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A CLASS OF FOURTH-ORDER HYPERBOLIC EQUATIONS WITH STRONGLY DAMPED AND NONLINEAR LOGARITHMIC TERMS

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Abstract. In this paper, we study a class of hyperbolic equations of the fourth order with strong damping and logarithmic source terms. Firstly, we prove the local existence of the weak solution by using the contraction mapping principle. Secondly, in the potential well framework, the global existence of weak solutions and the energy decay estimate are obtained. Finally, we give the blow up result of the solution at a finite time under the subcritical initial energy.

1. **Introduction.** In this paper, we study the following initial boundary value problem:

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u_t = |u|^{p-2}u \log |u|^k, \ x \in \Omega, t > 0, \\ u = \frac{\partial u}{\partial n} = 0 \ or \ u = \Delta u = 0, \ x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^n$ is a bound domain with smooth boundary $\partial\Omega$, the vector n is the unit outer normal to $\partial\Omega$ and k is a positive real number. p satisfies

$$2 < p < \begin{cases} \frac{2(n-2)}{n-4}, \ n > 4, \\ +\infty, \ n \leq 4. \end{cases} \tag{2}$$

The differential equations studied by many researchers are significant [3, 8, 36], especially the logarithmic nonlinear problem. The logarithmic nonlinear problem is applied to many branches of physics, such as nuclear physics, optics, and geophysics [5, 6, 18], and it appears naturally in inflation cosmology and supersymmetric filed theories, quantum mechanics and nuclear physics [4, 15]. Fourth-order differential equation with strong damping term has wide application in viscoelastic mechanics and quantum mechanics [7, 10, 34]. The strong damping term Δu_t indicates that the stress is proportional not only to the strain in the Hook law, but also to the strain rate in the linearized Kelvin-Voigt material.

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Górka [18] studied the one-dimensional Klein-Gordon equation with logarithmic source terms

$$u_{tt} - u_{xx} = -u + \varepsilon u \ln |u|^2,$$

by using Galerkin's method, logarithmic Sobolev inequality and compactness theorem, the existence of weak solutions is obtained. Gazenave and Haraux [9] considered the problem

$$u_{tt} - \Delta u = u \ln |u|^k, \tag{3}$$

they prove the existence and uniqueness of weak solutions in three dimensions. Later, in the case of infinite dimension in reference [25], Lian et al. modified the potential well method and combined with Sobolev inequality to obtain the global existence of the solution and the blow up result under the condition of different initial energy($E(0) < d$, $E(0) = d$, $E(0) > d$). When $u \ln |u|^k$ in problem (3) becomes $|u|^p \ln |u|$, the problem is also considered by Lian et al.[26], and they establish the global existence and finite time blow up of solutions at three different energy levels.

Hiramatus et al.[21] introduced the equation

$$u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln |u|, \tag{4}$$

to study the dynamics of Q-ball in theoretical physics. A numerical research was mainly carried out, but there was no theoretical research in that paper. For problem (4), Han [19] obtained the result of global existence of weak solution in three-dimensional bounded domain, and Zhang et al.[42] proved the energy decay estimate in infinite dimension case.

Al-Gharabli and Messaoudi [1] considered the Neumann problem of weakly damped wave equations with logarithmic source term

$$u_{tt} + \Delta^2 u + u + u_t = ku \ln |u|, \tag{5}$$

in two-dimensional bounded domain, they first obtained the existence of weak solutions by Galerkin method. Secondly, under the framework of potential well, they proved the global existence and exponential decay of weak solutions for all the conditions where$(u_0, u_1) \in H_0^2 \times L^2$ satisfy $I(u_0) > 0$ and $0 < E(0) < d$. In reference [2], $h(u_t)$ replaces u_t in equation (5), Al-Gharabli and Messaoudi proved the existence and energy decay of solutions in the two-dimensional case.

For hyperbolic equations with nonlinear damping terms, there have also been extensive studies in recent years. Messaoudi [33] studied the following equation with nonlinear damping terms and polynomial source terms

$$u_{tt} + \Delta^2 u + a|u_t|^{m-2} u_t = b|u|^{p-2} u, \tag{6}$$

where $a, b > 0$, $m, p > 2$. First, the author obtained the local existence of the solution by the contraction mapping principle. In addition, the global existence of the solution is proved under the condition of $m \geq p$, and the blow up results are obtained under the condition of $m < p$ and negative initial energy. After that, Wu and Tsai [38] generalized the result of [33]. Chen and Zhou [13] further studied the problem (6) and proved the global nonexistence of the solution with positive initial energy. Moreover, in the case of linear damping ($m = 2$), they obtained blow up result of the solution even if the initial energy disappears under certain conditions.

Liu [31] considered the equation

$$u_{tt} + \Delta^2 u + |u_t|^{m-2} u_t = |u|^{p-2} u \log |u|^k.$$

When $2 \leq m < p$ and the initial energy $E(0) < d$, the author obtained the global existence of weak solution and decay estimate. When the initial energy is negative, the author proved the blow-up results at finite time.

Gazzola and Squassina [17] studied the following damped semilinear wave equation

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = |u|^{p-2} u.$$

For the initial energy $E(0) < d$, the authors obtained the global existence of the weak solution. In addition, when $\omega = 0$, they obtained the finite time blow up result at any arbitrarily high initial energy.

On the basis of reference [17], Lian and Xu [28] studied the following semilinear wave equation with logarithmic source term

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = u \ln |u|,$$

the author studied the global existence, asymptotic behavior and the blow up results under the conditions of subcritical initial energy and critical initial energy respectively. Under the condition of $\omega = 0$ and $E(0) > 0$, the author obtained the blow up results at infinite time.

Recently, Yang et al.[40] investigated a class of fourth order strongly damped nonlinear wave equations

$$u_{tt} - \Delta u + \Delta^2 u - \alpha \Delta u_t = f(u),$$

they comprehensively investigated the global existence, long-time behavior and finite time blow up of the solution at three different initial energy levels. Zeng and Zhao [41] considered the Cauchy problem of a Keller-Segel type chemotaxis model with logarithmic sensitivity and logistic growth, and obtain similar results. In [14], Di considered the initial boundary value problem of the fourth order wave equation with an internal nonlinear source $|u|^{\rho} u$, they proved the global existence and uniqueness of the regular solution and the weak solution respectively, and studied the explicit decay rate estimation of energy. Liu and Zhou [32] considered the local well-posedness of solutions to the initial boundary value problem for fourth-order plate equations with Hardy-Hénon potential and polynomial nonlinearity, and also studied the global existence and finite time blow-up results of solutions.

After looking up these literatures on the dynamic behavior of logarithmic term, it is not difficult to find that the estimates of power-type nonlinear term cannot be directly generalized to logarithmic nonlinear term. When the logarithmic source term is $u \ln |u|$, the logarithmic Sobolev inequality is usually used to deal with such problem. In this paper, the non-linear logarithmic source term $|u|^{p-2} u \log |u|^k$ brings us some difficulties, here we cannot apply logarithmic Sobolev inequality. In addition, the hyperbolic equation is different from the parabolic equation. The parabolic equation with strong damping term has been extensively studied by many authors, and a large number of results have been obtained [11, 20, 27, 29, 37, 39, 43]. Specially, Chen and Xu [12] studied the initial boundary value problem of infinitely degenerate semilinear pseudo-parabolic equations with logarithmic nonlinear terms, and obtained the global existence, blow up and the asymptotic behavior of the solution. However, in this paper, for the fourth order hyperbolic equation, the emergence of logarithmic term and strong damping term prevent us from obtaining the blow up result of the solution. Here, we use the potential well method and some new techniques to obtain the existence of solution, estimate of energy decay, and blow up results.

This paper is organized as follows: In Section 2, we introduce some mathematical symbols, basic definitions and important lemmas needed for theorem proof. In Section 3, we prove the local existence of the weak solution of the problem (1). In Section 4, we give the global existence of the solution and energy decay. In the last Section, we obtain the result of the blow up at a finite time.

2. **Preliminaries and main results.** In this section, we first introduce some of the notation used in this paper. The norm of $L^p(\Omega)$ is denoted by $\|\cdot\|_p$, where $1 \leq p \leq \infty$. We define the following space for further discussion

$$\mathrm{H} = \begin{cases} H_0^2(\Omega) \text{ for } u = 0 \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \\ H_0^1(\Omega) \cap H^2(\Omega) \text{ for } u = 0 \text{ and } \Delta u = 0 \text{ on } \partial\Omega. \end{cases}$$

Naturally, by Poincaré's inequality [30], $\|\Delta \cdot\|_2$ is the equivalent norm of $\|\cdot\|_{\mathrm{H}}$. Besides, $\langle \cdot, \cdot \rangle$ represents the duality pairing between $H^{-2}(\Omega)$ and H.

Let us introduce some of the required functionals.

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 - \frac{1}{p}\int_\Omega |u|^p \log |u|^k dx + \frac{k}{p^2}\|u\|_p^p, \tag{7}$$

$$J(u) = \frac{1}{2}\|\Delta u\|_2^2 - \frac{1}{p}\int_\Omega |u|^p \log |u|^k dx + \frac{k}{p^2}\|u\|_p^p, \tag{8}$$

$$I(u) = \|\Delta u\|_2^2 - \int_\Omega |u|^p \log |u|^k dx. \tag{9}$$

By (8) and (9), we have

$$J(u) = \frac{1}{p} I(u) + \left(\frac{1}{2} - \frac{1}{p}\right)\|\Delta u\|_2^2 + \frac{k}{p^2}\|u\|_p^p. \tag{10}$$

We define the Nehari manifold

$$\mathcal{N} = \{u \in \mathrm{H}\backslash\{0\} : I(u) = 0\}.$$

The depth of potential well is defined as

$$d = \inf_{u \in \mathcal{N}} J(u), \tag{11}$$

by Lemma 2.8, we know that d satisfies

$$d \geq M := \left(\frac{p-2}{2p}\right) {r_*}^2, \tag{12}$$

where r_* is the positive constant defined in Lemma 2.7.

The potential well (stable set) W and the outer space of potential well (unstable set) V are defined as follows:

$$W := \{u \in \mathrm{H} |\, I(u) > 0, J(u) < d\} \cup \{0\},$$
$$V := \{u \in \mathrm{H} |\, I(u) < 0, J(u) < d\}.$$

Now, we give the definition of weak solution to problem(1).

Definition 2.1. The function $u = u(x,t)$ is called a weak solution of the problem(1) on $\Omega \times [0,T)$, if $u \in C([0,T], \mathrm{H}) \cap C^1([0,T], L^2(\Omega)) \cap C^2([0,T], H^{-2}(\Omega))$, $u_t \in L^2(0,T; H_0^1(\Omega))$, and there holds

$$\langle u_{tt}, v \rangle + \int_\Omega \Delta u \Delta v dx + \int_\Omega \nabla u_t \nabla v dx = \int_\Omega |u|^{p-2} u \log |u|^k v dx,$$

for any $v \in \mathrm{H}$, $t \in [0,T)$, where $u(x,0) = u_0(x)$ in H, $u_t(x,0) = u_1(x)$ in $L^2(\Omega)$.

Next, we state our main results of this paper as follows:

Theorem 2.2. *(Local existence) Suppose that $u_0 \in \mathrm{H}$, $u_1(x) \in L^2(\Omega)$. Then there exist a $T > 0$ such that problem(1) admits a unique weak solution u on $[0,T]$ satisfying*

$$u \in C\left([0,T], \mathrm{H}\right) \cap C^1\left([0,T], L^2(\Omega)\right) \cap C^2\left([0,T], H^{-2}(\Omega)\right).$$

Theorem 2.3. *(Global existence and decay estimate) Assume (2) holds, if $u_0 \in \mathrm{H}$, $u_1 \in L^2(\Omega)$, $E(0) \le M$ and $I(u_0) \ge 0$, then the problem(1) admits a global weak solution $u \in L^\infty(0,\infty;\mathrm{H})$ with $u_t \in L^2\left(0,\infty;H_0^1(\Omega)\right)$. Furthermore, there exists a positive constant $\mathcal{K}_0$ such that the energy functional $E(t)$ satisfies the following polynomial decay estimate:*

$$E(t) \le \frac{\mathcal{K}_0}{1+t},\ for\ all\ t \in [0,\infty). \tag{13}$$

In particular, if $E(0) < \min\left\{M, \frac{p-2}{2p}\left(\frac{e\mu}{kC_2{}^{p+\mu}}\right)^{\frac{2}{p+\mu-2}}\right\}$ and $0 < \mu < 2^* - p$, then there exist positive constants $\mathcal{K}_1$ and $\mathcal{K}_2$, such that the $E(t)$ satisfies the exponential decay estimate as follows:

$$E(t) \le \mathcal{K}_1 e^{-\mathcal{K}_2 t},\ for\ all\ t \in [0,\infty), \tag{14}$$

where M is the positive constant defined in (12).

Theorem 2.4. *(Blow up) Assume $u_0 \in \mathrm{H}\backslash\{0\}$, $u_1 \in L^2(\Omega)$ satisfy $E(0) < d$, $I(u_0) < 0$, then the solution u of problem (1) blows up in finite time.*

To prove our main results, we need to introduce some lemmas.

Let (2) holds, by Sobolev's embedding theorem [16], we know that $\|u\|_p \le C_p\|\Delta u\|_2$, where C_p is the optimal embedding constant of $\mathrm{H} \hookrightarrow L^p(\Omega)$, i.e. $C_p = \sup\limits_{u\in \mathrm{H}\backslash\{0\}} \frac{\|u\|_p}{\|\Delta u\|_2}$. We define

$$\alpha^* = \begin{cases} \frac{2n}{n-4} - p,\ n > 4, \\ \infty,\ n \le 4, \end{cases}$$

for any $\alpha \in [0,\alpha^*)$, then $\mathrm{H} \hookrightarrow L^{p+\alpha}(\Omega)$. And we denote $C_{p+\alpha}$ by C_*.

Lemma 2.5. *Let $u \in \mathrm{H} \setminus \{0\}$, then*
(1) $\lim\limits_{\lambda\to 0^+} J(\lambda u) = 0$, $\lim\limits_{\lambda\to+\infty} J(\lambda u) = -\infty$;
(2) There exists a unique $\lambda_* > 0$ such that $\frac{d}{d\lambda}J(\lambda u)\big|_{\lambda=\lambda^*} = 0$, and $J(\lambda u)$ is increasing on $\lambda \in (0,\lambda_*)$, decreasing on $\lambda \in (\lambda_*, +\infty)$;
(3) $I(\lambda u)\begin{cases} > 0, \lambda \in (0,\lambda_*) \\ < 0, \lambda \in (\lambda_*, +\infty) \end{cases}$ and $I(\lambda_* u) = 0$.

Proof. The proof of this lemma can refer to [24]. Here, we omit it. □

From Lemma 2.5, it is easy to see that Nehari manifold is not empty and the definition of d is meaningful.

Lemma 2.6. *Assume (2) holds. Let $u \in \mathrm{H} \setminus \{0\}$, $\alpha \in (0,\alpha^*)$ and*

$$r(\alpha) = \left(\frac{\alpha}{kC_*^{p+\alpha}}\right)^{\frac{1}{p+\alpha-2}},$$

then we have

(i) *if* $0 < \|\Delta u\|_2 \le r(\alpha)$*, then* $I(u) > 0$*.*

(ii) *if* $I(u) \le 0$*, then* $\|\Delta u\|_2 > r(\alpha)$*.*

Proof. For any constant $y > 0$, we have $\log y < y$. Combined with Sobolev embedding inequality, through a direct calculation, we obtain

$$\begin{aligned} I(u) &= \|\Delta u\|_2^2 - \int_\Omega |u|^p \log |u|^k dx \\ &> \|\Delta u\|_2^2 - \frac{k}{\alpha} \|u\|_{p+\alpha}^{p+\alpha} \\ &\ge \|\Delta u\|_2^2 - \frac{k}{\alpha} C_*^{p+\alpha} \|\Delta u\|_2^{p+\alpha} \\ &= \frac{k}{\alpha} C_*^{p+\alpha} \|\Delta u\|_2^2 \left(r(\alpha)^{p+\alpha-2} - \|\Delta u\|_2^{p+\alpha-2} \right). \end{aligned} \tag{15}$$

By the above inequality, it is easy to know that (i) and (ii) hold. □

Lemma 2.7. *Combined with the notation in Lamma 2.6, we have*

$$\begin{aligned} 0 < r_* :&= \sup_{\alpha \in (0,\alpha^*)} r(\alpha) = \sup_{\alpha \in (0,\alpha^*)} \left(\frac{\alpha}{k C_*^{p+\alpha}} \right)^{\frac{1}{p+\alpha-2}} \\ &\le r^* := \sup_{\alpha \in (0,\alpha^*)} \left(\frac{\alpha}{k B^{p+\alpha}} \right)^{\frac{1}{p+\alpha-2}} |\Omega|^{\frac{\alpha}{p(p+\alpha-2)}} \\ &< \infty, \end{aligned}$$

where, $|\Omega|$ *is the measure of* Ω*, and* $B = C_p$ *is the optimal embedding constant.*

Proof. Obviously, if r_* exists, we have $r_* > 0$. So we just have to prove $r(\alpha) < \rho(\alpha)$, r_* exists and $r_* < \infty$, where

$$\rho(\alpha) = \left(\frac{\alpha}{k B^{p+\alpha}} \right)^{\frac{1}{p+\alpha-2}} |\Omega|^{\frac{\alpha}{p(p+\alpha-2)}}.$$

For any $u \in \mathrm{H} \setminus \{0\}$, using the Hölder inequality, we have

$$\|u\|_p \le |\Omega|^{\frac{\alpha}{p(p+\alpha)}} \|u\|_{p+\alpha}.$$

Noticing that $C_* = C_{p+\alpha}$, $B = C_p$, we get

$$\begin{aligned} C_* &= \sup_{u \in \mathrm{H}\setminus\{0\}} \frac{\|u\|_{p+\alpha}}{\|\Delta u\|_2} \\ &\ge |\Omega|^{\frac{-\alpha}{p(p+\alpha)}} \sup_{u \in \mathrm{H}\setminus\{0\}} \frac{\|u\|_p}{\|\Delta u\|_2} \\ &\ge |\Omega|^{\frac{-\alpha}{p(p+\alpha)}} B. \end{aligned}$$

Hence,

$$\left(\frac{\alpha}{k C_*^{p+\alpha}} \right)^{\frac{1}{p+\alpha-2}} \le \left(\frac{\alpha}{k B^{p+\alpha}} \right)^{\frac{1}{p+\alpha-2}} |\Omega|^{\frac{\alpha}{p(p+\alpha-2)}},$$

that is, $r(\alpha) < \rho(\alpha)$.

Next,we prove r_* exists and $r_* < \infty$, the proof is divided into two cases.

Case1. If $n > 4$, then $\alpha \in (0, \alpha^*) = \left(0, \frac{2n}{n-4} - p\right)$. Since $\rho(\alpha)$ is continuous on $\left[0, \frac{2n}{n-4} - p\right]$, we have r_* exists and

$$r^* = \sup_{\alpha \in \left(0, \frac{2n}{n-4} - p\right)} \rho(\alpha) \le \max_{\alpha \in \left[0, \frac{2n}{n-4} - p\right]} \rho(\alpha) < \infty.$$

Case2. If $n \le 4$, then $\alpha \in (0, +\infty)$. We define

$$\begin{aligned} h(\alpha) := \log [\rho(\alpha)] = \left(\frac{1}{p + \alpha - 2}\right) [\log \alpha - \log k - (p + \alpha) \log B] \\ + \frac{\alpha}{p(p + \alpha - 2)} \log |\Omega|, \ \alpha \in (0, +\infty), \end{aligned}$$

thus,

$$h'(\alpha) = \frac{p^2 + p\alpha - 2p + p\alpha \log k - p\alpha \log \alpha + 2p\alpha \log B + p\alpha \log |\Omega| - 2\alpha \log |\Omega|}{p\alpha (p + \alpha - 2)^2}.$$

Let

$$g(\alpha) = p^2 + p\alpha - 2p + p\alpha \log k - p\alpha \log \alpha + 2p\alpha \log B + p\alpha \log |\Omega| - 2\alpha \log |\Omega|,$$

then,

$$\begin{aligned} g'(\alpha) &= p + p \log k - p \log \alpha - p + 2p \log B + p \log |\Omega| - 2 \log |\Omega| \\ &= p \log \frac{kB^2 |\Omega|^{1 - \frac{2}{p}}}{\alpha}, \end{aligned}$$

which shows that $g(\alpha)$ is strictly increasing on $\left(0, kB^2|\Omega|^{1-\frac{2}{p}}\right)$, and strictly decreasing on $\left(kB^2|\Omega|^{1-\frac{2}{p}}, \infty\right)$.

On the one hand, it is easy to see that

$$\lim_{\alpha \to 0^+} g(\alpha) = p^2 - 2p > 0.$$

On the other hand, we can get that

$$\lim_{\alpha \to +\infty} g(\alpha) = p^2 - 2p + p\alpha \left(1 + \log kB^2 |\Omega|^{1 - \frac{2}{p}} - \log \alpha\right) = -\infty.$$

Given the monotonicity of $g(\alpha)$, it is easy to see that there is a unique $\alpha_* \in \left(kB^2|\Omega|^{1-\frac{2}{p}}, \infty\right)$ such that $g(\alpha_*) = 0$. Hence, $g(\alpha) > 0$ for $\alpha \in (0, \alpha_*)$, $g(\alpha) < 0$ for $\alpha \in (\alpha_*, \infty)$, $h(\alpha)$ attains its maximum at $\alpha = \alpha_*$. Therefore,

$$r^* = \sup_{\alpha \in (0, +\infty)} \rho(\alpha) = e^{h(\alpha_*)} < \infty.$$

□

From Lemma 2.6 and Lemma 2.7, it is not difficult to get the following corollary.

Corollary 1. *Assume* (2) *holds. Let* $u \in \mathrm{H} \setminus \{0\}$, *we have*
(i) *if* $0 < \|\Delta u\|_2 \le r_*$, *then* $I(u) > 0$;
(ii) *if* $I(u) \le 0$, *then* $\|\Delta u\|_2 \ge r_*$,
where r_* *is defined in Lemma* 2.7.

Lemma 2.8. *Assume* (2) *holds, we have*

$$d \geq M := \left(\frac{p-2}{2p}\right) {r_*}^2, \tag{16}$$

where r_* *is defined in Lemma 2.7.*

Proof. By the definition of d, we know $u \in \mathcal{N}$, then $I(u) = 0$. Combined with (10) and (ii) of Corollary 1, we get

$$J(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u\|_2^2 + \frac{k}{p^2} \|u\|_p^p \geq \left(\frac{p-2}{2p}\right) {r_*}^2,$$

thus, (16) holds. □

From the previous definition of $E(t)$, we have the following energy equation

$$E(t) + \int_0^t \|\nabla u_t\|_2^2 \, d\tau = E(0). \tag{17}$$

Lemma 2.9. *If* $u_0 \in \mathrm{H}$, $u_1 \in L^2(\Omega)$, $p > 2$, $E(0) < d$ *and* u *is a weak solution of problem*(1) *on* $[0, T)$, *then*
 (i) *if* $I(u_0) > 0$, *then* $u \in W$;
 (ii) *if* $I(u_0) < 0$, *then* $u \in V$.

Proof. By the definition of $E(t)$, $J(u)$ with (17), we have

$$\frac{1}{2} \|u_t\|_2^2 + J(u) \leq \frac{1}{2} \|u_1\|_2^2 + J(u_0) < d. \tag{18}$$

(i) By contradiction, we assume that there exists $t_0 \in [0, T)$, such that $u(t) \in W$ on $[0, t_0)$and $u(t_0) \notin W$. By the continuity of $J(u)$ and $I(u)$, we have

$$J(u(t_0)) = d \text{ or } I(u(t_0)) = 0.$$

Obviously, $J(u(t_0)) = d$ is impossible. If $I(u(t_0)) = 0$ holds, by the definition of d, then $J(u(t_0)) \geq d$, which is contradictive with (18). Thus, $u \in W$. (ii) The proof is similar to (i), which we omit here. □

3. Local existence of weak solution.

In this part, we prove the local existence and uniqueness of weak solution. To prove the local existence of weak solution, firstly, we need to introduce the following lemmas.

Lemma 3.1. [22] *For any* $\varepsilon > 0$, *there exists a constant* $A > 0$, *such that the function*

$$j(s) = |s|^{p-2} \log|s|, \ p > 2$$

satisfies

$$|j(s)| \leq A + |s|^{p-2+\varepsilon}.$$

Here, for every $T > 0$, we consider the space

$$\mathcal{H} = C([0, T], \mathrm{H}) \cap C^1\left([0, T], L^2(\Omega)\right)$$

endowed with the norm

$$\|u\|_{\mathcal{H}}^2 = \max_{t \in [0,T]} \left(\|\Delta u\|_2^2 + \|u_t\|_2^2\right)$$

Lemma 3.2. *For every $T>0$, $u\in\mathcal{H}$ and every initial data $(u_0,u_1)\in \mathrm{H}\times L^2(\Omega)$, there exists a unique solution $v\in C\left([0,T],\mathrm{H}\right)\cap C^1\left([0,T],L^2(\Omega)\cap C^2\left([0,T],H^{-2}(\Omega)\right)\right.$ with $v_t\in L^2\left([0,T],H_0^1(\Omega)\right)$, which solves the linear problem*

$$\begin{cases} v_{tt}+\Delta^2 v-\Delta v_t=|u|^{p-2}u\log|u|^k, \ x\in\Omega, t>0,\\ v=\frac{\partial v}{\partial n}=0 \ \textit{or} \ v=\Delta v=0 \ x\in\partial\Omega, t\geq 0,\\ v\left(x,0\right)=u_0\left(x\right), v_t\left(x,0\right)=u_1\left(x\right), \ x\in\Omega. \end{cases} \tag{19}$$

Proof. Applying Garlerkin's method, for every $h\geq 1$, let $W_h=\operatorname{span}\{w_1,w_2,\cdots,w_h\}$, where $\{w_j\}$ is the orthonormal complete system of eigenfunctions of $-\Delta$ in H such that $\|w_j\|_2=1$ for all j. According to their multiplicity of

$$\Delta w_j+\lambda_j w_j=0,$$

we denote the related eigenvalues repeated by λ_j. Let

$$u_0^h=\sum_{j=1}^h\left(\int_\Omega u_0w_jdx\right)w_j,\ u_1^h=\sum_{j=1}^h\left(\int_\Omega u_1w_jdx\right)w_j,$$

so that $u_0^h\in W_h$, $u_1^h\in W_h$, $u_0^h\to u_0$ in H and $u_1^h\to u_1$ in $L^2(\Omega)$ as $h\to\infty$. For all $h\geq 1$, we seek h functions $\gamma_1^h,\cdots,\gamma_h^h\in C^2\left[0,T\right]$ such that

$$v_h\left(t\right)=\sum_{j=1}^h\gamma_j^h\left(t\right)w_j \tag{20}$$

solves the problem

$$\begin{cases} \int_\Omega\left[\ddot{v}_h(t)+\Delta^2 v_h-\Delta\dot{v}_h-|u|^{p-2}u\log|u|^k\right]\eta=0,\\ v_h(0)=u_0^h,\ \dot{v}_h(0)=u_1^h. \end{cases} \tag{21}$$

where $\eta\in W_h$ and $t\geq 0$. Taking $\eta=w_j$ for $j=1,\cdots,h$ in (21), we obtain the following Cauchy problem for a linear ordinary differential equation with unknown γ_j^h:

$$\begin{cases} \ddot{\gamma}_j^h(t)+\lambda_j^2\gamma_j^h(t)+\lambda_j\dot{\gamma}_j^h(t)=\psi_j\left(t\right),\\ \gamma_j^h(0)=\int_\Omega u_0w_jdx, \dot{\gamma}_j^h(0)=\int_\Omega u_1w_jdx. \end{cases} \tag{22}$$

where $\psi_j\left(t\right)=\int_\Omega|u|^{p-2}u\log|u|^k w_jdx\in C\left[0,T\right]$. For all j, the above Cauchy problem has a unique local solution $\gamma_j^h\in C^2\left[0,T\right]$, which implies a unique v_h defined by (20) satisfying (21).

Let $\eta=\dot{v}_h(t)$ in (21), integrating over $[0,t]\subset[0,T]$, we get

$$\begin{aligned} &\|\dot{v}_h(t)\|_2^2+\|\Delta v_h(t)\|_2^2+2\int_0^t\|\nabla\dot{v}_h(t)\|_2^2d\tau\\ =&\left\|u_1^h\right\|_2^2+\left\|\Delta u_0^h\right\|_2^2+2\int_0^t\int_\Omega|u|^{p-2}u\log|u|^k\dot{v}_hdxd\tau, \end{aligned} \tag{23}$$

for every $h\geq 1$. We estimate the last term in the right-hand side of (23). Using Hölder's inequality, we have

$$\begin{aligned} 2\int_0^t\int_\Omega|u|^{p-2}u\log|u|^k\dot{v}_hdxd\tau&\leq 2\int_0^t\int_\Omega\left||u|^{p-2}u\log|u|^k\right||\dot{v}_h|\,dxd\tau\\ &\leq 2\int_0^t\left\||u|^{p-2}u\log|u|^k\right\|_{\frac{p}{p-1}}\|\dot{v}_h\|_pd\tau. \end{aligned} \tag{24}$$

Using the fact $\left|x^{p-1}\log x\right| \leq (e\,(p-1))^{-1}$ for $0 < x < 1$, while $x^{-\mu}\log x \leq (e\mu)^{-1}$ for $x \geq 1$, $\mu > 0$. Choosing $\mu > 0$ such that $\frac{p(p-1+\mu)}{p-1} < \frac{2n}{n-2} < 2^* = \frac{2n}{n-4}$, by a direct calculation and Sobolev inequality, we have

$$\begin{aligned}
&\int_\Omega \left||u|^{p-2}u\log|u|^k\right|^{\frac{p}{p-1}}dx \\
=&\int_{\{x\in\Omega:|u|<1\}}\left||u|^{p-2}u\log|u|^k\right|^{\frac{p}{p-1}}dx+\int_{\{x\in\Omega:|u|\geq1\}}\left||u|^{p-2}u\log|u|^k\right|^{\frac{p}{p-1}}dx \\
\leq& k^{\frac{p}{p-1}}(e\,(p-1))^{-\frac{p}{p-1}}|\Omega|+k^{\frac{p}{p-1}}(e\mu)^{-\frac{p}{p-1}}\int_{\{x\in\Omega:|u|\geq1\}}|u|^{\frac{p(p-1+\mu)}{p-1}}dx \\
\leq& C+C\left\|\Delta u\right\|_2^{\frac{p(p-1+\mu)}{p-1}} \\
\leq& C,
\end{aligned} \tag{25}$$

here, it is needs to be noted that C in the text is a general constant, and the C in each row and even in the same row is differemt.

By the Sobolev embedding theorem, we have $\|\dot v_h\|_p \leq C\|\nabla\dot v_h\|_2$. Combined with Young's inequality, (24) yields

$$\begin{aligned}
2\int_0^t\int_\Omega |u|^{p-2}u\log|u|^k\dot v_h dxd\tau &\leq 2\int_0^t\left\||u|^{p-2}u\log|u|^k\right\|_{\frac{p}{p-1}}\|\dot v_h\|_p d\tau \\
&\leq 2C\int_0^t\|\nabla\dot v_h\|_2 d\tau \\
&\leq CT+\int_0^t\|\nabla\dot v_h\|_2^2\,d\tau.
\end{aligned} \tag{26}$$

Recalling the convergence of u_0^h and u_1^h, by (23) and (26), we obtain

$$\|\dot v_h(t)\|_2^2+\|\Delta v_h(t)\|_2^2+\int_0^t\|\nabla\dot v_h(t)\|_2^2 d\tau \leq C, \tag{27}$$

for every $h \geq 1$, where C is independent of h. By this uniform estimate, we have

$\{v_h\}$ is bounded in $L^\infty([0,T],\mathrm{H})$;

$\{\dot v_h\}$ is bounded in $L^\infty\left([0,T],L^2(\Omega)\right)\cap L^2\left([0,T],H_0^1(\Omega)\right)$;

$\{\ddot v_h\}$ is bounded in $L^2\left([0,T],H^{-2}(\Omega)\right)$.

Thus, up to a subsequence, we could pass to the limit in (21) satisfying above regularity. Then a weak local solution of problem (19) can be obtained.

Uniqueness follows arguing for contradiction, if v and w were two solutions of (19) which have the same initial date, by subtracting the equations and testing with $v_t - w_t$, we could get

$$\|v_t-w_t\|_2^2+\|\Delta v-\Delta w\|_2^2+2\int_0^t\int_\Omega|\nabla v_t-\nabla w_t|^2dxd\tau=0,$$

which yields $w = v$. The proof of the lemma is complete. □

Now, we begin to prove Theorem 2.2.

Proof of Theorem 2.2. For any $u \in \mathcal{H}$, $u_0 \in \mathrm{H}$, $u_1 \in L^2(\Omega)$, let

$$R^2 := 2\left(\|u_1\|_2^2+\|\Delta u_0\|_2^2\right).$$

For any $T > 0$, we consider

$$\mathcal{U}_T=\left\{u\in\mathcal{H}: u(0)=u_0, u_t(0)=u_1, \|u\|_{\mathcal{H}}\leq R\right\}.$$

By Lemma 3.2, for any $u \in \mathcal{U}_T$ we could define $v = \Phi(u)$, where v is the unique solution of problem (19). We claim that, for a suitable $T > 0$, Φ is a contractive map satisfying $\Phi(\mathcal{U}_T) \subseteq \mathcal{U}_T$. Given $u \in \mathcal{U}_T$, the corresponding solution $v = \Phi(u)$ satisfies the energy identity for all $t \in (0, T]$ as follows

$$\|v_t\|_2^2+\|\Delta v\|_2^2+2\int_0^t \|\nabla v_t\|_2^2 d\tau \le \|u_1\|_2^2+\|\Delta u_0\|_2^2+2\int_0^t \left\| |u|^{p-1}\log|u|^k \right\|_{\frac{p}{p-1}} \|v_t\|_p d\tau. \tag{28}$$

Now we estimate the last term of (28), we get

$$\begin{aligned} &2\int_0^t \left\| |u|^{p-1}\log|u|^k \right\|_{\frac{p}{p-1}} \|v_t\|_p d\tau \\ \le& C\int_0^t \left\| |u|^{p-1}\log|u|^k \right\|^2_{\frac{p}{p-1}} d\tau + 2\int_0^t \|\nabla v_t\|_2^2 d\tau \\ \le& C\int_0^t \left| k^{\frac{p}{p-1}}(e(p-1))^{-\frac{p}{p-1}}|\Omega| + C\|\Delta u\|_2^{\frac{p(p-1+\mu)}{p-1}} \right|^{\frac{2(p-1)}{p}} d\tau+2\int_0^t \|\nabla v_t\|_2^2 d\tau \\ \le& CT(1+R^{2(p-1+\mu)}) + 2\int_0^t \|\nabla v_t\|_2^2 d\tau. \end{aligned} \tag{29}$$

Substituting (29) into (28), we obtain

$$\|v_t\|_2^2 + \|\Delta v\|_2^2 \le \frac{R^2}{2} + CT(1+R^{2(p-1+\mu)}).$$

Choosing $T > 0$ small enough, such that $CT(1+R^{2(p-1+\mu)}) \le \frac{R^2}{2}$. Hence, $\|\Phi(u)\|_{\mathcal{H}} \le R$, that is $\Phi(\mathcal{U}_T) \subseteq \mathcal{U}_T$.

Next, we show that Φ is contractive in $\mathcal{U}_T$. Namely, For any $u_1, u_2 \in \mathcal{U}_T$, there exists $0 < \delta < 1$, such that

$$\|\Phi(u_1) - \Phi(u_2)\|_{\mathcal{H}} \le \delta\|u_1 - u_2\|_{\mathcal{H}}.$$

Let $u_1, u_2 \in \mathcal{U}_T$, $v_1 = \Phi(u_1)$, $v_2 = \Phi(u_2)$ and $z = v_1 - v_2$, then z is the unique solution to the following problem

$$\begin{cases} z_{tt} + \Delta^2 z - \Delta z_t = |u_1|^{p-2}u_1\log|u_1|^k - |u_2|^{p-2}u_2\log|u_2|^k, \\ z(x,0) = 0, \ \ z_t(x,0) = 0. \end{cases} \tag{30}$$

Multiplying both sides of (30) by z_t, and integrating over $(0,t) \times \Omega$, we get

$$\begin{aligned} &\|z_t\|_2^2 + \|\Delta z\|_2^2 + 2\int_0^t \|\nabla z_t\|_2^2 d\tau \\ =&2\int_0^t\int_\Omega \left(|u_1|^{p-2}u_1\log|u_1|^k - |u_2|^{p-2}u_2\log|u_2|^k\right) z_t dx d\tau. \end{aligned}$$

Using Lagrange Theorem, for $0 < \theta < 1$, combined with Lemma 3.1, we have

$$\begin{aligned} &\left| |u_1|^{p-2}u_1\log|u_1|^k - |u_2|^{p-2}u_2\log|u_2|^k \right| \\ =&k\left|1+(p-1)\log|\theta u_1 + (1-\theta)u_2|\right| |\theta u_1 + (1-\theta)u_2|^{p-2} |u_1 - u_2| \\ \le& k|\theta u_1 + (1-\theta)u_2|^{p-2}|u_1-u_2| + k(p-1)A|u_1-u_2| \\ &+ k(p-1)|\theta u_1 + (1-\theta)u_2|^{p-2+\varepsilon}|u_1-u_2| \\ \le& k|u_1+u_2|^{p-2}|u_1-u_2| + k(p-1)A|u_1-u_2| + k(p-1)|u_1+u_2|^{p-2+\varepsilon}|u_1-u_2|. \end{aligned}$$

Since $u_1, u_2 \in \mathcal{U}_T$, utilizing Hölder's inequality and Sobolev embedding theorem, we have

$$\begin{aligned}
&\int_\Omega \Big| |u_1+u_2|^{p-2} |u_1-u_2| \Big|^2 dx \\
&\le \left(\int_\Omega |u_1+u_2|^{2(p-1)} dx\right)^{\frac{p-2}{p-1}} \left(\int_\Omega |u_1-u_2|^{2(p-1)} dx\right)^{\frac{1}{p-1}} \\
&\le C\Big(\|u_1\|_{2(p-1)}^{2(p-1)} + \|u_2\|_{2(p-1)}^{2(p-1)}\Big)^{\frac{p-2}{p-1}} \|u_1-u_2\|_{2(p-1)}^2 \\
&\le C\left(\|u_1\|_{\mathcal{H}}^{2(p-1)} + \|u_2\|_{\mathcal{H}}^{2(p-1)}\right)^{\frac{p-2}{p-1}} \|u_1-u_2\|_{\mathcal{H}}^2 \\
&\le CR^{2(p-2)} \|u_1-u_2\|_{\mathcal{H}}^2.
\end{aligned} \tag{31}$$

Choosing $\varepsilon > 0$ small enough, such that $\bar{p} = 2(p-1) + \frac{2\varepsilon(p-1)}{p-2} < \frac{2n}{n-4}$, by a calculation similar to (31), we obtain

$$\begin{aligned}
&\int_\Omega \Big| |u_1+u_2|^{p-2+\varepsilon} |u_1-u_2| \Big|^2 dx \\
&\le \left(\int_\Omega |u_1+u_2|^{\frac{2(p-2+\varepsilon)(p-1)}{p-2}} dx\right)^{\frac{p-2}{p-1}} \left(\int_\Omega |u_1-u_2|^{2(p-1)} dx\right)^{\frac{1}{p-1}} \\
&\le C\left(\int_\Omega |u_1+u_2|^{2(p-1)+\frac{2\varepsilon(p-1)}{p-2}} dx\right)^{\frac{p-2}{p-1}} \|u_1-u_2\|_{2(p-1)}^2 \\
&\le C\Big(\|u_1\|_{\bar{p}}^{\bar{p}} + \|u_2\|_{\bar{p}}^{\bar{p}}\Big)^{\frac{p-2}{p-1}} \|u_1-u_2\|_{2(p-1)}^2 \\
&\le CR^{\frac{\bar{p}(p-2)}{p-1}} \|u_1-u_2\|_{\mathcal{H}}^2.
\end{aligned} \tag{32}$$

From the above calculation, we can deduce

$$\Big\| |u_1|^{p-2} u_1 \log|u_1|^k - |u_2|^{p-2} u_2 \log|u_2|^k \Big\|_2^2 \le C(R^{2(p-2)} + 1 + R^{\frac{\bar{p}(p-2)}{p-1}}) \|u_1-u_2\|_{\mathcal{H}}^2.$$

Hence, for some $\delta < 1$ as long as T is sufficiently small, we have

$$\|z_t\|_2^2 + \|\Delta z\|_2^2 \le CT(1 + R^{2(p-2)} + R^{\frac{\bar{p}(p-2)}{p-1}}) \|u_1-u_2\|_{\mathcal{H}}^2 < \delta \|u_1-u_2\|_{\mathcal{H}}^2.$$

That is,

$$\|\Phi(u_1) - \Phi(u_2)\|_{\mathcal{H}}^2 = \|v_1-v_2\|_{\mathcal{H}}^2 \le \delta \|u_1-u_2\|_{\mathcal{H}}^2.$$

So by the contraction mapping principle, we can conclude that problem (1) admits a unique solution. □

4. Global existence and energy decay estimate.

In this section, we prove Theorem 2.3, which is divided into 4 steps.

Proof of Theorem 2.3. **Step 1. Global existence for the case of $E(0) < M$ and $I(u_0) \ge 0$.**

By the definition of $E(t)$ and (10), we know that $0 \le J(u_0) \le E(0) < M$, and combine with Lemma 2.8, then we have respectively

(i) If $E(0) = 0$ and $I(u_0) \ge 0$, then this implies that $(u_0, u_1) = (0, 0)$, which is a trivial case;

(ii) If $0 < E(0) < M \le d$ and $I(u_0) = 0$, then it contradicts with the definition of potential depth d.

Hence, we only need to consider the case of $0 < E(0) < M \leq d$ and $I(u_0) > 0$.

Let $\{w_j(x)\}$ be a system of base functions in H. We construct the following approximate solutions to problem (1)

$$u_m(x,t) = \sum_{j=1}^{m} g_{jm}(t)\, w_j(x), m = 1, 2, ...,$$

satisfying

$$\langle u_{mtt}, w_j\rangle + (\Delta u_m, \Delta w_j) + (\nabla u_{mt}, \nabla w_j) = \int_\Omega |u_m|^{p-2} u_m \log |u_m|^k\, w_j dx, \quad j = 1, 2, ..., m \tag{33}$$

$$u_{0m} = u_m(x,0) = \sum_{j=1}^{m} g_{jm}(0)\, w_j(x) \to u_0 \text{ strongly in H}, \tag{34}$$

$$u_{1m} = u_{mt}(x,0) = \sum_{j=1}^{m} g_{jmt}(0)\, w_j(x) \to u_1 \text{ strongly in } L^2(\Omega). \tag{35}$$

Now, multiplying (33) by $g_{jmt}(t)$, summing for j, and integrating over $[0,t]$, we can compute

$$E_m(t) + \int_0^t \|\nabla u_{mt}\|_2^2\, d\tau = E_m(0),\ 0 \leq t < +\infty, \tag{36}$$

for sufficiently large m. Since $E(0) < M \leq d$ and $I(u_0) > 0$, by (34) and (35), for sufficiently large m, we conclude that $E_m(0) < M \leq d$ and $I(u_{0m}) > 0$. By the argument in the proof of Lemma 2.9, for sufficiently large m and $0 \leq t < \infty$, we have $u_m(t) \in W$. Hence, combined with (10), we have

$$d \geq M > E_m(t) > J(u_m) > \left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u_m\|_2^2 + \frac{k}{p^2} \|u_m\|_p^p,$$

for sufficiently large m and $0 \leq t < \infty$. So it follows that

$$\|\Delta u_m\|_2^2 < \frac{2pM}{p-2},\ \|u_m\|_p^p < \frac{p^2 M}{k}, \tag{37}$$

$$\int_0^t \|\nabla u_{mt}\|_2^2\, d\tau < M. \tag{38}$$

By (37), (38), there exist functions u and a subsequence of $\{u_m\}_{m=1}^\infty$ which we still denote it by $\{u_m\}_{m=1}^\infty$ such that

$$u_m \to u \text{ weakly star in } L^\infty(0,\infty;\mathrm{H}), \tag{39}$$

$$u_{mt} \to u_t \text{ weakly in } L^2\left(0,\infty; H_0^1(\Omega)\right), \tag{40}$$

By Aubin-Lions-Simon Lemma (see [35], Corollary 4), we get

$$u_m \to u \text{ strongly in } C\left(0,\infty; H_0^1(\Omega)\right),$$

so, $u_m \to u$ a.e. $(x,t) \in \Omega \times [0,\infty)$, $m \to +\infty$. This implies

$$|u_m|^{p-2} u_m \log |u_m|^k \to |u|^{p-2} u \log |u|^k,\ \text{ a.e. } (x,t) \in \Omega \times [0,+\infty). \tag{41}$$

On the other hand, from (41), (25) and Lions Lemma (see [30], Lemma 1.3, p.12), we have

$$|u_m|^{p-2} u_m \log |u_m|^k \to |u|^{p-2} u \log |u|^k \text{ weakly star in } L^\infty\left(0,\infty; L^{\frac{p}{p-1}}(\Omega)\right). \tag{42}$$

Integrating (33) with respect to t we obtain

$$\begin{aligned}&(u_{mt},w_j)+\int_0^t(\Delta u_m,\Delta w_j)d\tau+(\nabla u_m,\nabla w_j)\\&=(u_{1m},w_j)+(\nabla u_{0m},\nabla w_j)+\int_0^t\left(|u_m|^{p-2}u_m\log|u_m|^k,w_j\right)d\tau,\end{aligned}\tag{43}$$

therefore, up to a subsequence, by (39)-(42), we could pass to the limit in (43). Moreover, from(34) and (35), we get $u(x,0)=u_0$ in H and $u_t(x,0)=u_1$ in $L^2(\Omega)$. Then we have a global weak solution $u(x,t)$ to problem(1).

Step 2. Global existence for the case of $E(0)=M$ and $I(u_0)\geq 0$.

By Lemma 2.8, we know $d\geq M$. If $E(0)=M<d$, then the problem (1) has a global weak solution, which is similar to the proof of step 1. If $E(0)=M=d$, we consider two cases $I(u_0)=0$ and $I(u_0)>0$.

(i)$E(0)=M=d$, $I(u_0)=0$

From the definition of d, we have $J(u_0)\geq d$. However, $\frac{1}{2}\|u_1\|_2^2+J(u_0)=E(0)=d$, it follows that $J(u_0)<d$. So case (1) is impossible.

(ii)$E(0)=M=d$, $I(u_0)>0$

In order to prove the global existence result of problem (1), we first choose a sequence $\{\gamma_m\}_{m=1}^{\infty}\subset(0,1)$ such that $\lim\limits_{m\to\infty}\gamma_m=1$. Then we considering the following problem

$$\begin{cases}u_{tt}+\Delta^2u-\Delta u_t=|u|^{p-2}u\log|u|^k,\ (x,t)\in\Omega\times(0,T),\\u=\frac{\partial u}{\partial n}=0,(x,t)\in\partial\Omega\times(0,T),\\u(x,0)=u_{0m},u_t(x,0)=u_{1m},\ x\in\Omega,\end{cases}\tag{44}$$

where $u_{0m}=\gamma_mu_0$, $u_{1m}=\gamma_mu_1$. Since $I(u_0)>0$, it follows from Lemma 2.5 that $\lambda_*>1$.

Hence, we get

$$I(u_{0m})=I(\gamma_mu_0)>0,$$

$$J(u_{0m})=J(\gamma_mu_0)<J(u_0),$$

and

$$0<E_m(0)=\frac{1}{2}\|u_{1m}\|_2^2+J(u_{0m})<\frac{1}{2}\|u_1\|_2^2+J(u_0)=E(0)=M=d.$$

Using the similar arguments as previous step 1, we find that problem (44) admits a global weak solution u_m which satisfies

$$u_m\in L^\infty(0,\infty;\mathrm{H}),\ u_{mt}\in L^2\left(0,\infty;H_0^1(\Omega)\right)$$

and

$$\langle u_{mtt},\upsilon\rangle+(\Delta u_m,\Delta\upsilon)+(\nabla u_{mt},\nabla\upsilon)=\int_\Omega|u_m|^{p-2}u_m\log|u_m|^k\upsilon dx,\ j=1,2,...,m,$$

for any $\upsilon\in\mathrm{H}$, and for a.e. $0\leq t<\infty$. The remainder of the proof can be processed similarly as previous step 1. Hence, u is a global weak solution for problem (1).

Step 3. Polynomial decay estimate of energy for the case of $E(0)\leq M$ and $I(u_0)\geq 0$.

Firstly, by (10), (17) and $I(u) \geq 0$, we obtain

$$\begin{aligned} E(0) =& E(t) + \int_0^t \|\nabla u_t\|_2^2 d\tau \\ =& \frac{1}{2}\|u_t\|_2^2 + \frac{1}{p} I(u) + \frac{p-2}{2p}\|\Delta u\|_2^2 + \frac{k}{p^2}\|u\|_p^p + \int_0^t \|\nabla u_t\|_2^2 d\tau \\ \geq& \frac{1}{2}\|u_t\|_2^2 + \frac{p-2}{2p}\|\Delta u\|_2^2 + \frac{k}{p^2}\|u\|_p^p + \int_0^t \|\nabla u_t\|_2^2 d\tau. \end{aligned} \tag{45}$$

A combination of (45) and $E(0) \leq M$, we have

$$\int_0^t \|u_t\|_2^2 d\tau \leq \frac{1}{\lambda_1}\int_0^t \|\nabla u_t\|_2^2 d\tau < \frac{M}{\lambda_1}, \tag{46}$$

where λ_1 is the first eigenvalue of the following problem:

$$\begin{cases} -\Delta\phi(x) = \lambda\phi(x), x \in \Omega, \\ \phi(x) = 0, x \in \partial\Omega, \end{cases}$$

for $\phi(x) \in H_0^1(\Omega)$. Next, multiplying the first equation of problem (1) by u and integrating over $\Omega \times (0,t)$. Using Young inequality, we have

$$\begin{aligned} \int_0^t I(u)\, d\tau =& -\int_0^t (u_{tt}, u)\, d\tau - \int_0^t (\nabla u_t, \nabla u) d\tau \\ =& -\int_\Omega u_t u dx + \int_\Omega u_1 u_0 dx + \int_0^t \|u_t\|_2^2 d\tau + \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 \\ \leq& \frac{1}{2}\|u\|_{H_0^1}^2 + \frac{1}{2}\|u_t\|_2^2 + \int_0^t \|u_t\|_2^2 d\tau + \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|u_0\|_{H_0^1}^2 \\ \leq& \frac{C_1^2}{2}\|u\|_{\mathrm{H}}^2 + \frac{1}{2}\|u_t\|_2^2 + \int_0^t \|u_t\|_2^2 d\tau + \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|u_0\|_{H_0^1}^2, \end{aligned} \tag{47}$$

where C_1 stand by the best constant in the embedding $\mathrm{H} \hookrightarrow H_0^1(\Omega)$. Using (45), (46) and $u \in L^\infty(0,T;\mathrm{H})$, then (47) implies that

$$\int_0^t I(u)\, d\tau \leq C, \text{ for } 0 < t < +\infty. \tag{48}$$

From $I(u) \geq 0$, we know that there exists a $\lambda_* \geq 1$ such that $I(\lambda_* u) = 0$. On the other hand, we have

$$\begin{aligned} 0 = I(\lambda_* u) =& \lambda_*^2 \|\Delta u\|_2^2 - \lambda_*^p \int_\Omega |u|^p \log|u|^k dx - k\lambda_*^p \log\lambda_* \|u\|_p^p \\ =& \lambda_*^p I(u) - \left(\lambda_*^p - \lambda_*^2\right)\|\Delta u\|_2^2 - k\lambda_*^p \log\lambda_* \|u\|_p^p. \end{aligned}$$

Hence, we obtain

$$I(u) = \left(1 - \frac{1}{\lambda_*^{p-2}}\right)\|\Delta u\|_2^2 + k\log\lambda_* \|u\|_p^p.$$

Combining the above equation with (48), we obtain

$$\int_0^t \|\Delta u\|_2^2 d\tau \leq C, \tag{49}$$

and

$$\int_0^t \|u\|_p^p d\tau \leq C. \tag{50}$$

Differentiating $E(t)$ and using Eq.(1), we can compute

$$E'(t) = -\int_\Omega |\nabla u_t|^2 dx \le 0.$$

Since

$$\frac{d}{dt}\left[(1+t)E(t)\right] = (1+t)E'(t) + E(t) \le E(t), \tag{51}$$

then integrating (51) over $(0,t)$, we have

$$\begin{aligned}
&(1+t)E(t)\\
\le& E(0) + \int_0^t E(t)\,d\tau\\
=& E(0) + \frac{1}{2}\int_0^t \|u_t\|_2^2 d\tau + \frac{1}{p}\int_0^t I(u)d\tau + \left(\frac{1}{2}-\frac{1}{p}\right)\int_0^t \|\Delta u\|_2^2 d\tau\\
&+ \frac{k}{p^2}\int_0^t \|u\|_p^p d\tau.
\end{aligned} \tag{52}$$

Thus, applying $E(0) \le M$, (46), (48), (49) and (50) to (52), we can derive that there exists a positive constants $\mathcal{K}_0$ such that the energy functional $E(t)$ satisfies the following polynomial decay estimation:

$$E(t) \le \frac{\mathcal{K}_0}{1+t}, \text{ for all } t \in [0,+\infty).$$

Step 4. Exponential decay estimate of energy for the case of $E(0) < \min\left\{M, \frac{p-2}{2p}\left(\frac{e\mu}{kC_2{}^{p+\mu}}\right)^{\frac{2}{p+\mu-2}}\right\}$**.**

We define

$$L(t) = E(t) + \epsilon\int_\Omega u u_t dx + \frac{\epsilon}{2}\int_\Omega |\nabla u|^2 dx, \tag{53}$$

for any $0 \le t < \infty$, where ϵ is a positive constant to be specified later. By the Young inequality, we can easily know that there exist two positive constant α_1 and α_2 such that

$$\alpha_1 E(t) \le L(t) \le \alpha_2 E(t), \text{ for all } \ t \in [0,+\infty). \tag{54}$$

that is to say, $L(t)$ and $E(t)$ are equivalent.

By taking the time derivative of the function $L(t)$, using Eq.(1), we get

$$\begin{aligned}
L'(t) =& E'(t) + \epsilon\int_\Omega |u_t|^2 dx + \epsilon\int_\Omega u u_{tt} dx + \epsilon\int_\Omega \nabla u \nabla u_t dx\\
=& -\int_\Omega |\nabla u_t|^2 dx + \epsilon\int_\Omega |u_t|^2 dx - \epsilon\int_\Omega |\Delta u|^2 dx + \epsilon\int_\Omega |u|^p \log|u|^k dx\\
=& -\beta\epsilon E(t) + \frac{\beta\epsilon}{2}\|u_t\|_2^2 + \frac{\beta\epsilon}{2}\|\Delta u\|_2^2 - \frac{\beta\epsilon}{p}\int_\Omega |u|^p\log|u|^k dx + \frac{\beta\epsilon k}{p^2}\|u\|_p^p\\
&- \int_\Omega |\Delta u_t|^2 dx + \epsilon\int_\Omega |u_t|^2 dx - \epsilon\int_\Omega |\Delta u|^2 dx + \epsilon\int_\Omega |u|^p\log|u|^k dx\\
\le& -\beta\epsilon E(t) + \left(\frac{\beta\epsilon}{2} + \epsilon - \lambda_1\right)\|u_t\|_2^2 + \left(\frac{\beta\epsilon}{2} - \epsilon\right)\|\Delta u\|_2^2\\
&+ \left(\epsilon - \frac{\beta\epsilon}{p}\right)\int_\Omega |u|^p\log|u|^k dx + \frac{\beta\epsilon k}{p^2}\|u\|_p^p.
\end{aligned} \tag{55}$$

By virtue of the Sobolev embedding inequality and (45), we obtain

$$
\begin{aligned}
\int_{\Omega} |u|^{p} \log |u|^{k} \, dx \leq & \int_{\{x\in\Omega:|u|\geq 1\}} |u|^{p} \log |u|^{k} \, dx \\
\leq & k(e\mu)^{-1} \int_{\{x\in\Omega:|u|\geq 1\}} |u|^{p+\mu} dx \\
\leq & k(e\mu)^{-1} \|u\|_{p+\mu}^{p+\mu} \\
\leq & k(e\mu)^{-1} {C_2}^{p+\mu} \|\Delta u\|_{2}^{p+\mu} \\
\leq & k(e\mu)^{-1} {C_2}^{p+\mu} \left(\frac{2pE\,(0)}{p-2} \right)^{\frac{p+\mu-2}{2}} \|\Delta u\|_{2}^{2} \,,
\end{aligned}
\tag{56}
$$

and

$$
\|u\|_{p}^{p} \leq C_{3}^{p} \|\Delta u\|_{2}^{p} \leq C_{3}^{p} \left(\frac{2pE\,(0)}{p-2} \right)^{\frac{p-2}{2}} \|\Delta u\|_{2}^{2} \,, \tag{57}
$$

where C_2, C_3 are the Sobolev constant satisfying $\|u\|_{p+\mu} \leq C_2 \|\Delta u\|_2$, $\|u\|_p \leq C_3 \|\Delta u\|_2$. Substituting (56) and (57) into (55), we get

$$
\begin{aligned}
L'\,(t) \leq & - \beta\epsilon E\,(t) + \left(\frac{\beta\epsilon}{2} + \epsilon - \lambda_1 \right) \|u_t\|_{2}^{2} + \\
& \epsilon \Big\{ \frac{\beta}{2} + \frac{k\beta C_{3}^{p}}{p^2} \left(\frac{2pE\,(0)}{p-2} \right)^{\frac{p-2}{2}} + \frac{k{C_2}^{p+\mu}}{e\mu} \left(\frac{2pE\,(0)}{p-2} \right)^{\frac{p+\mu-2}{2}} - 1 \\
& - \frac{k\beta {C_2}^{p+\mu}}{ep\mu} \left(\frac{2pE\,(0)}{p-2} \right)^{\frac{p+\mu-2}{2}} \Big\} \|\Delta u\|_{2}^{2} \,.
\end{aligned}
\tag{58}
$$

Since $E\,(0) < \frac{p-2}{2p} \left(\frac{e\mu}{k{C_2}^{p+\mu}} \right)^{\frac{2}{p+\mu-2}}$, we have

$$
\frac{k{C_2}^{p+\mu}}{e\mu} \left(\frac{2pE\,(0)}{p-2} \right)^{\frac{p+\mu-2}{2}} - 1 < 0.
$$

Taking $\beta > 0$ small sufficiently such that

$$
\frac{\beta}{2} + \frac{k\beta C_{3}^{p}}{p^2} \left(\frac{2pE\,(0)}{p-2} \right)^{\frac{p-2}{2}} + \frac{k{C_2}^{p+\mu}}{e\mu} \left(\frac{2pE\,(0)}{p-2} \right)^{\frac{p+\mu-2}{2}} - 1 < 0.
$$

Now, choosing $\epsilon > 0$ small sufficiently such that

$$
\frac{\beta\epsilon}{2} + \epsilon - \lambda_1 < 0.
$$

Thus, combining with (54), we have

$$
L'\,(t) \leq -\beta\varepsilon E\,(t) \leq \frac{-\beta\varepsilon}{\alpha_2} L\,(t) \,. \tag{59}
$$

Integrating (59) over $(0,t)$, we can deduce that there exist $\mathcal{K}_1 = \frac{L(0)}{\alpha_1}$ and $\mathcal{K}_2 = \frac{\beta\varepsilon}{\alpha_2}$ such that

$$
E\,(t) \leq \mathcal{K}_1 e^{-\mathcal{K}_2 t}, \text{ for all } t \in [0, +\infty) \,.
$$

This completes the proof of Theorem 2.3. □

5. **Finite time blow up.** In this section, we prove Theorem 2.4, which implies that the solution u of problem (1) blow up in finite time. Firstly, we need to introduce the following lemma.

Lemma 5.1. [23] *Let $F(t)$ be a positive C^2 function satisfying the inequality*

$$F(t)F^{''}(t)-(1+\alpha)[F^{'}(t)]^2\geq 0,$$

for some $\alpha>0$. If $F(0)>0$ and $F^{'}(0)>0$, then there exists a time $T^*\leq\frac{F(0)}{\alpha F^{'}(0)}$ such that $\lim\limits_{t\to T^{*-}}F(t)=\infty$.

Now, let us prove the theorem 2.4.

Proof of Theorem 2.4. By contradiction, we suppose that u is global. For any $T>0$, we consider the auxiliary function $F:[0,T]\to R^+$ defined by

$$F\left(t\right)=\|u\|_2^2+\int_0^t\|\nabla u\|_2^2d\tau+\left(T-t\right)\|\nabla u_0\|_2^2+b(t+T_0)^2, \tag{60}$$

where $b>0$ and $T_0>0$, which will be specified later.

Obviously, $F\left(t\right)>0$ for any $t\in[0,T]$. Through a direct calculation, we obtain

$$F'\left(t\right)=2\int_\Omega uu_tdx+2\int_0^t\int_\Omega\nabla u\nabla u_tdxd\tau+2b(t+T_0),$$

$$\begin{aligned}F''\left(t\right)&=2\int_\Omega|u_t|^2dx+2\int_\Omega u_{tt}udx-2\int_\Omega u\Delta u_tdx+2b\\&=2\int_\Omega|u_t|^2dx+2\int_\Omega u[|u|^{p-2}u\log|u|^k-\Delta^2u]dx+2b\\&=2\int_\Omega|u_t|^2dx+2\int_\Omega|u|^p\log|u|^kdx-2\int_\Omega|\Delta u|^2dx+2b\\&=2\|u_t\|_2^2-2I\left(u\right)+2b.\end{aligned}$$

Using Schwarz's inequality and Young inequality, we have

$$\begin{aligned}\frac{(F'(t))^2}{4}&=\left(\int_\Omega uu_tdx+\int_0^t\int_\Omega\nabla u\nabla u_tdxd\tau+b(t+T_0)\right)^2\\&\leq\left(\|u\|_2^2+\int_0^t\|\nabla u\|_2^2d\tau+b(t+T_0)^2\right)\left(\|u_t\|_2^2+\int_0^t\|\nabla u_t\|_2^2d\tau+b\right)\\&\leq F\left(t\right)\left(\|u_t\|_2^2+\int_0^t\|\nabla u_t\|_2^2d\tau+b\right).\end{aligned}$$

Hence,

$$\begin{aligned}&F\left(t\right)F''\left(t\right)-\frac{p+2}{4}[F'\left(t\right)]^2\\\geq&F\left(t\right)\left(F''\left(t\right)-(p+2)\left(\|u_t\|_2^2+\int_0^t\|\nabla u_t\|_2^2d\tau+b\right)\right)\\=&F\left(t\right)\left(2\int_\Omega|u_t|^2dx+2\int_\Omega|u|^p\log|u|^kdx-2\int_\Omega|\Delta u|^2dx-(p+2)\|u_t\|_2^2\right.\\&\left.-(p+2)\int_0^t\|\nabla u_t\|_2^2d\tau-pb\right)\end{aligned}$$

$$=F\left(t\right)\left(-p\left\|u_t\right\|_2^2-2\left\|\Delta u\right\|_2^2+2\int_\Omega|u|^p\log|u|^k dx-(p+2)\int_0^t\left\|\nabla u_t\right\|_2^2 d\tau-pb\right). \tag{61}$$

Let

$$\xi\left(t\right)=-p\left\|u_t\right\|_2^2-2\left\|\Delta u\right\|_2^2+2\int_\Omega|u|^p\log|u|^k dx-(p+2)\int_0^t\left\|\nabla u_t\right\|_2^2 d\tau-pb,$$

by the definition of $E(t)$ and (17), we have

$$\begin{aligned}\xi\left(t\right)&=(p-2)\left\|\Delta u\right\|_2^2-2pE\left(t\right)-(p+2)\int_0^t\left\|\nabla u_t\right\|_2^2 d\tau+\frac{2k}{p}\left\|u\right\|_p^p-pb\\&=(p-2)\left\|\Delta u\right\|_2^2-2pE\left(0\right)+(p-2)\int_0^t\left\|\nabla u_t\right\|_2^2 d\tau+\frac{2k}{p}\left\|u\right\|_p^p-pb.\end{aligned} \tag{62}$$

From $I(u_0)<0$ and Lemma 2.9, we know $u\in V$, which implies that $I(u)<0$. By Lemma 2.5, there exists a $\lambda_*\in(0,1)$ such that $I\left(\lambda_* u\right)=0$. Hence, we have

$$\left(\frac{p-2}{2p}\right)\left\|\Delta u\right\|_2^2+\frac{k}{p^2}\left\|u\right\|_p^p\geq J\left(\lambda_* u\right)\geq d. \tag{63}$$

Combined with (63), we get

$$\begin{aligned}\xi\left(t\right)&=(p-2)\left\|\Delta u\right\|_2^2-2pE\left(0\right)+(p-2)\int_0^t\left\|\nabla u_t\right\|_2^2 d\tau+\frac{2k}{p}\left\|u\right\|_p^p-pb\\&\geq(p-2)\left\|\Delta u\right\|_2^2-2pE(0)+\frac{2k}{p}\left\|u\right\|_p^p-pb\\&\geq 2pd-2pE(0)-pb.\end{aligned} \tag{64}$$

Choosing $b>0$ sufficiently small such that $0<b\leq 2d-2E(0)$, we have $\xi(t)\geq 0$. Thus, by the above discussion, we obtain

$$F\left(t\right)F''\left(t\right)-\frac{p+2}{4}\left[F'\left(t\right)\right]^2\geq 0.$$

By the definition of $F(t)$, $F(0)=\|u_0\|_2^2+T\|\nabla u_0\|_2^2+bT_0^2>0$, we choose T_0 sufficiently large, which satisfies

$$T_0>\frac{(p-2)\left(\left\|u_0\right\|_2^2+\left\|u_1\right\|_2^2\right)+4\left\|\nabla u_0\right\|_2^2}{2\left(p-2\right)b}, \tag{65}$$

thus, $F'\left(0\right)=2bT_0+2\int_\Omega u_0u_1dx\geq 2bT_0-\left\|u_0\right\|_2^2-\left\|u_1\right\|_2^2>0$.

According to Lemma 5.1, we conclude that

$$\lim_{t\to T^{*-}}F(t)=\infty, \tag{66}$$

for

$$T^*\leq\frac{4F\left(0\right)}{\left(p-2\right)F'\left(0\right)}=\frac{2bT_0{}^2+2\left\|u_0\right\|_2^2+2T\left\|\nabla u_0\right\|_2^2}{(p-2)\left(bT_0+\int_\Omega u_0u_1dx\right)}.$$

Hence, we deduce that

$$T^*\leq\frac{2bT_0{}^2+2\left\|u_0\right\|_2^2}{(p-2)\left(bT_0+\int_\Omega u_0u_1dx\right)-2\left\|\nabla u_0\right\|_2^2}. \tag{67}$$

By (60), (66) and (67), we have

$$\lim_{t\to T^{*}-}\|u\|_2^2=\infty,$$

which contradicts the assumption of u being global. Hence, the solution u of problem (1) blow up in finite time. This completes the proof of Theorem 2.4. ☐

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