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VARIATIONS ON LYAPUNOV'S STABILITY CRITERION AND PERIODIC PREY-PREDATOR SYSTEMS

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Dedicated to Professor E. N. Dancer, on the occasion of his 75th birthday

Abstract. A classical stability criterion for Hill's equation is extended to more general families of periodic two-dimensional linear systems. The results are motivated by the study of mechanical vibrations with friction and periodic prey-predator systems.

1. **Introduction.** In Section 49 of his famous memoir [7], Lyapunov considered the linear equation

$$\ddot{x} + \alpha(t)x = 0, \tag{1}$$

where $\alpha(t)$ is T-periodic. After introducing a parameter ϵ and expanding the solutions in terms of ϵ, he proved that this equation is stable if $\alpha(t)$ is non-negative everywhere and the inequality below holds,

$$0 < T\int_0^T \alpha \le 4.$$

This result was the first stability criterion for an equation with periodic coefficients and it is in the origin of an extensive theory. See [9, 3] for more information.

In this paper we will use Lyapunov's criterion as a unifying theme and we will obtain related stability criteria for two families of linear equations having some unexpected connections. The first family is the dissipative Hill's equation

$$\ddot{x} + c\dot{x} + \alpha(t)x = 0, \tag{2}$$

where $c > 0$ is a constant. The second family is the linear prey-predator system

$$\dot{x}_1 = -a_{11}(t)x_1 - a_{12}(t)x_2, \;\; \dot{x}_2 = a_{21}(t)x_1 - a_{22}(t)x_2, \tag{3}$$

where all the coefficients $a_{ij}(t)$ are T-periodic and positive. After these results on linear equations we will look for applications to Lotka-Volterra prey-predator systems of the type

$$\dot{u} = u(a(t) - b(t)u - c(t)v), \;\; \dot{v} = v(d(t) + e(t)u - f(t)v), \;\; u > 0, v > 0, \tag{4}$$

where all the coefficients are T-periodic and $b(t)$, $c(t)$, $e(t)$ and $f(t)$ are positive. The insights in the paper by Dancer [4] will play an important role. Periodic solutions of period T are sometimes called coexistence states and the necessary and

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sufficient conditions for their existence are well understood, see [8]. However many questions on the stability properties of these solutions remain open. With the help of the linear criteria we will analyze two aspects. First we will present a sufficient condition for the uniqueness and asymptotic stability of the coexistence state. In a second part we will follow the point of view in [4] and the periodic solution of (4) will be understood as a solution of the reaction-diffusion p.d.e. system

$$\frac{\partial u}{\partial t} = r_1 \Delta_x u + u(a(t) - b(t)u - c(t)v), \ \frac{\partial v}{\partial t} = r_2 \Delta_x v + v(d(t) + e(t)u - f(t)v),$$
$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \times [0, \infty[,$$

where $u = u(x,t)$, $v = v(x,t)$ are functions defined on $\Omega \times [0,\infty[$, $\Omega \subset \mathbb{R}^m$ is a smooth bounded domain and the numbers r_1 and r_2 are positive. In some cases Turing instabilities can appear, meaning that the coexistence state is stable for the o.d.e but unstable for the p.d.e. This phenomenon was described in [4] and we will review it to show the connection with a well known phenomenon in dissipative Mechanics: some equations without friction of the type (1) are stable but after adding friction the new equation (2) is unstable.

In the previous discussions we have not paid attention to the regularity of the coefficients appearing in the equations. In most cases it will be sufficient to assume that they belong to the Banach space $L^1(\mathbb{R}/T\mathbb{Z})$, composed by all locally integrable and T-periodic functions. The associated norm is

$$||f||_{L^1(\mathbb{R}/T\mathbb{Z})} = \int_0^T |f|$$

and the average will be denoted by $\overline{f} = \frac{1}{T}\int_0^T f$. Sometimes the class of continuous and T-periodic functions, denoted by $C(\mathbb{R}/T\mathbb{Z})$, will appear. Also $C^\omega(\mathbb{R}/T\mathbb{Z})$, the class of real analytic and T-periodic functions.

2. **An oscillator with variable elasticity.** Consider the differential equation (1) where $\alpha \in L^1(\mathbb{R}/T\mathbb{Z})$. This equation appears in many physical contexts such as Hill's Lunar theory or Quantum Mechanics. Other mechanical and electric examples are described in Section 8.2 of [5]. As a simple interpretation (for positive α) we can think of a harmonic oscillator with non-constant elasticity coefficient. The variation in the elasticity will be produced by certain cyclic effects such as changes of temperature.

In the presence of linear friction the modified equation (2) is considered, where $c > 0$ is a constant parameter. The original equation (1) will be called stable in the dissipative sense if the equation (2) is asymptotically stable for each $c > 0$. This concept of stability is different from the traditional notion of Lyapunov stability. In the next Section we will construct a function α such that (1) is Lyapunov stable but (2) is unstable for some $c > 0$. At first sight this may seem counter-intuitive because in this case friction has a destabilizing effect.

Lyapunov's criterion can be adapted to dissipative stability. In this version of the criterion the function α can change sign and α^+ denotes its positive part.

Proposition 1. *Assume that $\alpha \in L^1(\mathbb{R}/T\mathbb{Z})$ satisfies*

$$\int_0^T \alpha \geq 0 \ \textit{and} \ 0 < T\int_0^T \alpha^+ \leq 4.$$

Then (1) is stable in the dissipative sense.

To prepare the proof of this result, let us introduce some terminology inspired by degree theory. The functions $\alpha_0, \alpha_1 \in L^1(\mathbb{R}/T\mathbb{Z})$ will be called homotopic if there exists a continuous family $\{\alpha_\lambda\}_{\lambda\in[0,1]}$ in $L^1(\mathbb{R}/T\mathbb{Z})$ such that the bi-parametric family of equations

$$\ddot{x} + c\dot{x} + \alpha_\lambda(t)x = 0, \quad \lambda \in [0,1],\ c > 0 \tag{5}$$

has no periodic solutions of period $2T$ excepting $x \equiv 0$.

The continuity of the family $\{\alpha_\lambda\}$ means that, for each $\lambda \in [0,1]$,

$$\lim_{h\to 0} ||\alpha_{\lambda+h} - \alpha_\lambda||_{L^1(\mathbb{R}/T\mathbb{Z})} = 0.$$

The use of the double period $2T$ will be essential to establish a link between homotopy and stability.

Lemma 2.1. *Assume that α_0 and α_1 are homotopic and the equation (1) is stable in the dissipative sense for $\alpha = \alpha_0$. Then the equation for $\alpha = \alpha_1$ is also stable in the dissipative sense.*

Proof. It is a consequence of general results (see [9, 2, 11]) but we present a sketch to show how these results are adapted to our concrete equation. Let $X(t)$ be the matrix solution of

$$\dot{X} = \begin{pmatrix} 0 & 1 \\ -\alpha(t) & -c \end{pmatrix} X, \quad X(0) = I,$$

where I denotes the 2×2 identity matrix. From Jacobi-Liouville formula we know that

$$\det X(T) = e^{-cT}.$$

Then it is not hard to deduce that (2) is asymptotically stable if and only if the trace of the monodromy matrix $X(T)$ satisfies

$$|\mathrm{tr}\ X(T)| < 1 + e^{-cT}.$$

In the case of equality, $|\mathrm{tr}\ X(T)| = 1 + e^{-cT}$, the equation (2) has a non-trivial $2T$-periodic solution. Let us now take into account the dependence of α with respect to the parameter. The continuity of $\{\alpha_\lambda\}$ implies that the function $\Delta(\lambda) = \mathrm{tr}\ X(T,\lambda)$, $\lambda \in [0,1]$, is continuous. From the assumptions, $|\Delta(0)| < 1 + e^{-cT}$ and $|\Delta(\lambda)| \neq 1 + e^{-cT}$. We conclude that $|\Delta(1)| < 1 + e^{-cT}$. □

Proof of Proposition 1. In view of the Lemma it will be sufficient to prove that the function α is homotopic to a constant function α_0 with $0 < \alpha_0 \leq \frac{4}{T^2}$. We consider the family

$$\alpha_\lambda = \lambda\alpha + (1-\lambda)\alpha_0, \quad \lambda \in [0,1]$$

and assume, by a contradiction argument, that $x(t)$ is a non-trivial $2T$-periodic solution of (5). We distinguish two cases.

Case (i). $x(t)$ **never vanishes**

We can divide (5) by $x(t)$ and integrate from $t = 0$ to $t = 2T$. After integration by parts,

$$-\int_0^{2T} \frac{\dot{x}^2}{x^2} = \int_0^{2T} (\frac{\ddot{x}}{x} + c\frac{\dot{x}}{x}) = \int_0^{2T} \alpha_\lambda \geq 0.$$

This is impossible unless $\dot{x} \equiv 0$. In such a case $x(t)$ should be constant and, from the equation, $\alpha_\lambda \equiv 0$. This is not compatible with the assumption $\int_0^T \alpha^+ > 0$.

Case (ii). $x(t)$ **vanishes somewhere**

Assume that $x(\tau) = 0$ for some $\tau \in \mathbb{R}$. The uniqueness for the initial value problem associated to (5) implies that $\dot{x}(\tau) \neq 0$. In consequence, $x(\tau) = x(\tau+2T) = 0$ and $\dot{x}(\tau) = \dot{x}(\tau + 2T) \neq 0$. The periodicity of $x(t)$ implies the existence of an intermediate zero $\hat{\tau} \in]\tau, \tau+2T[$. Taking either $\tau_1 = \tau$, $\tau_2 = \hat{\tau}$ or $\tau_1 = \hat{\tau}$, $\tau_2 = \tau+2T$, we can assume the existence of $\tau_1 < \tau_2 \leq \tau_1 + T$ such that $x(\tau_1) = x(\tau_2) = 0$. Let us now consider the function $y(t) = e^{\frac{c}{2}t}x(t)$. It satisfies

$$\ddot{y} + [\alpha_\lambda(t) - \frac{c^2}{4}]y = 0, \quad y(\tau_1) = y(\tau_2) = 0.$$

We multiply this equation by $y(t)$ and integrate by parts over the interval $I = [\tau_1, \tau_2]$, to obtain

$$\int_I \dot{y}^2 = \int_I (\alpha_\lambda - \frac{c^2}{4})y^2 < \int_I \alpha_\lambda^+ y^2.$$

Next we invoke the inequality of Sobolev type

$$||\dot{\varphi}||^2_{L^2(I)} \geq \frac{4}{|I|}||\varphi||^2_{L^\infty(I)},$$

valid for any function $\varphi \in H^1_0(I)$. See [18]. In particular, since the length of the interval satisfies $|I| \leq T$, for $\varphi = y$

$$\frac{4}{T}||y||^2_{L^\infty(I)} < (\int_I \alpha_\lambda^+)||y||^2_{L^\infty(I)}.$$

This last inequality is not compatible with the assumption $\int_0^T \alpha^+ \leq \frac{4}{T}$ because

$$\int_I \alpha_\lambda^+ \leq \int_0^T \alpha_\lambda^+ \leq \int_0^T (\lambda\alpha^+ + (1-\lambda)\alpha_0) \leq \frac{4}{T}.$$

□

Remark 1. Using the ideas of Zhang and Li in [18] this proof can be modified to obtain L^p-criteria ($p > 1$) for dissipative stability.

To finish this Section it may be worth to observe that dissipative stability is not sufficient to guarantee the asymptotic stability of the more general class of dissipative equations

$$\ddot{x} + c(t)\dot{x} + \alpha(t)x = 0, \tag{6}$$

where $c = c(t)$ is a positive and T-periodic function, say $c \in L^1(\mathbb{R}/T\mathbb{Z})$. We will construct an example in the next Section.

3. **Two examples.** First we will construct a function $\alpha(t)$ such that the equation (1) is stable in the Lyapunov sense but the equation (2) is unstable for some $c > 0$. The second example will show that dissipative stability does not extend to time dependent friction.

First construction. Let us take a sequence of non-negative functions $\delta_n \in L^1(\mathbb{R}/T\mathbb{Z})$ with the property

$$\int_0^T \delta_n\phi \to \phi(0) \text{ for each } \phi \in C(\mathbb{R}/T\mathbb{Z}).$$

We consider the equation

$$\ddot{x} + c\dot{x} + (\omega^2 + a\delta_n(t))x = 0, \tag{7}$$

and we are going to select positive constants ω_*, a_* and c_* such that, for large n, (7) is stable if $c = 0$, $\omega = \omega_*$, $a = a_*$ and unstable if $c = c_*$, $\omega = \omega_*$, $a = a_*$.

After the change of variables $y = e^{\frac{c}{2}t}x$ the equation is transformed to

$$\ddot{y} + (\beta^2 + a\delta_n(t))y = 0, \tag{8}$$

where $\beta = \sqrt{\omega^2 - \frac{c^2}{4}}$. We are assuming $\omega > \frac{c}{2}$. Let M_n be the monodromy matrix of (8) for the initial time $t_0 = 0$. The discriminant of (8) can be computed as the trace of M_n and it will be denoted by $D_n = \mathrm{tr}(M_n)$. Sometimes it will be convenient to interpret $D_n = D_n(c)$ as a function of c, for fixed ω and a. We are interested in the inequalities, for large n,

$$|D_n(0)| < 2, \quad D_n(c_*) > 2\cosh(\frac{c_*}{2}T). \tag{9}$$

At this point it is convenient to recall the proof of Lemma 2.1.

The sequence of functions δ_n converges in a weak sense to a periodic Dirac measure. This measure is denoted by $\delta = \delta(t)$ and it is defined rigorously as the functional on $C(\mathbb{R}/T\mathbb{Z})$,

$$\phi \mapsto \langle \delta, \phi \rangle = \phi(0).$$

Letting $n \to \infty$ at (8) we obtain

$$\ddot{y} + (\beta^2 + a\delta(t))y = 0. \tag{10}$$

This is an equation of the type considered in [10], although the notation has been changed. It can be interpreted as a classical equation with periodic impulses. Namely,

$$\ddot{y} + \beta^2 y = 0,\ t \neq nT,\ \ \dot{y}(nT+) = \dot{y}(nT-) - ay(nT),\ n \in \mathbb{Z}.$$

The monodromy matrix from $t_0 = 0-$ to $t_1 = T-$ is

$$M = \begin{pmatrix} \cos(\beta T) - a\frac{\sin(\beta T)}{\beta} & \frac{\sin(\beta T)}{\beta} \\ -\beta\sin(\beta T) - a\cos(\beta T) & \cos(\beta T) \end{pmatrix}.$$

By continuous dependence it is possible to prove that M_n converges to M. See [10] for more details. The discriminant of (10) is

$$D = 2\cos(\beta T) - a\frac{\sin(\beta T)}{\beta}.$$

Sometimes D will be interpreted as a function of certain parameters, $D = D(c)$, $D = D(c, \omega)$,... As a first step we fix the frequency $\omega = \omega_0$ by the formula

$$\omega_0 T = 2\pi.$$

After expanding in powers of c,

$$D(c) = 2 + \frac{aT^3}{32\pi^2}c^2 + \cdots, \quad 2\cosh(\frac{c}{2}T) = 2 + \frac{T^2}{4}c^2 + \cdots$$

We select a large number a_* and a small c_* so that $D(c_*) > 2\cosh(\frac{c_*}{2}T)$ if $a = a_*$. The parameters have been chosen appropriately in order to adjust the discriminants to the conditions $D(0, \omega_0) = 2$, $D(c_*, \omega_0) > 2\cosh(\frac{c_*}{2}T)$. The last step is to define the parameter $\omega_* = \omega_0 + \epsilon$ where ϵ is positive and small enough. Then $-2 < D(0, \omega_*) < 2$, $D(c_*, \omega_*) > 2\cosh(\frac{c_*}{2}T)$. The inequalities (9) are a consequence of the continuous dependence.

It is convenient to observe that the functions δ_n can be selected in the class of real analytic positive functions,

$$\delta_n \in C^\omega(\mathbb{R}/T\mathbb{Z}), \quad \delta_n(t) > 0 \text{ for each } t \in \mathbb{R}.$$

Also, the previous construction provides additional information on the Floquet multipliers, denoted by μ_1, μ_2. For $\omega = \omega_*$, $a = a_*$ the equation (7) is elliptic ($\mu_1 = \overline{\mu}_2$, $|\mu_1| = 1$, $\mu_1 \neq \pm 1$) if $c = 0$ and hyperbolic ($|\mu_1| < 1 < |\mu_2|$) if $c = c_*$.

Second construction. Before presenting a concrete example, it is convenient to perform some general computations on the equation (6). Let us split the function c as $c = \overline{c} + \tilde{c}$ with $\overline{c} = \frac{1}{T}\int_0^T c$. Then we can define the change of variables $y = e^{\frac{C(t)}{2}} x$ where $\dot{C} = \tilde{c}$. The equation (6) is transformed in

$$\ddot{y} + \overline{c}\dot{y} + [\alpha(t) - \frac{\overline{c}\tilde{c}(t)}{2} - \frac{\tilde{c}(t)^2}{4} - \frac{\dot{c}(t)}{2}]y = 0. \tag{11}$$

The stability properties of the equations (6) and (11) are the same because $C(t)$ is continuous and periodic. In particular this function and its derivative are bounded and the change of variables preserves stability and asymptotic stability.

In view of the previous computations we make a choice of $c(t)$ and $\alpha(t)$. Define

$$\overline{c} = 1, \quad \tilde{c}(t) = \epsilon \sin t, \quad \alpha(t) = \frac{\tilde{c}(t)}{2} + \frac{\dot{c}(t)}{2} = \frac{\epsilon}{2}(\sin t + \cos t),$$

where $\epsilon > 0$ will be adjusted. For the period $T = 2\pi$,

$$\int_0^T \alpha = 0, \quad T\int_0^T \alpha^+ = \pi\epsilon \int_0^{2\pi} (\sin t + \cos t)^+ = 2\sqrt{2}\pi\epsilon.$$

Then α satisfies the assumptions of Proposition 1 if $\pi\epsilon \leq \sqrt{2}$. Under this condition the equation

$$\ddot{x} + \frac{\epsilon}{2}(\sin t + \cos t)x = 0$$

is stable in the dissipative sense. In contrast, the equation with non-constant positive friction

$$\ddot{x} + (1 + \epsilon \sin t)\dot{x} + \frac{\epsilon}{2}(\sin t + \cos t)x = 0$$

is unstable.

To prove the instability we observe that this last equation is in the class (6) and the equivalent equation in the class (11) is

$$\ddot{y} + \dot{y} - \frac{\epsilon^2 \sin^2 t}{4} y = 0.$$

We will prove that there are unbounded solutions. Let $y(t)$ be the solution with initial conditions $y(0) = 1$, $\dot{y}(0) = 0$. The theory of differential inequalities for higher order equations (see Section 15 in [17]) implies that $y(t) \geq 1$, $\dot{y}(t) \geq 0$ for each $t \geq 0$. After integrating the equation over the interval $[0, t]$ we obtain

$$\dot{y}(t) + y(t) \geq 1 + \int_0^t \frac{\epsilon^2 \sin^2 s}{4} ds \to +\infty \ \text{ as } t \to +\infty.$$

4. **A class of linear systems in the plane.** Let us now consider the system (3) where the coefficients a_{ij} belong to $L^1(\mathbb{R}/T\mathbb{Z})$ and satisfy

$$\overline{a}_{11} \geq 0, \quad \overline{a}_{22} \geq 0 \tag{12}$$

$$a_{12}(t) \geq \delta, \quad a_{21}(t) \geq \delta \ \text{ a.e. } t \in \mathbb{R} \tag{13}$$

for some $\delta > 0$.

With the choice $x_1 = \dot{x}$, $x_2 = x$, the equations (1) and (2) are in this class when $\alpha(t)$ is positive.

Next we present an adaptation of Lyapunov's criterion to this setting.

Proposition 2. *In the previous conditions assume also that the inequality below holds*

$$\left(\int_0^T a_{12}\right)^{1/2} \left(\int_0^T a_{21}\right)^{1/2} + \frac{1}{2}\int_0^T |a_{11} - a_{22}| \leq 2. \tag{14}$$

Then the system (3) is stable. Moreover, it is asymptotically stable if $\overline{a}_{11} + \overline{a}_{22} > 0$.

In the case $a_{11} = a_{22} = 0$ the previous result is essentially contained in Lemma 5.2 of [14]. The proof in that paper employed the same ideas of the previous proof of Proposition 1. We will present a different proof for Proposition 2 which is somehow related to the proof of Lemma 3.4 in [12].

As in the previous Section it will be convenient to introduce homotopies. We consider families of systems

$$\dot{x} = A_\lambda(t)x, \quad \lambda \in [0, 1] \tag{15}$$

where $\{A_\lambda\}_{\lambda \in [0,1]}$ is a continuous matrix in $L^1(\mathbb{R}/T\mathbb{Z})$ whose coefficients

$$A_\lambda(t) = \begin{pmatrix} -a_{11}(t,\lambda) & -a_{12}(t,\lambda) \\ a_{21}(t,\lambda) & -a_{22}(t,\lambda) \end{pmatrix}$$

satisfy the conditions (12) and (13) for each λ.

The family $\{A_\lambda\}$ defines a homotopy when the system (15) has no $2T$-periodic solutions excepting $x \equiv 0$. The same type of reasoning as in Lemma 2.1 allows to prove that stability is preserved by homotopies. This is also the case for asymptotic stability whenever $\overline{a}_{11}(\cdot, \lambda) + \overline{a}_{22}(\cdot, \lambda) > 0$ for each $\lambda \in [0, 1]$.

Another tool for the proof will be the argument function associated to each non-trivial solution. This argument will be defined with respect to a system of elliptic-polar coordinates in $\mathbb{R}^2 \setminus \{0\}$,

$$x_1 = \sqrt{\mu} r \cos\theta, \quad x_2 = \frac{1}{\sqrt{\mu}} r \sin\theta,$$

where $\mu > 0$ is a parameter that will be determined later. Given $(x_1(t), x_2(t))$, non-trivial solution of (3), there is an absolutely continuous branch of the argument $\theta = \theta(t)$ satisfying the equation

$$\dot{\theta} = \mu a_{21}(t)\cos^2\theta + \frac{1}{\mu} a_{12}(t)\sin^2\theta + (a_{11}(t) - a_{22}(t))\cos\theta\sin\theta. \tag{16}$$

Note that $\theta(t)$ also depends continuously upon μ but this dependence will not be made explicit.

An important property of this argument is that its crossings with the lines $\theta = m\frac{\pi}{2}$, $m \in \mathbb{Z}$, are always positive. This means that $(t - t_0)(\theta(t) - m\frac{\pi}{2}) > 0$ if $\theta(t_0) = m\frac{\pi}{2}$ and $|t - t_0| > 0$ is small. In terms of the Cartesian coordinates this means that the solution $(x_1(t), x_2(t))$ crosses the axes in the counter-clockwise sense. To prove it assume for instance that $x_1(t_0) = 1$ and $x_2(t_0) = 0$. Then $\frac{d}{dt}(e^{A_{22}(t)}x_2(t)) = a_{21}(t)e^{A_{22}(t)}x_1(t) > 0$ almost everywhere in a small neighbourhood of t_0. Here A_{22} is a primitive of a_{22}. The conclusion follows because the function $e^{A_{22}(t)}x_2(t)$ is increasing around t_0.

We are ready for the proof.

Proof of Proposition 2. Let us start with the

Claim. *The system (3) has no $2T$-periodic solutions (excepting $x \equiv 0$) if the conditions (12), (13) and (14) hold.*

Let us assume, by a contradiction argument, that $(x_1(t), x_2(t))$ is a non-trivial $2T$-periodic solution. Then there exists an integer $k \in \mathbb{Z}$ such that for every $t \in \mathbb{R}$,

$$\theta(t + 2T) = \theta(t) + 2\pi k. \tag{17}$$

The above discussions on the crossing with the axes allow to deduce that k should be non-negative. We distinguish two cases.

Case i) $k = 0$

The solution $(x_1(t), x_2(t))$ must lie in one open quadrant, for otherwise some crossing with the axes should be negative. There are two possibilities, either $x_1(t) \cdot x_2(t) > 0$ for every $t \in \mathbb{R}$ or $x_1(t) \cdot x_2(t) < 0$. In the first case we divide the first equation by x_1 and integrate over a period to obtain

$$-\bar{a}_{11} = \frac{1}{T}\int_0^T a_{12}\frac{x_2}{x_1}.$$

This identity is not consistent with (12) and (13). In the second case we divide the second equation by x_2 and integrate in order to obtain a second inconsistent identity. We have proved that (17) cannot hold for $k = 0$.

Case ii) $k > 0$

The sets

$$\mathcal{C} = \{t \in \mathbb{R} : \ |\sin\theta(t)| < |\cos\theta(t)|\}, \ \ \mathcal{S} = \{t \in \mathbb{R} : \ |\cos\theta(t)| < |\sin\theta(t)|\}$$

have infinitely many connected components. In particular, for each $m \in \mathbb{Z}$ there are intervals $I_m =]t_0, t_1[$ and $J_m =]\tau_0, \tau_1[$ which are connected components of $\mathcal{C}$ and $\mathcal{S}$ respectively and satisfy

$$\theta(t_0) = (m - \frac{1}{4})\pi, \ \theta(t_1) = (m + \frac{1}{4})\pi, \ \theta(\tau_0) = (m + \frac{1}{4})\pi, \ \theta(\tau_1) = (m + \frac{3}{4})\pi.$$

Since the crossings with the axes are positive it is clear that these components are unique, although the sets $\mathcal{C}$ and $\mathcal{S}$ could have additional components of different nature. The condition (17) can be invoked to infer that the diameter of the set $I_m \cup J_m \cup I_{m+1} \cup J_{m+1}$ cannot be greater than $2T$.

From the equation (16) we deduce that the inequality below holds on the interval I_m,

$$\dot{\theta} < D_\mu(t)\cos^2\theta,$$

where $D_\mu := \mu a_{21} + \frac{1}{\mu}a_{12} + |a_{11} - a_{22}|$. Then

$$2 = \int_{(m-\frac{1}{4})\pi}^{(m+\frac{1}{4})\pi} \frac{d\theta}{\cos^2\theta} < \int_{I_m} D_\mu.$$

Analogous inequalities can be obtained on J_m. Therefore,

$$8 < \int_{I_m \cup J_m \cup I_{m+1} \cup J_{m+1}} D_\mu \le \int_0^{2T} D_\mu .$$

For the choice $\mu = \left(\frac{\int_0^T a_{12}}{\int_0^T a_{21}} \right)^{1/2}$ we conclude that

$$4 < 2 \left(\int_0^T a_{12} \right)^{1/2} \left(\int_0^T a_{21} \right)^{1/2} + \int_0^T |a_{11} - a_{22}|$$

and this is against (14). Note that this value of μ minimizes D_μ.

The claim has been proved and we are going to apply it to the family (15) with

$$A_\lambda(t) = (1-\lambda)A(t) + \lambda \overline{A},$$

where $\overline{A}$ is the averaged constant matrix $\begin{pmatrix} -\overline{a}_{11} & -\overline{a}_{12} \\ \overline{a}_{21} & -\overline{a}_{22} \end{pmatrix}$. The coefficients of A_λ satisfy (12) and (13). Also, from $|\overline{a}_{11} - \overline{a}_{22}| \le \frac{1}{T} \int_0^T |a_{11} - a_{22}|$ we observe that the assumption (14) also holds. From the claim we deduce that the family (15) has no $2T$-periodic solutions excepting $x \equiv 0$. The proof is complete because the system of constant coefficients $\dot{x} = \overline{A}x$ is stable in all cases and asymptotically stable when $\overline{a}_{11} + \overline{a}_{22} > 0$. □

5. **Asymptotic stability of coexistence states.** Consider the system

$$\dot{u} = u(a(t) - bu - cv), \quad \dot{v} = v(d(t) + eu - fv), \quad u > 0, v > 0, \tag{18}$$

with $a, d \in L^1(\mathbb{R}/T\mathbb{Z})$ and b, c, e, f positive constants. This particular situation allows to formulate the results in a more elegant way. At the end of the Section we will discuss the extension to general systems where all the coefficients are time dependent. It is convenient to stress that we will be interested in solutions lying in the first open quadrant, denoted by $\mathrm{int}(\mathbb{R}^2_+) =]0, \infty[\times]0, \infty[$.

Let $\mathcal{E} \in \mathbb{R}^2$ be the solution of the linear system

$$M\mathcal{E} = \begin{pmatrix} \overline{a} \\ \overline{d} \end{pmatrix}$$

with $\mathcal{E} = \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{pmatrix}$ and $M = \begin{pmatrix} b & c \\ -e & f \end{pmatrix}$. The point $\mathcal{E}$ can be interpreted as the equilibrium of the averaged system and it plays an important role in the dynamics of the periodic system. In fact it is well known that the system (18) has a T-periodic solution if and only if $\mathcal{E} \in \mathrm{int}(\mathbb{R}^2_+)$, see [6, 1]. Next we will impose an additional condition on $\mathcal{E}$ in order to guarantee the uniqueness and asymptotic stability of the periodic solution.

Theorem 5.1. *Assume that the equilibrium point satisfies* $\mathcal{E} \in \mathrm{int}(\mathbb{R}^2_+)$ *and*

$$T\left(\sqrt{ce\mathcal{E}_1\mathcal{E}_2} + \frac{1}{2}(b\mathcal{E}_1 + f\mathcal{E}_2)\right) \le 2. \tag{19}$$

Then the system (18) has a unique T-periodic solution and this solution is asymptotically stable.

Remark 2. a) In [1] the condition $\mathcal{E} \in \text{int}(\mathbb{R}^2_+)$ was reformulated in terms of the equivalent inequalities

$$\overline{a} > 0, \quad -\frac{e}{b} < \frac{\overline{d}}{\overline{a}} < \frac{f}{c}.$$

b) I do not know if the periodic solution given by the Theorem is always a global attractor. In [16] Tineo proved the existence of a globally asymptotically stable T-periodic solution when $\mathcal{E} \in \text{int}(\mathbb{R}^2_+)$ and some additional conditions on the coefficients hold. Note that, in constrast to (19), Tineo's condition is independent of the period.
c) The number 2 is optimal in the inequality (19). We will be more precise about this statement after proving the above result.

Proof of Theorem 5.1. The first step will be to observe that the equilibrium coincides with the average of any T-periodic solution. Given $(u(t), v(t))$, T-periodic solution of (18), we divide the first equation by u and the second by v. After integrating over a period we obtain the identity $M\begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} = \begin{pmatrix} \overline{a} \\ \overline{d} \end{pmatrix}$ and therefore $\begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} = \mathcal{E}$.

The next step will be to analyze the stability properties of the variational system

$$\dot{y}_1 = (a(t) - bu(t) - cv(t))y_1 - u(t)(by_1 + cy_2), \tag{20}$$

$$\dot{y}_2 = (d(t) + eu(t) - fv(t))y_2 + v(t)(ey_1 - fy_2). \tag{21}$$

As noticed in [4], the change of variables $y_1 = u(t)x_1$, $y_2 = v(t)x_2$ preserves the stability properties and transforms the linear system into

$$\dot{x}_1 = -bu(t)x_1 - cv(t)x_2, \quad \dot{x}_2 = eu(t)x_1 - fv(t)x_2. \tag{22}$$

This new system is in the class considered in Section 4. In order to apply Proposition 2 we observe that

$$\int_0^T |bu(t) - fv(t)|dt \leq T(b\overline{u} + f\overline{v})$$

and $\overline{u} = \mathcal{E}_1$, $\overline{v} = \mathcal{E}_2$. The condition (14) is implied by the inequality (19). At this point we know, by the linearization principle, that all T-periodic solutions of (18) are asymptotically stable. It remains to prove that there is only one of these solutions and this follows from a degree argument. Similar proofs can be found in the recent paper [15]. With respect to degree we follow the notation and terminology of [13]. Let $P : \text{int}(\mathbb{R}^2_+) \to \text{int}(\mathbb{R}^2_+)$ be the Poincaré map associated to (18). The condition $\mathcal{E} \in \text{int}(\mathbb{R}^2_+)$ implies the existence of Ω, an open and bounded subset of the plane, satisfying: $\overline{\Omega} \subset \text{int}(\mathbb{R}^2_+)$, all fixed points of P lie in Ω, $\deg(id - P, \Omega) = 1$. See [1] for more details. Asymptotically stable T-periodic solutions produce isolated fixed points of P whose fixed point index is one. In our situation this is the case for all fixed points. In particular there is a finite number of them, say $\xi_1, \dots, \xi_r \in \Omega$ with $I(P, \xi_k) = 1$ for each $k = 1, \dots, r$. From the additivity property of degree,

$$1 = \deg(id - P, \Omega) = \sum_{k=1}^{r} I(P, \xi_k),$$

and we must conclude that $r = 1$. □

Once we have completed the proof of the Theorem we can explain why the number 2 is optimal in the inequality (19). Let us fix $\delta > 0$ and select an analytic, positive and T-periodic function $\alpha_\delta(t)$ such that the equation (1) with $\alpha = \alpha_\delta$ is unstable and $T\int_0^T \alpha_\delta < 4+\delta$. This function exists because the number 4 is optimal in Lyapunov's criterion. Indeed it can be assumed that the equation

$$\ddot{x} + \alpha_\delta(t)x = 0 \tag{23}$$

is hyperbolic, meaning that the Floquet multipliers do not lie in $\mathbb{S}^1$. We consider a system of the type (18) with $c = e = 1$ and $b = f = \epsilon$. The functions $a(t)$ and $b(t)$ are adjusted so that $u(t) = 1$, $v(t) = \alpha_\delta(t)$ is a T-periodic solution of (18). Then $\mathcal{E}_1 = \overline{u} = 1$, $\mathcal{E}_2 = \overline{v} = \overline{\alpha}_\delta$. The linearized system is equivalent to (22),

$$\dot{x}_1 = -\epsilon x_1 - \alpha_\delta(t)x_2, \quad \dot{x}_2 = x_1 - \epsilon\alpha_\delta(t)x_2.$$

For $\epsilon = 0$ we obtain a system equivalent to the equation (23) with $x = x_2$. Due to the hyperbolicity of this equation we deduce that the perturbed system is unstable for small ϵ. The quantity appearing in (19) becomes

$$T(\overline{\alpha}_\delta^{1/2} + \frac{1}{2}\epsilon(1+\overline{\alpha}_\delta)) \to T\overline{\alpha}_\delta^{1/2}, \quad \text{as } \epsilon \to 0.$$

Since $T\overline{\alpha}_\delta^{1/2} < (4+\delta)^{1/2}$, we have constructed a system of the type (18) having an unstable T-periodic solution and such that the inequality (19) holds if 2 is replaced by $(4+\delta)^{1/2}$.

To finish this Section we notice that the previous techniques can be applied to a general prey-predator system of the type described by the equations in (4), where $a, d \in L^1(\mathbb{R}/T\mathbb{Z})$ and the coefficients b, c, e, f are positive functions in $C(\mathbb{R}/T\mathbb{Z})$. It is well known that the existence of a T-periodic solution of (4) is equivalent to the linear instability of the trivial and semi-trivial states. See [8] for more details. From now on we assume that $(u(t), v(t))$ is a T-periodic solution. The information on the average is now less precise. After integrating in (4) we obtain the identities

$$\overline{a} = \frac{1}{T}\int_0^T bu + \frac{1}{T}\int_0^T cv, \quad \overline{d} = -\frac{1}{T}\int_0^T eu + \frac{1}{T}\int_0^T fv.$$

In consequence the point $\begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}$ will belong to the set $\mathcal{C}$ composed by all points $\mathcal{E} = \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{pmatrix} \in \mathrm{int}(\mathbb{R}^2_+)$ satisfying the linear inequalities

$$b_L\mathcal{E}_1 + c_L\mathcal{E}_2 \le \overline{a} \le b_M\mathcal{E}_1 + c_M\mathcal{E}_2,$$

$$-e_M\mathcal{E}_1 + f_L\mathcal{E}_2 \le \overline{d} \le -e_L\mathcal{E}_1 + f_M\mathcal{E}_2.$$

Here $b_L = \min_{[0,T]} b(t)$, $b_M = \max_{[0,T]} b(t)$,...

In particular $\mathcal{C}$ is convex and non-empty. The same techniques of the previous proof can be applied to the system (4) if we assume that the inequality $T\Phi(\mathcal{E}_1, \mathcal{E}_2) \le 2$ is valid for each $\mathcal{E} \in \mathcal{C}$. Here Φ is the function

$$\Phi(\mathcal{E}_1, \mathcal{E}_2) := \sqrt{c_M e_M \mathcal{E}_1\mathcal{E}_2} + \frac{1}{2}(b_M\mathcal{E}_1 + f_M\mathcal{E}_2).$$

Taking into account that this function is increasing in each variable and the geometry of the set $\mathcal{C}$ it is not hard to observe the the maximum of Φ over $\mathcal{C}$ must be reached on a certain segment contained in the boundary. More precisely, $\max_{\mathcal{C}}\Phi = \max_{\mathcal{C}\cap\ell}\Phi$, where ℓ is the straight line with equation $b_L\mathcal{E}_1 + e_L\mathcal{E}_2 = \overline{a}$. In this way we have obtained an extension of Theorem 5.1.

Theorem 5.2. *Assume that the system (4) has a T-periodic solution and the inequality*

$$T\Phi(\mathcal{E}_1, \mathcal{E}_2) \le 2$$

holds for each $\mathcal{E} \in \mathcal{C} \cap \ell$. Then this T-periodic solution is unique and asymptotically stable.

6. **Turing instabilities for coexistence states.** In [4] Dancer constructed an example of a prey-predator system having a T-periodic solution which is asymptotically stable as a solution of the o.d.e. system but it becomes unstable when it is interpreted as a solution of a reaction-diffusion system. In this Section we will review Dancer's example and it will be observed that it is somehow linked to the phenomenon of dissipative instability described in the first Example of Section 3.

Following along the lines of the previous Section we consider the system (18) and the reaction-diffusion system

$$\frac{\partial u}{\partial t} = r_1 \Delta_x u + u(a(t) - bu - cv), \ \frac{\partial v}{\partial t} = r_2 \Delta_x v + v(d(t) + eu - fv),$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \times [0, \infty[,$$

where $u = u(x,t)$, $v = v(x,t)$ are functions defined on $\Omega \times [0,\infty[$, $\Omega \subset \mathbb{R}^m$ is a smooth bounded domain and the numbers r_1 and r_2 are positive.

Since the boundary conditions are of Neumann type, every T-periodic solution of (18) is also a periodic solution of the p.d.e. system. As we saw in the previous Section, the variational system associated to this solution is

$$\dot{Y} = A(t)Y, \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{24}$$

where $A(t)$ is the 2×2 periodic matrix defined by the equations (20) and (21).

Let $R = \operatorname{diag}(r_1, r_2)$ be the diagonal matrix determined by the diffusion coefficients and let us assume that for some $\lambda > 0$ the system

$$\dot{Y} = (A(t) - \lambda R)Y \tag{25}$$

has a Floquet multiplier outside the unit disk, $|\mu| > 1$. According to [4] we know that the solution $(u(t), v(t))$ will be unstable with respect to the p.d.e. system on some domain Ω. To explain why the system (25) plays a role we assume that $Y_*(t)$ is a non-trivial solution of (25) with $Y_*(t+T) = \mu Y_*(t)$. After selecting a domain Ω such that the Neumann problem

$$\Delta\phi + \lambda\phi = 0 \ \text{ in } \Omega, \ \frac{\partial \phi}{\partial n} = 0 \ \text{ on } \partial\Omega$$

has a non-trivial solution, we observe that $\xi(x,t) = \phi(x)Y_*(t)$ is a solution of the linearization of the parabolic system. Moreover, $\xi(x, t+T) = \mu\xi(x,t)$.

Motivated by the above discussions we construct a system of the type (18) having a T-periodic solution such that the Floquet multipliers of (24) satisfy $|\mu_i| < 1$, $i = 1, 2$ and the multipliers of (25) satisfy $|\mu_1| < 1 < |\mu_2|$ for some $\lambda > 0$.

The change of variables $y_1 = u(t)x_1$, $y_2 = v(t)x_2$ preserves the Floquet multipliers and transforms (24) into the system

$$\dot{X} = B(t)X, \ \ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{26}$$

where $B(t)$ is defined by (22). Similarly this change of variables transforms (25) into

$$\dot{X} = (B(t) - \lambda R)X. \tag{27}$$

Next we go back to the first example in Section 3 and select a positive, analytic and T-periodic function $\alpha(t)$ such that (1) is elliptic and there exists $c_* > 0$ such that $\ddot{x} + c^*\dot{x} + \alpha(t)x = 0$ is hyperbolic. To define the system (18) we take $b = f = \epsilon$ where $\epsilon > 0$ is a small parameter, $c = e = 1$. The coefficients $a(t)$ and $d(t)$ are computed after imposing that $u(t) \equiv 1$, $v(t) = \alpha(t)$ is a T-periodic solution of (18). Finally, the diffusion coefficients in the p.d.e. are $r_1 = \epsilon$, $r_2 = 1$. The system (26) is defined by the equations

$$\dot{x}_1 = -\epsilon x_1 - \alpha(t)x_2, \;\; \dot{x}_2 = x_1 - \epsilon\alpha(t)x_2. \tag{28}$$

The corresponding multipliers $\mu_{1,\epsilon}$, $\mu_{2,\epsilon}$ are the roots of a quadratic polynomial with real coefficients. These coefficients are continuous functions of the parameter ϵ. For $\epsilon = 0$ we obtain the equation (1) with $x = x_2$ and therefore the multipliers are complex conjugate numbers. In consequence this property is also valid for small ϵ, $\mu_{2,\epsilon} = \overline{\mu}_{1,\epsilon}$, $\mu_{1,\epsilon} \in \mathbb{C} \setminus \mathbb{R}$. The application of Jacobi-Liouville formula to the system (28) implies that

$$|\mu_{1,\epsilon}|^2 = \mu_{1,\epsilon} \cdot \mu_{2,\epsilon} = e^{-\epsilon(T + \int_0^T \alpha)} < 1.$$

Therefore the system (28) is asymptotically stable and the same can be said about (26).

Analogously the system (27) is defined by the equations

$$\dot{x}_1 = -\epsilon(1+\lambda)x_1 - \alpha(t)x_2, \;\; \dot{x}_2 = x_1 - (\epsilon\alpha(t) + \lambda)x_2. \tag{29}$$

For $\epsilon = 0$ and $\lambda = c^*$ we obtain the equation $\ddot{x} + c^*\dot{x} + \alpha(t)x = 0$ with $x = x_2$. We know that we are in the hyperbolic case and by continuous dependence the system (29) will have a multiplier outside the unit circle when ϵ is small and $\lambda = c^*$

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