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ON MATHEMATICAL ANALYSIS OF COMPLEX FLUIDS IN ACTIVE HYDRODYNAMICS

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Abstract. This is a survey article for this special issue providing a review of the recent results in the mathematical analysis of active hydrodynamics. Both the incompressible and compressible models are discussed for the active liquid crystals in the Landau-de Gennes Q-tensor framework. The mathematical results on the weak solutions, regularity, and weak-strong uniqueness are presented for the incompressible flows. The global existence of weak solution to the compressible flows is recalled. Other related results on the inhomogeneous flows, incompressible limits, and stochastic analysis are also reviewed.

1. **Introduction.** This article provides a survey of recent mathematical analysis on the complex fluids in active hydrodynamics. Active hydrodynamics describes the fluids with active constituent particles in a collective motion that constantly maintains out of equilibrium by the internal energy sources, which is quite generic in nature and has wide applications. For example, many biophysical systems are considered as active hydrodynamics, such as bacteria [10], microtubule bundles [59], dense suspensions of microswimmers [68]. Furthermore, the collective motion usually induces the particles with elongated shapes to demonstrate orientational ordering at high concentration. Thus, there are natural analogies with nematic liquid crystals, and hence a large class of active systems are referred to as active liquid crystals; see [5, 57, 25, 26, 52, 13] and the references therein for more information and discussions. There are different phases of liquid crystals, which can be distinguished by their distinct optical properties. One of the most common liquid crystal phases is nematic. In the nematic phase, rod-shaped or elongated organic molecules have long-range orientational order with their long axes approximately parallel. Therefore, the molecules flow freely as in a conventional liquid, but still

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maintain their long-range directional order [12, 62]. Active nematic systems are different from the typical passive counterparts since the constituent particles are active and the system is out of equilibrium. Consequently, active hydrodynamic systems are truly striking and leading to novel effects, such as the occurrence of giant density fluctuations [56, 47, 48], the spontaneous laminar flow [63, 45, 27], unconventional rheological properties [60, 24, 20], low Reynolds number turbulence [68, 26], and very different spatial and temporal patterns [47, 7, 58, 22, 46].

Although active liquid crystals are popular in physics and applications, a rigorous mathematical description of active nematics is relatively new. A common approach of modeling for active liquid crystals is to add phenomenological active terms to the hydrodynamic theories for nematic liquid crystals; see for example [55]. There are several classical models for nematic liquid crystals in the literature, such as the Doi-Onsager model [13], the Oseen-Frank model [49, 21], the Ericksen-Leslie model [15, 37], and the Landau-de Gennes model [12]. We refer the readers to [3, 4, 42] for the discussions of these models including their advantages and differences. The Landau-de Gennes theory is one of the most comprehensive models for nematic liquid crystals, where the state of a nematic liquid crystal is modeled by a symmetric traceless $d \times d$ matrix $Q \in \mathbb{M}^{d\times d}$, known as the Q-tensor order parameter. The Landau-de Gennes Q-tensor order parameter describes primary and secondary directions of nematic alignment along with variations in the degree of nematic order [42]. A nematic liquid crystal is said to be (i) isotropic when $Q = 0$, (ii) uniaxial when Q has a pair of equivalent non-zero eigenvalues, and (iii) biaxial if Q has three distinct eigenvalues. A model for the incompressible flow of active liquid crystal fluids in $\mathbb{R}^d$, $d = 2$ or 3 is the following (see [4, 23, 25]):

$$\begin{cases} \partial_t c + (u\cdot\nabla)c = D_0\Delta c, \\ \partial_t u + (u\cdot\nabla)u + \nabla p - \mu\Delta u = \nabla\cdot\sigma + \nabla\cdot\tau - \lambda\nabla\cdot(|Q|H), \\ \partial_t Q + (u\cdot\nabla)Q - S(\nabla u, Q) - \lambda|Q|D = \Gamma H, \\ \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

where $c = c(t,x) > 0$ is the concentration of active particles; $u = u(t,x) \in \mathbb{R}^d$ is the velocity field of the flow; $p = p(t,x) \in \mathbb{R}$ is the pressure; $Q = Q(t,x) \in \mathbb{M}^{d\times d}$ denotes the nematic tensor order parameter that is a traceless and symmetric $d\times d$ matrix; $D_0 > 0$ is the diffusion constant; $\mu > 0$ is the viscosity coefficient; $1/\Gamma > 0$ is the rotational viscosity; $\lambda \in \mathbb{R}$ denotes the nematic alignment parameter. Moreover, $H = H(c,Q)$ is the molecular tensor, namely,

$$H = K\Delta Q - \frac{k}{2}(c-c_*)Q + b(Q^2 - \frac{tr(Q^2)}{d}I_d) - c_*Qtr(Q^2),$$

which describes the relaxational dynamics of the nematic phase; it can be obtained from the Landau-de Gennes free energy, i.e., $H_{ij} = -\frac{\delta \mathbf{F}}{\delta Q_{ij}}$, where

$$\mathbf{F} = \int \Big(\frac{K}{2}|\nabla Q|^2 + \frac{k}{2}(c-c_*)tr(Q^2) - \frac{b}{3}tr(Q^3) + \frac{c_*}{4}|tr(Q^2)|^2\Big)\,dA, \tag{1.2}$$

$K > 0$ is the elastic constant for the elastic energy density, $k > 0$, $b \in \mathbb{R}$ are material-dependent constants, c_* denotes the critical concentration for the isotropic-nematic transition, and $I_d \in \mathbb{M}^{d\times d}$ denotes the identity matrix. Without loss of generality, we take $K = k = 1$. Besides, the matrix valued function

$$S(\nabla u, Q) = \xi D(Q + \frac{1}{d}I_d) + \xi(Q + \frac{1}{d}I_d)D - 2\xi(Q + \frac{1}{d}I_d)tr(Q\nabla u) + \Omega Q - Q\Omega$$

describes how the flow gradient rotates and stretches the director field, as well as the molecules can be tumbled and aligned by the flow, where

$$D = \frac{1}{2}(\nabla u + \nabla u^\top), \quad \text{and} \quad \Omega = \frac{1}{2}(\nabla u - \nabla u^\top)$$

are the symmetric and antisymmetric part of the strain tensor with $(\nabla u)_{ij} = \partial_j u_i$, and ξ is the liquid crystal material parameter that describes the relationship between the tumbling and aligning effects imposed by the shear flow on the liquid crystal directors. If the molecules only tumble in a shear flow but do not align, it tends to a simple case $\xi = 0$. The stress tensor $\sigma = (\sigma_{ij})$ consists of two parts:

$$\sigma = \sigma^r + \sigma^a,$$

where

$$\sigma^r = QH - HQ$$

is the elastic stress tensor from the nematic elasticity, and

$$\sigma^a = \sigma_* c^2 Q$$

is the active contribution due to the contractile ($\sigma_* > 0$) or extensile ($\sigma_* < 0$) stresses exerted by the active particles in the direction of the director field. The symmetric additional stress tensor is denoted by:

$$\tau = -\xi(Q + \frac{1}{d}I_d)H - \xi H(Q + \frac{1}{d}I_d) + 2\xi(Q + \frac{1}{d}I_d)tr(QH) - \nabla Q \odot \nabla Q,$$

where the symbol $\nabla Q \odot \nabla Q$ denotes the $d \times d$ matrix whose (i, j)-th term is given by $(\nabla Q \odot \nabla Q)_{ij} = \partial_i Q_{mn} \partial_j Q_{mn}$. Here we use the Einstein summation convention, i.e., the repeated indices are summed over, and $\partial_i = \frac{\partial}{\partial x_i}$.

Regarding modeling and analysis of the Ericksen-Leslie equations describing nematic liquid crystals we refer the readers to the works [33, 32, 44, 43, 38, 65, 35, 36, 66, 31, 40] and the survey papers [30, 42, 70, 41] as well as the references therein for more discussions on the physics and mathematical results. We now recall some analysis results for the Q-tensor system in the Beris-Edwards hydrodynamics framework. Paicu-Zarnescu [50, 51] proved the existence of global weak solutions to the full coupled Navier-Stokes and Q-tensor system in $\mathbb{R}^d, d = 2, 3$, and the existence of global regular solutions with sufficiently regular initial data in the two-dimensional case with $\xi = 0$ and ξ smallness hypothesis respectively. Wilkinson [69] obtained the existence of strictly physical global weak solutions on the two and three-dimensional torus and global strong solutions in dimension two over a certain singular potential proposed in Ball-Majumdar [3] with $\xi = 0$. Feireisl-Rocca-Schimperna-Zarnescu [18, 19] derived the global-in-time weak solutions of the nonisothermal Landau-de Gennes nematic liquid crystal flows in three-dimensional periodic space with Ball-Majumdar's singular free energy bulk potential for arbitrary physically relevant initial data for general ξ. For the initial-boundary value problems, we refer to [1, 2, 28, 29] where the existence of global weak solutions, the existence and uniqueness of local strong solutions were obtained. Furthermore, Wang-Xu-Yu [64] developed the existence and long-time dynamics of globally defined weak solutions for the coupled compressible Navier-Stokes and Q-tensor system. See [11, 34, 6, 14, 70] and the references therein for more results and discussions. All the above results are about the passive nematic liquid crystals without active terms and the concentration equation.

For the active systems, Chen-Majumdar-Wang-Zhang [8] analyzed active hydrodynamics in an incompressible Beris-Edwards framework and established the existence of global weak solutions in $\mathbb{R}^d, d = 2, 3$, and the higher regularity of the weak solutions and the weak-strong uniqueness in the two-dimensional case, under the assumption that the concentration of active particles is constant. For the inhomogeneous incompressible active liquid crystals, Lian-Zhang [39] obtained global weak solutions in a three-dimensional bounded domain. For the active system with non-constant particle concentration in the fluid, Chen-Majumdar-Wang-Zhang [9] analyzed the initial-boundary value problems for compressible active nematic liquid crystals in the three-dimensional space and proved the existence of global weak solutions for the active system by the three-level approximations and weak convergence argument. The incompressible limit was studied in [67]. Some stochastic analysis of active hydrodynamics was done in [53, 54].

More detailed survey on the analysis of active hydrodynamics will be given in the rest of the paper. In Section 2, we present the results on the incompressible flow of active liquid crystals from [8]. In Section 3, we present the results on the compressible flow of active liquid crystals from [9]. In Section 4, we review various other related results and discuss some open problems.

2. Incompressible flows: Weak solution, regularity, and weak-strong uniqueness.

For the incompressible flows of active liquid crystals, Chen-Majumdar-Wang-Zhang [8] established the existence of global weak solutions in $\mathbb{R}^d, d = 2, 3$, and the higher regularity and weak-strong uniqueness in the two-dimensional case, for the constant concentration of active particles. When the concentration of active particles changes, we consider the system (1.1) and can prove the same results by some modifications of the arguments of the paper [8]. Below we shall present the results of [8] but in the context of the system (1.1), and give an outline for the proof of the existence of weak solutions; see [8] for the more detailed arguments.

We rewrite system (1.1) as

$$\begin{cases} \partial_t c + (u\cdot\nabla)c = D_0\Delta c, \\ \partial_t u + (u\cdot\nabla)u + \nabla p - \mu\Delta u + \nabla\cdot(\nabla Q\odot\nabla Q) \\ \qquad = -\xi\nabla\cdot\big((Q+\frac{1}{d}I_d)H + H(Q+\frac{1}{d}I_d) - 2(Q+\frac{1}{d}I_d)tr(QH)\big) \\ \qquad\quad +\nabla\cdot\big(Q\Delta Q - \Delta Q Q + \sigma_* c^2 Q\big) - \lambda\nabla\cdot(|Q|H), \\ \partial_t Q + (u\cdot\nabla)Q - (\Omega Q - Q\Omega) - \lambda|Q|D \\ \qquad = \xi\big(D(Q+\frac{1}{d}I_d) + (Q+\frac{1}{d}I_d)D - 2(Q+\frac{1}{d}I_d)tr(Q\nabla u)\big) + \Gamma H, \\ \nabla\cdot u = 0, \end{cases} \tag{2.1}$$

where

$$H = \Delta Q - \frac{c-c_*}{2}Q + b(Q^2 - \frac{tr(Q^2)}{d}I_d) - c_* Q tr(Q^2),$$

and $D_0 > 0,\ \mu > 0,\ \Gamma > 0,\ c_* > 0,\ b, \sigma_*, \lambda, \xi \in \mathbb{R},\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$; and consider the following initial condition:

$$(c, u, Q)|_{t=0} = (c_0, u_0, Q_0)(x),\ \text{for } x \in \mathbb{R}^d, \tag{2.2}$$

with

$$c_0(x) - \hat{c} \in L^2(\mathbb{R}^d),\ 0 < \underline{c} \le c_0 \le \bar{c} < \infty,\ c_0 \to \hat{c}\ as\ x \to \infty, \tag{2.3}$$

$$u_0(x) \in L^2(\mathbb{R}^d),\ \nabla\cdot u_0 = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d), \tag{2.4}$$

$$Q_0(x) \in H^1(\mathbb{R}^d),\ Q_0 \in S_0^d\ a.e. \text{ in } \mathbb{R}^d. \tag{2.5}$$

For the sake of convenience, we shall use the same notation of [8]. We denote the Sobolev space by H^k for integer $k \geq 1$, with the norm $\|\cdot\|_{H^k}$ defined by

$$\|v\|_{H^k}^2 := \sum_{0\leq|\nu|\leq k} \|D^\nu v\|_{L^2}^2,$$

where $D^\nu := \partial_1^{\nu_1} \cdots \partial_d^{\nu_d}$ is the distributional derivative. The space H^{-k} is the dual space of H_0^k, with the norm:

$$\|v\|_{H^{-k}} := \sup_{\varphi\in H_0^k, \|\varphi\|_{H_0^k}\leq 1} |(v,\varphi)|,$$

where $(\cdot,\cdot)$ denotes the inner product in L^2. If a and b are vector functions, $(a,b) := \int_{\mathbb{R}^d} a(x)\cdot b(x)\,dx$ and if A and B are matrices, $(A,B) := \int_{\mathbb{R}^d} A : B\,dx$ with $A : B = tr(A^TB) = A_{ij}B_{ij}$. We can also write $A : B = tr(AB)$ if A or B is symmetric. We denote by $S_0^d \subset \mathbb{M}^{d\times d}$ the space of symmetric traceless Q-tensors in d-dimension:

$$S_0^d := \big\{Q \in \mathbb{M}^{d\times d};\ Q_{ij} = Q_{ji},\ tr(Q) = 0,\ i,j = 1,\cdots,d\big\},$$

with the Frobenius norm $|Q| := \sqrt{tr(Q^TQ)} = \sqrt{Q_{ij}Q_{ij}}$, and then define the Sobolev space for the Q-tensors:

$$H^1(\mathbb{R}^d, S_0^d) := \Big\{Q : \mathbb{R}^d \to S_0^d;\ \int_{\mathbb{R}^d} (|Q(x)|^2 + |\nabla Q(x)|^2)dx < \infty\Big\},$$

where $|\nabla Q|^2 := \partial_k Q_{ij}\partial_k Q_{ij}$. We also denote $|\Delta Q|^2 := \Delta Q_{ij}\Delta Q_{ij}$.

Denote the Landau-de Gennes free energy for the nematic liquid crystals by

$$\mathbf{F}(Q) := \int_{\mathbb{R}^d} \Big(\frac{1}{2}|\nabla Q|^2 + \frac{c-c_*}{4}|Q|^2 - \frac{b}{3}tr(Q^3) + \frac{c_*}{4}|Q|^4\Big)dx, \tag{2.6}$$

and the energy of the system (2.1) by

$$E(t) := \mathbf{F}(Q) + \int_{\mathbb{R}^d} \Big(\frac{1}{2}|c-\hat{c}|^2 + \frac{1}{2}|u|^2\Big)dx. \tag{2.7}$$

First we have the following basic energy estimate.

Proposition 1. *Let (c,u,Q) be a smooth solution of system (2.1) such that for any given $T > 0$,*

$$c - \hat{c} \in L^\infty(0,T; L^2(\mathbb{R}^d)) \cap L^2(0,T; H^1(\mathbb{R}^d)), \tag{2.8}$$

$$u \in L^\infty(0,T; L^2(\mathbb{R}^d)) \cap L^2(0,T; H^1(\mathbb{R}^d)), \tag{2.9}$$

$$Q \in L^\infty(0,T; H^1(\mathbb{R}^d)) \cap L^2(0,T; H^2(\mathbb{R}^d)). \tag{2.10}$$

Then we have

$$\begin{aligned} &\frac{d}{dt}E(t) + D_0\int_{\mathbb{R}^d} |\nabla c|^2dx + \frac{\mu}{2}\int_{\mathbb{R}^d} |\nabla u|^2dx + \Gamma\int_{\mathbb{R}^d} tr(H^2)dx \\ &\leq C(D_0,\bar{c},\sigma_*,\mu)\int_{\mathbb{R}^d} (|Q|^2 + |\nabla Q|^2 + |tr(Q\Delta Q)|)dx, \end{aligned} \tag{2.11}$$

for any $t \in (0,T)$.

Based on Proposition 1, Gronwall's inequality and assumption of lower bound of initial data $\underline{c}$ or the liquid crystal material parameter ξ, we obtain the following a priori estimates.

Proposition 2. *Let (c, u, Q) be a smooth solution of system (2.1)-(2.2) in $\mathbb{R}^d$, $d = 2, 3$. There exist some positive constants c^* and ξ_*, such that, if $\underline{c} > c^*$ or $|\xi| < \xi_*$, then for the initial data $(c_0, u_0, Q_0) \in L^\infty \times L^2 \times H^1$ and any $t > 0$, one has*

$$0 < \underline{c} \leq c \leq \bar{c} < \infty, \tag{2.12}$$

$$\|Q(t,\cdot)\|_{H^1}^2 \leq C_1 e^{C_2 t} \|Q_0\|_{H^1}^2, \tag{2.13}$$

and

$$\|u(t,\cdot)\|_{L^2}^2 + \frac{\mu}{2} \int_0^t \|\nabla u(s,\cdot)\|_{L^2}^2 ds \leq C_3(\|Q_0\|_{H^1}^2 + \|u_0\|_{L^2}^2) e^{C_2 t} + C_4, \tag{2.14}$$

where constants C_i, $i = 1, \cdots, 4$, depend on $D_0, \mu, \lambda, \Gamma, c_*$ and the initial data (c_0, u_0, Q_0).

Remark 1. For the passive system considered in [50], the hypothesis of small $|\xi|$ is necessary in $\mathbb{R}^d$. In our system, because of the varying concentration in the energy density, we can obtain the a priori estimates without smallness condition on ξ.

Next, we introduce the definition of weak solutions to the system (2.1) subject to the initial condition (2.2) in $\mathbb{R}^d$ for $d = 2, 3$.

Definition 2.1. Let the initial data (c_0, u_0, Q_0) satisfy (2.3)-(2.5). The triple (c, u, Q) is called a weak solution to the system (2.1)-(2.2) if

$$c - \hat{c} \in L^\infty_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^d)) \cap L^2_{loc}(\mathbb{R}_+; H^1(\mathbb{R}^d)), \tag{2.15}$$

$$u \in L^\infty_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^d)) \cap L^2_{loc}(\mathbb{R}_+; H^1(\mathbb{R}^d)), \tag{2.16}$$

$$Q \in L^\infty_{loc}(\mathbb{R}_+; H^1(\mathbb{R}^d)) \cap L^2_{loc}(\mathbb{R}_+; H^2(\mathbb{R}^d)), \tag{2.17}$$

and the weak formulation holds:

$$\begin{aligned}&-\int_0^\infty \int_{\mathbb{R}^d} c\partial_t \phi dx dt - \int_0^\infty \int_{\mathbb{R}^d} cu \cdot \nabla \phi dx dt + D_0 \int_0^\infty (\nabla c, \nabla \phi) dt \\ &= \int_{\mathbb{R}^d} c_0(x) \phi(0, x) dx,\end{aligned} \tag{2.18}$$

$$\begin{aligned}&-\int_0^\infty (u, \partial_t \varphi) dt - \int_0^\infty (u, u \cdot \nabla \varphi) dt + \mu \int_0^\infty (\nabla u, \nabla \varphi) dt - \int_{\mathbb{R}^d} u_0(x) \cdot \varphi(0, x) dx \\ &= \xi \int_0^\infty \big((Q + \frac{1}{d} I_d) H + H(Q + \frac{1}{d} I_d) - 2(Q + \frac{1}{d} I_d) tr(QH), \nabla \varphi\big) dt \\ &\quad + \int_0^\infty \big(\nabla Q \odot \nabla Q - (Q\Delta Q - \Delta Q Q) - \sigma_* c^2 Q + \lambda |Q| H, \nabla \varphi\big) dt,\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}&-\int_0^\infty (Q, \partial_t \psi) dt - \int_0^\infty (Q, u \cdot \nabla \psi) dt - \Gamma \int_0^\infty (\Delta Q, \psi) dt - \int_{\mathbb{R}^d} Q_0(x) : \psi(0, x) dx \\ &= \xi \int_0^\infty \big(D(Q + \frac{1}{d} I_d) + (Q + \frac{1}{d} I_d) D - 2(Q + \frac{1}{d} I_d) tr(Q\nabla u), \psi\big) dt \\ &\quad + \Gamma \int_0^\infty \big(- \frac{c - c_*}{2} Q + b\big(Q^2 - \frac{tr(Q^2)}{d} I_d\big) - c_* Q tr(Q^2), \psi\big) dt \\ &\quad + \int_0^\infty (\Omega Q - Q\Omega, \psi) dt + \lambda \int_0^\infty (|Q| D, \psi) dt,\end{aligned} \tag{2.20}$$

for all $\phi \in C^\infty([0,\infty) \times \mathbb{R}^d;\mathbb{R})$, $\varphi \in C^\infty([0,\infty) \times \mathbb{R}^d;\mathbb{R}^d)$ with $\nabla \cdot \varphi = 0$ and $\psi \in C^\infty([0,\infty) \times \mathbb{R}^d; S_0^d)$.

We now state the results on the system (2.1)-(2.2). The first result is the the existence of global weak solutions in two and three dimensions.

Theorem 2.2 (Existence of weak solutions)**.** *Let the initial data (c_0, u_0, Q_0) satisfy (2.3)-(2.5). Then there exist some positive constants c^* and ξ_*, such that if $\underline{c} > c^*$ or $|\xi| < \xi_*$ there exists a weak solution (c, u, Q) to the system (2.1)-(2.2) for $d = 2, 3$.*

The second result states that, in two-dimensional case, the system (2.1) has solutions with higher regularity, subject to the initial data with higher regularity.

Theorem 2.3 (Higher regularity)**.** *For $s > 0$ and the initial data $(c_0, u_0, Q_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2)$, there exist some positive constants c^* and ξ_*, such that if $\underline{c} > c^*$ or $|\xi| < \xi_*$, there exists a global solution (c, u, Q) of the system (2.1)-(2.2) satisfying*

$$
\begin{aligned}
&c - \hat{c} \in L^\infty_{loc}(\mathbb{R}_+; H^s(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2)),\\
&u \in L^\infty_{loc}(\mathbb{R}_+; H^s(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2)),\\
&Q \in L^\infty_{loc}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}_+; H^{s+2}(\mathbb{R}^2)),
\end{aligned}
$$

and

$$
\|c(t,\cdot) - \hat{c}\|^2_{H^s(\mathbb{R}^2)} + \|u(t,\cdot)\|^2_{H^s(\mathbb{R}^2)} + \|\nabla Q(t,\cdot)\|^2_{H^s(\mathbb{R}^2)} \le C,
$$

where the constant C depends only on $t, D_0, b, c_*, \mu, \Gamma, \lambda$ and c_0, u_0, Q_0. Moreover, if $\xi = 0$ the increase in time of the above norms can be made only doubly exponential.

The third result is the weak-strong uniqueness, stated in the following theorem.

Theorem 2.4 (Weak-strong uniqueness)**.** *For $s > 0$ and the initial data $(c_0, u_0, Q_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2)$, there exist some positive constants c^* and ξ_*, such that, if $\underline{c} > c^*$ or $|\xi| < \xi_*$, the weak solution of the system (2.1)-(2.2) in Theorem 2.2 and the strong solution in Theorem 2.3 are equal.*

Next we shall give an outline of the proof for the Theorem 2.2. We will construct the approximation system and obtain the uniform estimates to prove the global existence of the weak solutions for the system (2.1)-(2.2). The proof of Theorem 2.2 will be divided into three steps following [8]. Firstly, we construct regularized solutions $(c^{\varepsilon,n}, u^{\varepsilon,n}, Q^{\varepsilon,(n)})$ to the following approximate system (2.21) by the classical Friedrich's scheme. Secondly, we obtain some a priori estimates and pass to the limit as $n \to \infty$ to achieve the weak solution $(c^\varepsilon, u^\varepsilon, Q^\varepsilon)$ to the modified system (2.25). Finally, we finish the proof of Theorem 2.2 by passing to the limit as $\varepsilon \to 0$ in the modified system (2.25) with the uniform bounds.

Step 1. construction of the approximation system. Let $\chi \in C_0^\infty$ be a radial positive function such that $\int_{\mathbb{R}^d} \chi(y)dy = 1$ and we define R_ε as the convolution operator with kernel $\varepsilon^{-d}\chi(\varepsilon^{-1}\cdot)$. We also define $\mathcal{P}$ as the Leray projector onto divergence-free vector fields, *i.e.*,

$$
\mathcal{P}: \; L^2 \to H = \{v \in L^2 : \; \nabla \cdot v = 0\}
$$

and the mollifying operator J_n $(n = 1, 2, \cdots)$ as

$$
\mathcal{F}(J_n f)(\xi) := 1_{[2^{-n}, 2^n]}(|\xi|)\mathcal{F}(f)(\xi),
$$

where $\mathcal{F}$ is the Fourier transform. For any fixed $\varepsilon > 0$ and $n > 0$, by using the convolution operator R_ε, the Leray projector $\mathcal{P}$ and the mollifying operator $J_n,\ n = 1, 2, \cdots$, adding some regularizing terms to the system (2.1), we construct the following approximation system by the classical Friedrich's scheme, which is similar to [8] (from now on, we denote the solution $(c^{\varepsilon,n}, u^{\varepsilon,n}, Q^{\varepsilon,(n)})$ by $(c^n, u^n, Q^{(n)})$ for simplicity):

$$\partial_t c^n + J_n((R_\varepsilon u\cdot\nabla)c^n) = D_0\Delta c^n, \tag{2.21a}$$

$$\left.\begin{aligned}
&\partial_t u^n + J_n((R_\varepsilon u\cdot\nabla)u^n) - \mu\Delta u^n + \mathcal{P}\nabla\cdot J_nR_\varepsilon(\nabla Q^{(n)}\odot\nabla Q^{(n)})\\
=&-\varepsilon\mathcal{P}J_nR_\varepsilon\big(\nabla c^n(R_\varepsilon u^n\cdot\nabla c^n)\big)\\
&-\varepsilon\mathcal{P}J_nR_\varepsilon\big(\nabla Q^{(n)}(R_\varepsilon u^n\cdot\nabla Q^{(n)})|R_\varepsilon u^n\cdot\nabla Q^{(n)}|\big)\\
&+\varepsilon\mathcal{P}\nabla\cdot J_nR_\varepsilon(\nabla R_\varepsilon u^n|\nabla R_\varepsilon u^n|^2)\\
&-\xi\mathcal{P}\nabla\cdot J_nR_\varepsilon\big((Q^{(n)}+\frac{1}{d}I_d)J_n\bar{H}^{(n)} + J_n\bar{H}^{(n)}(Q^{(n)}+\frac{1}{d}I_d)\big)\\
&+2\xi\mathcal{P}\nabla\cdot J_nR_\varepsilon\big((Q^{(n)}+\frac{1}{d}I_d)tr(Q^{(n)}J_n\bar{H}^{(n)})\big)\\
&+\mathcal{P}\nabla\cdot J_nR_\varepsilon\big(Q^{(n)}\Delta Q^{(n)} - \Delta Q^{(n)}Q^{(n)} + \sigma_*(c^n)^2Q^{(n)}\big)\\
&-\lambda\mathcal{P}\nabla\cdot J_nR_\varepsilon(|Q^{(n)}|J_n\bar{H}^{(n)}),
\end{aligned}\right\}\tag{2.21b}$$

$$\left.\begin{aligned}
&\partial_t Q^{(n)} + J_n((R_\varepsilon u\cdot\nabla)Q^{(n)}) - J_n(R_\varepsilon\Omega^nQ^{(n)} - Q^{(n)}R_\varepsilon\Omega^n)\\
=&\ \xi J_n\big(R_\varepsilon D^n(Q^{(n)}+\frac{1}{d}I_d) + (Q^{(n)}+\frac{1}{d}I_d)R_\varepsilon D^n\big)\\
&-2\xi J_n\big((Q^{(n)}+\frac{1}{d}I_d)tr(Q^{(n)}\nabla R_\varepsilon u^n)\big)\\
&+\lambda J_n(|Q^{(n)}|R_\varepsilon D^n) + \Gamma J_n\bar{H}^{(n)},
\end{aligned}\right\}\tag{2.21c}$$

$$(c^n, u^n, Q^{(n)})|_{t=0} = (J_nR_\varepsilon c_0, J_nR_\varepsilon u_0, J_nR_\varepsilon Q_0), \tag{2.21d}$$

where

$$\bar{H}^{(n)} = \Delta Q^{(n)} - \frac{c^n - c_*}{2}Q^{(n)} + b\big((Q^{(n)})^2 - \frac{tr((Q^{(n)})^2)}{d}I_d\big) - c_*Q^{(n)}tr((Q^{(n)})^2),$$

and $(c^n, u^n, Q^{(n)}) \in C^1([0, T_n); \cap_{k=1}^{\infty}H^k)$. Then we have the existence and uniqueness of the approximate system, which can be regarded as a system of ordinary differential equations in L^2 as well as the conditions $(\mathcal{P}J_n)^2 = \mathcal{P}J_n,\ (J_n)^2 = J_n$. We also have that $Q^{(n)} = (Q^{(n)})^\top$ $a.e.$ in $[0, T_n] \times \mathbb{R}^d$.

Step 2. the limit as $n \to \infty$. Before we pass to the limit as $n \to \infty$, we need to derive the following a priori estimates for the system (2.21):

$$\begin{aligned}
&\frac{d}{dt}\big(E^n(t) + M\|Q^{(n)}\|_{L^2}^2\big) + D_0\|\nabla c^n\|_{L^2}^2 + \frac{\mu}{4}\|\nabla u^n\|_{L^2}^2 + \frac{c_*^2\Gamma}{2}\|J_n(Q^{(n)}|Q^{(n)}|^2)\|_{L^2}^2\\
&\quad+\frac{\Gamma}{2}\|\Delta Q^{(n)}\|_{L^2}^2 + \frac{\varepsilon}{2}\|R_\varepsilon u^n\cdot\nabla c^n\|_{L^2}^2 + \frac{\varepsilon}{2}\|R_\varepsilon u^n\cdot\nabla Q^{(n)}\|_{L^3}^3 + \frac{\varepsilon}{4}\|\nabla R_\varepsilon u^n\|_{L^4}^4\\
&\le C(\|u^n\|_{L^2}^2 + \|Q^{(n)}\|_{L^2}^2 + \|\nabla Q^{(n)}\|_{L^2}^2 + \|Q^{(n)}\|_{L^4}^4),
\end{aligned}\tag{2.22}$$

where $M = M(\underline{c}, b, c_*) > 0$ is a suitable large constant satisfying

$$0 \le \frac{M}{2}|Q^{(n)}|^2 + \frac{c_*}{8}|Q^{(n)}|^4 \le (M + \frac{c - c_*}{4})|Q^{(n)}|^2 - \frac{b}{3}tr((Q^{(n)})^3) + \frac{c_*}{4}|Q^{(n)}|^4, \tag{2.23}$$

and $C = C(c_*, \bar{c}, b, \Gamma, D_0, \sigma_*, \mu, \lambda, \varepsilon, M)$ is independent of n.

From the above estimates, by Gronwall's inequality, we can conclude the a priori bounds of the solution $(c^n, u^n, Q^{(n)})$ of the system (2.21) for any $T < \infty$ as the following:

$$\begin{aligned}
&\sup_n \|c^n - \hat{c}\|_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)} \le C,\ 0 < \underline{c} \le c^n \le \bar{c} < \infty,\\
&\sup_n \|u^n\|_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)} \le C,\\
&\sup_n \|Q^{(n)}\|_{L^\infty(0,T;H^1\cap L^4)\cap L^2(0,T;H^2)} + \sup_n \|J_n(Q^{(n)}|Q^{(n)}|^2)\|_{L^2(0,T;L^2)} \le C,\\
&\sup_n \|R_\varepsilon u^n \cdot \nabla c^n\|_{L^2(0,T;L^2)} \le C,\\
&\sup_n \|R_\varepsilon u^n \cdot \nabla Q^{(n)}\|_{L^3(0,T;L^3)} \le C,\\
&\sup_n \|\nabla R_\varepsilon u^n\|_{L^4(0,T;L^4)} \le C,
\end{aligned} \tag{2.24}$$

where C is independent of n.

Because of the symmetry properties of the Q-tensor $Q^{(n)}$, it remains to show $tr(Q^{(n)}) = 0$ to prove that $Q^{(n)} \in S_0^d$. We take the trace on both sides of the equation (2.21c) to obtain the following initial value problem:

$$\begin{cases}
\partial_t tr(Q^{(n)}) + J_n(R_\varepsilon u^n \cdot \nabla tr(Q^{(n)})) = \Gamma J_n \Delta tr(Q^{(n)}) - \Gamma J_n(\dfrac{c^n - c_*}{2} tr(Q^{(n)}))\\
\qquad - c_* \Gamma J_n(tr(Q^{(n)}) tr((Q^{(n)})^2)) - 2\xi J_n(tr(Q^{(n)}) tr(Q^{(n)} \nabla R_\varepsilon u^n)),\\
tr(Q^{(n)})|_{t=0} = J_n R_\varepsilon tr(Q_0) = 0,
\end{cases}$$

where we have used the fact that $(Q^{(n)})^\top = Q^{(n)}$, and $tr(Q^{(n)}\Omega^{(n)}) = tr(\Omega^{(n)} Q^{(n)})$. Then one has the following estimate:

$$\frac{d}{dt}\|tr(Q^{(n)})\|_{L^2}^2 \le (C_1 + C_2 \|\nabla R_\varepsilon u^n\|_{L^4}^{\frac{4}{4-d}}) \|tr(Q^{(n)})\|_{L^2}^2.$$

Hence, we conclude that $tr(Q^{(n)}) = 0$ by using Gronwall's inequality and the tracelessness of the initial condition of $Q^{(n)}$.

From the uniform energy estimate (2.22) with respect to n, we can conclude that $T_n = \infty$. Moreover, from the system (2.21) and the estimates (2.24), we can compute the bounds for $\partial_t(c^n, u^n, Q^{(n)})$ in some $L^1(0,T;H^{-N})$ space for sufficiently large N. Therefore, by the classical Aubin-Lions compactness lemma, as $n \to \infty$, we have

$$\begin{cases}
Q^{(n)} \rightharpoonup Q \text{ in } L^2(0,T;H^2),\ Q^{(n)} \to Q \text{ in } L^2(0,T;H^{2-\delta}_{loc}) \text{ for any } \delta > 0,\\
Q^{(n)}(t) \rightharpoonup Q(t) \text{ in } H^1,\ \text{for any } t > 0,\\
Q^{(n)} \rightharpoonup Q \text{ in } L^p(0,T;H^1),\ Q^{(n)} \to Q \text{ in } L^p(0,T;H^{1-\delta}_{loc}) \text{ for any } \delta > 0,\ p \ge 2,\\
c^n - \hat{c} \rightharpoonup c - \hat{c} \text{ in } L^2(0,T;H^1),\ c^n - \hat{c} \to c - \hat{c} \text{ in } L^2(0,T;H^{1-\delta}_{loc}),\ \text{for any } \delta > 0,\\
c^n(t) - \hat{c} \rightharpoonup c(t) - \hat{c} \text{ in } L^2,\ \text{for any } t > 0,\\
u^n \rightharpoonup u \text{ in } L^2(0,T;H^1),\ u^n \to u \text{ in } L^2(0,T;H^{1-\delta}_{loc}),\ \text{for any } \delta > 0,\\
u^n(t) \rightharpoonup u(t) \text{ in } L^2,\ \text{for any } t > 0.
\end{cases}$$

Hence we can pass to the limit as $n \to \infty$ to obtain a weak solution $(c^\varepsilon, u^\varepsilon, Q^\varepsilon)$ for the following modified system (from now on, we denote the solution $(c^\varepsilon, u^\varepsilon, Q^\varepsilon)$ by

(c, u, Q) for simplicity):

$$\begin{cases} \partial_t c + (R_\varepsilon u \cdot \nabla) c = D_0 \Delta c, \\ \partial_t u + \mathcal{P}(R_\varepsilon u \cdot \nabla) u - \mu \Delta u + \mathcal{P} \nabla \cdot R_\varepsilon (\nabla Q \odot \nabla Q) \\ = -\varepsilon \mathcal{P} R_\varepsilon \big(\nabla c (R_\varepsilon u \cdot \nabla c) \big) - \varepsilon \mathcal{P} R_\varepsilon \big(\nabla Q (R_\varepsilon u \cdot \nabla Q) | R_\varepsilon u \cdot \nabla Q | \big) \\ \qquad + \varepsilon \mathcal{P} \nabla \cdot R_\varepsilon \big(\nabla R_\varepsilon u | \nabla R_\varepsilon u |^2 \big) \\ \qquad - \xi \mathcal{P} \nabla \cdot R_\varepsilon \big((Q + \frac{1}{d} I_d) H + H (Q + \frac{1}{d} I_d) - 2 (Q + \frac{1}{d} I_d) tr(QH) \big) \\ \qquad + \mathcal{P} \nabla \cdot R_\varepsilon \big(Q \Delta Q - \Delta Q Q + \sigma_* c^2 Q \big) - \lambda \mathcal{P} \nabla \cdot R_\varepsilon (|Q| H), \\ \partial_t Q + (R_\varepsilon u \cdot \nabla) Q - (R_\varepsilon \Omega Q - Q R_\varepsilon \Omega) - \lambda |Q| R_\varepsilon D \\ = \xi \big(R_\varepsilon D (Q + \frac{1}{d} I_d) + (Q + \frac{1}{d} I_d) R_\varepsilon D - 2 (Q + \frac{1}{d} I_d) tr(Q \nabla R_\varepsilon u) \big) + \Gamma H, \\ (c, u, Q)|_{t=0} = (R_\varepsilon c_0, R_\varepsilon u_0, R_\varepsilon Q_0). \end{cases} \tag{2.25}$$

From the above analysis, we have the following proposition about the existence of weak solutions of the modified system (2.25).

Proposition 3. *Assume the initial data (c_0, u_0, Q_0) satisfies (2.3)-(2.5). Then there exists a weak solution $(c^\varepsilon, u^\varepsilon, Q^\varepsilon)$ to the modified system (2.25), for $d = 2, 3$, satisfying*

$$\begin{aligned} & c^\varepsilon - \hat{c} \in L^\infty_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^d)) \cap L^2_{loc}(\mathbb{R}_+; H^1(\mathbb{R}^d)), \\ & u^\varepsilon \in L^\infty_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^d)) \cap L^2_{loc}(\mathbb{R}_+; H^1(\mathbb{R}^d)), \\ & Q^\varepsilon \in L^\infty_{loc}(\mathbb{R}_+; H^1(\mathbb{R}^d)) \cap L^2_{loc}(\mathbb{R}_+; H^2(\mathbb{R}^d)). \end{aligned}$$

Step 3. the limit as $\varepsilon \to 0$. Now we will complete the proof of Theorem 2.2 by passing to the limit in system (2.25) as $\varepsilon \to 0$. Through the similar procedure in Step 2 and the classical Aubin-Lions lemma, we can conclude that there exists (c, u, Q) satisfying

$$\begin{cases} c - \hat{c} \in L^\infty_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^d)) \cap L^2_{loc}(\mathbb{R}_+; H^1(\mathbb{R}^d)), \\ u \in L^\infty_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^d)) \cap L^2_{loc}(\mathbb{R}_+; H^1(\mathbb{R}^d)), \\ Q \in L^\infty_{loc}(\mathbb{R}_+; H^1(\mathbb{R}^d)) \cap L^2_{loc}(\mathbb{R}_+; H^2(\mathbb{R}^d)), \end{cases}$$

such that, as $\varepsilon \to 0$, we have

$$\begin{cases} Q^\varepsilon \rightharpoonup Q \text{ in } L^2(0,T; H^2), \ Q^\varepsilon \to Q \text{ in } L^2(0,T; H^{2-\delta}_{loc}) \text{ for any } \delta > 0, \\ Q^\varepsilon(t) \rightharpoonup Q(t) \text{ in } H^1, \text{ for any } t > 0, \\ Q^\varepsilon \rightharpoonup Q \text{ in } L^p(0,T; H^1), \ Q^\varepsilon \to Q \text{ in } L^p(0,T; H^{1-\delta}_{loc}) \text{ for any } \delta > 0, \ p \geq 2, \\ c^\varepsilon - \hat{c} \rightharpoonup c - \hat{c} \text{ in } L^2(0,T; H^1), \ c^\varepsilon - \hat{c} \to c - \hat{c} \text{ in } L^2(0,T; H^{1-\delta}_{loc}), \text{ for any } \delta > 0, \\ c^\varepsilon(t) - \hat{c} \rightharpoonup c(t) - \hat{c} \text{ in } L^2, \text{ for any } t > 0, \\ u^\varepsilon \rightharpoonup u \text{ in } L^2(0,T; H^1), \ u^\varepsilon \to u \text{ in } L^2(0,T; H^{1-\delta}_{loc}), \text{ for any } \delta > 0, \\ u^\varepsilon(t) \rightharpoonup u(t) \text{ in } L^2, \text{ for any } t > 0. \end{cases}$$

Hence, by passing to the limit in system (2.25) as $\varepsilon \to 0$, we can obtain a weak solution (c, u, Q) of the system (2.1) satisfying (2.18)-(2.20). The proof of Theorem 2.2 is completed.

3. **Compressible flows: Weak solutions in a bounded domain.**

In this section we shall present the result taken directly from [9] on the existence of weak solutions for the compressible flows of active liquid crystals (c.f. [23, 25]) in a bounded domain $\mathcal{O} \subset \mathbb{R}^3$:

$$\begin{cases} \partial_t c + (u \cdot \nabla)c = D_0 \Delta c, \\ \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) - \mu \Delta u - (\nu + \mu)\nabla \mathrm{div}\, u = \nabla \cdot \tau + \nabla \cdot \sigma, \\ \partial_t Q + (u \cdot \nabla)Q + Q\Omega - \Omega Q = \Gamma H[Q, c], \end{cases} \tag{3.1}$$

where ρ is the density of the fluid, $u \in \mathbb{R}^3$ is the flow velocity, $P = \kappa \rho^\gamma$ denotes the pressure with adiabatic constant $\gamma > 1$ and constant $\kappa > 0$, and all the other variables are the same as in the system (1.1) of incompressible flows. Consider the initial-boundary value problem of (3.1) under the following initial condition:

$$(c, \rho, \rho u, Q)|_{t=0} = (c_0, \rho_0, m_0, Q_0)(x) \qquad \text{for } x \in \mathcal{O} \subset \mathbb{R}^3, \tag{3.2}$$

with

$$\begin{aligned} &c_0 \in H^1(\mathcal{O}), \qquad 0 < \underline{c} \le c_0 \le \bar{c} < \infty, \\ &Q_0 \in H^1(\mathcal{O}), \qquad Q_0 \in S_0^3 \quad a.e. \text{ in } \mathcal{O}, \end{aligned}$$

and the following boundary condition:

$$\nabla c \cdot \vec{n}|_{\partial\mathcal{O}} = 0, \qquad u|_{\partial\mathcal{O}} = 0, \qquad \nabla Q \cdot \vec{n}|_{\partial\mathcal{O}} = 0, \tag{3.3}$$

with the compatibility condition:

$$\rho_0 \in L^\gamma(\mathcal{O}), \quad \rho_0 \ge 0; \quad m_0 \in L^1(\mathcal{O}), \quad m_0 = 0 \text{ if } \rho_0 = 0; \quad \frac{|m_0|^2}{\rho_0} \in L^1(\mathcal{O}), \tag{3.4}$$

where $\vec{n}$ is the unit outward normal on $\partial\mathcal{O}$.

We shall construct the global finite-energy weak solution to (3.1)–(3.4) in the following sense:

Definition 3.1. For any $T > 0$, (c, ρ, u, Q) is a finite-energy weak solution of the initial-boundary value problem (3.1)–(3.4) if the following conditions are satisfied:

(i) $c > 0$, $c \in L^\infty(0, T; L^2(\mathcal{O})) \cap L^2(0, T; H^1(\mathcal{O}))$; $\rho \ge 0$, $\rho \in L^\infty(0, T; L^\gamma(\mathcal{O}))$; $u \in L^2(0, T; H_0^1(\mathcal{O}))$, $Q \in L^\infty(0, T; H^1(\mathcal{O})) \cap L^2(0, T; H^2(\mathcal{O}))$, and $Q \in S_0^3$ *a.e.* in $\mathcal{O}_T = [0, T] \times \mathcal{O}$.

(ii) The system (3.1) is valid in $\mathcal{D}'(\mathcal{O}_T)$. Moreover, the continuity equation is valid in $\mathcal{D}'(0, T; \mathbb{R}^3)$, if (ρ, u) are extended to be zero on $\mathbb{R}^3 \setminus \mathcal{O}$.

(iii) Energy $E(t)$ is locally integrable on $(0, T)$ and satisfies the energy inequality:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} E(t) + \frac{D_0}{2}\|\nabla c\|_{L^2}^2 + \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + (\nu + \mu)\|\mathrm{div}\, u\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2 + \frac{c_*^2 \Gamma}{2}\|Q\|_{L^6}^6 \\ &\quad \le C\big(\|u\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2 + \|Q\|_{L^2}^2 + \|Q\|_{L^4}^4\big) \qquad \text{in } \mathcal{D}'(0, T), \end{aligned}$$

where

$$E(t) := \int_{\mathcal{O}} \Big(\frac{1}{2}|c|^2 + \frac{1}{2}\rho|u|^2 + \frac{\rho^\gamma}{\gamma - 1} + \frac{1}{2}|Q|^2 + \frac{1}{2}|\nabla Q|^2 + \frac{c_*}{4}|Q|^4\Big)\mathrm{d}x.$$

(iv) The continuity equation is satisfied in the sense of renormalized solutions; that is, for any function $g \in C^1(\mathbb{R})$ with the property:

$$g'(z) \equiv 0 \qquad \text{for all } z \ge M \text{ for a sufficiently large constant } M,$$

the following holds

$$\partial_t g(\rho) + \mathrm{div}\,(g(\rho)u) + \big(g'(\rho)\rho - g(\rho)\big)\mathrm{div}\,u = 0 \qquad \text{in } \mathcal{D}'(0,T).$$

The main result on the existence of solutions can be stated as follows.

Theorem 3.2 ([9]). *Let $\gamma > \frac{3}{2}$ and $\mathcal{O} \subset \mathbb{R}^3$ be a bounded domain of the class $C^{2+\tau}$ for some $\tau > 0$. Assume that the initial data $(c_0, \rho_0, m_0, Q_0)(x)$ satisfies the compatibility condition (3.4). Then, for any $T > 0$, the initial-boundary value problem (3.1)−(3.3) admits a finite-energy weak solution $(c, \rho, u, Q)(t, x)$ on $\mathcal{O}_T$.*

Theorem 3.2 can be proved by the Faedo-Galerkin's method [61] with three levels of approximations in [9], as well as the weak convergence argument in the spirit of [17, 16]. The first level of approximation is to add the artificial pressure in order to increase the integrability of the density. The second level approximation is to add the artificial viscosity in the continuity equation for the higher regularity of the density. The third level approximation is the Faedo-Galerkin's approximation from the finite-dimensional to infinite-dimensional space. This approach was used to construct weak solutions to the compressible Q-tensor system in [64]. New difficulties arise from the concentration equation and its coupling with both the fluid and Q-tensor equations, and thus new techniques are needed.

The approximate problem for (3.1)−(3.3) is the following: for fixed $\delta > 0$ and $\varepsilon > 0$,

$$\partial_t c + (u\cdot\nabla)c = D_0\Delta c, \tag{3.5}$$

$$\partial_t \rho + \nabla\cdot(\rho u) = \varepsilon\Delta\rho, \tag{3.6}$$

$$\begin{aligned}\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla\big(\rho^\gamma + \delta\nabla\rho^\beta\big) + \varepsilon(\nabla\rho\cdot\nabla)u \\ = \mu\Delta u + (\nu+\mu)\nabla\mathrm{div}\,u + \nabla\cdot(\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q\odot\nabla Q) \\ + \nabla\cdot(Q\Delta Q - \Delta Q Q) + \sigma_*\nabla\cdot(c^2 Q),\end{aligned} \tag{3.7}$$

$$\partial_t Q + (u\cdot\nabla)Q + Q\Omega - \Omega Q = \Gamma H[Q, c], \tag{3.8}$$

subject to the modified initial condition:

$$c|_{t=0} = c_0 \in H^1(\mathcal{O}), \qquad 0 < \underline{c} \le c_0(x) \le \bar{c}, \tag{3.9}$$

$$\rho|_{t=0} = \rho_0 \in C^3(\bar{\mathcal{O}}), \qquad 0 < \underline{\varrho} \le \rho_0(x) \le \bar{\varrho}, \tag{3.10}$$

$$(\rho u)|_{t=0} = m_0(x) \in C^2(\bar{\mathcal{O}}), \tag{3.11}$$

$$Q|_{t=0} = Q_0(x) \in H^1(\mathcal{O}), \qquad Q_0 \in S_0^3 \ \ a.e. \text{ in } \mathcal{O}, \tag{3.12}$$

and the boundary condition:

$$\nabla c\cdot\vec{n}|_{\partial\mathcal{O}} = 0, \qquad \nabla\rho\cdot\vec{n}|_{\partial\mathcal{O}} = 0, \tag{3.13}$$

$$u|_{\partial\mathcal{O}} = 0, \qquad \frac{\partial Q}{\partial\vec{n}}|_{\partial\mathcal{O}} = 0, \tag{3.14}$$

where $\underline{c}$, $\bar{c}$, $\underline{\varrho}$, and $\bar{\varrho}$ are positive constants, and $\vec{n}$ is the unit outward normal on $\partial\mathcal{O}$. The classical Faedo-Galerkin method can be used to construct a solution (c_n, ρ_n, u_n, Q_n) of the initial-boundary value problem (3.5)–(3.14). We know that the family of smooth eigenfunctions $\{\psi_n\}_{n=1}^\infty$ of the Laplacian operator form an orthogonal basis of $H_0^1(\mathcal{O})$. Define a sequence of finite-dimensional spaces: $X_n = \mathrm{span}\{\psi_1, \psi_2, \cdots, \psi_n\}$ for $n \in \mathbb{N}$. First, it is shown that there is a unique solution $(\rho[u_n], c[u_n], Q[u_n])$ to the initial-boundary value problem (3.5)–(3.6) and (3.8) for any given $u_n \in C(0, T; X_n)$. Then, substituting $(\rho[u_n], c[u_n], Q[u_n])$ into the variational formulation of the momentum equation, one can obtain a local solution (ρ_n, c_n, u_n, Q_n) of the approximation system (3.5)–(3.14) on the time interval $[0, T_n]$ by using the contraction map theorem. Using some cancellation property of the system and some maximum principle argument, one can extend the local solution to a global solution by the uniform energy estimates of the system with respect to n and also obtain the existence of the first level approximation solution as $n \to \infty$. The next step is to let the artificial viscosity $\varepsilon \to 0$ to recover the original continuity equation, for which the convergence of the effective viscous flux sequence is applied to deal with the lack of regularity of the density sequence to retrieve the compactness results of the solutions. The last step is to pass to the limit of the vanishing artificial pressure sequence ($\delta \to 0$) to obtain a finite-energy weak solution of the original problem, including the vacuum case. The details of the proof can be found in [9].

4. **Other results and open problems.** *Inhomogeneous incompressible flow of active liquid crystals:* In [39] the incompressible flow of the active liquid crystals with inhomogeneous density was discussed in the Q-tensor framework. Global solutions are constructed by the Faedo-Galerkin method for the initial-boundary value problem. Two levels of approximations are used and the weak convergence is obtained through compactness estimates to obtain the existence of global weak solutions in a two or three dimensional bounded domain.

Incompressible limit: In [67] the connection between the compressible flows and the incompressible flows of liquid crystals was studied when the Mach number is low. The convergence of the weak solutions of the compressible model to the incompressible model is proved as the Mach number approaches zero based on the uniform estimates of the weak solutions and various compactness criteria.

Stochastic analysis: In [53, 54] the martingale solution and strong solution were obtained for the stochastic active liquid crystal system. The three-dimensional compressible flow of active nematic liquid crystals with the random force was studied in [53] and the global martingale solution via an approximation scheme was constructed. The strong solution to the compressible stochastic Navier-Stokes equations coupled with the Q-tensor system of active liquid crystals was established in [54] through the energy method up to a stopping time. The incompressible limit was also proved for the stochastic flows of active liquid crystals in [67].

Open problems: Many fundamental mathematical problems remain open for the active hydrodynamics, for example, the global existence of smooth solutions with large data and uniqueness for the compressible flows, large-time behavior of strong and weak solutions, singular limits of solutions. The stochastic analysis is widely open for the active hydrodynamics, for example, global strong solutions, qualitative behavior of solutions, noise effect on the stability of solutions, and so on.

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