ELECTRONIC RESEARCH ARCHIVE
Volume **29**, Number **6**, December **2021**

doi:10.3934/era.2021067
pp. **3889–3908**

OPTIMAL TIME-DECAY RATES OF THE COMPRESSIBLE NAVIER–STOKES–POISSON SYSTEM IN $\mathbb{R}^3$

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(Communicated by Xianpeng Hu)

Abstract. We are concerned with the Cauchy problem of the 3D compressible Navier–Stokes–Poisson system. Compared to the previous related works, the main purpose of this paper is two–fold: First, we prove the optimal decay rates of the higher spatial derivatives of the solution. Second, we investigate the influences of the electric field on the qualitative behaviors of solution. More precisely, we show that the density and high frequency part of the momentum of the compressible Navier–Stokes–Poisson system have the same L^2 decay rates as the compressible Navier–Stokes equation and heat equation, but the L^2 decay rate of the momentum is slower due to the effect of the electric field.

1. **Introduction.** We consider the following 3D compressible Navier-Stokes-Poisson (NSP) system:

$$\begin{cases} \rho_t + \nabla\cdot m = 0, \quad x\in\mathbb{R}^3,\ t>0, \\ m_t + \nabla\cdot\left(\frac{m\bigotimes m}{\rho}\right) + \nabla p(\rho) + \rho\nabla\phi = \mu\Delta\left(\frac{m}{\rho}\right) + (\mu+\nu)\nabla\left(\nabla\cdot\left(\frac{m}{\rho}\right)\right), \\ -\lambda^2\Delta\phi = \rho-\bar{\rho}, \quad \lim\limits_{|x|\to\infty}\phi(x,t)\to 0, \\ \rho(x,0)=\rho_0(x),\ m(x,0)=m_0(x), \quad x\in\mathbb{R}^3. \end{cases} \tag{1.1}$$

Here, the density $\rho>0$, m represents the momentum, $u=\frac{m}{\rho}$ stands for the velocity, and ϕ represents the electrostatic potential. The viscosity coefficients $\mu>0$ and ν satisfy $\frac{2}{3}\mu+\nu\geq 0$, the Debye length $\lambda>0$ and the pressure functions $p=p(\rho)$. $\bar{\rho}$ is the background doping profile, which is regarded as a positive constant for the sake of simplicity.

2020 *Mathematics Subject Classification.* 35B40, 76N10.

Key words and phrases. Navier–Stokes–Poisson equations, large time behavior, optimal decay rates.

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1.1. **History of the problem.** To go directly to the theme of this paper, let us give some explanations about the above model. For the existence and long time behavior of the solution to the compressible Navier–Stokes (NS) system (i.e., $\phi \equiv 0$ in (1.1)), one can refer to [10, 11, 12, 18, 19, 6] and references therein. For the compressible NSP system, Li–Matsumura–Zhang [14] proved that the density of the NSP system converges to its equilibrium state at the same L^2–rate $(1+t)^{-\frac{3}{4}}$ as the compressible Navier–Stokes (NS) system, but the momentum of the NSP system decays at the L^2–rate $(1+t)^{-\frac{1}{4}}$, which is slower than the L^2–rate $(1+t)^{-\frac{3}{4}}$ for the compressible NS system due to the effect of the electric field. And then they extended similar result to the non–isentropic case [27]. Wang [25] obtained the optimal asymptotic decay of solutions just by pure energy estimates, and particularly he proved that the density of the compressible NSP system decays at the L^2–rate $(1+t)^{-\frac{5}{4}}$, which is faster than the L^2–rate $(1+t)^{-\frac{3}{4}}$ for the NS system due to the effect of the electric field. Hao–Li [9], Tan–Wu [22], Chikami–Danchin [4], Bie–Wang–Yao [2] and Shi-Xu [21] also established the unique global solvability and the optimal decay rates in critical spaces. We mention that there are many results on the existence and long time behavior of the weak solutions or non–constant stationary solutions, see, for example [1, 5, 8, 28] and the references therein. For the compressible NSP system with external force, [16] investigated the existence and zero-electron-mass limit of strong solutions to the stationary with large external force and [17] proved that the strong solution existence to the boundary value problem in a bounded domain. For the bipolar Navier–Stokes–Poisson(BNSP) system, we refer to [7], [15], [24], [26], [29], [30], and references therein.

We remark that this paper is strongly motivated by Li–Matsumura–Zhang [14]. Their main results can be outlined as follows: Assume that $(\rho_0 - \bar{\rho}, m_0) \in H^l \cap L^1$, $l \geq 4$, and $\|(\rho_0 - \bar{\rho}, m_0)\|_{H^l \cap L^1}$ is sufficiently small. Then, the Cauchy problem (1.1) admits a global smooth solution (ρ, m, ϕ) satisfying:

$$\|\nabla^k(\rho - \bar{\rho})(t)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}} \|(\rho_0 - \bar{\rho}, m_0)\|_{H^l \cap L^1}, \text{ for } k = 0, 1, \tag{1.2}$$

$$\|\nabla^k(m, \nabla\phi)(t)\|_{L^2} \lesssim (1+t)^{-\frac{1}{4}-\frac{k}{2}} \|(\rho_0 - \bar{\rho}, m_0)\|_{H^l \cap L^1}, \text{ for } k = 0, 1, \tag{1.3}$$

and

$$\|(\rho - \bar{\rho}, m)(t)\|_{H^4} \lesssim \|(\rho_0 - \bar{\rho}, m_0)\|_{H^l \cap L^1}. \tag{1.4}$$

However, it is clear that (1.2) – (1.4) give no information on the decay rates of higher order spatial derivatives ($k \geq 2$) of the solution. The main purpose of this article is to give a clear answer to this issue.

1.2. **Main results.** In this paper, we first introduce some notations and conventions. We use $H^l(\mathbb{R}^3)$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^l}$ and use L^p, $1 \leq p \leq \infty$ to denote the usual $L^p(\mathbb{R}^3)$ spaces with norm $\|\cdot\|_{L^p}$. The notation $a \lesssim b$ means that $a \leq Cb$ for a universal positive constant which is only dependent on the parameters of the problem. For a radial function $\phi \in C_0^\infty(\mathbb{R}^3_\xi)$ such that $\phi(\xi) = 1$ when $|\xi| \leq 1$ and $\phi(\xi) = 0$ when $|\xi| \geq 2$, we define the low–frequency part and the high–frequency part of f by

$$f^l = \mathfrak{F}^{-1}[\phi(\xi)\widehat{f}], \text{ and } f^h = \mathfrak{F}^{-1}[(1-\phi(\xi))\widehat{f}].$$

If the Fourier transform of f exists, then $f = f^h + f^l$.

Now, we state out our main results in the following theorem:

Theorem 1.1. *Let $p'(\rho) > 0$ for $\rho > 0$. Assume that $(\rho_0 - \bar{\rho}, m_0) \in H^4(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, with $\delta_0 =: \|(\rho_0 - \bar{\rho}, m_0)\|_{H^3(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}$ small. Then for any $t \geq 0$, the Cauchy problem (1.1) has a unique global solution (ρ, m, ϕ) satisfying the following optimal time decay rates*

$$\|\nabla^k(\rho - \bar{\rho})\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{1.5}$$

$$\|\nabla^k(m, \nabla\phi)\|_{L^2} \lesssim (1+t)^{-\frac{1}{4}-\frac{k}{2}}, \tag{1.6}$$

and

$$\|\nabla^k(m^h, \nabla\phi^h)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{1.7}$$

for $2 \leq k \leq 4$.

Remark 1.2. Compared to the result (1.2)-(1.4) in [14], the L^2 decay rates of higher spatial derivatives from 2-order to 4-order of the solution $(\rho, m, \nabla\phi)$ in (1.5)-(1.7) are totally new. Furthermore, we only need the smallness of $H^3 \cap L^1$-norm of the initial data, while H^4-norm of the initial data may be arbitrarily large. Finally, by noticing that, for $2 \leq k \leq 4$, we can obtain the optimal decay rates of $\|\nabla^k(m^h, \nabla\phi^h)\|_{L^2}$ in (1.7), which are the same as those of the compressible Navier–Stokes equations and heat equation, and particularly faster than one of $\|\nabla^k(m, \nabla\phi)\|_{L^2}$ in (1.6). This implies that the electric field has no effect on the decay rates of the density and the high-frequency part of the momentum, but reduces the decay rate of the momentum.

Now, let us sketch the strategy of proving Theorem 1.1. Motivated by the results in (1.2)-(1.4), we will focus on deriving the L^2 time decay rates of higher spatial derivatives from 2-order to 4-order of the solution $(\rho, m, \nabla\phi)$. Our strategy can be outlined as follows. First, when we prove the Theorem 1.1, we encounter such a difficulty: we need the optimal decay rate of the second-order spatial derivative of the solution to deduce the decay rate of the third-order spatial derivative of the solution. In the same way, we also need the decay rate of the third-order spatial derivative of the solution to derive the decay rate of the fourth-order spatial derivative of the solution. Therefore, we will prove Theorem 1.1 through three cases. Notice that the optimal decay rates on low-frequency part of $(\rho, m, \nabla\phi)$ has been established by [3]. Therefore, we only need to derive the optimal decay rates on $\|\nabla^k(\rho^h, m^h, \nabla\phi^h)\|$, $k = 2, 3, 4$. For case 1, we hope to establish the L^2-time decay rates of second-order spatial derivative of $(\rho, m, \nabla\phi)$. We rewrite system (1.1) into (2.1), and deduce the energy inequality as follows:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |\nabla^2 n^h|^2 + |\nabla^2 u^h|^2 + |\nabla^2\nabla\phi^h|^2 \mathrm{d}x + (\mu_1 + \mu_2)\int_{\mathbb{R}^3} |\nabla^3 u^h|^2 \mathrm{d}x \\ &\lesssim \delta_0(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^2\nabla\phi^h\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2.\end{aligned} \tag{1.8}$$

However, it should be mentioned that in the process of deducing the energy inequality (1.8), we encounter the trouble term $I_2 = -\langle\nabla^2(\nabla n \cdot u)^h, \nabla^2 n^h\rangle$ in (2.7). If we estimate it directly, we will meet the norm $\|\nabla^3 n\|_{L^2}$ which however can not be controlled in our framework. To overcome this obstacle, we need to reduce the order of the density by splitting I_2 into three parts as follows:

$$\begin{aligned}-\langle\nabla^2(\nabla n \cdot u)^h, \nabla^2 n^h\rangle &= -\langle\nabla^2(\nabla n \cdot u) - \nabla^2(\nabla n \cdot u)^l, \nabla^2 n^h\rangle \\ &= -\langle\nabla^2(\nabla n^h \cdot u) + \nabla^2(\nabla n^l \cdot u) - \nabla^2(\nabla n \cdot u)^l, \nabla^2 n^h\rangle\end{aligned} \tag{1.9}$$

$$:= I_{21} + I_{22} + I_{23},$$

and then make full use of the benefit of the low–frequency and high–frequency decomposition technique to bound the terms I_{21}, I_{22} and I_{23} one by one. For I_{21}, it is clear that

$$\nabla^2(\nabla n^h \cdot u) = u \cdot \nabla^3 n^h + 2\nabla u \cdot \nabla^2 n^h + \nabla^2 u \cdot \nabla n^h,$$

and using integration by parts, we have

$$\langle u \cdot \nabla^3 n^h, \nabla^2 n^h \rangle = -\frac{1}{2}\int_{\mathbb{R}^3} \mathrm{div} u |\nabla^2 n^h|^2 \mathrm{d}x.$$

Thus we can deduce

$$I_2 \lesssim \delta_0(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^2 n^h\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2) \tag{1.10}$$

as in (2.13). Now, note that (1.8) only gives the dissipation estimate for u^h. In order to explore the dissipation estimates for n^h and $\nabla\phi^h$, we will employ the new interactive energy functional between u^h and ρ^h by using critical L^2 estimates, low-frequency and high-frequency decomposition. Particularly, we can get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \nabla u^h \nabla^2 n^h \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla^2 n^h|^2 - |\nabla^2 u^h|^2 + |\nabla^2 \nabla \phi^h|^2 \mathrm{d}x \\ &\lesssim \delta_0(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + \|\nabla^2 u^h\|_{L^2}^2 + \|\nabla^2 n^h\|_{L^2}^2) + (\|\nabla^2 n^h\|_{L^2}^2 + \|\nabla^3 u^h\|_{L^2}^2).\end{aligned} \tag{1.11}$$

Next, we choose a sufficiently large positive constant D_1 and then define the temporal energy functional as

$$\mathfrak{E}_1(\mathrm{t}) = \frac{D_1}{2}\left(\|\nabla^2 n^h\|_{L^2}^2 + \|\nabla^2 u^h\|_{L^2}^2 + \|\nabla^2 E^h\|_{L^2}^2\right) + \int_{\mathbb{R}^3} \nabla u^h \nabla^2 n^h \mathrm{d}x. \tag{1.12}$$

Noting that (1.12) is equivalent to $\|\nabla^2 n^h\|_{L^2}^2 + \|\nabla^2 u^h\|_{L^2}^2 + \|\nabla^2 E^h\|_{L^2}^2$. Multiplying (1.8) by D_1, adding the resulting inequality with (1.11), we obtain

$$\frac{d}{dt}\mathfrak{E}_1(t) + C_1\mathfrak{E}_1(t) \lesssim \|\nabla^2 n^l\|_{L^2}^2 + \|\nabla^3 u^l\|_{L^2}^2, \tag{1.13}$$

then the optimal L^2 time decay rates of $\|\nabla^2((\rho-\bar{\rho})^h, m^h, \nabla\phi^h)\|_{L^2}$ in (1.7) for $k=2$ can be deduced by virtue of Lemma 2.1, Lemma 2.2, Gronwall's argument. Furthermore, combining with the 2-order low-frequency decay estimates in Lemma 2.2, we get the decay rates of 2-order spatial derivatives of $(\rho, m, \nabla\phi)$ immediately. This implies that we complete the proof of Theorem 1.1 for $k=2$. We use similar methods to prove the Theorem 1.1 for $k=3$ and $k=4$ in case 2 and case 3 respectively.

2. **Proof of theorem 1.1.**

This section is devoted to prove the optimal L^2 time decay rates of solution stated in Theorem 1.1. First, we rewrite the system. Denoting $(\rho, u) = (1+n, \frac{m}{\rho})$ and $\nabla\phi = E$, then the Cauchy problem (1.1) can be reformulated as

$$\begin{cases} n_t + \nabla\cdot u = f_1, \\ u_t + \nabla n + E - \mu_1 \Delta u - \mu_2 \nabla(\nabla\cdot u) = -f_2 - f_3, \\ E = \nabla(-\Delta)^{-1} n, \quad \lim\limits_{|x|\to\infty} E \to 0, \\ n(x,0) = n_0(x) =: \rho_0(x) - 1, \quad u(x,0) = u_0(x) = m_0(x)/\rho_0(x), \quad x \in \mathbb{R}^3, \end{cases} \tag{2.1}$$

where

$$f_1 = -n\nabla \cdot u - \nabla n \cdot u,$$

$$f_2 = (u\cdot\nabla)u + \left(1 - \frac{p'(1+n)}{1+n}\right)\nabla n + \mu_1(\frac{n}{1+n})\Delta u + \mu_2(\frac{n}{1+n})\nabla(\nabla\cdot u),$$

$$f_3 = -n\nabla\phi = -nE,$$

and as in [14], we have taken $\bar{\rho} = 1$, $p'(1) = 1$, $\mu = \mu_1$, $(\mu+\nu) = \mu_2$ and $d = 1$ for simplicity. Notice that

$$f_2 \sim \mathcal{O}(1)(n\nabla n + u\nabla u + n\nabla^2 u),$$

and

$$\|\nabla^k E\|_{L^2} \lesssim \|\nabla^{k-1} n\|_{L^2},\ k \geq 1.$$

Second, the decay rates on the linearized system of (2.1) are given in the following lemma:

Lemma 2.1 *Assume that* $U_0 = (n_0, m_0) \in H^l(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, $l \geq 4$, *and denote* $(\bar{n}(t), \bar{m}(t)) =: \bar{U}(t)$. *Then, for* $0 \leq k \leq l$, *the solution* $(\bar{n}, \bar{m}, \bar{E})$ *of the linearized system of* (1.1) *with* $\bar{E} = \nabla(-\Delta)^{-1}\bar{n}$ *satisfies that*

$$\|\nabla^k \bar{n}(t)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|U_0\|_{L^1} + \|\nabla^k U_0\|_{L^2}), \tag{2.2}$$

$$\|\nabla^k (\bar{E}, \bar{m})(t)\|_{L^2} \lesssim (1+t)^{-\frac{1}{4}-\frac{k}{2}}(\|U_0\|_{L^1} + \|\nabla^k U_0\|_{L^2}). \tag{2.3}$$

Proof. See [14].

Third, we state the L^2-time decay estimates on the low-frequency part of the solution in the nonliear system (1.1).

Lemma 2.2. *Assume that the assumptions of Theorem 1.1 are in force, Then for* $0 \leq k \leq l$, *the following decay rates hold:*

$$\|\nabla^k n^l(t)\|_{L^2} \lesssim (1+t)^{-\frac{3+2k}{4}}, \tag{2.4}$$

$$\|\nabla^k (E^l, m^l)(t)\|_{L^2} \lesssim (1+t)^{-\frac{1+2k}{4}}. \tag{2.5}$$

□

Proof. See [3].

Next, we will prove Theorem 1.1 in three cases. Before giving the precise proofs of these three cases, we must emphasize that under the assumption of Theorem 1.1, [14] proved that the following estimate holds

$$\|(\rho - \bar{\rho}, m)(t)\|_{H^4} \lesssim \delta_0. \tag{2.6}$$

More details of (2.6), please refer to (3.62)-(3.64) on page 697 of reference [14].

Case 1. Proof of Theorem 1.1 for $k = 2$**.** Theorem 1.1 will be proved by the good properties of the low-frequency and high-frequency decomposition. The proof involves the following steps.

Step 1. High–frequency L^2 **energy estimate.** Taking

$$\langle \mathfrak{F}^{-1}(1-\phi(\xi))\nabla^2(2.1)_1, \nabla^2 n^h\rangle + \langle \mathfrak{F}^{-1}(1-\phi(\xi))\nabla^2(2.1)_2, \nabla^2 u^h\rangle,$$

and using integration by parts, yields directly

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\nabla^2 n^h|^2+|\nabla^2 u^h|^2+|\nabla^2 E^h|^2\mathrm{d}x+(\mu_1+\mu_2)\int_{\mathbb{R}^3}|\nabla^3 u^h|^2\mathrm{d}x\\&=-\langle\nabla^2(n\nabla\cdot u)^h,\nabla^2 n^h\rangle-\langle\nabla^2(\nabla n\cdot u)^h,\nabla^2 n^h\rangle-\langle\nabla E^h,\nabla^2(n\nabla\cdot u)^h\rangle\\&\quad-\langle\nabla E^h,\nabla^2(\nabla n\cdot u)^h\rangle-\langle\nabla^2(f_2+f_3)^h,\nabla^2 u^h\rangle:=\sum_{i=1}^{5}I_i.\end{aligned}\tag{2.7}$$

The right-hand side of (2.7) can be estimated one by one. For term I_1, due to Hölder's inequality, Young inequality, Sobolev interpolation theorem, it holds that

$$\begin{aligned}|I_1|&=|-\langle\nabla^2(n\nabla\cdot u)^h,\nabla^2 n^h\rangle|\\&\lesssim\|\nabla^2(n\nabla\cdot u)\|_{L^2}\|\nabla^2 n^h\|_{L^2}\\&\lesssim(\|\nabla u\|_{L^\infty}\|\nabla^2 n\|_{L^2}+\|n\|_{L^\infty}\|\nabla^3 u\|_{L^2})\|\nabla^2 n^h\|_{L^2}\\&\lesssim(\|\nabla u\|_{H^2}\|\nabla^2 n\|_{L^2}+\|n\|_{H^2}\|\nabla^3 u\|_{L^2})\|\nabla^2 n^h\|_{L^2}\\&\lesssim\delta_0(\|\nabla^2 n\|_{L^2}^2+\|\nabla^3 u\|_{L^2}^2).\end{aligned}\tag{2.8}$$

The second term I_2 can be rewritten as follows

$$\begin{aligned}I_2&=-\langle\nabla^2(\nabla n\cdot u)^h,\nabla^2 n^h\rangle\\&=-\langle\nabla^2(\nabla n\cdot u)-\nabla^2(\nabla n\cdot u)^l,\nabla^2 n^h\rangle\\&=-\langle\nabla^2(\nabla n^h\cdot u)+\nabla^2(\nabla n^l\cdot u)-\nabla^2(\nabla n\cdot u)^l,\nabla^2 n^h\rangle\\&:=I_{21}+I_{22}+I_{23}.\end{aligned}\tag{2.9}$$

It is easy to obtain

$$\nabla^2(\nabla n^h\cdot u)=u\cdot\nabla^3 n^h+2\nabla u\cdot\nabla^2 n^h+\nabla^2 u\cdot\nabla n^h.$$

Following from integration by parts, we have

$$\langle u\cdot\nabla^3 n^h,\nabla^2 n^h\rangle=-\frac{1}{2}\int_{\mathbb{R}^3}\mathrm{div}u|\nabla^2 n^h|^2\mathrm{d}x,$$

thus

$$\begin{aligned}|I_{21}|&\lesssim(\|\nabla u\|_{L^\infty}\|\nabla^2 n^h\|_{L^2}+\|\nabla n^h\|_{L^3}\|\nabla^2 u\|_{L^6})\|\nabla^2 n^h\|_{L^2}\\&\lesssim(\|\nabla u\|_{H^2}\|\nabla^2 n^h\|_{L^2}+\|\nabla n\|_{H^1}\|\nabla^3 u\|_{L^2})\|\nabla^2 n^h\|_{L^2}\\&\lesssim\delta_0(\|\nabla^2 n\|_{L^2}^2+\|\nabla^3 u\|_{L^2}^2).\end{aligned}\tag{2.10}$$

For the term I_{22}, we have

$$\begin{aligned}|I_{22}|&=|\langle\nabla^2(\nabla n^l\cdot u),\nabla^2 n^h\rangle|\\&\lesssim\|\nabla^2(\nabla n^l\cdot u)\|_{L^2}\|\nabla^2 n^h\|_{L^2}\\&\lesssim(\|u\|_{L^\infty}\|\nabla^3 n^l\|_{L^2}+\|\nabla n^l\|_{L^3}\|\nabla^2 u\|_{L^6})\|\nabla^2 n^h\|_{L^2}\\&\lesssim(\|u\|_{H^2}\|\nabla^2 n\|_{L^2}+\|\nabla n\|_{H^1}\|\nabla^3 u\|_{L^2})\|\nabla^2 n^h\|_{L^2}\\&\lesssim\delta_0(\|\nabla^2 n\|_{L^2}^2+\|\nabla^3 u\|_{L^2}^2+\|\nabla^2 n^h\|_{L^2}^2).\end{aligned}\tag{2.11}$$

For the term I_{23}, by Lemma A.4, Hölder's inequality, Young inequality, we have

$$\begin{aligned}|I_{23}| &= |\langle \nabla^2(\nabla n\cdot u)^l, \nabla^2 n^h\rangle| \\ &\lesssim \|\nabla^2(\nabla n\cdot u)^l\|_{L^2}\|\nabla^2 n^h\|_{L^2} \\ &\lesssim \|\nabla n\cdot u\|_{L^2}\|\nabla^2 n^h\|_{L^2} \\ &\lesssim \|u\|_{L^3}\|\nabla n\|_{L^6}\|\nabla^2 n^h\|_{L^2} \\ &\lesssim \|u\|_{H^1}\|\nabla^2 n\|_{L^2}\|\nabla^2 n^h\|_{L^2} \\ &\lesssim \delta_0(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^2 n^h\|_{L^2}^2),\end{aligned} \tag{2.12}$$

Substituting (2.10)-(2.12) into (2.9), we can conclude that

$$I_2 \lesssim \delta_0(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^2 n^h\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2) \tag{2.13}$$

For the term I_3 and I_4, similar to the proofs of (2.8) and (2.13), it holds that

$$\begin{aligned}|I_3| + |I_4| &= |\langle \nabla E^h, \nabla^2(n\nabla\cdot u)^h\rangle| + |\langle \nabla E^h, \nabla^2(\nabla n\cdot u)^h\rangle| \\ &\lesssim \|\nabla^2 E^h\|_{L^2}(\|\nabla^2(n\nabla\cdot u)\|_{L^2} + \|\nabla^2(\nabla n\cdot u)^h\|_{L^2}) \\ &\lesssim \delta_0(\|\nabla^2 E^h\|_{L^2}^2 + \|\nabla^2 n\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2),\end{aligned} \tag{2.14}$$

For the term I_5, making use of integration by parts, we obtain

$$\begin{aligned}I_5 =& -\langle \nabla^2(f_2+f_3)^h, \nabla^2 u^h\rangle \\ =&\langle \nabla(f_2+f_3)^h, \nabla^3 u^h\rangle \\ =&\langle \nabla(n\nabla n)^h + \nabla(u\nabla u)^h + \nabla(n\nabla^2 u)^h - \nabla(nE)^h, \nabla^3 u^h\rangle \\ :=& I_{51} + I_{52} + I_{53} + I_{54}.\end{aligned} \tag{2.15}$$

For the term I_{51}, we have

$$\begin{aligned}|I_{51}| &= |\langle \nabla(n\nabla n)^h, \nabla^3 u^h\rangle| \\ &\lesssim \|\nabla(n\nabla n)\|_{L^2}\|\nabla^3 u^h\|_{L^2} \\ &\lesssim (\|\nabla n\|_{L^3}\|\nabla n\|_{L^6} + \|n\|_{L^\infty}\|\nabla^2 n\|_{L^2})\|\nabla^3 u^h\|_{L^2} \\ &\lesssim (\|\nabla n\|_{H^1}\|\nabla^2 n\|_{L^2} + \|n\|_{H^2}\|\nabla^2 n\|_{L^2})\|\nabla^3 u^h\|_{L^2} \\ &\lesssim \delta_0(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^3 u^h\|_{L^2}^2).\end{aligned} \tag{2.16}$$

For the term I_{52}, it holds that

$$\begin{aligned}|I_{52}| &= |\langle \nabla(u\nabla u)^h, \nabla^3 u^h\rangle| \\ &\lesssim \|\nabla^2(u\nabla u)\|_{L^2}\|\nabla^3 u^h\|_{L^2} \\ &\lesssim (\|u\|_{L^\infty}\|\nabla^3 u\|_{L^2} + \|\nabla u\|_{L^3}\|\nabla^2 u\|_{L^6})\|\nabla^3 u^h\|_{L^2} \\ &\lesssim (\|u\|_{H^2}\|\nabla^3 u\|_{L^2} + \|\nabla u\|_{H^1}\|\nabla^3 u\|_{L^2})\|\nabla^3 u^h\|_{L^2} \\ &\lesssim \delta_0(\|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 u^h\|_{L^2}^2).\end{aligned} \tag{2.17}$$

For the term I_{53}, we get

$$\begin{aligned}|I_{53}| &= |\langle \nabla(n\nabla^2 u)^h, \nabla^3 u^h\rangle| \\ &\lesssim \|\nabla(n\nabla^2 u)\|_{L^2}\|\nabla^3 u^h\|_{L^2} \\ &\lesssim (\|n\|_{L^\infty}\|\nabla^3 u\|_{L^2} + \|\nabla^2 u\|_{L^3}\|\nabla n\|_{L^6})\|\nabla^3 u^h\|_{L^2} \\ &\lesssim (\|n\|_{H^2}\|\nabla^3 u\|_{L^2} + \|\nabla^2 u\|_{H^1}\|\nabla^2 n\|_{L^2})\|\nabla^3 u^h\|_{L^2} \\ &\lesssim \delta_0(\|\nabla^3 u\|_{L^2}^2 + \|\nabla^2 n\|_{L^2}^2).\end{aligned} \tag{2.18}$$

For the term I_{54}, we have

$$\begin{aligned}|I_{54}| &= |\nabla(nE)^h, \nabla^3 u^h\rangle| \\ &\lesssim \|\nabla^2(nE)\|_{L^2}\|\nabla^3 u^h\|_{L^2} \\ &\lesssim (\|E\|_{L^\infty}\|\nabla^2 n\|_{L^2} + \|n\|_{L^3}\|\nabla^2 E\|_{L^6})\|\nabla^3 u^h\|_{L^2} \\ &\lesssim (\|\nabla E\|_{H^1}\|\nabla^2 n\|_{L^2} + \|n\|_{H^1}\|\nabla^2 n\|_{L^2})\|\nabla^3 u^h\|_{L^2} \\ &\lesssim (\|n\|_{H^1}\|\nabla^2 n\|_{L^2} + \|n\|_{H^1}\|\nabla^2 n\|_{L^2})\|\nabla^3 u^h\|_{L^2} \\ &\lesssim \delta_0(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^3 u^h\|_{L^2}^2).\end{aligned} \tag{2.19}$$

Thus, one may deduce that

$$|I_5| \lesssim \delta_0(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2). \tag{2.20}$$

Substituting (2.8), (2.13)–(2.14) and (2.20) into (2.7) and using the smallness of δ_0, yield directly

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |\nabla^2 n^h|^2 + |\nabla^2 u^h|^2 + |\nabla^2 E^h|^2 \mathrm{d}x + (\mu_1+\mu_2)\int_{\mathbb{R}^3} |\nabla^3 u^h|^2 \mathrm{d}x \\ &\lesssim \delta_0(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^2 E^h\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2).\end{aligned} \tag{2.21}$$

Step 2. Dissipation of $\nabla^2 n^h$. Taking $\langle \mathfrak{F}^{-1}(1-\phi(\xi))\nabla(2.1)_2, \nabla^2 n^h\rangle$, it holds that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \nabla u^h \nabla^2 n^h \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla^2 n^h|^2 - |\nabla^2 u^h|^2 + |\nabla^2 E^h|^2 \mathrm{d}x \\ &= -\langle \nabla^2(n\nabla\cdot u)^h, \nabla u^h\rangle - \langle \nabla^2(\nabla n\cdot u)^h, \nabla u^h\rangle - \langle \nabla(f_2+f_3)^h\, \nabla^2 n^h\rangle \\ &\quad + (\mu_1+\mu_2)\langle \nabla^3 u^h, \nabla^2 n^h\rangle := I_6 + I_7 + I_8 + I_9.\end{aligned} \tag{2.22}$$

We should estimate the right-hand side of (2.22) one by one. For I_6, using a similar argument like (2.8), we have

$$|I_6| \lesssim \delta_0(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + \|\nabla^2 u^h\|_{L^2}^2). \tag{2.23}$$

For I_7, similar to the proof of (2.13), it holds that

$$|I_7| \lesssim \delta_0(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^2 u^h\|_{L^2}^2\|\nabla^3 u\|_{L^2}^2 + \|\nabla^2 n^h\|_{L^2}^2). \tag{2.24}$$

For I_8, as the proof of (2.19), we have

$$|I_8| \lesssim \delta_0(\|\nabla^2 n\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2. \tag{2.25}$$

For I_9, we obtain

$$\begin{aligned}|I_9| &\lesssim (\mu_1+\mu_2)\|\nabla^3 u^h\|_{L^2}\|\nabla^2 n^h\|_{L^2} \\ &\lesssim \frac{1}{4}\|\nabla^2 n^h\|_{L^2}^2 + \|\nabla^3 u^h\|_{L^2}^2.\end{aligned} \tag{2.26}$$

Putting (2.23)–(2.26)into (2.22) and using the smallness of δ_0, we conclude that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla u^h\nabla^2 n^h\mathrm{d}x+\int_{\mathbb{R}^3}|\nabla^2 n^h|^2+|\nabla^2 E^h|^2\mathrm{d}x\\&\lesssim\delta_0(\|\nabla^2 n\|_{L^2}^2+\|\nabla^3 u\|_{L^2}^2+\|\nabla^2 u^h\|_{L^2}^2+\|\nabla^2 n^h\|_{L^2}^2)+\frac{1}{4}\|\nabla^2 n^h\|_{L^2}^2\\&\quad+\|\nabla^3 u^h\|_{L^2}^2+\int_{\mathbb{R}^3}|\nabla^2 u^h|^2\mathrm{d}x.\end{aligned}\tag{2.27}$$

Step 3. Closing the estimates. Now, we are in a position to close the estimates. To do this, we choose sufficiently large time T_1 and positive constant D_1, and then define the temporary energy functional

$$\mathfrak{E}_1(\mathfrak{t})=\frac{D_1}{2}\left(\|\nabla^2 n^h\|_{L^2}^2+\|\nabla^2 u^h\|_{L^2}^2+\|\nabla^2 E^h\|_{L^2}^2\right)+\int_{\mathbb{R}^3}\nabla u^h\nabla^2 n^h\mathrm{d}x,\tag{2.28}$$

for $t\geq T_1$, it is noticed that $\mathfrak{E}_1(\mathfrak{t})$ is equivalent to $\|\nabla^2 n^h\|_{L^2}^2+\|\nabla^2 u^h\|_{L^2}^2+\|\nabla^2 E^h\|_{L^2}^2$ since D_1 is large enough. Substituting (2.21) and (2.27) into

$$D_1\times(2.21)+(2.27),$$

which together with the smallness of δ_0, for $t\geq T_1$, it holds that

$$\begin{aligned}&\frac{d}{dt}\mathfrak{E}_1(\mathfrak{t})+(\frac{3}{4}-\delta_0)\|\nabla^2 n^h\|_{L^2}^2\\&\qquad+[(\mu_1+\mu_2)D_1-2]\|\nabla^3 u^h\|_{L^2}^2+\|\nabla^2 E^h\|_{L^2}^2\lesssim\|\nabla^2 n^l\|_{L^2}^2+\|\nabla^3 u^l\|_{L^2}^2,\end{aligned}\tag{2.29}$$

where we have used the fact that T_1 is large enough. On other hand, it is clear that

$$(\frac{3}{4}-\delta_0)\|\nabla^2 n^h\|_{L^2}^2+[(\mu_1+\mu_2)D_1-2]\|\nabla^3 u^h\|_{L^2}^2+\|\nabla^2 E^h\|_{L^2}^2\geq C_1\mathfrak{E}_1(\mathfrak{t}).\tag{2.30}$$

Hence, by virtue of (2.4)-(2.5), (2.29)-(2.30) and Gronwall's inequality, we can arrive at

$$\|\nabla^2 n^h\|_{L^2}^2+\|\nabla^2 u^h\|_{L^2}^2+\|\nabla^2 E^h\|_{L^2}^2\lesssim(1+t)^{-\frac{7}{4}}.\tag{2.31}$$

Furthermore, combining with (2.4)-(2.5) and (2.31), the following estimates can be obtained:

$$\begin{aligned}\|\nabla^2 n\|_{L^2}^2&\lesssim\|\nabla^2 n^h\|_{L^2}^2+\|\nabla^2 n^l\|_{L^2}^2\\&\lesssim(1+t)^{-\frac{7}{4}},\end{aligned}\tag{2.32}$$

and

$$\begin{aligned}\|\nabla^2(u,E)\|_{L^2}^2&\lesssim\|\nabla^2(u^h,E^h)\|_{L^2}^2+\|\nabla^2(u^l,E^l)\|_{L^2}^2\\&\lesssim(1+t)^{-\frac{5}{4}}.\end{aligned}\tag{2.33}$$

The proof of Theorem 1.1 for $k=2$ is completed.

Case 2. Proof of Theorem 1.1 for $k=3$. Theorem 1.1 for $k=3$ will be proved as in Case 1 by the good properties of the low-frequency and high-frequency decomposition. The proof involves the following steps.

Step 1. High–frequency L^2 energy estimate. Taking

$$\langle\mathfrak{F}^{-1}(1-\phi(\xi))\nabla^3(2.1)_1,\nabla^3 n^h\rangle+\langle\mathfrak{F}^{-1}(1-\phi(\xi))\nabla^3(2.1)_2,\nabla^3 u^h\rangle,$$

and using integration by parts, which implies

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\nabla^3 n^h|^2+|\nabla^3 u^h|^2+|\nabla^3 E^h|^2\mathrm{d}x+(\mu_1+\mu_2)\int_{\mathbb{R}^3}|\nabla^4 u^h|^2\mathrm{d}x\\
&=-\langle\nabla^3(n\nabla\cdot u)^h,\nabla^3 n^h\rangle-\langle\nabla^3(\nabla n\cdot u)^h,\nabla^3 n^h\rangle-\langle\nabla^2 E^h,\nabla^3(n\nabla\cdot u)^h\rangle\\
&\quad-\langle\nabla^2 E^h,\nabla^3(\nabla n\cdot u)^h\rangle-\langle\nabla^3(f_2+f_3)^h,\nabla^3 u^h\rangle:=\sum_{i=1}^{5}J_i.
\end{aligned}\tag{2.34}$$

The five terms in the above equation can be estimated as follows. Firstly, for term J_1, using Hölder's inequality, Young inequality, Sobolev interpolation theorem, we obtain

$$\begin{aligned}
|J_1|&=|\langle\nabla^3(n\nabla\cdot u)^h,\nabla^3 n^h\rangle|\\
&\lesssim\|\nabla^3(n\nabla\cdot u)\|_{L^2}\|\nabla^3 n^h\|_{L^2}\\
&\lesssim(\|\nabla u\|_{L^\infty}\|\nabla^3 n\|_{L^2}+\|n\|_{L^\infty}\|\nabla^4 u\|_{L^2})\|\nabla^3 n^h\|_{L^2}\\
&\lesssim(\|\nabla u\|_{H^2}\|\nabla^3 n\|_{L^2}+\|n\|_{H^2}\|\nabla^4 u\|_{L^2})\|\nabla^3 n^h\|_{L^2}\\
&\lesssim\delta_0(\|\nabla^3 n\|_{L^2}^2+\|\nabla^3 n^h\|_{L^2}^2+\|\nabla^4 u\|_{L^2}^2),
\end{aligned}\tag{2.35}$$

The term J_2 can be rewritten as follows

$$\begin{aligned}
J_2=&-\langle\nabla^3(\nabla n\cdot u)^h,\nabla^3 n^h\rangle\\
=&-\langle\nabla^3(\nabla n\cdot u)-\nabla^3(\nabla n\cdot u)^l,\nabla^3 n^h\rangle\\
=&-\langle\nabla^3(\nabla n^h\cdot u)+\nabla^3(\nabla n^l\cdot u)-\nabla^3(\nabla n\cdot u)^l,\nabla^3 n^h\rangle\\
:=&J_{21}+J_{22}+J_{23}.
\end{aligned}\tag{2.36}$$

It is obvious that

$$\nabla^3(\nabla n^h\cdot u)=u\cdot\nabla^4 n^h+3\nabla u\cdot\nabla^3 n^h+3\nabla^2 u\cdot\nabla^2 n^h+\nabla n^h\cdot\nabla^3 u.$$

By virtue of integration by parts, we arrive at

$$\langle u\cdot\nabla^4 n^h,\nabla^3 n^h\rangle=-\frac{1}{2}\int_{\mathbb{R}^3}\mathrm{div}u|\nabla^3 n^h|^2\mathrm{d}x,$$

thus

$$\begin{aligned}
|J_{21}|&\lesssim(\|\nabla u\|_{L^\infty}\|\nabla^3 n^h\|_{L^2}+\|\nabla^2 u\|_{L^3}\|\nabla^2 n^h\|_{L^6}+\|\nabla n^h\|_{L^3}\|\nabla^3 u\|_{L^6})\|\nabla^3 n^h\|_{L^2}\\
&\lesssim(\|\nabla u\|_{H^2}\|\nabla^3 n^h\|_{L^2}+\|\nabla^2 u\|_{H^1}\|\nabla^3 n^h\|_{L^2}+\|\nabla n\|_{H^1}\|\nabla^4 u\|_{L^2})\|\nabla^3 n^h\|_{L^2}\\
&\lesssim\delta_0(\|\nabla^3 n^h\|_{L^2}^2+\|\nabla^4 u\|_{L^2}^2),
\end{aligned}\tag{2.37}$$

For the term J_{22}, we have

$$\begin{aligned}
|J_{22}|&=|\langle\nabla^3(\nabla n^l\cdot u),\nabla^3 n^h\rangle|\\
&\lesssim\|\nabla^3(\nabla n^l\cdot u)\|_{L^2}\|\nabla^3 n^h\|_{L^2}\\
&\lesssim(\|u\|_{L^\infty}\|\nabla^4 n^l\|_{L^2}+\|\nabla n^l\|_{L^3}\|\nabla^3 u\|_{L^6})\|\nabla^3 n^h\|_{L^2}\\
&\lesssim(\|u\|_{H^2}\|\nabla^3 n\|_{L^2}+\|\nabla n\|_{H^1}\|\nabla^4 u\|_{L^2})\|\nabla^3 n\|_{L^2}\\
&\lesssim\delta_0(\|\nabla^3 n\|_{L^2}^2+\|\nabla^4 u\|_{L^2}^2).
\end{aligned}\tag{2.38}$$

For the term J_{23}, with the help of LemmaA.1-Lemma A.3, Hölder's inequality, (1.3), (2.32)-(2.33) and Young inequality, we have

$$\begin{aligned}
|J_{23}| &= |\langle \nabla^3(\nabla n\cdot u)^l, \nabla^3 n^h\rangle| \\
&\lesssim \|\nabla(\nabla n\cdot u)\|_{L^2}\|\nabla^3 n^h\|_{L^2} \\
&\lesssim (\|u\|_{L^3}\|\nabla^2 n\|_{L^6} + \|\nabla u\|_{L^3}\|\nabla n\|_{L^6})\|\nabla^3 n^h\|_{L^2} \\
&\lesssim \|u\|_{H^1}\|\nabla^3 n\|_{L^2}\|\nabla^3 n^h\|_{L^2} + \|u\|_{L^2}^{\frac14}\|\nabla^2 u\|_{L^2}^{\frac34}\|\nabla^2 n\|_{L^2}\|\nabla^3 n^h\|_{L^2} \\
&\lesssim \delta_0(\|\nabla^3 n\|_{L^2}^2 + \|\nabla^3 n^h\|_{L^2}^2) + (1+t)^{-\frac14\times\frac14}(1+t)^{-\frac54\times\frac34}(1+t)^{-\frac74}\|\nabla^3 n^h\|_{L^2} \\
&\lesssim \delta_0(\|\nabla^3 n\|_{L^2}^2 + \|\nabla^3 n^h\|_{L^2}^2) + (1+t)^{-\frac94-\frac12}\|\nabla^3 n^h\|_{L^2} \\
&\lesssim \delta_0(\|\nabla^3 n\|_{L^2}^2 + \|\nabla^3 n^h\|_{L^2}^2) + (1+t)^{-\frac92} + (1+t)^{-1}\|\nabla^3 n^h\|_{L^2}^2.
\end{aligned} \tag{2.39}$$

Substituting (2.37)-(2.39) into (2.36), we can achieve

$$J_2 \lesssim \delta_0(\|\nabla^3 n\|_{L^2}^2 + \|\nabla^3 n^h\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2) + (1+t)^{-\frac92} + (1+t)^{-1}\|\nabla^3 n^h\|_{L^2}^2. \tag{2.40}$$

For the term J_3 and J_4, similarly to the proof of (2.35) and (2.38), it holds that

$$|J_3| + |J_4| \lesssim \delta_0(\|\nabla^3 E^h\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2) + (1+t)^{-\frac92} + (1+t)^{-1}\|\nabla^3 E^h\|_{L^2}^2, \tag{2.41}$$

For the term J_5, by integration by parts, we have

$$\begin{aligned}
J_5 =& -\langle \nabla^3(f_2+f_3)^h, \nabla^3 u^h\rangle \\
=&\langle \nabla^2(f_2+f_3)^h, \nabla^4 u^h\rangle \\
=&\langle \nabla^2(n\nabla n)^h + \nabla(u\nabla u)^h + \nabla(n\nabla^2 u)^h - \nabla(nE)^h, \nabla^4 u^h\rangle \\
:=&J_{51} + J_{52} + J_{53} + J_{54}.
\end{aligned} \tag{2.42}$$

For the term J_{51}, we have

$$\begin{aligned}
|J_{51}| &= |\langle \nabla^2(n\nabla n)^h, \nabla^4 u^h\rangle| \\
&\lesssim \|\nabla^2(n\nabla n)\|_{L^2}\|\nabla^4 u^h\|_{L^2} \\
&\lesssim (\|\nabla n\|_{L^3}\|\nabla^2 n\|_{L^6} + \|n\|_{L^\infty}\|\nabla^3 n\|_{L^2})\|\nabla^4 u^h\|_{L^2} \\
&\lesssim (\|\nabla n\|_{H^1}\|\nabla^3 n\|_{L^2} + \|n\|_{H^2}\|\nabla^3 n\|_{L^2})\|\nabla^4 u^h\|_{L^2} \\
&\lesssim \delta_0(\|\nabla^3 n\|_{L^2}^2 + \|\nabla^4 u^h\|_{L^2}^2).
\end{aligned} \tag{2.43}$$

For the term J_{52}, we have

$$\begin{aligned}
|J_{52}| &= |\langle \nabla^2(u\nabla u)^h, \nabla^4 u^h\rangle| \\
&\lesssim \|\nabla^3(u\nabla u)\|_{L^2}\|\nabla^4 u^h\|_{L^2} \\
&\lesssim (\|\nabla u\|_{L^\infty}\|\nabla^4 u\|_{L^2} + \|\nabla u\|_{L^3}\|\nabla^3 u\|_{L^6})\|\nabla^4 u^h\|_{L^2} \\
&\lesssim (\|\nabla u\|_{H^2}\|\nabla^4 u\|_{L^2} + \|\nabla u\|_{H^1}\|\nabla^4 u\|_{L^2})\|\nabla^4 u^h\|_{L^2} \\
&\lesssim \delta_0(\|\nabla^4 u\|_{L^2}^2 + \|\nabla^4 u^h\|_{L^2}^2).
\end{aligned} \tag{2.44}$$

For the term J_{53}, we have

$$\begin{aligned}|J_{53}| &= |\langle \nabla^2(n\nabla^2 u)^h, \nabla^4 u^h\rangle| \\ &\lesssim \|\nabla^2(n\nabla^2 u)\|_{L^2}\|\nabla^4 u^h\|_{L^2} \\ &\lesssim (\|n\|_{L^\infty}\|\nabla^4 u\|_{L^2} + \|\nabla^2 n\|_{L^6}\|\nabla^2 u\|_{L^3})\|\nabla^4 u^h\|_{L^2} \\ &\lesssim (\|n\|_{H^2}\|\nabla^4 u\|_{L^2} + \|\nabla^3 n\|_{L^2}\|\nabla^2 u\|_{H^1})\|\nabla^4 u^h\|_{L^2} \\ &\lesssim \delta_0(\|\nabla^4 u\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2).\end{aligned} \tag{2.45}$$

For the term J_{54}, we have

$$\begin{aligned}|J_{54}| &= |\nabla^2(nE)^h, \nabla^4 u^h\rangle| \\ &\lesssim \|\nabla^3(nE)\|_{L^2}\|\nabla^4 u^h\|_{L^2} \\ &\lesssim (\|E\|_{L^\infty}\|\nabla^3 n\|_{L^2} + \|n\|_{L^3}\|\nabla^3 E\|_{L^6})\|\nabla^4 u^h\|_{L^2} \\ &\lesssim (\|\nabla E\|_{H^1}\|\nabla^3 n\|_{L^2} + \|n\|_{H^1}\|\nabla^3 n\|_{L^2})\|\nabla^4 u^h\|_{L^2} \\ &\lesssim (\|n\|_{H^1}\|\nabla^3 n\|_{L^2} + \|n\|_{H^1}\|\nabla^3 n\|_{L^2})\|\nabla^4 u^h\|_{L^2} \\ &\lesssim \delta_0(\|\nabla^3 n\|_{L^2}^2 + \|\nabla^4 u^h\|_{L^2}^2).\end{aligned} \tag{2.46}$$

Thus, we can immediately to obtain

$$|J_5| \lesssim \delta_0(\|\nabla^3 n\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2). \tag{2.47}$$

Substitute (2.35), (2.40)–(2.41) and (2.47) into (2.34) and use the smallness of δ_0, we deduce that in fact

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |\nabla^3 n^h|^2 + |\nabla^3 u^h|^2 + |\nabla^3 E^h|^2 \mathrm{d}x + (\mu_1+\mu_2)\int_{\mathbb{R}^3} |\nabla^4 u^h|^2 \mathrm{d}x \\ &\lesssim \delta_0(\|\nabla^3 E^h\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2) \\ &\quad + (1+t)^{-\frac{9}{2}} + (1+t)^{-1}\|\nabla^3 n^h\|_{L^2}^2 + (1+t)^{-1}\|\nabla^3 E^h\|_{L^2}^2.\end{aligned} \tag{2.48}$$

Step 2. Dissipation of $\nabla^3 n^h$. Applying the operator $\nabla^2\mathfrak{F}^{-1}(1-\phi(\xi))$ to $(2.1)_2$, multiplying the resulting equality by $\nabla^3 n^h$, integrating over $\mathbb{R}^3$, it holds that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \nabla^2 u^h \nabla^3 n^h \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla^3 n^h|^2 - |\nabla^3 u^h|^2 + |\nabla^3 E^h|^2 \mathrm{d}x \\ &= -\langle \nabla^3(n\nabla\cdot u)^h, \nabla u^h\rangle - \langle \nabla^3(\nabla n\cdot u)^h, \nabla u^h\rangle - \langle \nabla^2(f_2+f_3)^h\, \nabla^3 n^h\rangle \\ &\quad + (\mu_1+\mu_2)\langle \nabla^4 u^h, \nabla^3 n^h\rangle := J_6 + J_7 + J_8 + J_9.\end{aligned} \tag{2.49}$$

For the term J_6, we have

$$|J_6| \lesssim \delta_0(\|\nabla^3 n\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2 + \|\nabla^3 u^h\|_{L^2}^2). \tag{2.50}$$

For the term J_7, similarly to the proof of (2.40), we obtain

$$|J_7| \lesssim \delta_0(\|\nabla^3 n\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2) + (1+t)^{-\frac{9}{2}} + (1+t)^{-1}\|\nabla^3 u^h\|_{L^2}^2. \tag{2.51}$$

For the term J_8, similarly to the proof of (2.42), it holds that

$$|J_8| \lesssim \delta_0(\|\nabla^3 n\|_{L^2}^2 + \|\nabla^4 u^h\|_{L^2}^2 + \|\nabla^3 u^h\|_{L^2}^2). \tag{2.52}$$

For J_9, we obtain

$$\begin{aligned}|J_9| &\lesssim (\mu_1+\mu_2)\|\nabla^4 u^h\|_{L^2}\|\nabla^3 n^h\|_{L^2} \\ &\lesssim \frac{1}{4}\|\nabla^3 n^h\|_{L^2}^2 + \|\nabla^4 u^h\|_{L^2}^2.\end{aligned} \tag{2.53}$$

Substituting (2.50)–(2.53)into (2.49) and making use of the smallness of δ_0, we conclude that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^2u^h\nabla^3n^h\mathrm{d}x+\int_{\mathbb{R}^3}|\nabla^3n^h|^2+|\nabla^3E^h|^2\mathrm{d}x\\&\lesssim\delta_0(\|\nabla^3n\|_{L^2}^2+\|\nabla^3n^h\|_{L^2}^2+\|\nabla^3u^h\|_{L^2}^2+\|\nabla^4u\|_{L^2}^2)+(1+t)^{-\frac{9}{2}}\\&\qquad+(1+t)^{-1}\|\nabla^3u^h\|_{L^2}^2+\frac{1}{4}\|\nabla^3n^h\|_{L^2}^2+\|\nabla^4u^h\|_{L^2}^2+\int_{\mathbb{R}^3}|\nabla^3u^h|^2dx.\end{aligned}\tag{2.54}$$

Step 3. Closing the estimates. Now, let us close the estimates. For any $t\geq 0$, we define the temporary energy functional as follows

$$\mathfrak{E}_2(\mathfrak{t})=\frac{D_2}{2}\left(\|\nabla^3n^h\|_{L^2}^2+\|\nabla^3u^h\|_{L^2}^2+\|\nabla^3E^h\|_{L^2}^2\right)+\int_{\mathbb{R}^3}\nabla^2u^h\nabla^3n^h\mathrm{d}x,\tag{2.55}$$

for $t\geq T_2$, where it is noticed that $\mathfrak{E}_2(\mathfrak{t})$ is equivalent to $\|\nabla^3n^h\|_{L^2}^2+\|\nabla^3u^h\|_{L^2}^2+\|\nabla^3E^h\|_{L^2}^2$ if we choose sufficiently large time T_2 and positive constant D_2. Putting (2.48) and (2.54) into

$$D_2\times(2.48)+(2.54),$$

for $t\geq T_2$, δ_0 is small enough implies

$$\begin{aligned}&\frac{d}{dt}\mathfrak{E}_2(\mathfrak{t})+(\frac{3}{4}-\delta_0)\|\nabla^3n^h\|_{L^2}^2\\&\qquad+[(\mu_1+\mu_2)D_2-2]\|\nabla^4u^h\|_{L^2}^2+\|\nabla^3E^h\|_{L^2}^2\lesssim\|\nabla^3n^l\|_{L^2}^2+\|\nabla^4u^l\|_{L^2}^2,\end{aligned}\tag{2.56}$$

where we have used the fact that T_2 is large enough. On other hand, it is clear that

$$(\frac{3}{4}-\delta_0)\|\nabla^3n^h\|_{L^2}^2+[(\mu_1+\mu_2)D_2-2]\|\nabla^4u^h\|_{L^2}^2+\|\nabla^3E^h\|_{L^2}^2\geq C_2\mathfrak{E}_2(\mathfrak{t}).\tag{2.57}$$

Thus, in view of (2.4)-(2.5), (2.56)-(2.57) and Gronwall's argument, we have

$$\|\nabla^3n^h\|_{L^2}^2+\|\nabla^3u^h\|_{L^2}^2+\|\nabla^3E^h\|_{L^2}^2\lesssim(1+t)^{-\frac{9}{4}}.\tag{2.58}$$

Furthermore, utilizing (2.4)-(2.5) and (2.58), it is easy to have

$$\begin{aligned}\|\nabla^3n\|_{L^2}^2&\lesssim\|\nabla^3n^h\|_{L^2}^2+\|\nabla^3n^l\|_{L^2}^2\\&\lesssim(1+t)^{-\frac{9}{4}},\end{aligned}\tag{2.59}$$

$$\begin{aligned}\|\nabla^3(u,E)\|_{L^2}^2&\lesssim\|\nabla^3(u^h,E^h)\|_{L^2}^2+\|\nabla^3(u^l,E^h)\|_{L^2}^2\\&\lesssim(1+t)^{-\frac{7}{4}}.\end{aligned}\tag{2.60}$$

This complete the proof of Theorem 1.1 for $k=3$.

Case 3. Proof of Theorem 1.1 for $k=4$**.** taking the similar argument as in Case 2, we can prove Theorem 1.1 for $k=4$ by the following steps.

Step 1. High–frequency L^2 energy estimate. Taking

$$\langle\mathfrak{F}^{-1}(1-\phi(\xi))\nabla^4(2.1)_1,\nabla^4n^h\rangle+\langle\mathfrak{F}^{-1}(1-\phi(\xi))\nabla^4(2.1)_2,\nabla^4u^h\rangle,$$

and applying integration by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\nabla^4 n^h|^2+|\nabla^4 u^h|^2+|\nabla^4 E^h|^2\mathrm{d}x+(\mu_1+\mu_2)\int_{\mathbb{R}^3}|\nabla^5 u^h|^2\mathrm{d}x\\
&=-\langle\nabla^4(n\nabla\cdot u)^h,\nabla^4 n^h\rangle-\langle\nabla^4(\nabla n\cdot u)^h,\nabla^4 n^h\rangle-\langle\nabla^3 E^h,\nabla^4(n\nabla\cdot u)^h\rangle\\
&\quad-\langle\nabla^3 E^h,\nabla^4(\nabla n\cdot u)^h\rangle-\langle\nabla^4(f_2+f_3)^h,\nabla^4 u^h\rangle:=\sum_{i=1}^{5}K_i.
\end{aligned}\tag{2.61}$$

We will estimate the right-hand side of the above equation term by term. First, for term K_1, from Hölder's inequality, Young inequality, Sobolev interpolation theorem, it leads to

$$\begin{aligned}
|K_1|&=|\langle\nabla^4(n\nabla\cdot u)^h,\nabla^4 n^h\rangle|\\
&\lesssim\|\nabla^4(n\nabla\cdot u)\|_{L^2}\|\nabla^4 n^h\|_{L^2}\\
&\lesssim(\|\nabla u\|_{L^\infty}\|\nabla^4 n\|_{L^2}+\|n\|_{L^\infty}\|\nabla^5 u\|_{L^2})\|\nabla^4 n^h\|_{L^2}\\
&\lesssim(\|\nabla u\|_{H^2}\|\nabla^4 n\|_{L^2}+\|n\|_{H^2}\|\nabla^5 u\|_{L^2})\|\nabla^4 n^h\|_{L^2}\\
&\lesssim\delta_0(\|\nabla^4 n\|_{L^2}^2+\|\nabla^4 n^h\|_{L^2}^2+\|\nabla^5 u\|_{L^2}^2),
\end{aligned}\tag{2.62}$$

For the term K_2, we can rewrite it as

$$\begin{aligned}
K_2=&-\langle\nabla^4(\nabla n\cdot u)^h,\nabla^4 n^h\rangle\\
=&-\langle\nabla^4(\nabla n\cdot u)-\nabla^4(\nabla n\cdot u)^l,\nabla^4 n^h\rangle\\
=&-\langle\nabla^4(\nabla n^h\cdot u)+\nabla^4(\nabla n^l\cdot u)-\nabla^4(\nabla n\cdot u)^l,\nabla^4 n^h\rangle\\
:=&K_{21}+K_{22}+K_{23}.
\end{aligned}\tag{2.63}$$

Notice that

$$\nabla^4(\nabla n^h\cdot u)=u\cdot\nabla^5 n^h+4\nabla u\cdot\nabla^4 n^h+6\nabla^2 u\cdot\nabla^3 n^h+4\nabla^3 u\cdot\nabla^2 n^h+\nabla^4 u\cdot\nabla n^h.$$

From integration by parts, one has

$$\langle u\nabla^5 n^h,\nabla^3 n^h\rangle=-\frac{1}{2}\int_{\mathbb{R}^3}\mathrm{div}u|\nabla^4 n^h|^2\mathrm{d}x.$$

Note that

$$\langle\nabla^3 u\nabla^2 n^h,\nabla^4 n^h\rangle=\langle\nabla^3 u^h\nabla^2 n^h,\nabla^4 n^h\rangle+\langle\nabla^3 u^l\nabla^2 n^h,\nabla^4 n^h\rangle,$$

and

$$\langle\nabla^4 u\nabla n^h,\nabla^4 n^h\rangle=\langle\nabla^4 u^h\nabla n^h,\nabla^4 n^h\rangle+\langle\nabla^4 u^l\nabla n^h,\nabla^4 n^h\rangle.$$

Thus, we can have the following estimate

$$\begin{aligned}
|K_{21}|\lesssim&(\|\nabla u\|_{L^\infty}\|\nabla^4 n^h\|_{L^2}+\|\nabla^2 u\|_{L^3}\|\nabla^3 n^h\|_{L^6}+\|\nabla^2 n^h\|_{L^3}\|\nabla^3 u^h\|_{L^6}\\
&+\|\nabla^2 n^h\|_{L^3}\|\nabla^3 u^l\|_{L^6}+\|\nabla n^h\|_{L^3}\|\nabla^4 u^h\|_{L^6}+\|\nabla^4 u^l\|_{L^3}\|\nabla n^h\|_{L^6})\|\nabla^4 n^h\|_{L^2}\\
\lesssim&(\|\nabla u\|_{H^2}\|\nabla^4 n^h\|_{L^2}+\|\nabla^2 u\|_{H^1}\|\nabla^4 n^h\|_{L^2}+\|\nabla^2 n\|_{H^1}\|\nabla^4 u^h\|_{L^2}\\
&+\|\nabla^2 n\|_{H^1}\|\nabla^4 u\|_{L^2}+\|\nabla n\|_{H^1}\|\nabla^5 u^h\|_{L^2}+\|u\|_{H^1}\|\nabla^4 n^h\|_{L^2})\|\nabla^4 n^h\|_{L^2}\\
\lesssim&\delta_0(\|\nabla^4 n\|_{L^2}^2+\|\nabla^5 u\|_{L^2}^2),
\end{aligned}\tag{2.64}$$

For the term K_{22}, we have

$$\begin{aligned}|K_{22}| &= |\langle \nabla^4(\nabla n^l\cdot u), \nabla^4 n^h\rangle| \\ &\lesssim \|\nabla^4(\nabla n^l\cdot u)\|_{L^2}\|\nabla^4 n^h\|_{L^2} \\ &\lesssim (\|u\|_{L^\infty}\|\nabla^5 n^l\|_{L^2} + \|\nabla n^l\|_{L^3}\|\nabla^4 u\|_{L^6})\|\nabla^4 n^h\|_{L^2} \\ &\lesssim (\|u\|_{H^2}\|\nabla^4 n\|_{L^2} + \|n\|_{H^1}\|\nabla^5 u\|_{L^2})\|\nabla^4 n^h\|_{L^2} \\ &\lesssim \delta_0(\|\nabla^4 n\|_{L^2}^2 + \|\nabla^5 u\|_{L^2}^2 + \|\nabla^4 n^h\|_{L^2}^2).\end{aligned} \tag{2.65}$$

For the term K_{23}, we achieve

$$\begin{aligned}|K_{23}| &= |\langle \nabla^4(\nabla n\cdot u)^l, \nabla^4 n^h\rangle| \\ &\lesssim \|\nabla^4(\nabla n\cdot u)^l\|_{L^2}\|\nabla^4 n^h\|_{L^2} \\ &\lesssim \|\nabla^2(\nabla n\cdot u)\|_{L^2}\|\nabla^4 n^h\|_{L^2} \\ &\lesssim (\|u\|_{L^3}\|\nabla^3 n\|_{L^6} + \|\nabla n\|_{L^6}\|\nabla^2 u\|_{L^3})\|\nabla^4 n^h\|_{L^2} \\ &\lesssim (\|u\|_{H^1}\|\nabla^4 n\|_{L^2} + \|\nabla^2 n\|_{L^2}\|\nabla u\|_{L^2}^{\frac14}\|\nabla^3 u\|_{L^2}^{\frac34})\|\nabla^4 n^h\|_{L^2} \\ &\lesssim \delta_0(\|\nabla^4 n\|_{L^2}^2 + \|\nabla^4 n^h\|_{L^2}^2) + (1+t)^{-\frac{11}{4}-\frac12}\|\nabla^4 n^h\|_{L^2} \\ &\lesssim \delta_0(\|\nabla^4 n\|_{L^2}^2 + \|\nabla^4 n^h\|_{L^2}^2) + (1+t)^{-\frac{11}{2}} + (1+t)^{-1}\|\nabla^4 n^h\|_{L^2}^2.\end{aligned} \tag{2.66}$$

Substituting (2.64)-(2.66) into (2.63), we can arrive at

$$K_2 \lesssim \delta_0(\|\nabla^4 n\|_{L^2}^2 + \|\nabla^4 n^h\|_{L^2}^2 + \|\nabla^5 u\|_{L^2}^2) + (1+t)^{-\frac{11}{2}} + (1+t)^{-1}\|\nabla^4 n^h\|_{L^2}^2. \tag{2.67}$$

For the term K_3 and K_4, similarly to the proof of (2.62) and (2.65), it holds that

$$|K_3| + |K_4| \lesssim \delta_0(\|\nabla^4 E^h\|_{L^2}^2 + \|\nabla^4 n\|_{L^2}^2 + \|\nabla^5 u\|_{L^2}^2) + (1+t)^{-\frac{11}{2}} + (1+t)^{-1}\|\nabla^4 E^h\|_{L^2}^2, \tag{2.68}$$

For the term K_5, we have

$$\begin{aligned}K_5 =&\langle \nabla^3(f_2+f_3)^h, \nabla^5 u^h\rangle \\ =&\langle \nabla^3(f_2+f_3)^h, \nabla^5 u^h\rangle \\ =&\langle \nabla^3(n\nabla n)^h + \nabla(u\nabla u)^h + \nabla(n\nabla^3 u)^h - \nabla(nE)^h, \nabla^5 u^h\rangle \\ :=&K_{51} + K_{52} + K_{53} + K_{54}.\end{aligned} \tag{2.69}$$

For the term K_{51}, we have

$$\begin{aligned}|K_{51}| &= |\langle \nabla^3(n\nabla n)^h, \nabla^5 u^h\rangle| \\ &\lesssim \|\nabla^3(n\nabla n)\|_{L^2}\|\nabla^5 u^h\|_{L^2} \\ &\lesssim (\|\nabla n\|_{L^3}\|\nabla^3 n\|_{L^6} + \|n\|_{L^\infty}\|\nabla^4 n\|_{L^2})\|\nabla^5 u^h\|_{L^2} \\ &\lesssim (\|\nabla n\|_{H^1}\|\nabla^4 n\|_{L^2} + \|n\|_{H^2}\|\nabla^4 n\|_{L^2})\|\nabla^5 u^h\|_{L^2} \\ &\lesssim \delta_0(\|\nabla^4 n\|_{L^2}^2 + \|\nabla^5 u^h\|_{L^2}^2).\end{aligned} \tag{2.70}$$

For the term K_{52}, we have

$$\begin{aligned}
|K_{52}| &= |\langle \nabla^3(u\nabla u)^h, \nabla^5 u^h\rangle| \\
&\lesssim \|\nabla^4(u\nabla u)\|_{L^2}\|\nabla^5 u^h\|_{L^2} \\
&\lesssim (\|u\|_{L^\infty}\|\nabla^5 u\|_{L^2} + \|\nabla u\|_{L^3}\|\nabla^4 u\|_{L^6})\|\nabla^5 u^h\|_{L^2} \\
&\lesssim (\|u\|_{H^2}\|\nabla^5 u\|_{L^2} + \|\nabla u\|_{H^1}\|\nabla^5 u\|_{L^2})\|\nabla^5 u^h\|_{L^2} \\
&\lesssim \delta_0(\|\nabla^5 u\|_{L^2}^2 + \|\nabla^5 u^h\|_{L^2}^2).
\end{aligned} \tag{2.71}$$

For the term K_{53}, we obtain

$$\begin{aligned}
|K_{53}| &= |\langle \nabla^3(n\nabla^2 u)^h, \nabla^5 u^h\rangle| \\
&\lesssim \|\nabla^3(n\nabla^2 u)\|_{L^2}\|\nabla^5 u^h\|_{L^2} \\
&\lesssim (\|n\|_{L^\infty}\|\nabla^5 u\|_{L^2} + \|\nabla^2 u\|_{L^3}\|\nabla^3 n\|_{L^6})\|\nabla^5 u^h\|_{L^2} \\
&\lesssim (\|n\|_{H^2}\|\nabla^5 u\|_{L^2} + \|\nabla^2 u\|_{H^1}\|\nabla^4 n\|_{L^2})\|\nabla^5 u^h\|_{L^2} \\
&\lesssim \delta_0(\|\nabla^5 u\|_{L^2}^2 + \|\nabla^4 n\|_{L^2}^2).
\end{aligned} \tag{2.72}$$

For the term K_{54}, we have

$$\begin{aligned}
|K_{54}| &= |\nabla^3(nE)^h, \nabla^5 u^h\rangle| \\
&\lesssim \|\nabla^4(nE)\|_{L^2}\|\nabla^5 u^h\|_{L^2} \\
&\lesssim (\|E\|_{L^\infty}\|\nabla^4 n\|_{L^2} + \|n\|_{L^3}\|\nabla^4 E\|_{L^6})\|\nabla^5 u^h\|_{L^2} \\
&\lesssim (\|\nabla E\|_{H^1}\|\nabla^4 n\|_{L^2} + \|n\|_{H^1}\|\nabla^4 n\|_{L^2})\|\nabla^5 u^h\|_{L^2} \\
&\lesssim (\|n\|_{H^1}\|\nabla^4 n\|_{L^2} + \|n\|_{H^1}\|\nabla^4 n\|_{L^2})\|\nabla^5 u^h\|_{L^2} \\
&\lesssim \delta_0\|\nabla^4 n\|_{L^2}^2 + \|\nabla^5 u^h\|_{L^2}^2.
\end{aligned} \tag{2.73}$$

Thus, we can arrive at

$$|K_5| \lesssim \delta_0(\|\nabla^4 n\|_{L^2}^2 + \|\nabla^5 u\|_{L^2}^2). \tag{2.74}$$

Substituting (2.62), (2.67)–(2.68) and (2.74) into (2.61) and using the smallness of δ_0, we conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} |\nabla^4 n^h|^2 + |\nabla^4 u^h|^2 + |\nabla^4 E^h|^2 \mathrm{d}x + (\mu_1+\mu_2)\int_{\mathbb{R}^3} |\nabla^5 u^h|^2 \mathrm{d}x \\
&\lesssim \delta_0(\|\nabla^4 E^h\|_{L^2}^2 + \|\nabla^5 u\|_{L^2}^2 + \|\nabla^4 n\|_{L^2}^2) \\
&\quad + (1+t)^{-\frac{9}{2}} + (1+t)^{-1}\|\nabla^4 n^h\|_{L^2}^2 + (1+t)^{-1}\|\nabla^4 E^h\|_{L^2}^2.
\end{aligned} \tag{2.75}$$

Step 2. Dissipation of $\nabla^4 n^h$. Applying the operator $\nabla^3\mathfrak{F}^{-1}(1-\phi(\xi))$ to $(2.1)_2$, multiplying the resulting equality by $\nabla^4 n^h$, integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \nabla^3 u^h \nabla^4 n^h \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla^4 n^h|^2 - |\nabla^4 u^h|^2 + |\nabla^4 E^h|^2 \mathrm{d}x \\
&= -\langle \nabla^4(n\nabla\cdot u)^h, \nabla^3 u^h\rangle - \langle \nabla^4(\nabla n\cdot u)^h, \nabla^3 u^h\rangle - \langle \nabla^4(f_2+f_3)^h, \nabla^4 n^h\rangle \\
&\quad + (\mu_1+\mu_2)\langle \nabla^5 u^h, \nabla^4 n^h\rangle := K_6 + K_7 + K_8 + K_9.
\end{aligned} \tag{2.76}$$

For the term K_6, we have

$$|K_6| \lesssim \delta_0(\|\nabla^4 n\|_{L^2}^2 + \|\nabla^5 u\|_{L^2}^2 + \|\nabla^4 u^h\|_{L^2}^2). \tag{2.77}$$

For the term K_7, similarly to the proof of (2.67), we obtain

$$\begin{aligned}|K_7| \lesssim{} & \delta_0(\|\nabla^4 n\|_{L^2}^2 + \|\nabla^4 n^h\|_{L^2}^2 \\ & + \|\nabla^4 u^h\|_{L^2}^2 + \|\nabla^5 u\|_{L^2}^2) + (1+t)^{-\frac{11}{2}} + (1+t)^{-1}\|\nabla^4 u^h\|_{L^2}^2.\end{aligned} \tag{2.78}$$

For the term K_8, similarly to the proof of (2.69), it holds that

$$|K_8| \lesssim \delta_0(\|\nabla^4 n\|_{L^2}^2 + \|\nabla^5 u\|_{L^2}^2 + \|\nabla^4 n^h\|_{L^2}^2). \tag{2.79}$$

For J_9, we have

$$\begin{aligned}|K_9| &\lesssim (\mu_1+\mu_2)\|\nabla^5 u^h\|_{L^2}\|\nabla^4 n^h\|_{L^2} \\ &\lesssim \frac{1}{4}\|\nabla^4 n^h\|_{L^2}^2 + \|\nabla^5 u^h\|_{L^2}^2.\end{aligned} \tag{2.80}$$

Substituting (2.77)–(2.80) into (2.76) and using the smallness of δ_0, we dedeuce that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \nabla^3 u^h \nabla^4 n^h \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla^4 n^h|^2 + |\nabla^4 E^h|^2 \mathrm{d}x \\ &\lesssim \delta_0(\|\nabla^4 n\|_{L^2}^2 + \|\nabla^5 u\|_{L^2}^2 + \|\nabla^4 u^h\|_{L^2}^2 + \|\nabla^4 n^h\|_{L^2}^2) + (1+t)^{-\frac{11}{2}} \\ &\quad + (1+t)^{-1}\|\nabla^4 u^h\|_{L^2}^2 + \frac{1}{4}\|\nabla^4 n^h\|_{L^2}^2 + \|\nabla^5 u^h\|_{L^2}^2 + \int_{\mathbb{R}^3}|\nabla^4 u^h|^2 \mathrm{d}x.\end{aligned} \tag{2.81}$$

Step 3. Closing the estimates. Now, we are in a position to close the estimates. To do this, we define the temporal energy functional

$$\mathfrak{E}_3(\mathfrak{t}) = \frac{D_3}{2}\left(\|\nabla^4 n^h\|_{L^2}^2 + \|\nabla^4 u^h\|_{L^2}^2 + \|\nabla^4 E^h\|_{L^2}^2\right) + \int_{\mathbb{R}^3} \nabla^3 u^h \nabla^4 n^h \mathrm{d}x, \tag{2.82}$$

for $t \ge T_3$, where it is noticed that $\mathfrak{E}_3(\mathfrak{t})$ is equivalent to $\|\nabla^4 n^h\|_{L^2}^2 + \|\nabla^4 u^h\|_{L^2}^2 + \|\nabla^4 E^h\|_{L^2}^2$ if we choose sufficiently large time T_3 and positive constant D_3. Substituting (2.75) and (2.81) into

$$D_3 \times (2.75) + (2.81),$$

and using the smallness of δ_0, for $t \ge T_3$, we have

$$\begin{aligned}&\frac{d}{dt}\mathfrak{E}_3(\mathfrak{t}) + (\frac{3}{4}-\delta_0)\|\nabla^4 n^h\|_{L^2}^2 \\ &\qquad + [(\mu_1+\mu_2)D_3 - 2]\|\nabla^5 u^h\|_{L^2}^2 + \|\nabla^4 E^h\|_{L^2}^2 \lesssim \|\nabla^4 n^l\|_{L^2}^2 + \|\nabla^5 u^l\|_{L^2}^2,\end{aligned} \tag{2.83}$$

since T_3 is large enough. On other hand, it is clear that

$$(\frac{3}{4}-\delta_0)\|\nabla^4 n^h\|_{L^2}^2 + [(\mu_1+\mu_2)D_3 - 2]\|\nabla^5 u^h\|_{L^2}^2 + \|\nabla^4 E^h\|_{L^2}^2 \ge C_3\mathfrak{E}_3(\mathfrak{t}). \tag{2.84}$$

Therefore, together with (2.4)-(2.5), (2.83)-(2.84) and Gronwall's argument, we have

$$\|\nabla^4 n^h\|_{L^2}^2 + \|\nabla^4 u^h\|_{L^2}^2 + \|\nabla^4 E^h\|_{L^2}^2 \lesssim (1+t)^{-\frac{11}{4}}. \tag{2.85}$$

Consequently, we have from (2.4)-(2.5) and (2.85) that

$$\begin{aligned}\|\nabla^4 n\|_{L^2}^2 &\lesssim \|\nabla^4 n^h\|_{L^2}^2 + \|\nabla^4 n^l\|_{L^2}^2 \\ &\lesssim (1+t)^{-\frac{11}{4}},\end{aligned} \tag{2.86}$$

$$\begin{aligned}\|\nabla^4 (u,E)\|_{L^2}^2 &\lesssim \|\nabla^4 (u^h,E^h)\|_{L^2}^2 + \|\nabla^4 (u^l,E^h)\|_{L^2}^2 \\ &\lesssim (1+t)^{-\frac{9}{4}}.\end{aligned} \tag{2.87}$$

This completes the proof of Theorem 1.1 for $k = 4$.

Therefore, we complete the proof of Theorem 1.1. □

Appendix A. Analytic tools. We will use the Sobolev interpolation of Gagliardo-Nirenberg inequality.

Lemma A.1. *Let $0 \le i, j \le k$; then we have*

$$\|\nabla^i f\|_{L^p} \lesssim \|\nabla^j f\|_{L^q}^{1-\theta} \|\nabla^k f\|_{L^r}^{\theta},$$

where θ satisfies

$$\frac{i}{3} - \frac{1}{p} = \left(\frac{j}{3} - \frac{1}{q}\right)(1-\theta) + \left(\frac{k}{3} - \frac{1}{r}\right)\theta.$$

Lemma A.2. *Let $k \ge 1$ be an integer; then one has*

$$\|\nabla^k (fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|\nabla^k g\|_{L^{p_2}} + \|\nabla^k f\|_{L^{p_3}} \|g\|_{L^{p_4}},$$

where $p, p_2, p_3 \in [1, +\infty]$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Finally, we introduce the lemma concerning the estimate for the low-frequency part and the high-frequency part of f.

Lemma A.3. *If $2 \le p \le \infty$, $f \in L^p(\mathbb{R}^3)$, then we have*

$$\|f^l\|_{L^p} + \|f^h\|_{L^p} \lesssim \|f\|_{L^p}.$$

Proof. For $2 \le p \le \infty$, by Young inequality's for convolutions, for the low-frequency, we have

$$\|f^l\|_{L^p} \lesssim \|\mathcal{F}^{-1}\phi\|_{L^1} \|f\|_{L^p} \lesssim \|f\|_{L^p},$$

and hence

$$\|f^h\|_{L^p} \lesssim \|f\|_{L^p} + \|f^l\|_{L^p} \lesssim \|f\|_{L^p}.$$

□

Acknowledgments. Guochun Wu's research was in part supported by National Natural Science Foundation of China (No. 11701193, 11671086), Natural Science Foundation of Fujian Province (No. 2018J05005, 2017J01562), Program for Innovative Research Team in Science and Technology in Fujian Province University Quanzhou High–Level Talents Support Plan (No. 2017ZT012). Yinghui Zhang' research is partially supported by Guangxi Natural Science Foundation (No. 2019JJG110003, 2019AC20214), and National Natural Science Foundation of China (No. 11771150, 11571280, 11301172 and 11226170.)

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Received June 2021; revised July 2021; early access September 2021.

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