ELECTRONIC RESEARCH ARCHIVE
Volume **29**, Number **6**, December **2021**

doi:10.3934/era.2021064
pp. **3833–3851**

INITIAL BOUNDARY VALUE PROBLEM OF A CLASS OF MIXED PSEUDO-PARABOLIC KIRCHHOFF EQUATIONS

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(Communicated by Runzhang Xu)

Abstract. In this paper, we consider the initial boundary value problem for a mixed pseudo-parabolic Kirchhoff equation. Due to the comparison principle being invalid, we use the potential well method to give a threshold result of global existence and non-existence for the sign-changing weak solutions with initial energy $J(u_0) \le d$. When the initial energy $J(u_0) > d$, we find another criterion for the vanishing solution and blow-up solution. Our interest also lies in the discussion of the exponential decay rate of the global solution and life span of the blow-up solution.

1. **Introduction.** In this paper, we consider the following initial boundary value problem

$$\begin{cases} u_t - k\Delta u_t - M(\|\nabla u\|_p^p)\Delta_p u = |u|^{q-1}u - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega |u|^{q-1}u dx, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0, \qquad\qquad\qquad\qquad\qquad\quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \qquad\qquad\qquad\qquad\quad x \in \Omega, \end{cases} \tag{1}$$

where $u(x,t) : \Omega \times (0,T) \to \mathbb{R}$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p \ge 2$, $2p - 1 < q \le p^* - 1$, $\Omega \subset \mathbb{R}^n (n \ge 1)$ is a bounded domain with smooth boundary, ν is the unit outside normal vector on $\partial\Omega$, $M(s) = a + bs$ with $a > 0$ and $b > 0$, $u_0(x) \in W_N^{1,p}(\Omega)$ with $W_N^{1,p}(\Omega) = \left\{\phi \in W^{1,p}(\Omega) : \frac{\partial \phi}{\partial \nu}|_{\partial\Omega} = 0, \int_\Omega \phi dx = 0\right\}$. Integrating the first equation of (1) with respect to x,, we have that $\int_\Omega u dx = \int_\Omega u_0 dx = 0$.

Like the name in [27], we refer to (1) as the mixed pseudo-parabolic Kirchhoff equation, which with the combination of $M(\cdot)$ and p-Laplacian, can be used to describe the motion of a non-stationary fluid or gas in the nonhomogeneous and anisotropic medium [11], the growth and movement of biological species [13]... Especially, if p varies according to (x,t), then this type of problem can be applied to electrorheological fluids, nonlinear elastic and image restoration [34, 35, 25, 36, 9]. Equation (1) feathers several non-local mechanism. In the first instance, we choose

2020 *Mathematics Subject Classification.* Primary: 35K70, 35B40; Secondary: 35K35.
Key words and phrases. Pseudo-parabolic, Kirchhoff equation, global existence, blow-up.
The first author is supported by NSFC grant 11871134, 12171166.
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the diffusion coefficient $M(\cdot)$ as a typical case of Kirchhoff form [16], which expresses the dependence on the global information in the environment instead of expressing the information at a local location. In this sense, $M(\cdot)$ can describe a possible change in the global state of the population density, fluid or gas caused by the corresponding motion in the considered medium. Further the pseudo-parabolic viscosity Δu_t brings about an equivalent equation of (1)

$$u_t - \mathcal{B}M(\|\nabla u\|_p^p)\Delta_p u = \mathcal{B}(|u|^{q-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-1}udx),$$

where $\mathcal{B} = (I - \Delta)^{-1}$ is a nonlocal operator [33]. According to the above two non-local effects, equations like (1) have had a high profile in the study of many phenomena such as biological species dynamics, nonlinear elasticity, non-stationary fluid, image recovery,... (see [28, 1, 5, 6, 21] and references therein). The third non-local term comes from the source $|u|^{q-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-1}udx$, which leads to the conservation property $\int_\Omega u = 0$, and points out that the solutions may change sign. Hence the diffusion equations with such source usually model the phenomena in population dynamics and biological sciences where the total mass of a chemical or an organism is conserved [3].

The aim of this work is to reveal how the initial energy have an impact on the properties of sign changing solutions to (1). It is worth mentioning that several significant works have focused on such problems for nonlinear parabolic equations, where local well-posedness, global existence and non-existence, asymptotic behaviors of solutions are investigated. In details, we refer to Zhou et. al [10, 37] for the p-Laplace equation, Han et. al [13, 17] for the Kirchhoff equation, Su and Xu [32] for the pseudo-parabolic equation with localized source and arbitrary initial energy. For the non-local source case, we refer to some very recent related references, e.g. [24] for the threshold results of the global existence and non-existence for the sign-changing weak solutions of thin film equation; [14] and [8] for the Kirchhoff type problem with non-local source $\frac{1}{|\Omega|}\int_\Omega |u|^{q-1}udx$; [15] for the finite time blow-up of solutions with non-positive initial energy $J(u_0)$ and non-local source $\frac{1}{|\Omega|}\int_\Omega |u|^q dx$ with $q > 1$; [18] for the well-posedness of pseudo-parabolic equation with singular potential term at three initial energy levels, the logarithmic nonlinearity in [7] and the non-local source $\frac{1}{|\Omega|}\int_\Omega |u|^{q-1}udx$ in [30]. As far as we know, there are few research works concerned with the sign-changing solutions for the mixed pseudo-parabolic Kirchhoff equation. Due to the comparison principle being invalid for the sign-changing solutions and the interaction of multi-nonlocal factors, the research for (1) is more complicated. Inspired by the ideas of above papers, we combine the modified potential well method, the classical Galerkin method and the energy method to give a threshold result for the global existence and non-existence of the sign-changing weak solutions. The potential well theory is first proposed by Payne and Sattinger[22, 26]. It is useful to study the long time behaviors of solutions to many evolution equations and be improved by Liu et al.[32, 19, 20]. Moreover, learning from [12, 29, 23], we study the decay rate of the global solution and the life span of the finite time blow-up solution.

In this paper, we consider the weak solutions as follows:

Definition 1.1. A function $u(x,t)$ is called a weak solution to (1) on $\Omega \times [0,T)$, if $u(x,0) = u_0(x) \in W_N^{1,p}(\Omega)$, $u \in L^\infty(0,T;W_N^{1,p}(\Omega))$ with $u_t \in L^2(0,T;W_N^{1,2}(\Omega))$

and satisfies

$$(u_t,\ \varphi)+k(\nabla u_t,\ \nabla\varphi)+M(\|\nabla u\|_p^p)(|\nabla u|^{p-2}\nabla u,\ \nabla\varphi) = (|u|^{q-1}u-\frac{1}{|\Omega|}\int_\Omega |u|^{q-1}udx,\ \varphi),$$

for any $\varphi\in W_N^{1,p}(\Omega)$.

We use the expressions $\|\cdot\|_p=\|\cdot\|_{L^p(\Omega)}$ and $(u,v)=\int_\Omega u(x)v(x)dx$ throughout the paper. Using the potential well theory [26, 19], we introduce the potential energy functional

$$J(u)=\frac{a}{p}\|\nabla u\|_p^p+\frac{b}{2p}\|\nabla u\|_p^{2p}-\frac{1}{q+1}\|u\|_{q+1}^{q+1} \tag{2}$$

and the Nehari functional:

$$I(u)=a\|\nabla u\|_p^p+b\|\nabla u\|_p^{2p}-\|u\|_{q+1}^{q+1}=-\frac{1}{2}\frac{d}{dt}(\|u\|_2^2+k\|\nabla u\|_2^2). \tag{3}$$

(2) and (3) imply that

$$J(u)=\frac{1}{q+1}I(u)+\left(\frac{a}{p}-\frac{a}{q+1}\right)\|\nabla u\|_p^p+\left(\frac{b}{2p}-\frac{b}{q+1}\right)\|\nabla u\|_p^{2p}, \tag{4}$$

$$\frac{d}{dt}J(u)=-\|u_t\|_2^2-k\|\nabla u_t\|_2^2. \tag{5}$$

For any $\delta>0$, the modified Nehari functional can be defined as

$$I_\delta(u)=\delta(a\|\nabla u\|_p^p+b\|\nabla u\|_p^{2p})-\|u\|_{q+1}^{q+1}. \tag{6}$$

Then we can define the Nehari manifold and the potential wells

$$\begin{aligned}
&\mathscr{N}=\{u\in W_N^{1,p}(\Omega): I(u)=0,\ \|\nabla u\|_p\neq 0\},\\
&W=\{u\in W_N^{1,p}(\Omega): J(u)<d,\ I(u)>0\}\bigcup\{0\},\\
&V=\{u\in W_N^{1,p}(\Omega): J(u)<d,\ I(u)<0\},\\
&\mathscr{N}_\delta=\{u\in W_N^{1,p}(\Omega): I_\delta(u)=0,\ \|\nabla u\|_p\neq 0\},\\
&W_\delta=\{u\in W_N^{1,p}(\Omega): J(u)<d(\delta),\ I_\delta(u)>0\}\bigcup\{0\},\\
&V_\delta=\{u\in W_N^{1,p}(\Omega): J(u)<d(\delta),\ I_\delta(u)<0\},
\end{aligned}$$

where $d(\delta)$ is the depth of the potential well W_δ and

$$d=d(1)=\inf\{J(u): u\in\mathscr{N}\},\qquad d(\delta)=\inf\{J(u): u\in\mathscr{N}_\delta\}. \tag{7}$$

It is worth pointing out that the nonlinear terms in (1) make the local existence of solutions non-trivial. It is delightful that there are some important works on the local well-posedness of parabolic Kirchhoff type problems involving fractional Laplacian or p-Laplacian [11, 31]. By using the argument similar to the above references, (1) admits local weak solutions, thus all the statements in the following are for the weak solutions in Definition 1.1.

Theorem 1.2. *Let $u_0\in W_N^{1,p}(\Omega)$ with $J(u_0)<d$ and $I(u_0)>0$. Then (1) admits a global weak solution u. Further there exists $C>0$ such that $\|u\|_2^2+k\|\nabla u\|_2^2\leq [(\|u_0\|_2^2+k\|\nabla u_0\|_2^2)^{1-p}+Ct]^{-\frac{1}{p-1}}$. In addition, the weak solution is unique when it is bounded.*

Theorem 1.3. *Let $u_0 \in W_N^{1,p}(\Omega)$ with $J(u_0) < d$ and $I(u_0) < 0$. Then the weak solution of (1) blows up in finite time, namely there exists $T > 0$, such that*

$$\lim_{t\to T^-}\int_0^t (\|u\|_2^2 + k\|\nabla u\|_2^2)d\tau = +\infty.$$

Theorem 1.4. *Let $u_0 \in W_N^{1,p}(\Omega)$, $J(u_0) = d$ and $I(u_0) \geq 0$. Then (1) admits a unique global weak solution u satisfying $I(u) \geq 0$. Moreover if $I(u) > 0$, then there exists a constant $C > 0$ and $t_0 > 0$ such that $\|u\|_2^2 + k\|\nabla u\|_2^2 \leq \left[(\|u(t_0)\|_2^2 + k\|\nabla u(t_0)\|_2^2)^{1-p} + C(t-t_0)\right]^{-\frac{1}{p-1}}$. Otherwise the solution vanishes in a limited time.*

Theorem 1.5. *Let $u_0 \in W_N^{1,p}(\Omega)$, $J(u_0) = d$ and $I(u_0) < 0$. Then the weak solution of (1) blows up in finite time, namely there exists $T > 0$ such that*

$$\lim_{t\to T^-}\int_0^t (\|u\|_2^2 + k\|\nabla u\|_2^2)d\tau = +\infty.$$

Theorem 1.6. (Life span) *Let $u_0 \in W_N^{1,p}(\Omega)$, $J(u_0) \leq d$ and $I(u_0) < 0$. Then we have the following life span estimation of the blow-up solution in Theorem 1.3 and Theorem 1.5*
(i) *If $J(u_0) < 0$, then $T \leq \frac{\|u_0\|_2^2 + k\|\nabla u_0\|_2^2}{(1-q^2)J(u_0)}$.*
(ii) *If $0 \leq J(u_0) \leq d$, then*

$$T \leq \frac{4\|u_0\|_2^2 + 4k\|\nabla u_0\|_2^2}{(q-1)^2(\frac{a(q+1-p)}{p(q-1)}\|\nabla u(t_0)\|_p^p + \frac{b(q+1-2p)}{2p(q-1)}\|\nabla u(t_0)\|_p^{2p} - J(u_0))} + t_0,$$

where t_0 satisfies (20).

When $J(u_0) > d$, we invoke the ideas in [37, 17, 32] and introduce the following sets.

$$\begin{aligned}
\mathcal{N}_+ &= \{u \in W_N^{1,p}(\Omega) : I(u) > 0\},\\
\mathcal{N}_- &= \{u \in W_N^{1,p}(\Omega) : I(u) < 0\},\\
J^s &= \{u \in W_N^{1,p}(\Omega) : J(u) < s\}, \quad \text{for any } s > d,\\
\mathcal{N}^s &= \mathcal{N} \cap J^s = \{u \in \mathcal{N} : \frac{a(q+1-p)}{p(q+1)}\|\nabla u\|_p^p + \frac{b(q+1-2p)}{2p(q+1)}\|\nabla u\|_p^{2p} < s\},\\
\lambda_s &= \inf\{\|u\|_2^2 + k\|\nabla u\|_2^2 : u \in \mathcal{N}^s\},\\
\Lambda_s &= \sup\{\|u\|_2^2 + k\|\nabla u\|_2^2 : u \in \mathcal{N}^s\},\\
\mathcal{B} &= \{u_0 \in W_N^{1,p}(\Omega) : \text{the solution of (1) blows up in finite time}\},\\
\mathcal{G} &= \{u_0 \in W_N^{1,p}(\Omega) : \text{the solution of (1) is global in time}\},\\
\mathcal{G}_0 &= \{u_0 \in W_N^{1,p}(\Omega) : u(t) \to 0 \text{ in } W_N^{1,p}(\Omega), t \to +\infty\}.
\end{aligned}$$

Theorem 1.7. *Assume $J(u_0) > d$.*
(i) *If $u_0 \in \mathcal{N}_+$, $\|u_0\|_2^2 + k\|\nabla u_0\|_2^2 \leq \lambda_{J(u_0)}$, then $u_0 \in \mathcal{G}_0$.*
(ii) *If $u_0 \in \mathcal{N}_-$, $\|u_0\|_2^2 + k\|\nabla u_0\|_2^2 \geq \Lambda_{J(u_0)}$, then $u_0 \in \mathcal{B}$.*

Theorem 1.8. $\lambda_s \geq \begin{cases} \left[\dfrac{a}{\beta^{q+1}}\kappa^{p-\theta(q+1)}\right]^{\frac{2}{(1-\theta)(q+1)}}, & p > \dfrac{n}{n+2}(q+1); \\ \left[\dfrac{a}{\beta^{q+1}}\tilde{\kappa}^{p-\theta(q+1)}\right]^{\frac{2}{(1-\theta)(q+1)}}, & p < \dfrac{n}{n+2}(q+1), \end{cases}$

and

$$\Lambda_s \leq (1+k)|\Omega|^{\frac{p-2}{p}}\tilde{\kappa}^2,$$

where $\theta = (\frac{1}{2} - \frac{1}{q+1})(\frac{1}{2} - \frac{1}{p} + \frac{1}{n})^{-1}$*,* κ *is the unique positive solution of* $f(y) = d$ *and* $\tilde{\kappa}$ *is the unique positive solution of* $f(y) = s$ *with*

$$f(y) = \frac{b(q+1-2p)}{2p(q+1)}y^{2p} + \frac{a(q+1-p)}{p(q+1)}y^p, \quad y \in \mathbb{R}. \tag{8}$$

The paper is arranged as follows. In section 2, we give some important lemmas. We prove Theorem 1.2, 1.3, 1.4 and 1.5 in Section 3. Section 4 is devoted to Theorem 1.6. At last, we investigate the supercritical initial energy case, namely Theorem 1.7 and 1.8 in Section 5.

2. **Several crucial lemmas.**

In this section, we state some lemmas that are essential for proving the major theorems.

Lemma 2.1. *For any* $u \in W_N^{1,p}(\Omega)$ *with* $\|\nabla u\|_p \neq 0$*, there hold*
(i) $\lim\limits_{\lambda\to 0} J(\lambda u) = 0, \ \lim\limits_{\lambda\to+\infty} J(\lambda u) = -\infty.$
(ii) There exists a unique $\lambda^* > 0$*, such that* $\frac{d}{d\lambda}J(\lambda u)|_{\lambda=\lambda^*} = 0$*, namely* $\lambda^* u \in \mathscr{N}$*. Furthermore* $\frac{d}{d\lambda}J(\lambda u) > 0$ *on* $(0, \lambda^*)$*,* $\frac{d}{d\lambda}J(\lambda u) < 0$ *on* (λ^*, ∞)*, namely* $J(\lambda u)$ *takes the maximum at* $\lambda = \lambda^*$*.*

Proof. (i) For any $u \in W_N^{1,p}(\Omega)$ and $\lambda > 0$,

$$J(\lambda u) = \lambda^p \frac{a}{p}\|\nabla u\|_p^p + \lambda^{2p}\frac{b}{2p}\|\nabla u\|_p^{2p} - \frac{\lambda^{q+1}}{q+1}\|u\|_{q+1}^{q+1}. \tag{9}$$

Since $q+1 > 2p$, thus $\lim\limits_{\lambda\to 0} J(\lambda u) = 0, \ \lim\limits_{\lambda\to+\infty} J(\lambda u) = -\infty.$

(ii) Derivative $J(\lambda u)$ with respect to λ, we have

$$\begin{aligned} \frac{d}{d\lambda}J(\lambda u) &= a\lambda^{p-1}\|\nabla u\|_p^p + b\lambda^{2p-1}\|\nabla u\|_p^{2p} - \lambda^q\|u\|_{q+1}^{q+1} \\ &= \lambda^q\left(\frac{a}{\lambda^{q+1-p}}\|\nabla u\|_p^p + \frac{b}{\lambda^{q+1-2p}}\|\nabla u\|_p^{2p} - \|u\|_{q+1}^{q+1}\right). \end{aligned}$$

Set $g(\lambda) = \frac{a}{\lambda^{q+1-p}}\|\nabla u\|_p^p + \frac{b}{\lambda^{q+1-2p}}\|\nabla u\|_p^{2p} - \|u\|_{q+1}^{q+1}$, then

$$\lim_{\lambda\to 0} g(\lambda) = +\infty, \ \lim_{\lambda\to+\infty} g(\lambda) < 0,$$
$$g'(\lambda) = -\frac{a(q+1-p)}{\lambda^{q+2-p}}\|\nabla u\|_p^p - \frac{b(q+1-2p)}{\lambda^{q+2-2p}}\|\nabla u\|_p^{2p} < 0.$$

Therefore there exists a unique $\lambda^* > 0$ such that $g(\lambda^*) = 0$, namely $\frac{d}{d\lambda}J(\lambda u)|_{\lambda=\lambda^*} = 0$. It is easily to find that $J(\lambda u)$ is strictly increasing on $(0, \lambda^*]$, strictly decreasing on (λ^*, ∞), and takes the maximum at $\lambda = \lambda^*$. □

Lemma 2.2. *For any* $u \in W_N^{1,p}(\Omega)$ *with* $\|\nabla u\|_p \neq 0$*,* $r(\delta) = (\frac{\delta a}{S^{q+1}})^{\frac{1}{q+1-p}}$*, where* S *is the embedding coefficient of the Sobolev space inequality* $\|u\|_{q+1} \leq S\|\nabla u\|_p$*, there hold*
(i) If $0 < \|\nabla u\|_p \leq r(\delta)$*, then* $I_\delta(u) > 0$*;*

(ii) If $I_\delta(u) < 0$, then $\|\nabla u\|_p > r(\delta)$;
(iii) If $I_\delta(u) = 0$, then $\|\nabla u\|_p = 0$ or $\|\nabla u\|_p > r(\delta)$.

Proof. (i) The Sobolev embedding inequality and $0 < \|\nabla u\|_p \leq r(\delta)$ indicate that

$$\begin{aligned}\|u\|_{q+1}^{q+1} &\leq S^{q+1}\|\nabla u\|_p^{q+1} \leq S^{q+1} r^{q+1-p}(\delta)\|\nabla u\|_p^p \\ &= \delta a\|\nabla u\|_p^p < \delta a\|\nabla u\|_p^p + \delta b\|\nabla u\|_p^{2p},\end{aligned}$$

which means $I_\delta(u) > 0$.

(ii) can be directly derived from (i).

(iii) If $\|\nabla u\|_p = 0$, then $I_\delta(u) = 0$. If $I_\delta(u) = 0$ and $\|\nabla u\|_p \neq 0$, then $\delta a\|\nabla u\|_p^p < \|u\|_{q+1}^{q+1} \leq S^{q+1}\|\nabla u\|_p^{q+1}$, namely $\|\nabla u\|_p > r(\delta)$. □

Lemma 2.3. *$d(\delta)$ satisfies*
(i) $\lim_{\delta\to 0^+} d(\delta) = 0$, $\lim_{\delta\to+\infty} d(\delta) = -\infty$;
(ii) $d(\delta)$ is monotonically increased on $0 < \delta \leq 1$, monotonically decreased on $\delta > 1$, and the maximum is obtained at $\delta = 1$.

Proof. (i) For any $\lambda u \in \mathscr{N}_\delta$, we have

$$\delta a\|\nabla u\|_p^p + \lambda^p \delta b\|\nabla u\|_p^{2p} = \lambda^{q+1-p}\|u\|_{q+1}^{q+1},$$

which indicates

$$\delta = \frac{\lambda^{q+1-p}\|u\|_{q+1}^{q+1}}{a\|\nabla u\|_p^p + b\lambda^p\|\nabla u\|_p^{2p}}. \tag{10}$$

A directly computation on (10) show that λ increases as δ increases, δ increases as λ increases and $\lim_{\delta\to0^+}\lambda(\delta) = 0$, $\lim_{\delta\to+\infty}\lambda(\delta) = +\infty$. Thus from the definition of $d(\delta)$ and Lemma 2.1, we can get

$$\begin{gathered}0 \leq \lim_{\delta\to0^+} d(\delta) \leq \lim_{\delta\to0^+} J(\lambda u) = \lim_{\lambda\to0^+} J(\lambda u) = 0, \\ \lim_{\delta\to+\infty} d(\delta) \leq \lim_{\delta\to+\infty} J(\lambda u) = \lim_{\lambda\to+\infty} J(\lambda u) = -\infty.\end{gathered}$$

Therefore $\lim_{\delta\to0^+} d(\delta) = 0$ and $\lim_{\delta\to+\infty} d(\delta) = -\infty$.

(ii) Assume $0 < \delta' < \delta'' \leq 1$ or $1 < \delta'' < \delta'$. Let $h(\lambda) = J(\lambda(\delta)u)$ with $\lambda(\delta)u \in \mathscr{N}_\delta$, then

$$h'(\lambda) = \lambda^{p-1}a(1-\delta)\|\nabla u\|_p^p + \lambda^{2p-1}b(1-\delta)\|\nabla u\|_p^{2p}.$$

For any $u \in \mathscr{N}_{\delta''}$ with $\lambda(\delta'') = 1$, set $v = \lambda(\delta')u \in \mathscr{N}_{\delta'}$. If $0 < \delta' < \delta'' \leq 1$, since $\lambda(\delta)$ increases as δ increases, then

$$\begin{aligned}J(u) - J(v) = h(1) - h(\lambda(\delta')) &= \int_{\lambda(\delta')}^1 h'(\lambda)d\lambda \\ &= \int_{\lambda(\delta')}^1 [\lambda^{p-1}a(1-\delta)\|\nabla u\|_p^p + \lambda^{2p-1}b(1-\delta)\|\nabla u\|_p^{2p}]d\lambda \\ &> 0.\end{aligned}$$

Therefore for any $u \in \mathscr{N}_{\delta''}$, there exists $v \in \mathscr{N}_{\delta'}$ such that $J(u) > J(v)$, which leads to $d(\delta'') > d(\delta')$. The case for $1 < \delta'' < \delta'$ is similarly. □

Lemma 2.4. *For any $u \in W_N^{1,p}(\Omega)$ with $0 < J(u) < d$, the sign of $I_\delta(u)$ doesn't change for $\delta_1 < \delta < \delta_2$, where $\delta_1 < 1 < \delta_2$ are the two roots of $d(\delta) = J(u)$.*

Proof. If the sign of $I_\delta(u)$ changed for $\delta_1 < \delta < \delta_2$, then there exists $\delta_0 \in (\delta_1, \delta_2)$ such that $I_{\delta_0}(u) = 0$. Thus $u \in \mathscr{N}_{\delta_0}$ and $d(\delta_0) \le J(u)$. According to Lemma 2.3, $d(\delta_0) > d(\delta_1) = d(\delta_2) = J(u)$, which is a contradiction. □

Lemma 2.5. *Assume that u is a weak solution of (1) with $0 < J(u_0) < d$ on $\Omega \times [0, T)$, $\delta_1 < 1 < \delta_2$ are two roots of $d(\delta) = J(u_0)$.*
(i) If $I(u_0) > 0$, then $u(x,t) \in W_\delta,\ \delta_1 < \delta < \delta_2,\ 0 < t < T$.
(ii) If $I(u_0) < 0$, then $u(x,t) \in V_\delta,\ \delta_1 < \delta < \delta_2,\ 0 < t < T$.

Proof. (i) We first prove $u_0(x) \in W_\delta$ with $\delta_1 < \delta < \delta_2$. On the one hand, since $I(u_0) > 0$ and Lemma 2.4, we have $I_\delta(u_0) > 0$. On the other hand, Lemma 2.3 leads to $J(u_0) = d(\delta_1) = d(\delta_2) < d(\delta)$ with $\delta_1 < \delta < \delta_2$. In what follows we prove that $u(x,t) \in W_\delta$ with $\delta_1 < \delta < \delta_2$ on $0 < t < T$. Suppose that there are $t_0 \in (0, T)$ and $\delta_0 \in (\delta_1, \delta_2)$, such that $u \in W_\delta,\ \delta_1 < \delta < \delta_2,\ 0 < t < t_0,\ u(x, t_0) \in \partial W_{\delta_0}$, then we can get

$$I_{\delta_0}(u(t_0)) = 0,\ \|\nabla u\|_p \neq 0 \text{ or } J(u(t_0)) = d(\delta_0).$$

Due to $\frac{d}{dt}J(u) \le 0$, then $J(u(t_0)) \le J(u_0) < d(\delta_0)$. We only need to consider the first case, namely $u(t_0) \in \mathscr{N}_{\delta_0}$, which indicates $J(u(t_0)) \ge d(\delta_0)$. This is a contradiction.

(ii) The proof is similar to (i). □

Lemma 2.6. *If $u_0 \in W_N^{1,p}(\Omega)$, $J(u_0) = d$, $I(u_0) > 0$, then W is an invariant set. If $u_0 \in W_N^{1,p}(\Omega)$, $J(u_0) = d$, $I(u_0) < 0$, then V is an invariant set.*

Proof. Let T be the maximum existence time of the solution. If there exists $t_0 \in (0, T)$, such that $I(u) > 0$, $t \in [0, t_0)$ and $I(u(t_0)) = 0$. Due to $-I(u) = (u_t, u) + k(\nabla u_t, \nabla u) < 0$, we get $\int_0^t (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau > 0$, $t \in (0, t_0)$. Then

$$J(u(t_0)) = J(u_0) - \int_0^{t_0} \|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2 d\tau < d.$$

It is known from $I(u(t_0)) = 0$ and the definition of d (7) that $J(u(t_0)) \ge d$, which is a contradiction. Using the same method, we can prove the second part of this lemma. □

3. $J(u_0) \le d$.

In this section, we deal with the global existence and blowing-up of the weak solution to (1) under the condition $J(u_0) \le d$.

Proof of Theorem 1.2. From Lemma 2.1, for any $u \in W_N^{1,p}(\Omega)$ with $\|\nabla u\|_p \neq 0$, there hold $\lim\limits_{\lambda \to 0} J(\lambda u) = 0$, and there exists a unique $\lambda^* > 0$ such that $\frac{d}{d\lambda}J(\lambda u)|_{\lambda=\lambda^*}$ $= 0$, $\frac{d}{d\lambda}J(\lambda u) > 0$ on $(0, \lambda^*)$, $\frac{d}{d\lambda}J(\lambda u) < 0$ on (λ^*, ∞). Combined with $\frac{d}{d\lambda}J(\lambda u) = \frac{1}{\lambda}I(\lambda u)$, there exists a λ_* such that $J(\lambda_* u) < d$ and $I(\lambda_* u) > 0$. Let $\lambda_* u$ be the new u, then we have found the initial value u_0 that satisfies the problem setting. In addition, according to (4), $I(u_0) > 0$ and $2p < q + 1$, we have $J(u_0) > 0$.

Step1. Global existence.

Let $\{\phi_j(x)\}_{j=1}^\infty$ be the orthogonal base in $W_N^{1,p}(\Omega)$, which is also orthogonal in $L^2(\Omega)$. Construct the approximate solution $u^m(x,t)$ of (1) as follows

$$u^m(x,t) = \sum_{j=1}^m \alpha_j^m(t)\phi_j(x), \qquad \alpha_j^m(t) = (u^m, \phi_j), \qquad m = 1, 2, ...$$

which satisfy

$$
\begin{aligned}
&(u_t^m,\ \phi_j)+k(\nabla u_t^m,\ \nabla\phi_j)+M(\|\nabla u^m\|_p^p)(|\nabla u^m|^{p-2}\nabla u^m,\ \nabla\phi_j)\\
=&(|u^m|^{q-1}u^m-\frac{1}{|\Omega|}\int_\Omega|u^m|^{q-1}u^m dx,\ \phi_j),
\end{aligned}
\tag{11}
$$

$$
u^m(x,0)=\sum_{j=1}^m\alpha_j^m(0)\phi_j(x)\to u_0(x)\quad\text{in}\quad W_N^{1,p}(\Omega). \tag{12}
$$

Multiplying (11) by $\frac{d}{dt}\alpha_j^m(t)$, summing for j from 1 to m and integrating with respect to time, we can obtain

$$
J(u^m(x,0))=J(u^m(x,t))+\int_0^t(\|u_\tau^m\|_2^2+k\|\nabla u_\tau^m\|_2^2)d\tau,\quad\forall t>0.
$$

By (12), we have $J(u^m(x,0))\to J(u_0)<d$. Hence for sufficiently large m, there holds

$$
J(u^m(x,t))+\int_0^t(\|u_\tau^m\|_2^2+k\|\nabla u_\tau^m\|_2^2)d\tau=J(u^m(x,0))<d,\quad\forall t>0.
$$

From (12) again, we have $I(u^m(x,0))\to I(u_0)>0$. Hence for sufficiently large m, there holds $u^m(x,0)\in W$. Then by Lemma 2.5, $u^m(x,t)\in W$ and

$$
\int_0^t(\|u_\tau^m\|_2^2+k\|\nabla u_\tau^m\|_2^2)d\tau+\frac{a(q+1-p)}{p(q+1)}\|\nabla u^m\|_p^p+\frac{b(q+1-2p)}{2p(q+1)}\|\nabla u^m\|_p^{2p}<d,
$$

for all $t>0$. Thus

$$
\begin{aligned}
&\int_0^t(\|u_\tau^m\|_2^2+k\|\nabla u_\tau^m\|_2^2)d\tau<d,\quad \|\nabla u^m\|_p^p<\frac{dp(q+1)}{a(q+1-p)},\\
&\||u^m|^{q-1}\cdot u^m\|_{\frac{q+1}{q}}=\|u^m\|_{q+1}^q\le S^q\|\nabla u^m\|_p^q<S^q\left(\frac{dp(q+1)}{a(q+1-p)}\right)^{\frac{q}{p}}.
\end{aligned}
$$

Then there exists a positive constant C, such that $\|M(\|\nabla u^m\|_p^p)|\nabla u^m|^{p-2}\cdot\nabla u^m\|_{\frac{p}{p-1}}<C$. By the diagonal method and the Aubin-Lion's compactness embedding theorem, there exist u and a subsequence of $\{u^m\}_{m=1}^\infty$ (still represented by $\{u^m\}_{m=1}^\infty$) such that

$$
\begin{aligned}
&u_t^m\rightharpoonup u_t \text{ in } L^2(0,\infty;L^2(\Omega)),\\
&u^m\overset{*}{\rightharpoonup}u \text{ in } L^\infty(0,\infty;W_N^{1,p}(\Omega)),\\
&u^m\to u \text{ strongly in } L^2(\Omega\times(0,T)),\text{ a.e. in }\Omega\times(0,T),\\
&|u^m|^{q-1}\cdot u^m\overset{*}{\rightharpoonup}|u|^{q-1}\cdot u \text{ in } L^\infty(0,\infty;L^{\frac{q+1}{q}}(\Omega)),\\
&M(\|\nabla u^m\|_p^p)|\nabla u^m|^{p-2}\cdot\nabla u^m\overset{*}{\rightharpoonup}\xi \text{ in } L^\infty(0,\infty;L^{\frac{p}{p-1}}(\Omega)).
\end{aligned}
$$

Similar to the process of [17], we can prove $\xi=M(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u$. For fixed j, let $m\to+\infty$ in (11) to get

$$
\begin{aligned}
&(u_t,\ \phi_j)+k(\nabla u_t,\ \nabla\phi_j)+M(\|\nabla u\|_p^p)(|\nabla u|^{p-2}\nabla u,\ \nabla\phi_j)\\
=&(|u|^{q-1}u-\frac{1}{|\Omega|}\int_\Omega|u|^{q-1}u dx,\ \phi_j).
\end{aligned}
$$

Then from Definition 1.1 $u(x,t)$ is a global weak solution of (1).

Step2. Uniqueness.

Assume (1) has two global bounded weak solution u and v. Set $w = u - v$, then w satisfies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx + \frac{k}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2dx + M(\|\nabla u\|_p^p)\|\nabla u\|_p^p + M(\|\nabla v\|_p^p)\|\nabla v\|_p^p \\
=&M(\|\nabla u\|_p^p)\int_\Omega |\nabla u|^{p-2}\nabla u\nabla vdx + M(\|\nabla v\|_p^p)\int_\Omega |\nabla v|^{p-2}\nabla v\nabla udx \\
&+ \int_\Omega q|\theta u + (1-\theta)v|^{q-1}w^2dx
\end{aligned}$$

with $0 < \theta < 1$ and $w(x, 0) = 0$. Using the Young inequality, we can get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx + \frac{k}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2dx + M(\|\nabla u\|_p^p)\|\nabla u\|_p^p + M(\|\nabla v\|_p^p)\|\nabla v\|_p^p \\
\le&M(\|\nabla u\|_p^p)\frac{p-1}{p}\|\nabla u\|_p^p + M(\|\nabla u\|_p^p)\frac{1}{p}\|\nabla v\|_p^p + M(\|\nabla v\|_p^p)\frac{p-1}{p}\|\nabla v\|_p^p \\
&+ M(\|\nabla v\|_p^p)\frac{1}{p}\|\nabla u\|_p^p + \int_\Omega q|\theta u + (1-\theta)v|^{q-1}w^2dx,
\end{aligned}$$

which with the form of $M(s)$ indicates that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx + \frac{k}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2dx + \frac{b}{p}(\|\nabla u\|_p^p - \|\nabla v\|_p^p)^2 \\
\le&\int_\Omega q|\theta u + (1-\theta)v|^{q-1}w^2dx.
\end{aligned}$$

Thus we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx \le \int_\Omega q|\theta u + (1-\theta)v|^{q-1}w^2dx \le C\int_\Omega w^2dx,$$

where C is a positive constant depending only on q and the bound of u, v. Therefore by the Gronwall inequality, we have $u = v$.

Step3. Progressive estimation.

According to $u_0 \in W$ and Lemma 2.5, we have $u(x, t) \in W_\delta$, $\delta_1 < \delta < \delta_2$, where $\delta_1 < 1 < \delta_2$ are two roots of $d(\delta) = J(u_0)$. Furthermore, from the Hölder inequality and the Poincaré inequality, there exist positive constants C^* and C_*, such that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + k\|\nabla u\|_2^2) &= -I_{\delta_1}(u) + a(\delta_1 - 1)\|\nabla u\|_p^p + b(\delta_1 - 1)\|\nabla u\|_p^{2p} \\
&\le b(\delta_1 - 1)\|\nabla u\|_p^{2p} \\
&\le \frac{b(\delta_1 - 1)}{C^{*2p}}\|\nabla u\|_2^{2p} \\
&\le (\delta_1 - 1)\gamma(\|u\|_2^{2p} + k^p\|\nabla u\|_2^{2p})
\end{aligned}$$

with $\gamma = \min\left\{\frac{b}{2k^pC^{*2p}},\ \frac{bC_*^{2p}}{2C^{*2p}}\right\}$. Since there exists $K_p > 0$ for each p, such that $K_p(a^p + b^p) \ge (a + b)^p$ with non-negative a and b, then

$$\frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + k\|\nabla u\|_2^2) \le (\delta_1 - 1)\frac{\gamma}{K_p}\left(\|u\|_2^2 + k\|\nabla u\|_2^2\right)^p, \tag{13}$$

which implies $\|u\|_2^2 + k\|\nabla u\|_2^2 \le \left[\left(\|u_0\|_2^2 + k\|\nabla u_0\|_2^2\right)^{1-p} + (1-\delta_1)(p-1)\frac{2\gamma}{K_p}t\right]^{-\frac{1}{p-1}}$.

□

Proof of Theorem 1.3. Assume u is a global solution of (1). Let

$$H(t)=\int_0^t(\|u\|_2^2+k\|\nabla u\|_2^2)d\tau+(T^*-t)(\|u_0\|_2^2+k\|\nabla u_0\|_2^2),\quad t\in[0,T^*],$$

where T^* is a sufficiently large time. Then $H(t)\geq 0$ with $t\in[0,T^*]$. By a direct computation, we can get

$$H'(t)=\|u\|_2^2+k\|\nabla u\|_2^2-\|u_0\|_2^2-k\|\nabla u_0\|_2^2, \tag{14}$$

$$H''(t)=2(u_t,u)+2k(\nabla u_t,\nabla u)=-2I(u), \tag{15}$$

and

$$\begin{aligned}(H'(t))^2&=4\left[\int_0^t((u_\tau,u)+k(\nabla u_\tau,\nabla u))d\tau\right]^2\\&\leq 4\left[\int_0^t(\|u_\tau\|_2^2+k\|\nabla u_\tau\|_2^2)d\tau\right]\left[\int_0^t(\|u\|_2^2+k\|\nabla u\|_2^2)d\tau\right]\\&\leq 4H(t)\left[\int_0^t(\|u_\tau\|_2^2+k\|\nabla u_\tau\|_2^2)d\tau\right].\end{aligned}$$

Therefore we can deduce that

$$H''(t)H(t)-\frac{q+1}{2}(H'(t))^2\geq H(t)\left[-2I(u)-2(q+1)\int_0^t\|u_\tau\|_2^2+k\|\nabla u_\tau\|_2^2d\tau\right]. \tag{16}$$

Set

$$\xi(t)=-2I(u)-2(q+1)\int_0^t(\|u_\tau\|_2^2+k\|\nabla u_\tau\|_2^2)d\tau,$$

which with the definition of $J(u)$ indicates

$$\xi(t)=-2(q+1)J(u_0)+\frac{2a(q+1-p)}{p}\|\nabla u\|_p^p+\frac{b(q+1-2p)}{p}\|\nabla u\|_p^{2p}.$$

When $J(u_0)\leq 0$, then (5) means $J(u)\leq 0$, which with (4) leads to $I(u)<0$. By Lemma 2.2, there holds $\|\nabla u\|_p>r(1)$. Therefore

$$\xi(t)>\sigma_1>0\quad\text{with}\quad\sigma_1=\frac{2a(q+1-p)}{p}r^p(1). \tag{17}$$

When $0<J(u_0)<d$ and $I(u_0)<0$, then Lemma 2.5 implies $I_{\delta_2}(u)\leq 0$ and $\|\nabla u\|_p\geq r(\delta_2)>0$ with $\delta_1<1<\delta_2$ being the two roots of $J(u_0)=d(\delta)$. Thus from (15), we find that

$$\begin{aligned}H''(t)&=2a(\delta_2-1)\|\nabla u\|_p^p+2b(\delta_2-1)\|\nabla u\|_p^{2p}-2I_{\delta_2}(u)\\&\geq 2a(\delta_2-1)r^p(\delta_2),\end{aligned}$$

which with (14) guarantees

$$\|u\|_2^2+k\|\nabla u\|_2^2\geq H'(t)\geq 2a(\delta_2-1)r^p(\delta_2)t.$$

Thus there exists a $T_*>0$ such that (17) is established for $t\geq T_*$.

Substituting (17) into (16), we can deduce that

$$H''(t)H(t)-\frac{q+1}{2}(H'(t))^2>\sigma_1H(t).$$

Then

$$[H^{\frac{1-q}{2}}(t)]''\leq\frac{\sigma_1(1-q)}{2}[H^{\frac{1-q}{2}}(t)]^{\frac{q+1}{q-1}},\quad t\in[T_*,T^*].$$

Let $y(t) = H^{\frac{1-q}{2}}(t)$,

$$y''(t) \le \frac{\sigma_1(1-q)}{2}[y(t)]^{\frac{q+1}{q-1}}, \quad t \in [T_*, T^*].$$

Then there is $T \in (T_*, T^*)$ such that $\lim\limits_{t\to T^-} y(t) = 0$, which means $\lim\limits_{t\to T^-} H(t) = +\infty$. □

Proof of Theorem 1.4. Since $J(u_0) = d$, $u_0 \neq 0$. Set $\lambda_s = 1 - \frac{1}{s}$, $s = 1, 2, \ldots$ and consider the following initial value problem:

$$\begin{cases} u_t - k\Delta u_t - M(\|\nabla u\|_p^p)\Delta_p u = |u|^{q-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-1}u dx, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = \lambda_s u_0(x), & x \in \Omega. \end{cases}$$

According to $I(u_0) \ge 0$ and Lemma 2.1, there exists a unique $\lambda^* \ge 1$ such that $I(\lambda^* u_0) = 0$. Notice that $\lambda_s < 1 \le \lambda^*$, then $I(\lambda_s u_0) > 0$, $J(\lambda_s u_0) < J(u_0) = d$. By Theorem 1.2 and Lemma 2.5, for any s, there exists a unique global weak solution $u^s \in L^\infty(0,\infty; W_N^{1,p}(\Omega))$, $u_t^s \in L^2(0,\infty; W_N^{1,2}(\Omega))$ such that $u^s \in W$ and

$$\int_0^t (\|u_\tau^s\|_2^2 + k\|\nabla u_\tau^s\|_2^2)d\tau + J(u^s) = J(\lambda_s u_0) < d, \quad 0 \le t < +\infty.$$

Since $I(u^s) > 0$, then we have

$$\int_0^t (\|u_\tau^s\|_2^2 + k\|\nabla u_\tau^s\|_2^2)d\tau + \frac{a(q+1-p)}{p(q+1)}\|\nabla u^s\|_p^p + \frac{b(q+1-2p)}{2p(q+1)}\|\nabla u^s\|_p^{2p} < d.$$

Therefore

$$\int_0^t (\|u_\tau^s\|_2^2 + k\|\nabla u_\tau^s\|_2^2)d\tau < d,$$

$$\|\nabla u^s\|_p^p < \frac{dp(q+1)}{a(q+1-p)},$$

$$\||u^s|^{q-1}u^s\|_{\frac{q+1}{q}} = \|u^s\|_{q+1}^q \le S^q\|\nabla u^s\|_p^q < S^q(\frac{dp(q+1)}{a(q+1-p)})^{\frac{q}{p}}.$$

Similar to the proof of Theorem 1.2, (1) has a unique global weak solution $u \in L^\infty(0,\infty; W_N^{1,p}(\Omega))$, $u_t \in L^2(0,\infty; W_N^{1,2}(\Omega))$ with $I(u) \ge 0$ and $J(u) < d$.

If $I(u) > 0$, $0 < t < +\infty$, it can be seen from $\frac{d}{dt}(\|u\|_2^2 + k\|\nabla u\|_2^2) = -2I(u) < 0$ that $u_t \neq 0$. Further there exists $t_0 > 0$ such that

$$0 < J(u(t_0)) = J(u_0) - \int_0^{t_0} (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2)d\tau = d_1 < d.$$

Taking t_0 as the initial time, as it can be seen from Lemma 2.5, $u \in W_\delta$, $\delta_1 < \delta < \delta_2$ for $t > t_0$, where δ_1 and δ_2 are the two roots of $d(\delta) = J(u(t_0))$. Therefore we have $I_{\delta_1}(u) \ge 0$ for $t > t_0$. Similar to the proof of (13), we have

$$\frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + k\|\nabla u\|_2^2) \le \frac{(\delta_1 - 1)\gamma}{K_p}(\|u\|_2^2 + k\|\nabla u\|_2^2)^p$$

with $\gamma = \min\left\{\frac{b}{2k^pC^{*2p}}, \frac{bC_*^{2p}}{2C^{*2p}}\right\}$. Then we can get

$$\|u\|_2^2 + k\|\nabla u\|_2^2 \le \left[(\|u(t_0)\|_2^2 + k\|\nabla u(t_0)\|_2^2)^{1-p} + \frac{2(1-\delta_1)(p-1)\gamma}{K_p}(t - t_0)\right]^{-\frac{1}{p-1}}.$$

If there exists $t^* > 0$, such that $I(u) > 0,\ 0 < t < t^*,\ I(u(t^*)) = 0$, then

$$\int_0^{t^*} (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2)d\tau > 0,$$

$$J(u(t^*)) = d - \int_0^{t^*} (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2)d\tau < d.$$

By the definition of d, we know $u(t^*) = 0$. Thus for all $t > t^*$, $u = 0$, which means the weak solution of (1) distinguishes at a finite time. □

Proof of Theorem 1.5. Similar to Theorem 1.3, we get

$$\begin{aligned}
&H''(t)H(t) - \frac{q+1}{2}(H'(t))^2 \\
\geq &\left[\frac{2a(q+1-p)}{p}\|\nabla u\|_p^p + \frac{b(q+1-2p)}{p}\|\nabla u\|_p^{2p} - 2(q+1)J(u_0)\right] H(t).
\end{aligned}$$

From $J(u_0) = d > 0$, $I(u_0) < 0$ and Lemma 2.6, there exists $t_0 > 0$ such that $I(u(t)) < 0$, $0 < t < t_0$. Then (14) leads to $H''(t) > 0$ and $\|u_t\|_2^2 + k\|\nabla u_t\|_2^2 \neq 0$ for $0 < t < t_0$. Therefore $J(u(t_0)) = d - \int_0^{t_0}(\|u_\tau\|^2 + k\|\nabla u_\tau\|^2)d\tau = d_1 < d$. We choose t_0 as the initial time and complete the proof according to Theorem 1.3. □

4. **Life span.**

For the solutions that have been discussed in Section 3, we here further establish life span estimation of finite time blow-up solution without additional restriction on the initial data in Theorem 1.3.

Proof Theorem 1.6. Let $\theta(t) = \frac{1}{2}\|u\|_2^2 + \frac{k}{2}\|\nabla u\|_2^2,\ \eta(t) = -J(u)$.

(i) If $J(u_0) < 0$, then we have $\theta(0) > 0$, $\eta(0) > 0$, $\eta'(t) = \|u_t\|_2^2 + k\|\nabla u_t\|_2^2 \geq 0$ and $\eta(t) > 0$. From a simple computation, we can find that

$$\begin{aligned}
\theta'(t) &= -I(u) \\
&= -(q+1)J(u) + \frac{a(q+1-p)}{p}\|\nabla u\|_p^p + \frac{b(q+1-2p)}{2p}\|\nabla u\|_p^{2p} \\
&> (q+1)\eta(t) > 0, \qquad (18)\\
\theta(t)\eta'(t) &= \frac{1}{2}(\|u\|_2^2 + k\|\nabla u\|_2^2)\cdot(\|u_t\|_2^2 + k\|\nabla u_t\|_2^2) \\
&\geq \frac{1}{2}((u_t, u) + k(\nabla u_t, \nabla u))^2 \\
&= \frac{1}{2}(\theta'(t))^2 > \frac{q+1}{2}\theta'(t)\eta(t),
\end{aligned}$$

which implies

$$\left[\eta(t)\cdot\theta^{\frac{-q-1}{2}}(t)\right]' = \theta^{-1-\frac{q+1}{2}}(t)\left[\theta(t)\eta'(t) - \frac{q+1}{2}\eta(t)\theta'(t)\right] > 0. \qquad (19)$$

Thus (18) and (19) lead to

$$0 < \eta(0)\theta^{\frac{-q-1}{2}}(0) \leq \eta(t)\theta^{\frac{-q-1}{2}}(t) \leq \frac{1}{q+1}\theta'(t)\theta^{\frac{-q-1}{2}}(t) = \frac{2}{1-q^2}\left[\theta^{\frac{1-q}{2}}(t)\right]',$$

which further indicates that

$$0 \leq \theta^{\frac{1-q}{2}}(t) \leq \frac{1-q^2}{2}\theta^{\frac{-q-1}{2}}(0)\eta(0)t + \theta^{\frac{1-q}{2}}(0).$$

Thus we can deduce that $T \leq \frac{\|u_0\|_2^2 + k\|\nabla u_0\|_2^2}{(1-q^2)J(u_0)}$.

(ii) For $0 \le J(u_0) \le d$, if there exists $t^* > 0$ such that $J(u(t^*)) < 0$, we can get the upper bound estimation of T by (i). Therefore we only need to consider the case $0 \le J(u) \le d$ for any $t \in (0, T)$. According to Lemma 2.4 and Lemma 2.6, we have $I(u) < 0$ with $0 < t < T$, which means $\theta'(t) = -I(u) > 0$. Since u blows up in finite time and $\theta(t)$ increases with respect to time, there exists $0 < t_0 < T$ such that

$$\min_{t\in[t_0,T)} \left(\frac{2a(q+1-p)}{p} \|\nabla u(t)\|_p^p + \frac{b(q+1-2p)}{p} \|\nabla u(t)\|_p^{2p} \right) - 2(q+1)J(u_0) > 0. \tag{20}$$

Let

$$F(t) = \int_{t_0}^{t} (\|u\|_2^2 + k\|\nabla u\|_2^2) d\tau + (T-t)(\|u_0\|_2^2 + k\|\nabla u_0\|_2^2) + \beta((t-t_0)+\sigma)^2$$

with $t_0 \le t < T$ and positive constants β and σ to be determined later. By a direct computation, we can get

$$\begin{aligned} F'(t) &= \int_{t_0}^{t} \frac{d}{d\tau}(\|u\|_2^2 + k\|\nabla u\|_2^2) d\tau + 2\beta((t-t_0)+\sigma), \\ F''(t) &= \frac{d}{dt}(\|u\|_2^2 + k\|\nabla u\|_2^2) + 2\beta \\ &= -2(q+1)J(u(t_0)) + 2(q+1)\int_{t_0}^{t} (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau \\ &\quad + \frac{2a(q+1-p)}{p}\|\nabla u\|_p^p + \frac{b(q+1-2p)}{p}\|\nabla u\|_p^{2p} + 2\beta. \end{aligned}$$

Then $F(t_0) > 0$, $F'(t_0) > 0$ and $F'(t) > 0$ for $t \in [t_0, T)$, provided that β or σ is large enough. For any $\rho > 0$, we have

$$\begin{aligned} & F''(t)F(t) - \rho(F'(t))^2 \\ =& F''(t)F(t) + 4\rho\Big[\left(\int_{t_0}^{t} (\|u\|_2^2 + k\|\nabla u\|_2^2) d\tau + \beta((t-t_0)+\sigma)^2 \right) \cdot \\ & \qquad \left(\int_{t_0}^{t} (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau + \beta \right) \\ & - \left(\int_{t_0}^{t} [(u_\tau, u) + k(\nabla u_\tau, \nabla u)] d\tau + \beta((t-t_0)+\sigma) \right)^2 \\ & - (F(t) - (T-t)(\|u_0\|_2^2 + k\|\nabla u_0\|_2^2)) \left(\int_{t_0}^{t} (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau + \beta \right) \Big] \\ =& F''(t)F(t) + 4\rho(T-t)(\|u_0\|_2^2 + k\|\nabla u_0\|_2^2) \left(\int_{t_0}^{t} (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau + \beta \right) \\ & + 4\rho\zeta(t) - 4\rho F(t) \left(\int_{t_0}^{t} (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau + \beta \right), \end{aligned}$$

where

$$\begin{aligned} \zeta(t) = & \left(\int_{t_0}^{t} (\|u\|_2^2 + k\|\nabla u\|_2^2) d\tau + \beta((t-t_0)+\sigma)^2 \right) \cdot \\ & \qquad \left(\int_{t_0}^{t} (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau + \beta \right) \end{aligned}$$

$$-\left(\int_{t_0}^{t}[(u_\tau,u)+k(\nabla u_\tau,\nabla u)]d\tau+\beta((t-t_0)+\sigma)\right)^2\geq 0,\quad t\in[t_0,T).$$

Thus

$$\begin{aligned}&F''(t)F(t)-\rho(F'(t))^2\\ \geq&F(t)\left[F''(t)-4\rho\beta-4\rho\int_{t_0}^{t}(\|u_\tau\|_2^2+k\|\nabla u_\tau\|_2^2)d\tau\right]\\ =&F(t)\left[-2(q+1)J(u(t_0))+2(q+1-2\rho)\int_{t_0}^{t}(\|u_\tau\|_2^2+k\|\nabla u_\tau\|_2^2)d\tau\right.\\ &\left.+\frac{2a(q+1-p)}{p}\|\nabla u\|_p^p+\frac{b(q+1-2p)}{p}\|\nabla u\|_p^{2p}+2\beta-4\rho\beta\right].\end{aligned}$$

Take $\rho=\frac{q+1}{2}$ and using (5), the above inequality can be reduced to

$$\begin{aligned}&F''(t)F(t)-\frac{q+1}{2}(F'(t))^2\\ \geq&F(t)\Big[\frac{2a(q+1-p)}{p}\|\nabla u\|_p^p+\frac{b(q+1-2p)}{p}\|\nabla u\|_p^{2p}\\ &-2(q+1)J(u(t_0))-2(q+1)\beta\Big].\end{aligned}$$

Since (20), we can get

$$F''(t)F(t)-\frac{q+1}{2}(F'(t))^2\geq 0\quad\text{with}\quad t_0<t<T,$$

provided that

$$\beta\in\left(0,\frac{a(q+1-p)}{p(q-1)}\|\nabla u(t_0)\|_p^p+\frac{b(q+1-2p)}{2p(q-1)}\|\nabla u(t_0)\|_p^{2p}-J(u_0)\right].\qquad(21)$$

Set $G(t)=F^{1-\frac{q+1}{2}}(t)$ with $t\in[t_0,T)$, then

$$\begin{aligned}G'(t)&=\left(1-\frac{q+1}{2}\right)F^{-\frac{q+1}{2}}(t)F'(t)\leq 0,\\ G''(t)&=\left(1-\frac{q+1}{2}\right)\cdot F^{-1-\frac{q+1}{2}}(t)\left[-\frac{q+1}{2}(F'(t))^2+F''(t)F(t)\right]\leq 0,\\ G(t)&\leq G(t_0)+G'(t_0)(t-t_0).\end{aligned}$$

Because of $G(t_0)>0$ and $G'(t_0)<0$, we have

$$t-t_0\leq-\frac{G(t_0)}{G'(t_0)}=\frac{(T-t_0)(\|u_0\|_2^2+k\|\nabla u_0\|_2^2)+\beta\sigma^2}{(q-1)\beta\sigma}$$

with $t\in(t_0,T)$. For fixed β_0 satisfies (21), taking $\sigma\in\left(\frac{\|u_0\|_2^2+k\|\nabla u_0\|_2^2}{(q-1)\beta_0},+\infty\right)$, then

$$T-t_0\leq\frac{\beta_0\sigma^2}{(q-1)\beta_0\sigma-(\|u_0\|_2^2+k\|\nabla u_0\|_2^2)}.$$

Define

$$T_{\beta_0}(\sigma)\frac{\beta_0\sigma^2}{(q-1)\beta_0\sigma-(\|u_0\|_2^2+k\|\nabla u_0\|_2^2)}$$

with $\sigma \in \left(\frac{\|u_0\|_2^2+k\|\nabla u_0\|_2^2}{(q-1)\beta_0}, +\infty\right)$. We can find that $T_{\beta_0}(\sigma)$ takes the minimum at

$$\sigma = \frac{2\|u_0\|_2^2 + 2k\|\nabla u_0\|_2^2}{(q-1)\beta_0},$$

which indicates that

$$T - t_0 \leq \inf_{\sigma \in \left(\frac{\|u_0\|_2^2+k\|\nabla u_0\|_2^2}{(q-1)\beta_0}, +\infty\right)} T_{\beta_0}(\sigma) = \frac{4\|u_0\|_2^2 + 4k\|\nabla u_0\|_2^2}{(q-1)^2\beta_0}.$$

Combining the above inequality with (21), we can get

$$\begin{aligned} T - t_0 &\leq \inf_{\beta_0 \in \left(0, \frac{a(q+1-p)}{p(q-1)}\|\nabla u(t_0)\|_p^p + \frac{b(q+1-2p)}{2p(q-1)}\|\nabla u(t_0)\|_p^{2p} - J(u_0)\right]} \frac{4\|u_0\|_2^2 + 4k\|\nabla u_0\|_2^2}{(q-1)^2\beta_0} \\ &= \frac{4\|u_0\|_2^2 + 4k\|\nabla u_0\|_2^2}{(q-1)^2(\frac{a(q+1-p)}{p(q-1)}\|\nabla u(t_0)\|_p^p + \frac{b(q+1-2p)}{2p(q-1)}\|\nabla u(t_0)\|_p^{2p} - J(u_0))}. \end{aligned}$$

□

5. $J(u_0) > d$**.** This section is devoted to proving Theorem 1.7 and Theorem 1.8 to investigate the conditions that ensure the global existence or finite time blowing-up of solution to (1) when $J(u_0) > d$.

Proof of Theorem 1.7. (i) If $u_0 \in \mathcal{N}_+$ and $\|u_0\|_2^2 + k\|\nabla u_0\|_2^2 \leq \lambda_{J(u_0)}$, then we assert that $u(t) \in \mathcal{N}_+$, $0 \leq t < T(u_0)$ with $T(u_0)$ being the maximum existence time of the solution. Otherwise there exists $t_0 \in (0, T(u_0))$ such that $u(t) \in \mathcal{N}_+$, $0 \leq t < t_0$ and $u(t_0) \in \mathcal{N}$. Furthermore, (5) indicates that $J(u(t_0)) < J(u_0)$, which with the definition of J^s leads to $u(t_0) \in J^{J(u_0)}$. Thus $u(t_0) \in \mathcal{N}^{J(u_0)}$. According to the definition of $\lambda_{J(u_0)}$, we can get

$$\|u(t_0)\|_2^2 + k\|\nabla u(t_0)\|_2^2 \geq \lambda_{J(u_0)}. \tag{22}$$

It can be seen from $u(t) \in \mathcal{N}_+$ with $0 \leq t < t_0$ that

$$I(u) = -\frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + k\|\nabla u\|_2^2) > 0, \quad 0 < t < t_0.$$

Then we have

$$\|u(t_0)\|_2^2 + k\|\nabla u(t_0)\|_2^2 < \|u_0\|_2^2 + k\|\nabla u_0\|_2^2 \leq \lambda_{J(u_0)},$$

which contradicts (22). Hence $u(t) \in \mathcal{N}_+$, $0 \leq t < T(u_0)$.

Using (4) and (5), there holds

$$\begin{aligned} J(u_0) \geq J(u) &= \frac{1}{q+1}I(u) + (\frac{a}{p} - \frac{a}{q+1})\|\nabla u\|_p^p + (\frac{b}{2p} - \frac{b}{q+1})\|\nabla u\|_p^{2p} \\ &> \frac{a(q+1-p)}{p(q+1)}\|\nabla u\|_p^p, \end{aligned}$$

which means $\|\nabla u\|_p^p \leq \frac{p(q+1)J(u_0)}{a(q+1-p)}$ and further $T(u_0) = +\infty$. Define the ω − limit set of u_0 by $\omega(u_0) = \bigcap_{t\geq 0} \overline{\{u(\cdot, s) : s \geq t\}}$. Then for any $\omega \in \omega(u_0)$, we have

$$\|\omega\|_2^2 + k\|\nabla\omega\|_2^2 < \|u_0\|_2^2 + k\|\nabla u_0\|_2^2 \leq \lambda_{J(u_0)}, \quad J(\omega) \leq J(u_0).$$

So that $\omega(u_0) \cap \mathcal{N} = \emptyset$, which with the convergence result in [4] leads to $\omega(u_0) = \{0\}$, namely $u_0 \in \mathcal{G}_0$.

(ii) If $u_0 \in \mathcal{N}_-$, $\|u_0\|_2^2 + k\|\nabla u_0\|_2^2 \geq \Lambda_{J(u_0)}$, then similar to (i), we can get $u(t) \in \mathcal{N}_-$, $u(t) \in J^{J(u_0)}$ for $0 \leq t < T(u_0)$. If $T(u_0) = \infty$, then for any $\omega \in \omega(u_0)$, we conclude that

$$\|\omega\|_2^2 + k\|\nabla \omega\|_2^2 > \Lambda_{J(u_0)},\ J(\omega) \leq J(u_0).$$

Then $\omega(u_0) \cap \mathcal{N} = \emptyset$, which with the convergence result in [4] leads to $\omega(u_0) = \{0\}$. However due to $u \in \mathcal{N}_-$, we have

$$a\|\nabla u\|_p^p < a\|\nabla u\|_p^p + b\|\nabla u\|_p^{2p} < \|u\|_{q+1}^{q+1} \leq S^{q+1}\|\nabla u\|_p^{q+1},$$

which means $\|\nabla u\|_p \geq \left(\frac{a}{S^{q+1}}\right)^{\frac{1}{q+1-p}}$. It is a contradiction. Then $T(u_0) < +\infty$ and $u_0 \in \mathcal{B}$. □

Proof of Theorem 1.8. The definitions of λ_s and Λ_s indicate that $\lambda_s \leq \Lambda_s$. By the definition of the potential well depth d (7), we have

$$\begin{aligned} d &= \inf_{u \in \mathscr{N}} J(u) \\ &= \inf_{u \in \mathscr{N}} \left[\frac{a(q+1-p)}{p(q+1)}\|\nabla u\|_p^p + \frac{b(q+1-2p)}{2p(q+1)}\|\nabla u\|_p^{2p}\right] \\ &= \inf_{u \in \mathscr{N}} f(\|\nabla u\|_p), \end{aligned}$$

where $f(\cdot)$ is given in (8). Since $f(\cdot)$ is strictly increasing on $[0, +\infty)$ and $f(0) = 0$, there exists a unique

$$\kappa = \left(\frac{-a(q+1-p) + \sqrt{a^2(q+1-p)^2 + 2db(q+1-2p)p(q+1)}}{b(q+1-2p)}\right)^{1/p}$$

such that $f(\kappa) = d$. Then for any $u \in \mathscr{N}$, there is

$$\|\nabla u\|_p \geq \kappa > 0. \tag{23}$$

By the Gagliardo–Nirenberg inequality [2], we get

$$\|u\|_{q+1} \leq \beta\|u\|_2^{(1-\theta)}\|\nabla u\|_p^{\theta},$$

where β is a positive constant and $\theta(\frac{1}{2} - \frac{1}{p} + \frac{1}{n}) = \frac{1}{2} - \frac{1}{q+1}$. Then it follows from the above inequality that for any $u \in \mathscr{N}$

$$a\|\nabla u\|_p^p \leq \|u\|_{q+1}^{q+1} \leq \beta^{q+1}\|u\|_2^{(1-\theta)(q+1)}\|\nabla u\|_p^{\theta(q+1)},$$

which says

$$a\|\nabla u\|_p^{p-\theta(q+1)} \leq \beta^{q+1}\|u\|_2^{(1-\theta)(q+1)}. \tag{24}$$

Moreover by the definition of $\mathscr{N}^s$, it is known that if $u \in \mathscr{N}^s$,

$$f(u) - s < 0,$$

which implies

$$\|\nabla u\|_p \leq \tilde{\kappa} = \left(\frac{-a(q+1-p) + \sqrt{a^2(q+1-p)^2 + 2sb(q+1-2p)p(q+1)}}{b(q+1-2p)}\right)^{1/p}. \tag{25}$$

For the lower bound of λ_s, we divide into two cases to discuss.

Case 1. $p-\theta(q+1)\geq 0$, namely $p>\frac{n}{n+2}(q+1)$, then using (23) and (24), we have

$$\begin{aligned}\lambda_s &= \inf_{u\in\mathscr{N}^s}\left\{\|u\|_2^2+k\|\nabla u\|_2^2\right\}\\ &\geq \inf_{u\in\mathscr{N}}\left\{\|u\|_2^2+k\|\nabla u\|_2^2\right\}\\ &\geq \inf_{u\in\mathscr{N}}\left[\frac{a}{\beta^{q+1}}\|\nabla u\|_p^{p-\theta(q+1)}\right]^{\frac{2}{(1-\theta)(q+1)}}\\ &\geq \left[\frac{a}{\beta^{q+1}}\kappa^{p-\theta(q+1)}\right]^{\frac{2}{(1-\theta)(q+1)}}.\end{aligned}$$

Case 2. $p-\theta(q+1)<0$, namely $p<\frac{n}{n+2}(q+1)$, then using (23) and (25), we have

$$\begin{aligned}\lambda_s &= \inf_{u\in\mathscr{N}^s}\left\{\|u\|_2^2+k\|\nabla u\|_2^2\right\}\\ &\geq \sup_{u\in\mathscr{N}}\left[\frac{a}{\beta^{q+1}}\|\nabla u\|_p^{p-\theta(q+1)}\right]^{\frac{2}{(1-\theta)(q+1)}}\\ &\geq \left[\frac{a}{\beta^{q+1}}\tilde{\kappa}^{\frac{p-\theta(q+1)}{p}}\right]^{\frac{2}{(1-\theta)(q+1)}}.\end{aligned}$$

For the upper bound of Λ_s, using the Hölder inequality and (25), we have

$$\begin{aligned}\Lambda_s &= \sup_{u\in\mathscr{N}^s}\{\|u\|_2^2+k\|\nabla u\|_2^2\}\\ &\leq \sup_{u\in\mathscr{N}_s}(1+k)|\Omega|^{\frac{p-2}{p}}\|\nabla u\|_p^2\\ &\leq (1+k)|\Omega|^{\frac{p-2}{p}}\tilde{\kappa}^2.\end{aligned}$$

□

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Received January 2021; revised July 2021; early access September 2021.

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