analysis below (2.1), it is reasonable to guess that the corresponding Keller-Osserman type condition for the Hessian quotient equation $\frac{S_k}{S_l}(D^2u) = f(u)$ may has the following form

$$\int^{\infty}\left(\int_0^{\tau} f^{k-l}(t)dt\right)^{-\frac{1}{k-l+1}} d\tau = \infty.$$

This kind of conditions for Hessian quotient equations will be discussed independently in a sequel.

Next, we need to use the Taylor's expansion and pick the right value of α to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that the inequality (1) has entire positive solutions. We will deduce the contradiction.

Taking $k = 1$ and $l = 0$, we have

$$\begin{aligned}
&\Delta u - (na(x) - b(x))|Du|^2 \\
=&S_1(A^g u) = \frac{S_1}{S_0}(A^g u) \geq \left[\frac{S_k}{S_l}(A^g u)\right]^{\frac{1}{k-l}} \geq e^{\frac{\alpha}{k-l}u} \\
=&1 + \frac{\alpha}{k-l}u + \frac{\left(\frac{\alpha}{k-l}\right)^2}{2!}u^2 + \cdots + \frac{\left(\frac{\alpha}{k-l}\right)^n}{n!}u^n + \frac{e^{\theta}}{(n+1)!}\left(\frac{\alpha}{k-l}\right)^{n+1}u^{n+1} \\
\geq&1 + \frac{\alpha}{k-l}u + \frac{\left(\frac{\alpha}{k-l}\right)^2}{2!}u^2 + \cdots + \frac{\left(\frac{\alpha}{k-l}\right)^n}{n!}u^n,
\end{aligned} \tag{44}$$

where (39) and (1) are used to obtain the first and second inequalities, respectively, Taylor's expansion for some $\theta \in \left(0, \frac{\alpha}{k-l}u\right)$ is used in the last equality, the last inequality holds since $\frac{e^{\theta}}{(n+1)!}\left(\frac{\alpha}{k-l}\right)^{n+1}u^{n+1}$ is positive. We next split the proof into the following two cases of α and n:

(*i*) $\inf\limits_{x\in\mathbb{R}^n}(na(x) - b(x)) \geq 0, \alpha > 0$,

(*ii*) $\inf\limits_{x\in\mathbb{R}^n}(na(x) - b(x)) > 0, \alpha \geq 0$.

In case (i), we get from (44) that

$$\begin{aligned}\Delta u-(na(x)-b(x))Du|^2 \geq & 1+\frac{\alpha}{k-l}u+\frac{\left(\frac{\alpha}{k-l}\right)^2}{2!}u^2+\cdots+\frac{\left(\frac{\alpha}{k-l}\right)^n}{n!}u^n \\ \geq & \frac{\left(\frac{\alpha}{k-l}\right)^2}{2!}u^2.\end{aligned} \tag{45}$$

From case (ii) in Lemma 2.1, $\Delta u-(na(x)-b(x))|Du|^2 \geq cu^2$ (with a positive constant c) has no entire positive solution. So we get a contradiction in case (i).

In case (ii), we get from (44) that

$$\Delta u-(na(x)-b(x))|Du|^2 \geq 1+\frac{\alpha}{k-l}u+\frac{\left(\frac{\alpha}{k-l}\right)^2}{2!}u^2+\cdots+\frac{\left(\frac{\alpha}{k-l}\right)^n}{n!}u^n \geq 1. \tag{46}$$

From case (i) in Lemma 2.1, $\Delta u-(na(x)-b(x))|Du|^2 \geq 1$ has no entire positive solution. So we get a contradiction in case (ii).

We have completed the proof of Theorem 1.1. □

3.2. **System of conformal Hessian quotient inequalities.** In this section, we consider the nonexistence result of positive admissible solution pair of the system of conformal Hessian quotient inequalities (6).

Proof of Theorem 1.2. Suppose that the system (6) have entire positive (k,k')-admissible solution pair (u,v) in $\mathbb{R}^n$. We will deduce the contradiction. We next split the proof into the following two cases:

(*i*) $\inf\limits_{x\in\mathbb{R}^n}(na(x)-b(x)) \geq 0$, $\alpha>0$, $\beta>0$ and (8) hold;

(*ii*) $\inf\limits_{x\in\mathbb{R}^n}(na(x)-b(x)) > 0$, $\alpha\geq 0$ and $\beta \geq 0$ hold.

In case (i), Using Taylor's expansion, we have that

$$e^v=1+v+\frac{1}{2!}v^2+\cdots+\frac{1}{n!}v^n+\frac{e^\theta}{(n+1)!}v^{n+1} \geq v, \tag{47}$$

where $0<\theta<v$. Hence, we have

$$e^{\alpha v}=(e^v)^\alpha \geq v^\alpha, \tag{48}$$

for $\alpha>0$. In the same way, we have

$$e^{\beta u}=(e^u)^\beta \geq u^\beta, \tag{49}$$

for $\beta>0$. So we can derive from (6) to get

$$\begin{cases} \frac{S_k}{S_l}\left(A^g u\right) \geq v^\alpha, \\ \frac{S_{k'}}{S_{l'}}\left(A^g v\right) \geq u^\beta. \end{cases} \tag{50}$$

Using Maclaurin's inequality (39), the system (50) can be converted to the following system of inequalities:

$$\begin{cases} \Delta u-(na(x)-b(x))|Du|^2 \geq v^{\frac{\alpha}{k-l}}, \\ \Delta v-(na(x)-b(x))|Dv|^2 \geq u^{\frac{\beta}{k'-l'}}. \end{cases} \tag{51}$$

Since we only consider the situation that $\frac{\alpha}{k-l}=\frac{\beta}{k'-l'}=p\geq 1$ in this case, by adding the two inequalities in (51), we get

$$\Delta(u+v) \geq (na(x)-b(x))\left(|Du|^2+|Dv|^2\right)+v^p+u^p. \tag{52}$$

Setting $w = u + v$, we get

$$\Delta w \geq \frac{1}{2}(na(x) - b(x))|Dw|^2 + 2^{1-p}w^p, \tag{53}$$

where Cauchy's inequality and Jensen's inequality are used. By taking $m(x) = \frac{1}{2}(na(x) - b(x))$ in (13), there is only a coefficient difference between (53) and (13). Hence, the nonexistence of (53) can be proved in the same way as Section 2, which contracts with the existence of the solution pair (u, v).

In case (ii), since $\alpha \geq 0$ and $u, v > 0$, we get from (6) that

$$\frac{S_k}{S_l}(A^g u) \geq e^{\alpha v} \geq 1 = e^{0 \cdot u}. \tag{54}$$

It contradicts with the conclusion of Theorem 1.1 under the assumption $\inf_{x \in \mathbb{R}^n}(na(x) - b(x)) > 0, \alpha \geq 0$.

We have completed the proof of Theorem 1.2. □

Remark 3. It is interesting to ask whether condition (8) can be removed or not in Theorem 1.2.

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E-mail address: jfd2001@163.com
E-mail address: 20191215008@nuist.edu.cn
E-mail address: ashijuhua@163.com