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WEIGHTED FOURTH ORDER ELLIPTIC PROBLEMS IN THE UNIT BALL

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Dedicated to Professor Norman Dancer on the occasion of his 75th birthday

Abstract. Existence and uniqueness of positive radial solutions of some weighted fourth order elliptic Navier and Dirichlet problems in the unit ball B are studied. The weights can be singular at $x = 0 \in B$. Existence of positive radial solutions of the problems is obtained via variational methods in the weighted Sobolev spaces. To obtain the uniqueness results, we need to know exactly the asymptotic behavior of the solutions at the singular point $x = 0$.

1. **Introduction.** We study structure of positive radial solutions of the weighted fourth order elliptic problems

$$(N) \qquad \begin{cases} \Delta(|x|^\alpha \Delta u) = |x|^l u^p & \text{in } B, \\ u = \Delta u = 0, & \text{on } \partial B \end{cases}$$

and

$$(D) \qquad \begin{cases} \Delta(|x|^\alpha \Delta u) = |x|^l u^p & \text{in } B, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial B, \end{cases}$$

where $N \geq 5$, B is the unit ball of $\mathbb{R}^N$, $\alpha, l \in \mathbb{R}$, $1 \leq p < p_s$ with

$$p_s := \frac{N' + 4 + 2\tau}{N' - 4} \ (> 1) \tag{1.1}$$

and

$$2N > N' := N + \alpha > 4, \quad \tau := l - \alpha > -4. \tag{1.2}$$

It is known from (1.2) that $\alpha \in (4 - N, N)$.

The weighted second order equations of the form

$$-\text{div}(|x|^\alpha \nabla u) = |x|^l u^p \quad \text{in } \Omega \subseteq \mathbb{R}^N, \ (N \geq 3)$$

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with $p \geq 1$ and

$$N' := N + \alpha > 2, \quad \tau := l - \alpha > -2 \tag{1.3}$$

have been studied by many authors, see, for example, [1, 3, 4, 5, 6, 7, 8, 9, 11, 14, 15, 16, 17] and the references therein. It is known from [12] that under the basic assumption (1.3), the problem

$$\begin{cases} -\text{div}(|x|^\alpha \nabla u) = |x|^l u^p & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases}$$

where $1 \leq p < p^s := \frac{N'+2+2\tau}{N'-2}$ and B is the unit ball of $\mathbb{R}^N$, admits a positive radial solution $u \in C^2(B\backslash\{0\}) \cap C^0(\overline{B})$. It is known from [10] that u is the unique positive radial solution in $C^2(B\backslash\{0\}) \cap C^0(\overline{B})$.

In a recent paper [13], under the basic assumption (1.2), the authors established the embedding: $D_0^{2,\alpha}(\Omega) \hookrightarrow L_l^\kappa(\Omega)$, where Ω is a bounded domain of class C^2 and $0 \in \Omega$, $D_0^{2,\alpha}(\Omega)$ is the completion of $C_c^\infty(\Omega\backslash\{0\})$ with respect to the Hilbertian norm

$$\|\phi\|_{2,\alpha}^2 = \int_\Omega |x|^\alpha (\Delta\phi)^2 dx$$

and $L_l^\kappa(\Omega)$ is the space of functions ψ such that $|x|^{\frac{l}{\kappa}}|\psi| \in L^\kappa(\Omega)$ with the norm

$$\|\psi\|_{L_l^\kappa(\Omega)} = \Big(\int_\Omega |x|^l |\psi|^\kappa dx \Big)^{\frac{1}{\kappa}}.$$

It is known from [13] that

$$D_0^{2,\alpha}(\Omega) \hookrightarrow L_l^\kappa(\Omega), \quad \kappa \in [1, p_s + 1] \tag{1.4}$$

provided $\alpha \geq \frac{(N-4)}{4}\tau$ and this embedding is compact for $\kappa \in [1, p_s + 1)$;

$$D_0^{2,\alpha}(\Omega) \hookrightarrow L_l^\kappa(\Omega), \quad \kappa \in [1, p_* + 1], \tag{1.5}$$

provided $\alpha < \frac{(N-4)}{4}\tau$ and this embedding is compact for $\kappa \in [1, p_* + 1)$ and $p_* = \frac{N+4}{N-4}$. The embedding in (1.4) with $\kappa = p_s + 1$ and the embedding in (1.5) with $\kappa = p_* + 1$ are corresponding to the generalized Caffarelli-Kohn-Nirenberg inequalities for the fourth order case. Some special results related to the Caffarelli-Kohn-Nirenberg inequality in the fourth order case can be found in [2].

If $\Omega = B$, we can obtain

$$D_{0,rad}^{2,\alpha}(B) \hookrightarrow L_l^\kappa(B), \quad \kappa \in [1, p_s + 1] \tag{1.6}$$

provided that the basic assumption (1.2) holds only and this embedding is compact for $\kappa \in [1, p_s + 1)$, where $D_{0,rad}^{2,\alpha}(B) = \{\phi \in D_0^{2,\alpha}(B) : \ \phi(x) = \phi(|x|)\}$.

Set

$$\mathcal{O}_N = D_{rad}^{2,\alpha}(B) \cap D_{0,rad}^{1,\alpha}(B), \quad \mathcal{O}_D = D_{0,rad}^{2,\alpha}(B).$$

Using the embedding given in (1.6) and the variational method as in [13] in the spaces $\mathcal{O}_N$ and $\mathcal{O}_D$ respectively, we can obtain nontrivial nonnegative radial solutions $u_N \in \mathcal{O}_N$ and $u_D \in \mathcal{O}_D$ for problems (N) and (D) with $1 \leq p < p_s$ respectively.

Let

$$v_N(r) := -r^\alpha \Delta u_N(r), \quad v_D(r) := -r^\alpha \Delta u_D(r).$$

In this paper, we first study the regularity of (u_D, v_D) and (u_N, v_N) at $x = 0$. We will see that in some cases, $x = 0$ can be a removable singular point of (u_D, v_D) and (u_N, v_N) and in some other cases, $x = 0$ is a nonremovable singular point of (u_N, v_N) and (u_D, v_D). Then, we establish the asymptotic expansions of (u_N, v_N)

and (u_D, v_D) at $x = 0$ and obtain more detailed regularity information of (u_N, v_N) and (u_D, v_D) at $x = 0$. Using these asymptotic behaviors at $x = 0$, we also obtain uniqueness results for problems (N) and (D). The main results of this paper are the following theorems.

Theorem 1.1. *Let $N \geq 5$, $l, \alpha \in \mathbb{R}$ satisfy (1.2), $u_N \in D^{2,\alpha}_{rad}(B) \cap D^{1,\alpha}_{0,rad}(B)$ and $u_D \in D^{2,\alpha}_{0,rad}(B)$ be nontrivial nonnegative solutions of (N) and (D) respectively. Then,*

(1) If $\alpha \in (4-N, 2)$ and $l > -2$, we have

$$u_N, u_D \in C^4(B\backslash\{0\}) \cap C^0(\overline{B}), \quad v_N, v_D \in C^2(B\backslash\{0\}) \cap C^0(\overline{B}).$$

(2) If $\alpha \in (4-N, 2)$ and $l \in (\alpha - 4, -2]$, we have

$$u_N, u_D \in C^4(B\backslash\{0\}) \cap C^0(\overline{B}), \quad v_N, v_D \in C^2(B\backslash\{0\}).$$

(3) If $\alpha \in [2, N)$ and $\tau > -4$, we have

$$u_N, u_D \in C^4(B\backslash\{0\}), \quad v_N, v_D \in C^2(B\backslash\{0\}) \cap C^0(\overline{B}).$$

Theorem 1.2. *Assume that the assumptions of Theorem 1.1 hold and*

$$q = \frac{(N' + 4 + 2\tau) - (N' - 4)p}{2} > 0.$$

Then,

(1) If $\alpha \in (4-N, 2)$ and $l > -2$, we have

$$q + \frac{(N'-4)}{2}p > \frac{(N-\alpha)}{2} > \frac{(N'-4)}{2}$$

and, for r near 0,

$$u_N(r) = a_1 + a_2 r^{2-\alpha} + a_3 r^{[q+(p-1)\frac{(N'-4)}{2}]} + o(r^{[q+(p-1)\frac{(N'-4)}{2}]}),$$
$$u_D(r) = b_1 + b_2 r^{2-\alpha} + b_3 r^{[q+(p-1)\frac{(N'-4)}{2}]} + o(r^{[q+(p-1)\frac{(N'-4)}{2}]}),$$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are constants and $a_1 = u_N(0) > 0$, $b_1 = u_D(0) > 0$,

$$v_N(r) = c_1 + c_2 r^{[q+\frac{(N'-4)}{2}p - \frac{(N-\alpha)}{2}]} + c_3 r^{[q+\frac{(N'-4)}{2}(p-1)]} + o(r^{[q+\frac{(N'-4)}{2}(p-1)]}),$$
$$v_D(r) = d_1 + d_2 r^{[q+\frac{(N'-4)}{2}p - \frac{(N-\alpha)}{2}]} + d_3 r^{[q+\frac{(N'-4)}{2}(p-1)]} + o(r^{[q+\frac{(N'-4)}{2}(p-1)]}),$$

where $c_1, c_2, c_3, d_1, d_2, d_3$ are constants and $c_1 = v_N(0) > 0$, $d_1 = v_D(0) > 0$.

(2) If $\alpha \in (4-N, 2)$ and $l = -2$, we have

$$q + \frac{(N'-4)}{2}p = \frac{(N-\alpha)}{2} > \frac{(N'-4)}{2}$$

and, for r near 0,

$$u_N(r) = e_1 + e_2 r^{2-\alpha} \ln r + e_3 r^{2-\alpha} + o(r^{2-\alpha}),$$
$$u_D(r) = f_1 + f_2 r^{2-\alpha} \ln r + f_3 r^{2-\alpha} + o(r^{2-\alpha}),$$

where $e_1, e_2, e_3, f_1, f_2, f_3$ are constants and $e_1 = u_N(0) > 0$, $e_2 \neq 0$; $f_1 = u_D(0) > 0$, $f_2 \neq 0$,

$$v_N(r) = -g_1 \ln r + g_2 + g_3 r^{2-\alpha} \ln r + o(r^{2-\alpha} \ln r),$$
$$v_D(r) = -h_1 \ln r + h_2 + h_3 r^{2-\alpha} \ln r + o(r^{2-\alpha} \ln r),$$

where $g_1, g_2, g_3, h_1, h_2, h_3$ are constants and $g_1 = \lim_{r \to 0} \frac{v_N(r)}{-\ln r} > 0$, $h_1 = \lim_{r \to 0} \frac{v_D(r)}{-\ln r} > 0$.

(3) If $\alpha \in (4-N, 2)$ and $l \in (\alpha-4, -2)$, we have

$$\frac{(N'-4)}{2} < q + \frac{(N'-4)}{2}p < \frac{(N-\alpha)}{2}.$$

Moreover,

$$2q + \frac{(N'-4)}{2}(2p-1) > \frac{(N-\alpha)}{2}$$

provided $\tau > -3 - \frac{\alpha}{2}$,

$$2q + \frac{(N'-4)}{2}(2p-1) = \frac{(N-\alpha)}{2}$$

provided $\tau = -3 - \frac{\alpha}{2}$,

$$2q + \frac{(N'-4)}{2}(2p-1) < \frac{(N-\alpha)}{2}$$

provided $\tau \in (-4, -3 - \frac{\alpha}{2})$.

Then, for $\tau > -3 - \frac{\alpha}{2}$ and r near 0,

$$u_N(r) = i_1 + i_2 r^{q+\frac{(N'-4)}{2}(p-1)} + i_3 r^{2-\alpha} + o(r^{2-\alpha}),$$

$$u_D(r) = j_1 + j_2 r^{q+\frac{(N'-4)}{2}(p-1)} + j_3 r^{2-\alpha} + o(r^{2-\alpha}),$$

where $i_1, i_2, i_3, j_1, j_2, j_3$ are constants and $i_1 = u_N(0) > 0$, $j_1 = u_D(0) > 0$,

$$v_N(r) = k_1 r^{[q+\frac{(N'-4)}{2}p - \frac{(N-\alpha)}{2}]} + k_2 + k_3 r^{[2q+\frac{(N'-4)}{2}(2p-1)-\frac{(N-\alpha)}{2}]} + o(r^{[2q+\frac{(N'-4)}{2}(2p-1)-\frac{(N-\alpha)}{2}]}),$$

$$v_D(r) = l_1 r^{[q+\frac{(N'-4)}{2}p - \frac{(N-\alpha)}{2}]} + l_2 + l_3 r^{[2q+\frac{(N'-4)}{2}(2p-1)-\frac{(N-\alpha)}{2}]} + o(r^{[2q+\frac{(N'-4)}{2}(2p-1)-\frac{(N-\alpha)}{2}]}),$$

where $k_1, k_2, k_3, l_1, l_2, l_3$ are constants and $k_1 > 0$, $l_1 > 0$.

For $\tau = -3 - \frac{\alpha}{2}$ and r near 0,

$$u_N(r) = i_1 + i_2 r^{q+\frac{(N'-4)}{2}(p-1)} + i_3 r^{2-\alpha}\ln r + O(r^{2-\alpha}),$$

$$u_D(r) = j_1 + j_2 r^{q+\frac{(N'-4)}{2}(p-1)} + j_3 r^{2-\alpha}\ln r + O(r^{2-\alpha}),$$

where $i_1, i_2, i_3, j_1, j_2, j_3$ are constants and $i_1 = u_N(0) > 0$, $j_1 = u_D(0) > 0$,

$$v_N(r) = k_1 r^{[q+\frac{(N'-4)}{2}p - \frac{(N-\alpha)}{2}]} + k_2 \ln r + k_3 + o(1),$$

$$v_D(r) = l_1 r^{[q+\frac{(N'-4)}{2}p - \frac{(N-\alpha)}{2}]} + l_2 \ln r + l_3 + o(1),$$

where $k_1, k_2, k_3, l_1, l_2, l_3$ are constants and $k_1 > 0$, $l_1 > 0$.

For $\tau \in (-4, -3 - \frac{\alpha}{2})$ and r near 0,

$$u_N(r) = i_1 + i_2 r^{q+\frac{(N'-4)}{2}(p-1)} + i_3 r^{2[q+\frac{(N'-4)}{2}(p-1)]} + o(r^{2[q+\frac{(N'-4)}{2}(p-1)]}),$$

$$u_D(r) = j_1 + j_2 r^{q+\frac{(N'-4)}{2}(p-1)} + j_3 r^{2[q+\frac{(N'-4)}{2}(p-1)]} + o(r^{2[q+\frac{(N'-4)}{2}(p-1)]}),$$

where $i_1, i_2, i_3, j_1, j_2, j_3$ are constants and $i_1 = u_N(0) > 0$, $j_1 = u_D(0) > 0$.

For any $n \ge 3$ and $-\frac{(\alpha+4n-2)}{n} < \tau < -\frac{(\alpha+4n-6)}{(n-1)}$,

$$(n-1)q + \frac{(N'-4)}{2}[(n-1)p-(n-2)] < \frac{(N-\alpha)}{2} < nq + \frac{(N'-4)}{2}[np-(n-1)],$$

$$v_N(r) = k_1 r^{\mu_1} + k_2 r^{\mu_2} + \ldots + k_{n-1} r^{\mu_{n-1}} + k_n + k_{n+1} r^{\mu_{n+1}} + o(r^{\mu_{n+1}}),$$

where $k_1, k_2, \ldots, k_n, k_{n+1}$ *are constants with* $k_1 > 0$*,*

$$\mu_1 = q + \frac{(N'-4)}{2}p - \frac{(N-\alpha)}{2}, \;\; \mu_2 = 2q + \frac{(N'-4)}{2}(2p-1) - \frac{(N-\alpha)}{2}, \ldots,$$

$$\mu_{n-1} = (n-1)q + \frac{(N'-4)}{2}[(n-1)p-(n-2)] - \frac{(N-\alpha)}{2}, \;\; \mu_n = 0,$$

$$\mu_{n+1} = nq + \frac{(N'-4)}{2}[np-(n-1)] - \frac{(N-\alpha)}{2},$$

$$v_D(r) = l_1 r^{\nu_1} + l_2 r^{\nu_2} + \ldots + l_{n-1} r^{\nu_{n-1}} + l_n + l_{n+1} r^{\nu_{n+1}} + o(r^{\nu_{n+1}}),$$

where $l_1, l_2, \ldots, l_n, l_{n+1}$ *are constants with* $l_1 > 0$ *and*

$$\nu_1 = q + \frac{(N'-4)}{2}p - \frac{(N-\alpha)}{2}, \;\; \nu_2 = 2q + \frac{(N'-4)}{2}(2p-1) - \frac{(N-\alpha)}{2}, \ldots,$$

$$\nu_{n-1} = (n-1)q + \frac{(N'-4)}{2}[(n-1)p-(n-2)] - \frac{(N-\alpha)}{2}, \;\; \nu_n = 0,$$

$$\nu_{n+1} = nq + \frac{(N'-4)}{2}[np-(n-1)] - \frac{(N-\alpha)}{2}.$$

For $\tau = -\frac{(\alpha+4n-2)}{n}$*,*

$$nq + \frac{(N'-4)}{2}[np-(n-1)] = \frac{(N-\alpha)}{2},$$

$$v_N(r) = k_1 r^{\mu_1} + k_2 r^{\mu_2} + \ldots + k_{n-1} r^{\mu_{n-1}} + k_n \ln r + k_{n+1} + k_{n+2} r^{\mu_{n+1}} + o(r^{\mu_{n+1}}),$$

where $k_1, k_2, \ldots, k_n, k_{n+1}, k_{n+2}$ *are constants with* $k_1 > 0$*,*

$$\mu_1 = q + \frac{(N'-4)}{2}p - \frac{(N-\alpha)}{2}, \;\; \mu_2 = 2q + \frac{(N'-4)}{2}(2p-1) - \frac{(N-\alpha)}{2}, \ldots,$$

$$\mu_{n-1} = (n-1)q + \frac{(N'-4)}{2}[(n-1)p-(n-2)] - \frac{(N-\alpha)}{2}, \;\; \mu_n = 0,$$

$$\mu_{n+1} = (n+1)q + \frac{(N'-4)}{2}[(n+1)p-n] - \frac{(N-\alpha)}{2},$$

$$v_D(r) = l_1 r^{\nu_1} + l_2 r^{\nu_2} + \ldots + l_{n-1} r^{\nu_{n-1}} + l_n \ln r + l_{n+1} + l_{n+2} r^{\nu_{n+1}} + o(r^{\nu_{n+1}}),$$

where $l_1, l_2, \ldots, l_n, l_{n+1}, l_{n+2}$ *are constants with* $l_1 > 0$ *and*

$$\nu_1 = q + \frac{(N'-4)}{2}p - \frac{(N-\alpha)}{2}, \;\; \nu_2 = 2q + \frac{(N'-4)}{2}(2p-1) - \frac{(N-\alpha)}{2}, \ldots,$$

$$\nu_{n-1} = (n-1)q + \frac{(N'-4)}{2}[(n-1)p-(n-2)] - \frac{(N-\alpha)}{2}, \;\; \nu_n = 0,$$

$$\nu_{n+1} = (n+1)q + \frac{(N'-4)}{2}[(n+1)p-n] - \frac{(N-\alpha)}{2}.$$

(4) If $\alpha = 2$ *and* $\tau > -4$*, we have*

$$\frac{(N'-4)}{2} = \frac{(N-\alpha)}{2} = \frac{(N-2)}{2}.$$

Then, for r *near 0,*

$$u_N(r) = -m_1 \ln r + m_2 + m_3 |\ln r|^p r^{[q+\frac{(N-2)}{2}(p-1)]} + o(|\ln r|^p r^{[q+\frac{(N-2)}{2}(p-1)]}),$$

$$u_D(r) = -n_1 \ln r + n_2 + n_3 |\ln r|^p r^{[q+\frac{(N-2)}{2}(p-1)]} + o(|\ln r|^p r^{[q+\frac{(N-2)}{2}(p-1)]}),$$

where $m_1, m_2, m_3, n_1, n_2, n_3$ *are constants and* $m_1 > 0$*,* $n_1 > 0$*,*

$$v_N(r) = p_1 + p_2 |\ln r|^p r^{[q+\frac{(N-2)}{2}(p-1)]} + o(|\ln r|^p r^{[q+\frac{(N-2)}{2}(p-1)]}),$$

$$v_D(r) = q_1 + q_2|\ln r|^p r^{[q+\frac{(N-2)}{2}(p-1)]} + o(|\ln r|^p r^{[q+\frac{(N-2)}{2}(p-1)]}),$$

where p_1, p_2, q_1, q_2 *are constants and* $p_1 = v_N(0) > 0$, $q_1 = v_D(0) > 0$.

(5) If $\alpha \in (2, N)$ *and* $\tau > -4$, *we have*

$$\frac{(N-\alpha)}{2} < \frac{(N'-4)}{2}.$$

Then, for $1 \le p < \frac{(4+\tau)}{\alpha-2}$ *and* r *near 0,*

$$u_N(r) = \kappa_1 r^{2-\alpha} + \kappa_2 + \kappa_3 r^{[q+\frac{(N-\alpha)}{2}p-\frac{(N'-4)}{2}]} + o(r^{[q+\frac{(N-\alpha)}{2}p-\frac{(N'-4)}{2}]}),$$

$$u_D(r) = \theta_1 r^{2-\alpha} + \theta_2 + \theta_3 r^{[q+\frac{(N-\alpha)}{2}p-\frac{(N'-4)}{2}]} + o(r^{[q+\frac{(N-\alpha)}{2}p-\frac{(N'-4)}{2}]}),$$

where $\kappa_1, \kappa_2, \kappa_3, \theta_1, \theta_2, \theta_3$ *are constants and* $\kappa_1 > 0$, $\theta_1 > 0$,

$$v_N(r) = \tau_1 + \tau_2 r^{[q+\frac{(N-\alpha)}{2}(p-1)]} + \tau_3 r^{[q+\frac{(N-\alpha)}{2}(p-1)+\alpha-2]} + o(r^{[q+\frac{(N-\alpha)}{2}(p-1)+\alpha-2]}),$$

$$v_D(r) = \sigma_1 + \sigma_2 r^{[q+\frac{(N-\alpha)}{2}(p-1)]} + \sigma_3 r^{[q+\frac{(N-\alpha)}{2}(p-1)+\alpha-2]} + o(r^{[q+\frac{(N-\alpha)}{2}(p-1)+\alpha-2]}),$$

where $\tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2, \sigma_3$ *are constants and* $\tau_1 = v_N(0) > 0$, $\sigma_1 = v_D(0) > 0$.

For $p = \frac{(4+\tau)}{\alpha-2}$,

$$u_N(r) = \kappa_1 r^{2-\alpha} + \kappa_2 \ln r + \kappa_3 + o(1),$$
$$u_D(r) = \theta_1 r^{2-\alpha} + \theta_2 \ln r + \theta_3 + o(1),$$

where $\kappa_1, \kappa_2, \kappa_3, \theta_1, \theta_2, \theta_3$ *are constants and* $\kappa_1 > 0$, $\theta_1 > 0$,

$$v_N(r) = \tau_1 + \tau_2 r^{\alpha-2} + \tau_3 r^{[q+\frac{(N-\alpha)}{2}(p-1)+\alpha-2]} \ln r + o(r^{[q+\frac{(N-\alpha)}{2}(p-1)+\alpha-2]} \ln r),$$

$$v_D(r) = \sigma_1 + \sigma_2 r^{\alpha-2} + \sigma_3 r^{[q+\frac{(N-\alpha)}{2}(p-1)+\alpha-2]} \ln r + o(r^{[q+\frac{(N-\alpha)}{2}(p-1)+\alpha-2]} \ln r),$$

where $\tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2, \sigma_3$ *are constants and* $\tau_1 = v_N(0) > 0$, $\sigma_1 = v_D(0) > 0$.

For $\frac{(4+\tau)}{\alpha-2} < p < \frac{6+\alpha+2\tau}{2(\alpha-2)}$,

$$u_N(r) = \kappa_1 r^{2-\alpha} + \kappa_2 r^{[q+\frac{(N-\alpha)}{2}p-\frac{(N'-4)}{2}]} + \kappa_3 + o(1),$$

$$u_D(r) = \theta_1 r^{2-\alpha} + \theta_2 r^{[q+\frac{(N-\alpha)}{2}p-\frac{(N'-4)}{2}]} + \theta_3 + o(1),$$

where $\kappa_1, \kappa_2, \kappa_3, \theta_1, \theta_2, \theta_3$ *are constants and* $\kappa_1 > 0$, $\theta_1 > 0$,

$$v_N(r) = \tau_1 + \tau_2 r^{[q+\frac{(N-\alpha)}{2}(p-1)]} + \tau_3 r^{2[q+\frac{(N-\alpha)}{2}(p-1)]} + o(r^{2[q+\frac{(N-\alpha)}{2}(p-1)]}),$$

$$v_D(r) = \sigma_1 + \sigma_2 r^{[q+\frac{(N-\alpha)}{2}(p-1)]} + \sigma_3 r^{2[q+\frac{(N-\alpha)}{2}(p-1)]} + o(r^{2[q+\frac{(N-\alpha)}{2}(p-1)]}),$$

where $\tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2, \sigma_3$ *are constants and* $\tau_1 = v_N(0) > 0$, $\sigma_1 = v_D(0) > 0$.

For $p = \frac{6+\alpha+2\tau}{2(\alpha-2)}$,

$$u_N(r) = \kappa_1 r^{2-\alpha} + \kappa_2 r^{[q+\frac{(N-\alpha)}{2}p-\frac{(N'-4)}{2}]} + \kappa_3 \ln r + \kappa_4 + o(1),$$

$$u_D(r) = \theta_1 r^{2-\alpha} + \theta_2 r^{[q+\frac{(N-\alpha)}{2}p-\frac{(N'-4)}{2}]} + \theta_3 \ln r + \theta_4 + o(1),$$

where $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \theta_1, \theta_2, \theta_3, \theta_4$ *are constants and* $\kappa_1 > 0$, $\theta_1 > 0$,

$$v_N(r) = \tau_1 + \tau_2 r^{[q+\frac{(N-\alpha)}{2}(p-1)]} + \tau_3 r^{\alpha-2} + o(r^{\alpha-2}),$$

$$v_D(r) = \sigma_1 + \sigma_2 r^{[q+\frac{(N-\alpha)}{2}(p-1)]} + \sigma_3 r^{\alpha-2} + o(r^{\alpha-2}),$$

where $\tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2, \sigma_3$ *are constants and* $\tau_1 = v_N(0) > 0$, $\sigma_1 = v_D(0) > 0$.

For $p > \frac{6+\alpha+2\tau}{2(\alpha-2)}$,

$$u_N(r) = \kappa_1 r^{\mu_1} + \kappa_2 r^{\mu_2} + \ldots + \kappa_{n-1} r^{\mu_{n-1}} + \kappa_n + \kappa_{n+1} r^{\mu_{n+1}} + o(r^{\mu_{n+1}}),$$

or

$$u_N(r) = \kappa_1 r^{\mu_1} + \kappa_2 r^{\mu_2} + \ldots + \kappa_{n-1} r^{\mu_{n-1}} + \kappa_n \ln r + \kappa_{n+1} + o(1),$$

$$u_D(r) = \theta_1 r^{\nu_1} + \theta_2 r^{\nu_2} + \ldots + \theta_{n-1} r^{\nu_{n-1}} + \theta_n + \theta_{n+1} r^{\nu_{n+1}} + o(r^{\nu_{n+1}}),$$

or

$$u_D(r) = \theta_1 r^{\nu_1} + \theta_2 r^{\nu_2} + \ldots + \theta_{n-1} r^{\nu_{n-1}} + \theta_n \ln r + \theta_{n+1} + o(1),$$

where $\mu_1, \mu_2, \ldots, \mu_n, \mu_{n+1}$ *and* $\nu_1, \nu_2, \ldots, \nu_n, \nu_{n+1}$ *are constants depending on* N, p, l, α,

$$\mu_1 = 2 - \alpha, \quad \mu_2 = q + \frac{(N-\alpha)}{2} p - \frac{(N'-4)}{2}, \quad \ldots$$

with $n \geq 4$,

$$\mu_1 < \mu_2 < \ldots < \mu_{n-1} < 0 (= \mu_n) < \mu_{n+1},$$

$$\nu_1 = 2 - \alpha, \quad \nu_2 = q + \frac{(N-\alpha)}{2} p - \frac{(N'-4)}{2}, \quad \ldots$$

with

$$\nu_1 < \nu_2 < \ldots < \nu_{n-1} < 0 (= \nu_n) < \nu_{n+1},$$

$\kappa_1, \kappa_2, \ldots, \kappa_n, \kappa_{n+1}, \theta_1, \theta_2, \ldots, \theta_n, \theta_{n+1}$ *are constants and* $\kappa_1 > 0$, $\theta_1 > 0$,

$$v_N(r) = \tau_1 + \tau_2 r^{[q + \frac{(N-\alpha)}{2}(p-1)]} + \tau_3 r^{2[q + \frac{(N-\alpha)}{2}(p-1)]} + o(r^{2[q + \frac{(N-\alpha)}{2}(p-1)]}),$$

$$v_D(r) = \sigma_1 + \sigma_2 r^{[q + \frac{(N-\alpha)}{2}(p-1)]} + \sigma_3 r^{2[q + \frac{(N-\alpha)}{2}(p-1)]} + o(r^{2[q + \frac{(N-\alpha)}{2}(p-1)]}),$$

where $\tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2, \sigma_3$ *are constants and* $\tau_1 = v_N(0) > 0$, $\sigma_1 = v_D(0) > 0$.

Remark 1. It is seen from Theorem 1.2 that, for $\alpha \in (4-N, 2)$ and $l > -2$, (u_N, v_N) and (u_D, v_D) are Hölder continuous at $x = 0$. For $\alpha \in (4-N, 2)$ and $l \in (\alpha - 4, -2]$, u_N and u_D are Hölder continuous at $x = 0$, but $x = 0$ is a nonremovable singular point of v_N and v_D. For $\alpha \in [2, N)$ and $\tau > -4$, v_N and v_D are Hölder continuous at $x = 0$, but $x = 0$ is a nonremovable singular point of u_N and u_D.

Theorem 1.3. *Let* $N \geq 5$, $\alpha \in (4-N, 2)$ *and* $\tau = l - \alpha > -4$, $1 \leq p < p_s$. *Then, problem (N) admits a unique positive radial solution* $u_N \in C^4(B\backslash\{0\}) \cap C^0(\overline{B})$ *and problem (D) admits a unique positive radial solution* $u_D \in C^4(B\backslash\{0\}) \cap C^0(\overline{B})$.

Combining Theorems 1.1, 1.2 and 1.3, we have the following corollary.

Corollary 1. *Let* $N \geq 5$, $\sigma > -4$ *and* $1 \leq p < \frac{N+4+2\sigma}{N-4}$. *Then, problem*

$$\begin{cases} \Delta^2 u = |x|^\sigma u^p & \text{in } B, \\ u = \Delta u = 0 & \text{on } \partial B \end{cases}$$

admits a unique positive radial solution

$$u_N \in \begin{cases} C^4(B\backslash\{0\}) \cap C^0(\overline{B}) & \text{for } -4 < \sigma \leq -2, \\ C^4(B\backslash\{0\}) \cap C^{2,2+\sigma}(B) \cap C^2(\overline{B}) & \text{for } -2 < \sigma < -1, \\ C^4(B\backslash\{0\}) \cap C^3(B) \cap C^2(\overline{B}) & \text{for } \sigma = -1, \\ C^4(B\backslash\{0\}) \cap C^{3,1+\sigma}(B) \cap C^2(\overline{B}) & \text{for } -1 < \sigma < 0, \\ C^4(B) \cap C^2(\overline{B}) & \text{for } \sigma \geq 0, \end{cases}$$

and problem

$$\begin{cases} \Delta^2 u = |x|^\sigma u^p & \text{in } B, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B \end{cases}$$

admits a unique positive radial solution

$$u_D \in \begin{cases} C^4(B\backslash\{0\}) \cap C^0(\overline{B}) & \textit{for } -4 < \sigma \le -2, \\ C^4(B\backslash\{0\}) \cap C^{2,2+\sigma}(B) \cap C^2(\overline{B}) & \textit{for } -2 < \sigma < -1, \\ C^4(B\backslash\{0\}) \cap C^3(B) \cap C^2(\overline{B}) & \textit{for } \sigma = -1, \\ C^4(B\backslash\{0\}) \cap C^{3,1+\sigma}(B) \cap C^2(\overline{B}) & \textit{for } -1 < \sigma < 0, \\ C^4(B) \cap C^2(\overline{B}) & \textit{for } \sigma \ge 0. \end{cases}$$

We only show that u_N and u_D are the unique positive radial solutions of problems (N) and (D) in $C^4(B\backslash\{0\}) \cap C^0(\overline{B})$ respectively for $\alpha \in (4-N, 2)$ and $\tau > -4$ in this paper, since we know from Theorem 1.1 that, for $\alpha \in (4-N,2)$ and $l > -2$, $u_N \in C^4(B\backslash\{0\}) \cap C^0(\overline{B})$ and $v_N \in C^2(B\backslash\{0\}) \cap C^0(\overline{B})$; $u_D \in C^4(B\backslash\{0\}) \cap C^0(\overline{B})$ and $v_D \in C^2(B\backslash\{0\}) \cap C^0(\overline{B})$. For $\alpha \in (4-N,2)$ and $l \in (\alpha-4,-2]$, $u_N \in C^4(B\backslash\{0\}) \cap C^0(\overline{B})$ and $v_N \in C^2(B\backslash\{0\})$; $u_D \in C^4(B\backslash\{0\}) \cap C^0(\overline{B})$ and $v_D \in C^2(B\backslash\{0\})$. Moreover, $r=0$ is a nonremovable singular point of v_N and v_D. We think that u_N and u_D are also the unique positive radial solutions of problems (N) and (D) in $D^{2,\alpha}_{rad}(B) \cap D^{1,\alpha}_{0,rad}(B)$ and $D^{2,\alpha}_{0,rad}(B)$ respectively for $\alpha \in [2,N)$ and $\tau > -4$. We can use the main idea in the proof of the case $\alpha \in (4-N,2)$ and $l \in (\alpha-4,-2]$ to obtain the uniqueness result for this case. To obtain the uniqueness result for $\alpha \in [2,N)$ and $\tau > -4$, we need to know the asymptotic expansion up to arbitrary order of u_N or u_D at $r=0$ and show that if u_1 and u_2 are two positive radial solutions of problem (N) or problem (D), then $(u_1-u_2)(r)$ can not oscillate near $r=0$, i.e. there are sufficiently small positive ϵ_1, ϵ_2, ϵ_3 such that one of the three cases occurs: (i) $(u_1-u_2)(r) > 0$ for $r \in (0,\epsilon_1)$, (ii) $(u_1-u_2)(r) \equiv 0$ for $r \in (0,\epsilon_2)$, (iii) $(u_1-u_2)(r) < 0$ for $r \in (0,\epsilon_3)$. Different from the case $\alpha \in (4-N,2)$ and $l \in (\alpha-4,-2]$, there is an extra difficulty for the case $\alpha \in [2,N)$ and $\tau > -4$. The extra difficulty arises from the nonlinearities u_N^p and u_D^p, since it is known from Theorem 1.2 that $r=0$ is a nonremovable singular point of u_N and u_D. Therefore, we only need a little more detailed asymptotic expansion of v_N or v_D at $r=0$ for the case $\alpha \in (4-N,2)$ and $l \in (\alpha-4,-2]$ (note that the parameters N, l, α play important roles in the asymptotic expansions), but we need the asymptotic expansion up to arbitrary order of u_N or u_D at $r=0$ for the case $\alpha \in [2,N)$ and $\tau > -4$ (note that the parameters N, l, α, p play important roles in the asymptotic expansions). Of course, we can obtain the asymptotic behaviors up to arbitrary orders of u_N and u_D at $r=0$ by arguments similar to those in the proof of Theorem 1.2. We leave the details to the readers.

In section 2, we obtain some basic properties of (u_N, v_N) and (u_D, v_D). In section 3, we obtain the regularity of (u_N, v_N) and (u_D, v_D) at $x=0$ given in Theorem 1.1. In section 4, we establish more delicate asymptotic expansions of (u_N, v_N) and (u_D, v_D) at $x=0$ for different cases of the parameters l, α, p. In the final section, we obtain the uniqueness result for positive radial solutions of problems (N) and (D) and give the proof of Theorem 1.3.

2. Preliminaries.

In this section, we obtain some basic properties of $(u_N(r), v_N(r))$ and $(u_D(r), v_D(r))$, where

$$v_N(r) := -r^{\alpha}\Delta u_N(r), \quad v_D(r) := -r^{\alpha}\Delta u_D(r).$$

For $(u(r),v(r)) = (u_N(r),v_N(r))$ or $(u_D(r),v_D(r))$, (u,v) satisfies the system of equations

$$\begin{cases} -\Delta u = r^{-\alpha} v & \text{in } B, \\ -\Delta v = r^l u^p & \text{in } B. \end{cases} \tag{2.1}$$

The facts $u \in D_0^{2,\alpha}(B)$ and the embedding $D_0^{2,\alpha}(B) \hookrightarrow L_l^{\kappa}(B)$ for $1 \le \kappa \le p_s + 1$ imply that

$$\int_B |x|^l u^p dx < \infty, \quad \int_B |x|^{-\alpha} v^2 dx < \infty \tag{2.2}$$

and

$$\int_B |x|^{-\alpha}|v| dx \le \Big(\int_B |x|^{-\alpha} v^2 dx\Big)^{\frac{1}{2}} \Big(\int_B |x|^{-\alpha} dx\Big)^{\frac{1}{2}} < \infty, \tag{2.3}$$

since $\alpha < N$ and $p < p_s$. Therefore, for any sufficiently small $\epsilon > 0$,

$$-\int_{B_\epsilon} |x|^l u^p dx = \int_{B_\epsilon} \Delta v dx = C\epsilon^{N-1} v'(\epsilon), \;\; -\int_{B_\epsilon} |x|^{-\alpha} v dx = \int_{B_\epsilon} \Delta u dx = C\epsilon^{N-1} u'(\epsilon).$$

These, (2.2) and (2.3) imply that

$$r^{N-1}u'(r) \to 0, \quad r^{N-1}v'(r) \to 0 \quad \text{as } r \to 0. \tag{2.4}$$

On the other hand, if we write the system (2.1) to the form:

$$\begin{cases} -(r^{N-1}u'(r))' = r^{N-\alpha-1}v(r) & \text{in } (0,1), \\ -(r^{N-1}v')' = r^{N'+\tau-1}u^p & \text{in } (0,1), \end{cases} \tag{2.5}$$

we obtain from the second limit in (2.4) and the second equation of (2.5) that

$$v'(r) < 0 \quad \text{for } r \in (0,1]. \tag{2.6}$$

Moreover, if $(u,v) = (u_N, v_N)$, we easily see $v(1) = 0$ and (2.6) implies

$$v > 0 \quad \text{for } r \in [0,1). \tag{2.7}$$

It follows from (2.7), the first limit in (2.4) and the first equation of (2.5) that

$$u'(r) < 0 \quad \text{for } r \in (0,1]. \tag{2.8}$$

Since $u(1) = 0$, we obtain

$$u(r) > 0 \quad \text{for } r \in [0,1). \tag{2.9}$$

Therefore, for $(u,v) = (u_N, v_N)$, we have

$$u(r) > 0, \;\; v(r) > 0 \quad \text{for } r \in [0,1) \tag{2.10}$$

and

$$u'(r) < 0, \;\; v'(r) < 0 \quad \text{for } r \in (0,1]. \tag{2.11}$$

We now consider the case of $(u,v) = (u_D, v_D)$. We obtain from the second limit in (2.4) and the second equation of (2.5) that

$$v'(r) < 0 \quad \text{for } r \in (0,1]. \tag{2.12}$$

Since $\int_B |x|^{-\alpha} v dx = Cu'(1) = 0$, we easily know that $v(r)$ changes sign in $(0,1)$. We also know from (2.12) that there is a unique $r_D \in (0,1)$ such that $v(r_D) = 0$, $v(r) > 0$ for $r \in (0, r_D)$ and $v(r) < 0$ for $r \in (r_D, 1]$. We now claim

$$u'(r) < 0 \quad \text{for } r \in (0,1). \tag{2.13}$$

We obtain from the first limit in (2.4), the first equation of (2.5) and the fact $v(r) > 0$ for $r \in (0, r_D)$ that

$$u'(r) < 0 \quad \text{for } r \in (0, r_D].$$

Suppose our claim (2.13) does not hold, we see that there is $\tilde{r} \in (r_D, 1)$ such that $u'(\tilde{r}) = 0$. Since $u'(1) = 0$, integrating the first equation of (2.5) in $(\tilde{r}, 1)$, we obtain

$$\int_{\tilde{r}}^1 s^{N-\alpha-1} v(s) ds = 0.$$

This contradicts $v(r) < 0$ for $r \in (r_D, 1]$. This implies that our claim (2.13) holds. Since $u(1) = 0$, we obtain from (2.13) that $u(r) > 0$ for $r \in [0, 1)$. Therefore, for $(u, v) = (u_D, v_D)$, we have

$$u(r) > 0 \text{ for } r \in [0,1), \quad v(r) > 0 \text{ for } r \in [0, r_D), \quad v(r) < 0 \text{ for } r \in (r_D, 1] \quad (2.14)$$

and

$$u'(r) < 0, \ \ v'(r) < 0 \quad \text{for } r \in (0,1). \quad (2.15)$$

Arguments as the above also imply that for both $(u, v) = (u_N, v_N)$ and $(u, v) = (u_D, v_D)$,

$$u \in C^2(0,1), \quad v \in C^2(0,1).$$

3. **Regularity of (u_N, v_N) and (u_D, v_D) at $x = 0$: Proof of Theorem 1.1.** In this section, we consider the regularity of (u_N, v_N) and (u_D, v_D) at $x = 0$ and present the proof of Theorem 1.1. Since the parameters α and l play important roles in the regularity of u_N, v_N, u_D, v_D at $x = 0$, we will consider several cases of (α, l) separately. We will see that when $\alpha \in (4 - N, 2)$ and $l > -2$, $x = 0$ is a removable singular point of both (u_N, v_N) and (u_D, v_D); when $\alpha \in (4 - N, 2)$ and $l \in (\alpha - 4, -2]$, $x = 0$ is a removable singular point of u_N and u_D, but it is a nonremovable singular point of v_N and v_D; when $\alpha \in [2, N)$ and $\tau > -4$, $x = 0$ is a nonremovable singular point of u_N and u_D, but it is a removable singular point of v_N and v_D.

For $(u(r), v(r)) = (u_N(r), v_N(r))$ or $(u_D(r), v_D(r))$, we have

$$\int_0^1 r^{N'-1-2\alpha} v^2(r) dr < \infty. \quad (3.1)$$

We also obtain from the embedding $D_0^{2,\alpha}(B) \hookrightarrow L_l^{p_s+1}(B)$ that

$$\int_0^1 r^{N'+\tau-1} u^{\frac{2(N'+\tau)}{N'-4}}(r) dr < \infty. \quad (3.2)$$

This implies that, for r near 0,

$$u(r) = o(r^{-\frac{N'-4}{2}}), \quad v(r) = o(r^{-\frac{N-\alpha}{2}}). \quad (3.3)$$

We only show $(3.3)_1$ by using (3.2). The proof of $(3.3)_2$ is similar to that of $(3.3)_1$ by using (3.1). We easily see from (3.2) that

$$\int_0^r s^{N'+\tau-1} u^{\frac{2(N'+\tau)}{N'-4}}(s) ds = o(1) \quad \text{for } r \text{ near } 0.$$

Using the fact that $u(r)$ is decreasing, we have that

$$\int_0^r s^{N'+\tau-1} u^{\frac{2(N'+\tau)}{N'-4}}(s) ds \geq \frac{r^{N'+\tau}}{N'+\tau} u^{\frac{2(N'+\tau)}{N'-4}}(r).$$

This implies

$$r^{N'+\tau} u^{\frac{2(N'+\tau)}{N'-4}}(r) = o(1)$$

and

$$u(r) = o(r^{-\frac{N'-4}{2}}) \quad \text{as } r \to 0.$$

Proof of Theorem 1.1. We only consider (u_N, v_N), another case can be studied similarly. For convenience, we use (u, v) to replace (u_N, v_N). Let

$$w(t) = r^{\frac{N'-4}{2}} u(r), \quad z(t) = r^{\frac{N-\alpha}{2}} v(r), \quad t = -\ln r.$$

A simple calculation and (3.3) imply that $(w(t), z(t))$ satisfies the problem

$$\begin{cases} w_{tt} + (\alpha-2)w_t - \frac{(N-\alpha)(N'-4)}{4} w + z = 0 & t \in (0,\infty), \\ z_{tt} + (2-\alpha)z_t - \frac{(N-\alpha)(N'-4)}{4} z + e^{-qt} w^p = 0 & t \in (0,\infty), \\ w(t), z(t) \to 0 & \text{as } t \to \infty, \end{cases} \tag{3.4}$$

where

$$q = \frac{[N'+4+2\tau] - [N'-4]p}{2} > 0 \quad \text{for } 1 \le p < p_s.$$

The characteristic equations of the first and the second equations of the system (3.4) are

$$\lambda^2 + (\alpha-2)\lambda - \frac{(N-\alpha)(N'-4)}{4} = 0 \tag{3.5}$$

and

$$\sigma^2 + (2-\alpha)\sigma - \frac{(N-\alpha)(N'-4)}{4} = 0 \tag{3.6}$$

respectively. The two roots of (3.5) and (3.6) are

$$\lambda_1 = -\frac{(N'-4)}{2}, \quad \lambda_2 = \frac{(N-\alpha)}{2}$$

and

$$\sigma_1 = -\frac{(N-\alpha)}{2}, \quad \sigma_2 = \frac{(N'-4)}{2}$$

respectively. Therefore, it follows from the second equation of (3.4) that, for any $T \gg 1$ and $t > T$,

$$\begin{aligned} z(t) &= A_1 e^{-\frac{(N-\alpha)}{2}t} + A_2 e^{\frac{(N'-4)}{2}t} + B_1 \int_T^t e^{-\frac{(N-\alpha)}{2}(t-s)}[-e^{-qs}(w(s))^p]ds \\ &\quad - B_2 \int_t^\infty e^{\frac{(N'-4)}{2}(t-s)}[-e^{-qs}(w(s))^p]ds \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} w(t) &= C_1 e^{-\frac{(N'-4)}{2}t} + C_2 e^{\frac{(N-\alpha)}{2}t} + D_1 \int_T^t e^{-\frac{(N'-4)}{2}(t-s)}[-z(s)]ds \\ &\quad - D_2 \int_t^\infty e^{\frac{(N-\alpha)}{2}(t-s)}[-z(s)]ds, \end{aligned} \tag{3.8}$$

where A_1, A_2, C_1, C_2 are generic constants, B_1, B_2 are constants depending on σ_1, σ_2 and D_1, D_2 are constants depending on λ_1, λ_2. Since $z(t), w(t) \to 0$ as $t \to \infty$, we have $A_2 = 0$ and $C_2 = 0$. Then, for $t > T$,

$$\begin{aligned} z(t) &= A_1 e^{-\frac{(N-\alpha)}{2}t} + B_1 \int_T^t e^{-\frac{(N-\alpha)}{2}(t-s)}[-e^{-qs}(w(s))^p]ds \\ &\quad - B_2 \int_t^\infty e^{\frac{(N'-4)}{2}(t-s)}[-e^{-qs}(w(s))^p]ds \end{aligned} \tag{3.9}$$

and

$$w(t) = C_1 e^{-\frac{(N'-4)}{2}t} + D_1 \int_T^t e^{-\frac{(N'-4)}{2}(t-s)}[-z(s)]ds - D_2 \int_t^\infty e^{\frac{(N-\alpha)}{2}(t-s)}[-z(s)]ds. \tag{3.10}$$

We consider three cases for α: (i) $\alpha \in (4-N, 2)$, (ii) $\alpha = 2$, (iii) $\alpha \in (2, N)$.

For the case (i), we see

$$\frac{(N'-4)}{2} < \frac{(N-\alpha)}{2}.$$

We claim that, for $t > T$,

$$w(t) = (C+o(1))e^{-\frac{(N'-4)}{2}t}, \quad z(t) = \begin{cases} (A+o(1))e^{-\frac{(N-\alpha)}{2}t} & \text{for } l > -2, \\ (B+o(1))te^{-\frac{(N-\alpha)}{2}t} & \text{for } l = -2, \\ (B+o(1))e^{-(q+\frac{(N'-4)}{2}p)t} & \text{for } l < -2 \end{cases} \tag{3.11}$$

where A, B, C are some constants.

To show (3.11), we first notice that

$$\begin{cases} q + \frac{(N'-4)}{2}p > \frac{(N-\alpha)}{2} & \text{for } l > -2, \\ q + \frac{(N'-4)}{2}p = \frac{(N-\alpha)}{2} & \text{for } l = -2, \\ q + \frac{(N'-4)}{2}p < \frac{(N-\alpha)}{2} & \text{for } l < -2. \end{cases} \tag{3.12}$$

We see also from (3.9) that

$$z(t) = O(e^{-\min\{\frac{(N-\alpha)}{2}, q\}t}) \tag{3.13}$$

provided $q \neq \frac{(N-\alpha)}{2}$ and

$$z(t) = O(te^{-\frac{(N-\alpha)}{2}t}) \tag{3.14}$$

provided $q = \frac{(N-\alpha)}{2}$. An important fact is that if $q \geq \frac{(N-\alpha)}{2}$, then $l > -2$ must hold.

If (3.14) holds, using the fact that $\frac{(N'-4)}{2} < \frac{(N-\alpha)}{2}$, we directly see from (3.10) that

$$w(t) = (C+o(1))e^{-\frac{(N'-4)}{2}t} \tag{3.15}$$

for some constant C. Substituting (3.15) to (3.9) and using the fact that $q = \frac{(N-\alpha)}{2}$, we obtain

$$z(t) = (A+o(1))e^{-\frac{(N-\alpha)}{2}t}, \tag{3.16}$$

for some constant A.

If (3.13) holds with $q > \frac{(N-\alpha)}{2}$, we directly see from (3.9) that

$$z(t) = (A+o(1))e^{-\frac{(N-\alpha)}{2}t}, \tag{3.17}$$

for some constant A. Substituting (3.17) to (3.10) and using the fact that $\frac{(N-\alpha)}{2} > \frac{(N'-4)}{2}$, we obtain

$$w(t) = (C+o(1))e^{-\frac{(N'-4)}{2}t}, \tag{3.18}$$

for some constant C.

If (3.13) holds with $\frac{(N'-4)}{2} < q < \frac{(N-\alpha)}{2}$, we see from (3.9) that

$$z(t) = O(e^{-qt}). \tag{3.19}$$

Substituting (3.19) to (3.10), we see that

$$w(t) = (C + o(1))e^{-\frac{(N'-4)}{2}t}, \tag{3.20}$$

for some constant C. Then, for $l > -2$, we substitute (3.20) to (3.9) and use (3.12) to obtain

$$z(t) = (A + o(1))e^{-\frac{(N-\alpha)}{2}t}, \tag{3.21}$$

for some constant A. For $l = -2$, we substitute (3.20) to (3.9) and use (3.12) to obtain

$$z(t) = (B + o(1))te^{-\frac{(N-\alpha)}{2}t}, \tag{3.22}$$

for some constant B. For $\alpha - 4 < l < -2$, we substitute (3.20) to (3.9) and use (3.12) to obtain

$$z(t) = (B + o(1))e^{-(q+\frac{(N'-4)}{2}p)t}, \tag{3.23}$$

for some constant B.

If (3.13) holds with $q = \frac{(N'-4)}{2} < \frac{(N-\alpha)}{2}$, we see from (3.9) that

$$z(t) = O(e^{-qt}). \tag{3.24}$$

We substitute (3.24) to (3.10) to obtain

$$w(t) = O(te^{-\frac{(N'-4)}{2}t}). \tag{3.25}$$

Substituting (3.25) to (3.9) and using (3.12), we have that, for some constant A,

$$z(t) = \begin{cases} (A + o(1))e^{-\frac{(N-\alpha)}{2}t} & \text{for } l > -2, \\ O(t^{p+1}e^{-\frac{(N-\alpha)}{2}t}) & \text{for } l = -2, \\ O(t^p e^{-(q+\frac{(N'-4)}{2}p)t}) & \text{for } l < -2. \end{cases} \tag{3.26}$$

Substituting (3.26) to (3.10), we obtain

$$w(t) = (C + o(1))e^{-\frac{(N'-4)}{2}t}, \tag{3.27}$$

for some constant C. Substituting (3.27) to (3.9) and using (3.12), we obtain

$$z(t) = \begin{cases} (A + o(1))e^{-\frac{(N-\alpha)}{2}t} & \text{for } l > -2, \\ (B + o(1))te^{-\frac{(N-\alpha)}{2}t} & \text{for } l = -2, \\ (B + o(1))e^{-(q+\frac{(N'-4)}{2}p)t} & \text{for } l < -2 \end{cases} \tag{3.28}$$

for some constants A and B.

If (3.13) holds with $q < \frac{(N'-4)}{2} < \frac{(N-\alpha)}{2}$, we see from (3.9) that

$$z(t) = O(e^{-qt}). \tag{3.29}$$

We substitute (3.29) to (3.10) to obtain

$$w(t) = O(e^{-qt}). \tag{3.30}$$

Let $q_1 = q(1 + p)$. Substituting (3.30) to (3.9), we have

$$\begin{aligned} z(t) &= A_1 e^{-\frac{(N-\alpha)}{2}t} + B_1 \int_T^t e^{-\frac{(N-\alpha)}{2}(t-s)} O(e^{-q_1 s})ds \\ &\quad - B_2 \int_t^\infty e^{\frac{(N'-4)}{2}(t-s)} O(e^{-q_1 s})ds. \end{aligned} \tag{3.31}$$

Using q_1 to replace q and arguments similar to the above, we obtain:

For $q_1 \geq \frac{(N-\alpha)}{2}$,

$$z(t) = (A + o(1))e^{-\frac{(N-\alpha)}{2}t}, \quad w(t) = (C + o(1))e^{-\frac{(N'-4)}{2}t} \tag{3.32}$$

for some constants A and C.

For $\frac{(N'-4)}{2} \leq q_1 < \frac{(N-\alpha)}{2}$,

$$w(t) = (C + o(1))e^{-\frac{(N'-4)}{2}t}, \tag{3.33}$$

for some constant C and

$$z(t) = \begin{cases} (A + o(1))e^{-\frac{(N-\alpha)}{2}t} & \text{for } l > -2, \\ (B + o(1))te^{-\frac{(N-\alpha)}{2}t} & \text{for } l = -2, \\ (B + o(1))e^{-(q+\frac{(N'-4)}{2}p)t} & \text{for } l < -2 \end{cases} \tag{3.34}$$

for some constants A and B.

We only need to consider the case $q_1 < \frac{(N'-4)}{2}$. It follows from (3.31) that

$$z(t) = O(e^{-q_1 t}). \tag{3.35}$$

We substitute (3.35) to (3.10) to obtain

$$w(t) = O(e^{-q_1 t}). \tag{3.36}$$

Let $q_2 = q + q_1 p = q(1 + p + p^2)$. Substituting (3.36) to (3.9), we have

$$\begin{aligned} z(t) \quad = \quad & A_1 e^{-\frac{(N-\alpha)}{2}t} + B_1 \int_T^t e^{-\frac{(N-\alpha)}{2}(t-s)} O(e^{-q_2 s})ds \\ & -B_2 \int_t^\infty e^{\frac{(N'-4)}{2}(t-s)} O(e^{-q_2 s})ds. \end{aligned} \tag{3.37}$$

Arguments similar to the above that:

For $q_2 \geq \frac{(N-\alpha)}{2}$,

$$z(t) = (A + o(1))e^{-\frac{(N-\alpha)}{2}t}, \quad w(t) = (C + o(1))e^{-\frac{(N'-4)}{2}t} \tag{3.38}$$

for some constants A and C.

For $\frac{(N'-4)}{2} \leq q_2 < \frac{(N-\alpha)}{2}$,

$$w(t) = (C + o(1))e^{-\frac{(N'-4)}{2}t}, \tag{3.39}$$

for some constant C and

$$z(t) = \begin{cases} (A + o(1))e^{-\frac{(N-\alpha)}{2}t} & \text{for } l > -2, \\ (B + o(1))te^{-\frac{(N-\alpha)}{2}t} & \text{for } l = -2, \\ (B + o(1))e^{-(q+\frac{(N'-4)}{2}p)t} & \text{for } l < -2 \end{cases} \tag{3.40}$$

for some constants A and B.

We only need to consider the case $q_2 < \frac{N'-4}{2}$. For any $p \geq 1$, we can find a least positive integer $m := m(p)$ such that

$$q_m = q(1 + p + p^2 + \ldots + p^m) \geq \frac{(N'-4)}{2}.$$

Using an iteration argument, we have that:

For $q_m \geq \frac{(N-\alpha)}{2}$,

$$z(t) = (A + o(1))e^{-\frac{(N-\alpha)}{2}t}, \quad w(t) = (C + o(1))e^{-\frac{(N'-4)}{2}t} \tag{3.41}$$

for some constants A and C.

For $\frac{(N'-4)}{2} \leq q_m < \frac{(N-\alpha)}{2}$,

$$w(t) = (C + o(1))e^{-\frac{(N'-4)}{2}t}, \tag{3.42}$$

for some constant C and

$$z(t) = \begin{cases} (A + o(1))e^{-\frac{(N-\alpha)}{2}t} & \text{for } l > -2, \\ (B + o(1))te^{-\frac{(N-\alpha)}{2}t} & \text{for } l = -2, \\ (B + o(1))e^{-(q+\frac{(N'-4)}{2}p)t} & \text{for } l < -2 \end{cases} \tag{3.43}$$

for some constants A and B.

All the arguments above imply that our claim (3.11) holds. This completes the proof of the case (i).

For the case (ii), we have

$$\frac{(N'-4)}{2} = \frac{(N-\alpha)}{2}, \quad q = \frac{(N+2+2l)-(N-2)p}{2} > 0, \quad l > -2.$$

We claim that, for $t > T$,

$$z(t) = (A + o(1))e^{-\frac{(N-2)}{2}t}, \quad w(t) = (C + o(1))te^{-\frac{(N-2)}{2}t}, \tag{3.44}$$

for some constants A and C.

It is seen from (3.9) and (3.10) that

$$\begin{aligned} z(t) \quad = \quad & A_1 e^{-\frac{(N-2)}{2}t} + B_1 \int_T^t e^{-\frac{(N-2)}{2}(t-s)}[-e^{-qs}(w(s))^p]ds \\ & -B_2 \int_t^\infty e^{\frac{(N-2)}{2}(t-s)}[-e^{-qs}(w(s))^p]ds \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} w(t) \quad = \quad & C_1 e^{-\frac{(N-2)}{2}t} + D_1 \int_T^t e^{-\frac{(N-2)}{2}(t-s)}[-z(s)]ds \\ & -D_2 \int_t^\infty e^{\frac{(N-2)}{2}(t-s)}[-z(s)]ds. \end{aligned} \tag{3.46}$$

We see also from (3.45) that

$$z(t) = O(e^{-\min\{\frac{(N-2)}{2},q\}t}) \tag{3.47}$$

provided $q \neq \frac{(N-2)}{2}$ and

$$z(t) = O(te^{-\frac{(N-2)}{2}t}) \tag{3.48}$$

provided $q = \frac{(N-2)}{2}$.

If (3.48) holds with $q = \frac{(N-2)}{2}$, we substitute (3.48) to (3.46) to obtain

$$w(t) = O(t^2 e^{-\frac{(N-2)}{2}t}). \tag{3.49}$$

Substituting (3.49) to (3.45) and noticing $q = \frac{(N-2)}{2}$, we obtain

$$z(t) = (A + o(1))e^{-\frac{(N-2)}{2}t}, \tag{3.50}$$

for some constant A. Substituting (3.50) to (3.46), we obtain

$$w(t) = (C + o(1))te^{-\frac{(N-2)}{2}t}, \tag{3.51}$$

for some constant C.

If $q > \frac{(N-2)}{2}$, we directly see from (3.45) that

$$z(t) = (A + o(1))e^{-\frac{(N-2)}{2}t}, \tag{3.52}$$

for some constant A. Substituting (3.52) to (3.46), we obtain

$$w(t) = (C + o(1))te^{-\frac{(N-2)}{2}t}, \tag{3.53}$$

for some constant C.

If $q < \frac{(N-2)}{2}$, we see from (3.45) that

$$z(t) = O(e^{-qt}). \tag{3.54}$$

Substituting (3.54) to (3.46), we obtain

$$w(t) = O(e^{-qt}). \tag{3.55}$$

Let $q_1 = q(1 + p)$. Then, it follows from (3.45) that

$$\begin{aligned} z(t) \quad = \quad & A_1 e^{-\frac{(N-2)}{2}t} + B_1 \int_T^t e^{-\frac{(N-2)}{2}(t-s)} O(e^{-q_1 s}) ds \\ & -B_2 \int_t^\infty e^{\frac{(N-2)}{2}(t-s)} O(e^{-q_1 s}) ds. \end{aligned} \tag{3.56}$$

An iteration argument as in the proof of the case (i) implies that our claim (3.44) holds.

For the case (iii), we have

$$\frac{(N'-4)}{2} > \frac{(N-\alpha)}{2}.$$

We claim that, for $t > T$,

$$z(t) = (A + o(1))e^{-\frac{(N-\alpha)}{2}t}, \quad w(t) = (C + o(1))e^{-\frac{(N-\alpha)}{2}t}, \tag{3.57}$$

for some constants A and C.

We see also from (3.9) that

$$z(t) = O(e^{-\min\{\frac{(N-\alpha)}{2}, q\}t}) \tag{3.58}$$

provided $q \neq \frac{(N-\alpha)}{2}$ and

$$z(t) = O(te^{-\frac{(N-\alpha)}{2}t}) \tag{3.59}$$

provided $q = \frac{(N-\alpha)}{2}$.

If (3.59) holds with $q = \frac{(N-\alpha)}{2}$, we substitute (3.59) to (3.10) and obtain

$$w(t) = O(te^{-\frac{(N-\alpha)}{2}t}) \tag{3.60}$$

Substituting (3.60) to (3.9) and noticing $q = \frac{(N-\alpha)}{2}$, we obtain

$$z(t) = (A + o(1))e^{-\frac{(N-\alpha)}{2}t}, \tag{3.61}$$

for some constant A. Substituting (3.61) to (3.10) and noticing $\frac{(N'-4)}{2} > \frac{(N-\alpha)}{2}$, we obtain

$$w(t) = (C + o(1))e^{-\frac{(N-\alpha)}{2}t}, \tag{3.62}$$

for some constant C.

If $q > \frac{(N-\alpha)}{2}$, we directly obtain from (3.9) that

$$z(t) = (A + o(1))e^{-\frac{(N-\alpha)}{2}t}, \tag{3.63}$$

for some constant A. Substituting (3.63) to (3.10) and noticing $\frac{(N'-4)}{2} > \frac{(N-\alpha)}{2}$, we obtain

$$w(t) = (C + o(1))e^{-\frac{(N-\alpha)}{2}t}, \tag{3.64}$$

for some constant C.

We now consider the case $q < \frac{(N-\alpha)}{2}$. It is known from (3.58) that, for $t > T$,

$$z(t) = O(e^{-qt}). \tag{3.65}$$

Substituting (3.65) to (3.10) and noticing $q < \frac{(N-\alpha)}{2} < \frac{(N'-4)}{2}$, we obtain

$$w(t) = O(e^{-qt}). \tag{3.66}$$

Let $q_1 = q(1+p)$. Substituting (3.66) to (3.9), we have

$$\begin{aligned} z(t) \quad = \quad & A_1 e^{-\frac{(N-\alpha)}{2}t} + B_1 \int_T^t e^{-\frac{(N-\alpha)}{2}(t-s)} O(e^{-q_1 s}) ds \\ & -B_2 \int_t^\infty e^{\frac{(N'-4)}{2}(t-s)} O(e^{-q_1 s}) ds. \end{aligned} \tag{3.67}$$

An iteration argument as in the proof of the case (i) implies that our claim (3.57) holds.

We obtain the conclusions of Theorem 1.1 from the claims (3.11), (3.44) and (3.57). The proof of Theorem 1.1 is complete. □

4. **Asymptotic behaviors of (u_N, v_N) and (u_D, v_D) at $x = 0$: Proof of Theorem 1.2.** In this section, we will obtain more detailed asymptotic behaviors of (u_N, v_N) and (u_D, v_D) at $x = 0$. Of course, the parameters α and l also play important roles in the asymptotic behaviors of (u_N, v_N) and (u_D, v_D) at $x = 0$. As in Section 3, we also need to consider several cases of (α, l) separately. We still only consider $(u(r), v(r)) = (u_N(r), v_N(r))$. The study of another case $(u(r), v(r)) = (u_D(r), v_D(r))$ is similar.

Proof of Theorem 1.2. Let $(w(t), z(t))$ be given in (3.4). We will continue the study of Section 3. We also consider three cases here: (i) $\alpha \in (4-N, 2)$, (ii) $\alpha = 2$, (iii) $\alpha \in (2, N)$.

For the case (i), we see from (3.11) that, for $t > T$,

$$w(t) = (C + o(1))e^{-\frac{(N'-4)}{2}t}, \quad z(t) = \begin{cases} (A + o(1))e^{-\frac{(N-\alpha)}{2}t} & \text{for } l > -2, \\ (B + o(1))te^{-\frac{(N-\alpha)}{2}t} & \text{for } l = -2, \\ (B + o(1))e^{-(q + \frac{(N'-4)}{2}p)t} & \text{for } l < -2 \end{cases} \tag{4.1}$$

where A, B, C are some constants. We now write the system (3.4) to a fourth order equation of $w(t)$:

$$w^{(4)} - \Big[\frac{(N-\alpha)^2 + (N'-4)^2}{4}\Big] w'' + \Big[\frac{(N-\alpha)(N'-4)}{4}\Big]^2 w = e^{-qt} w^p. \tag{4.2}$$

The characteristic equation of (4.2) is

$$\lambda^4 - \Big[\frac{(N-\alpha)^2 + (N'-4)^2}{4}\Big] \lambda^2 + \Big[\frac{(N-\alpha)(N'-4)}{4}\Big]^2 = 0 \tag{4.3}$$

and the four roots of (4.3) are: $\pm\frac{(N-\alpha)}{2}$ and $\pm\frac{(N'-4)}{2}$. Therefore, for $t > T \gg 1$, we have

$$\begin{aligned} w(t) \quad = \quad & A_1 e^{-\frac{(N'-4)}{2}t} + A_2 e^{-\frac{(N-\alpha)}{2}t} + A_3 e^{\frac{(N'-4)}{2}t} + A_4 e^{\frac{(N-\alpha)}{2}t} \\ & +B_1 \int_T^t e^{-\frac{(N'-4)}{2}(t-s)} e^{-qs} w^p(s)ds + B_2 \int_T^t e^{-\frac{(N-\alpha)}{2}(t-s)} e^{-qs} w^p(s)ds \\ & -B_3 \int_t^\infty e^{\frac{(N'-4)}{2}(t-s)} e^{-qs} w^p(s)ds \\ & -B_4 \int_t^\infty e^{\frac{(N-\alpha)}{2}(t-s)} e^{-qs} w^p(s)ds, \end{aligned} \tag{4.4}$$

where A_1, A_2, A_3, A_4 are generic constants, B_1, B_2, B_3, B_4 are constants only depending on the four roots of (4.3). Since $w(t) \to 0$ as $t \to \infty$, we have $A_3 = A_4 = 0$ and

$$\begin{aligned} w(t) \quad = \quad & A_1 e^{-\frac{(N'-4)}{2}t} + A_2 e^{-\frac{(N-\alpha)}{2}t} \\ & +B_1 \int_T^t e^{-\frac{(N'-4)}{2}(t-s)} e^{-qs} w^p(s)ds + B_2 \int_T^t e^{-\frac{(N-\alpha)}{2}(t-s)} e^{-qs} w^p(s)ds \\ & -B_3 \int_t^\infty e^{\frac{(N'-4)}{2}(t-s)} e^{-qs} w^p(s)ds \\ & -B_4 \int_t^\infty e^{\frac{(N-\alpha)}{2}(t-s)} e^{-qs} w^p(s)ds. \end{aligned} \tag{4.5}$$

For $\alpha \in (4-N, 2)$ and $l > -2$, we see from the proof of Theorem 1.1 that

$$\frac{(N'-4)}{2} < \frac{(N-\alpha)}{2}, \quad q + \frac{(N'-4)}{2}p > \frac{(N-\alpha)}{2}. \tag{4.6}$$

Then, we obtain from (4.5) and (4.6) that

$$w(t) = c_1 e^{-\frac{(N'-4)}{2}t} + c_2 e^{-\frac{(N-\alpha)}{2}t} + c_3 e^{-[q+\frac{(N'-4)}{2}p]t} + O(e^{-[q+\frac{(N'-4)}{2}p+2-\alpha]t}), \tag{4.7}$$

where c_1, c_2, c_3 are constants. The third and fourth terms in the right hand side of (4.7) come from the nonlinearity $e^{-qs}w^p(s)$ in (4.5). The fact that $u(0) > 0$ implies $c_1 = u(0) > 0$. Note that

$$\begin{aligned} & e^{-qt}\Big(c_1 e^{-\frac{(N'-4)}{2}t} + c_2 e^{-\frac{(N-\alpha)}{2}t}\Big)^p \\ &= Ce^{-[q+\frac{(N'-4)}{2}p]t} + De^{-[q+\frac{(N'-4)}{2}p+\frac{(N-\alpha)}{2}-\frac{(N'-4)}{2}]t} + o(e^{-[q+\frac{(N'-4)}{2}p+\frac{(N-\alpha)}{2}-\frac{(N'-4)}{2}]t}) \\ &= Ce^{-[q+\frac{(N'-4)}{2}p]t} + De^{-[q+\frac{(N'-4)}{2}p+2-\alpha]t} + o(e^{-[q+\frac{(N'-4)}{2}p+2-\alpha]t}), \end{aligned}$$

since, for any $b > a > 0$, $\varrho \geq 1$ and $t > T \gg 1$,

$$\begin{aligned} (e^{-at} + e^{-bt})^\varrho &= e^{-\varrho at}(1 + e^{-(b-a)t})^\varrho \\ &= e^{-\varrho at}\Big(1 + B_1 e^{-(b-a)t} + B_2 e^{-2(b-a)t} \\ &\qquad + \ldots + B_k e^{-k(b-a)t} + o(e^{-k(b-a)t})\Big). \end{aligned} \tag{4.8}$$

Substituting (4.7) to (3.9), we have that

$$z(t) = d_1 e^{-\frac{(N-\alpha)}{2}t} + d_2 e^{-[q+\frac{(N'-4)}{2}p]t} + d_3 e^{-[q+\frac{(N'-4)}{2}p+2-\alpha]t} + o(e^{-[q+\frac{(N'-4)}{2}p+2-\alpha]t}). \tag{4.9}$$

The fact that $v(0) > 0$ implies $d_1 = v(0) > 0$. We obtain the identities in (1) of Theorem 1.2 via (4.7) and (4.9).

For $\alpha \in (4-N,2)$ and $l=-2$, we see from the proof of Theorem 1.1 that

$$\frac{(N'-4)}{2} < \frac{(N-\alpha)}{2}, \quad q+\frac{(N'-4)}{2}p = \frac{(N-\alpha)}{2}. \tag{4.10}$$

We obtain from (4.5) and (4.10) that

$$w(t) = c_1 e^{-\frac{(N'-4)}{2}t} + c_2 t e^{-\frac{(N-\alpha)}{2}t} + c_3 e^{-\frac{(N-\alpha)}{2}t} + O(te^{-[(2-\alpha)+\frac{(N-\alpha)}{2}]t}), \tag{4.11}$$

where c_1, c_2, c_3 are constants. The fact $u(0)>0$ implies $c_1 = u(0) > 0$. On the other hand, it follows from (4.1) that

$$z(t) = (B+o(1))te^{-\frac{(N-\alpha)}{2}t}$$

and the facts $u(0)=c_1>0$ and $-\Delta v = r^l u^p = r^{-2}u^p$ imply $B>0$ and $v(r) = -(B+o(1))\ln r$. Substituting (4.11) to (3.9), we have that

$$z(t) = d_1 te^{-\frac{(N-\alpha)}{2}t} + d_2 e^{-\frac{(N-\alpha)}{2}t} + d_3 te^{-[\frac{(N-\alpha)}{2}+2-\alpha]t} + o(te^{-[\frac{(N-\alpha)}{2}+2-\alpha]t}), \tag{4.12}$$

where $d_1 = B > 0$. The facts $d_1>0$ and $-\Delta u = r^{-\alpha}v$ also imply $c_2 \neq 0$. We obtain the identities in (2) of Theorem 1.2 via (4.11) and (4.12).

For $\alpha \in (4-N,2)$ and $l \in (\alpha-4,-2)$, simple calculations imply

$$\frac{(N'-4)}{2} < q+\frac{(N'-4)}{2}p < \frac{(N-\alpha)}{2} \tag{4.13}$$

and

$$2q+\frac{(N'-4)}{2}(2p-1) > \frac{(N-\alpha)}{2} \quad \text{for } \tau > -3-\tfrac{\alpha}{2}, \tag{4.14}$$

$$2q+\frac{(N'-4)}{2}(2p-1) = \frac{(N-\alpha)}{2} \quad \text{for } \tau = -3-\tfrac{\alpha}{2}, \tag{4.15}$$

$$2q+\frac{(N'-4)}{2}(2p-1) < \frac{(N-\alpha)}{2} \quad \text{for } \tau \in (-4,-3-\tfrac{\alpha}{2}). \tag{4.16}$$

For $\tau > -3-\frac{\alpha}{2}$, we obtain from (4.5), (4.13) and (4.14) that

$$w(t) = c_1 e^{-\frac{(N'-4)}{2}t} + c_2 e^{-[q+\frac{(N'-4)}{2}p]t} + c_3 e^{-\frac{(N-\alpha)}{2}t} + O(e^{-[2q+\frac{(N'-4)}{2}(2p-1)]t}), \tag{4.17}$$

where c_1, c_2, c_3 are constants and $c_1>0$. Note that

$$\begin{aligned} &e^{-qt}\Big(c_1 e^{-\frac{(N'-4)}{2}t} + c_2 e^{-[q+\frac{(N'-4)}{2}p]t}\Big)^p \\ &\quad = Ce^{-[q+\frac{(N'-4)}{2}p]t} + De^{-[2q+\frac{(N'-4)}{2}(2p-1)]t} + O(e^{-[3q+\frac{(N'-4)}{2}(3p-2)]t}). \end{aligned}$$

Substituting (4.17) to (3.9), we have that

$$z(t) = d_1 e^{-[q+\frac{(N'-4)}{2}p]t} + d_2 e^{-\frac{(N-\alpha)}{2}t} + d_3 e^{-[2q+\frac{(N'-4)}{2}(2p-1)]t} + o(e^{-[2q+\frac{(N'-4)}{2}(2p-1)]t}), \tag{4.18}$$

where d_1, d_2, d_3 are constants and $d_1>0$.

For $\tau = -3-\frac{\alpha}{2}$, we obtain from (4.5), (4.13) and (4.15) that

$$w(t) = c_1 e^{-\frac{(N'-4)}{2}t} + c_2 e^{-[q+\frac{(N'-4)}{2}p]t} + c_3 te^{-\frac{(N-\alpha)}{2}t} + O(e^{-\frac{(N-\alpha)}{2}t}), \tag{4.19}$$

where c_1, c_2, c_3 are constants and $c_1>0$. Substituting (4.19) to (3.9), we have that

$$z(t) = d_1 e^{-[q+\frac{(N'-4)}{2}p]t} + d_2 te^{-\frac{(N-\alpha)}{2}t} + d_3 e^{-\frac{(N-\alpha)}{2}t} + o(e^{-\frac{(N-\alpha)}{2}t}), \tag{4.20}$$

where d_1, d_2, d_3 are constants and $d_1>0$.

For $\tau \in (-4, -3-\frac{\alpha}{2})$, we obtain from (4.5), (4.13) and (4.16) that

$$w(t) = c_1 e^{-\frac{(N'-4)}{2}t} + c_2 e^{-[q+\frac{(N'-4)}{2}p]t} + c_3 e^{-[2q+\frac{(N'-4)}{2}(2p-1)]t} + o(e^{-[2q+\frac{(N'-4)}{2}(2p-1)]t}), \tag{4.21}$$

where c_1, c_2, c_3 are constants and $c_1 > 0$.

For any positive integer $n \geq 3$ and $-\frac{(\alpha+4n-2)}{n} < \tau < -\frac{(\alpha+4n-6)}{(n-1)}$,

$$(n-1)q + \frac{(N'-4)}{2}[(n-1)p-(n-2)] < \frac{(N-\alpha)}{2} < nq + \frac{(N'-4)}{2}[np-(n-1)].$$

Substituting (4.21) to (3.9), we have

$$z(t) = d_1 e^{-\vartheta_1 t} + d_2 e^{-\vartheta_2 t} + \ldots + d_{n-1} e^{-\vartheta_{n-1} t} + d_n e^{-\vartheta_n t} + O(e^{-\vartheta_{n+1} t}), \tag{4.22}$$

where $d_1, d_2, \ldots, d_n$ are constants with $d_1 > 0$ and

$$\vartheta_1 = q + \frac{(N'-4)}{2}p, \quad \vartheta_2 = 2q + \frac{(N'-4)}{2}(2p-1), \ldots,$$
$$\vartheta_{n-1} = (n-1)q + \frac{(N'-4)}{2}[(n-1)p-(n-2)], \quad \vartheta_n = \frac{(N-\alpha)}{2},$$
$$\vartheta_{n+1} = nq + \frac{(N'-4)}{2}[np-(n-1)].$$

For $\tau = -\frac{(\alpha+4n-2)}{n}$, we have

$$nq + \frac{(N'-4)}{2}[np-(n-1)] = \frac{(N-\alpha)}{2}.$$

Substituting (4.21) to (3.9), we have

$$z(t) = d_1 e^{-\vartheta_1 t} + d_2 e^{-\vartheta_2 t} + \ldots + d_{n-1} e^{-\vartheta_{n-1} t} + d_n t e^{-\vartheta_n t} + d_{n+1} e^{-\vartheta_n t} + O(e^{-\vartheta_{n+1} t}), \tag{4.23}$$

where $d_1, d_2, \ldots, d_n$ are constants with $d_1 > 0$ and

$$\vartheta_1 = q + \frac{(N'-4)}{2}p, \quad \vartheta_2 = 2q + \frac{(N'-4)}{2}(2p-1), \ldots,$$
$$\vartheta_{n-1} = (n-1)q + \frac{(N'-4)}{2}[(n-1)p-(n-2)], \quad \vartheta_n = \frac{(N-\alpha)}{2},$$
$$\vartheta_{n+1} = (n+1)q + \frac{(N'-4)}{2}[(n+1)p-n].$$

We obtain the identities in (3) of Theorem 1.2 via (4.17) and (4.18); (4.19) and (4.20); (4.21) and (4.22) or (4.23).

For $\alpha = 2$ and $\tau > -4$, we easily see that

$$\frac{(N'-4)}{2} = \frac{(N-\alpha)}{2} = \frac{(N-2)}{2}. \tag{4.24}$$

Moreover, the four roots of (4.3) are $\pm\frac{(N-2)}{2}$, $\pm\frac{(N-2)}{2}$. Then,

$$\begin{aligned} w(t) \;=\;& A_1 t e^{-\frac{(N-2)}{2}t} + A_2 e^{-\frac{(N-2)}{2}t} + t\int_T^t e^{-\frac{(N-2)}{2}(t-s)} e^{-qs} w^p(s)ds \\ & - \int_T^t s e^{-\frac{(N-2)}{2}(t-s)} e^{-qs} w^p(s)ds - t\int_t^\infty e^{\frac{(N-2)}{2}(t-s)} e^{-qs} w^p(s)ds \\ & + \int_t^\infty s e^{\frac{(N-2)}{2}(t-s)} e^{-qs} w^p(s)ds. \end{aligned} \tag{4.25}$$

We obtain from (4.25), (4.24) that, for $t > T \gg 1$,

$$w(t) = c_1 t e^{-\frac{(N-2)}{2}t} + c_2 e^{-\frac{(N-2)}{2}t} + c_3 t^p e^{-[q+\frac{(N-2)}{2}p]t} + o(t^p e^{-[q+\frac{(N-2)}{2}p]t}), \quad (4.26)$$

where c_1, c_2, c_3 are constants. Substituting (4.26) to (3.9), we obtain

$$z(t) = d_1 e^{-\frac{(N-2)}{2}t} + d_2 t^p e^{-[q+\frac{(N-2)}{2}p]t} + o(t^p e^{-[q+\frac{(N-2)}{2}p]t}), \quad (4.27)$$

where d_1, d_2 are constants and $v(0) = d_1 > 0$. It follows from $-\Delta u = r^{-\alpha} v = r^{-2} v$ and $d_1 > 0$ that $c_1 > 0$ in (4.26).

We obtain the identities in (4) of Theorem 1.2 via (4.26) and (4.27).

For $\alpha \in (2, N)$ and $\tau > -4$, we have

$$\frac{(N-\alpha)}{2} < \frac{(N'-4)}{2}. \quad (4.28)$$

Moreover, simple calculations imply

$$q + \frac{(N-\alpha)}{2}p \begin{cases} > \frac{(N'-4)}{2} & \text{for } 1 \le p < \frac{(4+\tau)}{(\alpha-2)}, \\ = \frac{(N'-4)}{2} & \text{for } p = \frac{(4+\tau)}{(\alpha-2)}, \\ < \frac{(N'-4)}{2} & \text{for } p > \frac{(4+\tau)}{(\alpha-2)}, \end{cases} \quad (4.29)$$

$$2q + \frac{(N-\alpha)}{2}(2p-1) \begin{cases} > \frac{(N'-4)}{2} & \text{for } 1 \le p < \frac{(6+\alpha+2\tau)}{2(\alpha-2)}, \\ = \frac{(N'-4)}{2} & \text{for } p = \frac{(6+\alpha+2\tau)}{2(\alpha-2)}, \\ < \frac{(N'-4)}{2} & \text{for } p > \frac{(6+\alpha+2\tau)}{2(\alpha-2)}. \end{cases} \quad (4.30)$$

For $1 \le p < \frac{(4+\tau)}{(\alpha-2)}$, we obtain from (4.5), (4.28) and (4.29) that

$$w(t) = c_1 e^{-\frac{(N-\alpha)}{2}t} + c_2 e^{-\frac{(N'-4)}{2}t} + c_3 e^{-[q+\frac{(N-\alpha)}{2}p]t} + o(e^{-[q+\frac{(N-\alpha)}{2}p]t}), \quad (4.31)$$

where c_1, c_2, c_3 are constants. Substituting (4.31) to (3.9), we obtain

$$z(t) = d_1 e^{-\frac{(N-\alpha)}{2}t} + d_2 e^{-[q+\frac{(N-\alpha)}{2}p]t} + d_3 e^{-[q+\frac{(N-\alpha)}{2}p+\alpha-2]t} + o(e^{-[q+\frac{(N-\alpha)}{2}p+\alpha-2]t}), \quad (4.32)$$

where d_1, d_2, d_3 are constants and $v(0) = d_1 > 0$. The facts $d_1 > 0$ and $-\Delta u = r^{-\alpha} v$ imply $c_1 > 0$ in (4.31).

For $p = \frac{(4+\tau)}{(\alpha-2)}$, we obtain from (4.5), (4.28) and (4.29) that

$$w(t) = c_1 e^{-\frac{(N-\alpha)}{2}t} + c_2 t e^{-\frac{(N'-4)}{2}t} + c_3 e^{-\frac{(N'-4)}{2}t} + o(e^{-\frac{(N'-4)}{2}t}), \quad (4.33)$$

where c_1, c_2, c_3 are constants. Substituting (4.33) to (3.9), we obtain

$$z(t) = d_1 e^{-\frac{(N-\alpha)}{2}t} + d_2 e^{-\frac{(N'-4)}{2}t} + d_3 t e^{-[q+\frac{(N-\alpha)}{2}p+\alpha-2]t} + o(t e^{-[q+\frac{(N-\alpha)}{2}p+\alpha-2]t}), \quad (4.34)$$

where d_1, d_2, d_3 are constants and $d_1 > 0$. Similarly, we have $c_1 > 0$ in (4.33).

For $\frac{(4+\tau)}{(\alpha-2)} < p < \frac{(6+\alpha+2\tau)}{2(\alpha-2)}$, we obtain from (4.5), (4.28) and (4.30) that

$$w(t) = c_1 e^{-\frac{(N-\alpha)}{2}t} + c_2 e^{-[q+\frac{(N-\alpha)}{2}p]t} + c_3 e^{-\frac{(N'-4)}{2}t} + o(e^{-\frac{(N'-4)}{2}t}), \quad (4.35)$$

where c_1, c_2, c_3 are constants. Substituting (4.35) to (3.9), we obtain

$$z(t) = d_1 e^{-\frac{(N-\alpha)}{2}t} + d_2 e^{-[q+\frac{(N-\alpha)}{2}p]t} + d_3 e^{-[2q+\frac{(N-\alpha)}{2}(2p-1)]t} + o(e^{-[2q+\frac{(N-\alpha)}{2}(2p-1)]t}), \quad (4.36)$$

where d_1, d_2, d_3 are constants and $d_1 > 0$. Similarly, we have $c_1 > 0$ in (4.35).

For $p = \frac{(6+\alpha+2\tau)}{2(\alpha-2)}$, we obtain from (4.5), (4.28) and (4.30) that

$$w(t) = c_1 e^{-\frac{(N-\alpha)}{2}t} + c_2 e^{-[q+\frac{(N-\alpha)}{2}p]t} + c_3 t e^{-\frac{(N'-4)}{2}t} + O(e^{-\frac{(N'-4)}{2}t}), \tag{4.37}$$

where c_1, c_2, c_3 are constants. Substituting (4.37) to (3.9), we obtain

$$z(t) = d_1 e^{-\frac{(N-\alpha)}{2}t} + d_2 e^{-[q+\frac{(N-\alpha)}{2}p]t} + d_3 e^{-[2q+\frac{(N-\alpha)}{2}(2p-1)]t} + o(e^{-[2q+\frac{(N-\alpha)}{2}(2p-1)]t}), \tag{4.38}$$

where d_1, d_2, d_3 are constants and $d_1 > 0$. Similarly, we have $c_1 > 0$ in (4.37).

For $p > \frac{(6+\alpha+2\tau)}{2(\alpha-2)}$, we obtain from (4.5), (4.28) and (4.30) that there are an integer $n \geq 4$ and $\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}, \vartheta_n, \vartheta_{n+1}$ with $\vartheta_1 = \frac{(N-\alpha)}{2}, \vartheta_2 = q + \frac{(N-\alpha)}{2}p, \vartheta_3 = 2q + \frac{(N-\alpha)}{2}(2p-1), \dots$ and

$$\vartheta_1 < \vartheta_2 < \dots < \vartheta_{n-1} < \frac{(N'-4)}{2} (= \vartheta_n) < \vartheta_{n+1}$$

such that

$$w(t) = c_1 e^{-\vartheta_1 t} + c_2 e^{-\vartheta_2 t} + \dots + c_{n-1} e^{-\vartheta_{n-1} t} + c_n e^{-\frac{(N'-4)}{2}t} + o(e^{-\frac{(N'-4)}{2}t}), \tag{4.39}$$

or

$$w(t) = c_1 e^{-\vartheta_1 t} + c_2 e^{-\vartheta_2 t} + \dots + c_{n-1} e^{-\vartheta_{n-1} t} + c_n t e^{-\frac{(N'-4)}{2}t} + c_{n+1} e^{-\frac{(N'-4)}{2}t} + o(e^{-\frac{(N'-4)}{2}t}), \tag{4.40}$$

where $c_1, c_2, \dots, c_n, c_{n+1}$ are constants. Substituting (4.39) or (4.40) to (3.9), we obtain

$$z(t) = d_1 e^{-\vartheta_1 t} + d_2 e^{-\vartheta_2 t} + \dots + d_{n-1} e^{-\vartheta_{n-1} t} + o(e^{-\vartheta_{n-1} t}), \tag{4.41}$$

where $\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}$ are given in (4.39), $d_1, d_2, \dots, d_{n-1}$ are constants and $d_1 > 0$. Similarly, we have $c_1 > 0$ in (4.39) or (4.40).

We obtain the identities in (5) of Theorem 1.2 via (4.31) and (4.32), (4.33) and (4.34), (4.35) and (4.36), (4.37) and (4.38), (4.39) or (4.40), (4.41).

The proof of Theorem 1.2 is complete. □

5. **Uniqueness results for problems (N) and (D): Proof of Theorem 1.3.** In this section, we consider the uniqueness results for positive radial solutions of problems (N) and (D) and present the proof of Theorem 1.3. We only consider the case for $\alpha \in (4-N, 2)$ and $\tau > -4$. We expect that u_N and u_D are also the unique positive radial solutions of problems (N) and (D) in $D^{2,\alpha}_{rad}(B) \cap D^{1,\alpha}_{0,rad}(B)$ and $D^{2,\alpha}_{0,rad}(B)$ respectively for $\alpha \in [2, N)$ and $\tau > -4$.

Proof of Theorem 1.3. We first consider the case $\alpha \in (4-N, 2)$ and $l > -2$ for problem (N). Let $u_1, u_2 \in D^{2,\alpha}_{rad}(B) \cap D^{1,\alpha}_{0,rad}(B)$ be two positive radial solutions of problem (N) for $1 \leq p < p_s$. It follows from Theorem 1.1 that $u_1, u_2 \in C^4(B\backslash\{0\}) \cap C^0(\overline{B})$ and $v_1, v_2 \in C^2(B\backslash\{0\}) \cap C^0(\overline{B})$.

Setting $\lambda^{\frac{(4+\tau)}{(p-1)}} = \frac{u_1(0)}{u_2(0)}$ and defining the function

$$u_3(s) = \lambda^{\frac{(4+\tau)}{(p-1)}} u_2(\lambda s) \quad s \in [0, \lambda^{-1}],$$

clearly we have u_3 satisfies the problem

$$\begin{cases} (s^{N-1}(s^\alpha \Delta u_3)')' = s^{N+l-1} u_3^p & \text{for } s \in (0, \frac{1}{\lambda}), \\ u_3(\frac{1}{\lambda}) = (\Delta u_3)(\frac{1}{\lambda}) = 0 \end{cases} \tag{5.1}$$

and

$$u_3(0) = u_1(0). \tag{5.2}$$

On the other hand, we see that $u_1(s) :\equiv u_1(r)$, $s = r$ satisfies the problem

$$\begin{cases} (s^{N-1}(s^\alpha \Delta u_1)')' = s^{N+l-1} u_1^p & \text{for } s \in (0,1), \\ u_1(1) = (\Delta u_1)(1) = 0. \end{cases} \tag{5.3}$$

Let $v_1(s) = -s^\alpha (\Delta u_1)(s)$, $v_3(s) = -s^\alpha (\Delta u_3)(s)$. Then,

$$\begin{cases} -\Delta u_1 = s^{-\alpha} v_1 & \text{for } s \in (0,1), \\ -\Delta v_1 = s^l u_1^p & \text{for } s \in (0,1), \end{cases} \tag{5.4}$$

and

$$\begin{cases} -\Delta u_3 = s^{-\alpha} v_3 & \text{for } s \in (0,\lambda^{-1}), \\ -\Delta v_3 = s^l u_3^p & \text{for } s \in (0,\lambda^{-1}). \end{cases} \tag{5.5}$$

It follows from (2.11) that

$$u_1'(s) < 0 \quad v_1'(s) < 0 \qquad \text{for } s \in (0,1] \tag{5.6}$$

and arguments similar to those in the proof of (2.11) imply that

$$u_3'(s) < 0 \quad v_3'(s) < 0 \qquad \text{for } s \in (0,\lambda^{-1}]. \tag{5.7}$$

Moreover, it follows from (2.10) that

$$u_1(s) > 0, \;\; v_1(s) > 0 \qquad \text{for } s \in [0,1) \tag{5.8}$$

and

$$u_3(s) > 0, \;\; v_3(s) > 0 \qquad \text{for } s \in [0,\lambda^{-1}). \tag{5.9}$$

Let $R(\lambda) = \min\{1, \lambda^{-1}\}$. We now show

$$v_1(0) = v_3(0). \tag{5.10}$$

On the contrary, without loss of generality, we assume $v_1(0) > v_3(0)$. We will show

$$v_1(s) > v_3(s) \quad \text{for } s \in [0, R(\lambda)]. \tag{5.11}$$

Suppose that there is $s_0 \in (0, R(\lambda)]$ such that

$$v_1(s) > v_3(s) \quad \text{for } s \in [0,s_0) \text{ and } \; v_1(s_0) = v_3(s_0). \tag{5.12}$$

Then, we have that

$$-(s^{N-1}(u_1 - u_3)'(s))' = s^{N-\alpha-1}(v_1(s) - v_3(s)) > 0 \quad \text{for } s \in [0, s_0).$$

The fact that $s^{N-1}(u_1 - u_3)'(s) \to 0$ as $s \to 0$ implies $(u_1 - u_3)'(s) < 0$ for $s \in (0, s_0]$. Since $(u_1 - u_3)(0) = 0$, we obtain $(u_1 - u_3)(s) < 0$ for $s \in (0, s_0]$. This implies

$$(s^{N-1}(v_1 - v_3)'(s))' = -s^{N+l-1}(u_1^p - u_3^p) > 0 \quad \text{for } s \in (0, s_0].$$

Since $\lim_{s\to 0} s^{N-1}(v_1 - v_3)'(s) = 0$, we obtain

$$(v_1 - v_3)'(s) > 0 \quad \text{for } s \in (0, s_0].$$

The fact that $(v_1 - v_3)(s_0) = 0$ implies

$$(v_1 - v_3)(s) < 0 \;\; \text{for } s \in (0, s_0),$$

which contradicts (5.12). This implies that (5.11) holds. Since

$$(v_1 - v_3)(R(\lambda)) = \begin{cases} v_1(\lambda^{-1}) & \text{if } \lambda > 1, \\ 0 & \text{if } \lambda = 1, \\ -v_3(1) & \text{if } \lambda < 1, \end{cases}$$

we deduce that necessarily $\lambda > 1$. Now as the above, we show that $(u_1 - u_3)(s) < 0$ for $s \in (0, R(\lambda)]$. Since

$$(u_1 - u_3)(R(\lambda)) = \begin{cases} u_1(\lambda^{-1}) & \text{if } \lambda > 1, \\ 0 & \text{if } \lambda = 1, \\ -u_3(1) & \text{if } \lambda < 1, \end{cases}$$

we deduce that necessarily $\lambda < 1$ and obtain a contradiction. The case $v_1(0) < v_3(0)$ can be handled in the same way. Thus, (5.10) is proved.

Now, we see from (5.4) and (5.5) that

$$(u_1 - u_3)(s) = -\int_0^s t^{1-N} \int_0^t \rho^{N-\alpha-1}(v_1 - v_3)(\rho) d\rho dt, \tag{5.13}$$

and

$$(v_1 - v_3)(s) = -\int_0^s t^{1-N} \int_0^t \rho^{N+l-1}(u_1^p - u_3^p) d\rho dt, \tag{5.14}$$

where we use the facts $(u_1 - u_3)(0) = 0$ and $(v_1 - v_3)(0) = 0$. Then, we see from (5.13) and (5.14) that, for sufficiently small $\epsilon > 0$ and $s \in [0, \epsilon]$,

$$|(u_1 - u_3)(s)| \leq \frac{s^{2-\alpha}}{(N-\alpha)(2-\alpha)} \max_{s \in [0,\epsilon]} |(v_1 - v_3)(s)|, \tag{5.15}$$

$$|(v_1 - v_3)(s)| \leq \frac{C s^{2+l}}{(N+l)(2+l)} \max_{s \in [0,\epsilon]} |(u_1 - u_3)(s)|, \tag{5.16}$$

where $C = p \max_{s \in [0, \frac{1}{2}]} \max\{u_1^{p-1}(s), u_3^{p-1}(s)\}$. Therefore,

$$\max_{s \in [0,\epsilon]} |(u_1 - u_3)(s)| \leq \frac{\epsilon^{2-\alpha}}{(N-\alpha)(2-\alpha)} \max_{s \in [0,\epsilon]} |(v_1 - v_3)(s)|, \tag{5.17}$$

$$\max_{s \in [0,\epsilon]} |(v_1 - v_3)(s)| \leq \frac{C\epsilon^{2+l}}{(N+l)(2+l)} \max_{s \in [0,\epsilon]} |(u_1 - u_3)(s)|. \tag{5.18}$$

Combining (5.17) and (5.18), we have

$$\max_{s \in [0,\epsilon]} |(u_1 - u_3)(s)| \leq \frac{C\epsilon^{4+\tau}}{(N-\alpha)(2-\alpha)(N+l)(2+l)} \max_{s \in [0,\epsilon]} |(u_1 - u_3)(s)|. \tag{5.19}$$

Choosing ϵ such that (note that $4 + \tau > 0$)

$$\frac{C\epsilon^{4+\tau}}{(N-\alpha)(2-\alpha)(N+l)(2+l)} < 1,$$

we see that

$$(u_1 - u_3)(s) \equiv 0 \;\text{ for } s \in [0, \epsilon].$$

The ODE theory implies

$$u_1(s) \equiv u_3(s) \;\text{ for } s \in [0, R(\lambda)].$$

This also implies $\lambda = 1$ and

$$u_1(r) \equiv u_2(r) \quad \text{for } r \in [0, 1].$$

The proof of uniqueness result for $\alpha \in (4 - N, 2)$ and $l > -2$ of problem (N) is completed.

We now consider the case $\alpha \in (4 - N, 2)$ and $l = -2$ for problem (N). Let $u_1, u_2 \in D^{2,\alpha}_{rad}(B) \cap D^{1,\alpha}_{0,rad}(B)$ be two positive radial solutions of problem (N) for $1 \leq p < p_s$. It follows from Theorems 1.1 and 1.2 that $u_1, u_2 \in C^4(B\backslash\{0\}) \cap C^0(\overline{B})$, but $r = 0$ is a nonremovable singular point of $v_1, v_2 \in C^2(B\backslash\{0\})$.

Setting $\lambda^{\frac{(4+\tau)}{(p-1)}} = \frac{u_1(0)}{u_2(0)}$ and defining the function

$$u_3(s) = \lambda^{\frac{(4+\tau)}{(p-1)}} u_2(\lambda s) \quad s \in [0, \lambda^{-1}],$$

clearly we have u_3 satisfies the problem

$$\begin{cases} (s^{N-1}(s^\alpha \Delta u_3)')' = s^{N+l-1} u_3^p & \text{for } s \in (0, \frac{1}{\lambda}), \\ u_3(\frac{1}{\lambda}) = (\Delta u_3)(\frac{1}{\lambda}) = 0 \end{cases} \tag{5.20}$$

and

$$u_3(0) = u_1(0). \tag{5.21}$$

Let $v_1(s) = -s^\alpha (\Delta u_1)(s)$, $v_3(s) = -s^\alpha(\Delta u_3)(s)$. Then,

$$\begin{cases} -\Delta u_1 = s^{-\alpha} v_1 & \text{for } s \in (0,1), \\ -\Delta v_1 = s^l u_1^p & \text{for } s \in (0,1), \end{cases} \tag{5.22}$$

and

$$\begin{cases} -\Delta u_3 = s^{-\alpha} v_3 & \text{for } s \in (0,\lambda^{-1}), \\ -\Delta v_3 = s^l u_3^p & \text{for } s \in (0,\lambda^{-1}). \end{cases} \tag{5.23}$$

Moreover,

$$u_1'(s) < 0, \quad v_1'(s) < 0 \text{ for } s \in (0,1], \quad u_3'(s) < 0, \ v_3'(s) < 0 \text{ for } s \in (0, \lambda^{-1}], \tag{5.24}$$

and

$$u_1(s) > 0, \quad v_1(s) > 0 \text{ for } s \in (0,1), \quad u_3(s) > 0, \ v_3(s) > 0 \text{ for } s \in (0, \lambda^{-1}). \tag{5.25}$$

On the other hand, noticing from Theorem 1.2 that the asymptotic expansion of $v_1(s)$ at $s = 0$ is similar to that of $v_3(s)$, the only difference is the coefficients in the expansions, we can conclude that one of the following three cases occurs: (i) $\lim_{s\to 0}(v_1 - v_3)(s) > 0$ (maybe $+\infty$), (ii) $\lim_{s\to 0}(v_1 - v_3)(s) = 0$, (iii) $\lim_{s\to 0}(v_1 - v_3)(s) < 0$ (maybe $-\infty$). Note that if the case (i) occurs, there is a sufficiently small $\epsilon > 0$ such that $v_1(s) > v_3(s)$ for $s \in (0, \epsilon)$; if the case (iii) occurs, there is a sufficiently small $\tilde{\epsilon} > 0$ such that $v_1(s) < v_3(s)$ for $s \in (0, \tilde{\epsilon})$. Arguments similar to those in the proof of the case $\alpha \in (4-N, 2)$ and $l > -2$ imply that (i) and (iii) can not occur. Suppose (i) occurs, we can show that (5.11) holds and a contradiction is derived. A contradiction is also derived for (iii). Then, (ii) must hold. It follows from the asymptotic behavior obtained in Theorem 1.2 and $\lim_{s\to 0}(v_1 - v_3)(s) = 0$, we see that, for s near 0,

$$(v_1 - v_3)(s) = O(s^{2-\alpha} \ln s) \quad \text{for } l = -2. \tag{5.26}$$

Substituting (5.26) to

$$(u_1 - u_3)(s) = -\int_0^s t^{1-N} \int_0^t \rho^{N-\alpha-1}(v_1 - v_3)(\rho) d\rho dt,$$

we obtain

$$(u_1 - u_3)(s) = O(s^{4-2\alpha} \ln s), \tag{5.27}$$

where we use $(u_1 - u_3)(0) = 0$. Note $3 - 2\alpha > -1$ since $\alpha < 2$. Substituting (5.27) to

$$(v_1 - v_3)(s) = -\int_0^s t^{1-N} \int_0^t \rho^{N+l-1}(u_1^p - u_3^p) d\rho dt,$$

we obtain

$$(v_1 - v_3)(s) = O(s^{4-2\alpha} \ln s) = O(s^{2(2-\alpha)} \ln s), \tag{5.28}$$

where we use $(v_1 - v_3)(s) \to 0$ as $s \to 0$. Note that $u_1^p - u_3^p = p\xi^{p-1}(u_1 - u_3)$, where $\xi(s)$ is between $u_1(s)$ and $u_3(s)$. Repeating the same procedure, we obtain that, for any large positive integer n,

$$(u_1 - u_3)(s) = O(s^{n(2-\alpha)} \ln s). \tag{5.29}$$

Note that a simple calculation implies that the constants arising from the integrations do not make any trouble here. This also implies that, for s near 0,

$$(u_1 - u_3)(s) \equiv 0. \tag{5.30}$$

The ODE theory implies $(u_1 - u_3)(s) \equiv 0$ for $s \in [0, R(\lambda)]$. Therefore, we obtain $\lambda = 1$ and

$$u_1(r) \equiv u_2(r) \quad \text{for } r \in [0, 1].$$

The proof of uniqueness result for $\alpha \in (4 - N, 2)$ and $l = -2$ of problem (N) is completed.

We now consider the case $\alpha \in (4-N, 2)$ and $l \in (\alpha - 4, -2)$ for problem (N). Let $u_1, u_2 \in D^{2,\alpha}_{rad}(B) \cap D^{1,\alpha}_{0,rad}(B)$ be two positive radial solutions of problem (N) for $1 \le p < p_s$. It follows from Theorems 1.1 and 1.2 that $u_1, u_2 \in C^4(B\backslash\{0\}) \cap C^0(\overline{B})$, but $r = 0$ is a nonremovable singular point of $v_1, v_2 \in C^2(B\backslash\{0\})$.

Setting $\lambda^{\frac{(4+\tau)}{(p-1)}} = \frac{u_1(0)}{u_2(0)}$ and defining the function

$$u_3(s) = \lambda^{\frac{(4+\tau)}{(p-1)}} u_2(\lambda s) \quad s \in [0, \lambda^{-1}]$$

as the above, we will show $\lambda = 1$.

Let $(u_1(s), v_1(s))$ and $(u_3(s), v_3(s))$ be defined as in the proof of the case $\alpha \in (4 - N, 2)$ and $l = -2$. It follows from (3) of Theorem 1.2 that, for any $n \ge 2$ and $-\frac{(\alpha+4n-2)}{n} < \tau < -\frac{(\alpha+4n-6)}{(n-1)}$ (note $\tau < -(\alpha + 2)$ is equivalent to $l < -2$),

$$(n-1)q + \frac{(N'-4)}{2}[(n-1)p - (n-2)] < \frac{(N-\alpha)}{2} < nq + \frac{(N'-4)}{2}[np - (n-1)]$$

and, for s near 0,

$$v_1(s) = k_1 s^{\mu_1} + k_2 s^{\mu_2} + \ldots + k_{n-1} s^{\mu_{n-1}} + k_n + k_{n+1} s^{\mu_{n+1}} + o(s^{\mu_{n+1}}),$$
$$v_3(s) = l_1 s^{\mu_1} + l_2 s^{\mu_2} + \ldots + l_{n-1} s^{\mu_{n-1}} + l_n + l_{n+1} s^{\mu_{n+1}} + o(s^{\mu_{n+1}}),$$

where $k_1, k_2, \ldots, k_n, k_{n+1}, l_1, l_2, \ldots, l_n, l_{n+1}$ are constants with $k_1 > 0$ and $l_1 > 0$,

$$\mu_1 = q + \frac{(N'-4)}{2}p - \frac{(N-\alpha)}{2} < 0,$$
$$\mu_2 = 2q + \frac{(N'-4)}{2}(2p-1) - \frac{(N-\alpha)}{2} < 0, \ldots,$$
$$\mu_{n-1} = (n-1)q + \frac{(N'-4)}{2}[(n-1)p - (n-2)] - \frac{(N-\alpha)}{2} < 0, \quad \mu_n = 0,$$
$$\mu_{n+1} = nq + \frac{(N'-4)}{2}[np - (n-1)] - \frac{(N-\alpha)}{2} > 0.$$

From the expansions of $v_1(s)$ and $v_3(s)$, we can conclude that one of the following three cases occurs: (i) $\lim_{s\to 0}(v_1 - v_3)(s) > 0$ (maybe $+\infty$), (ii) $\lim_{s\to 0}(v_1 - v_3)(s) = 0$, (iii) $\lim_{s\to 0}(v_1 - v_3)(s) < 0$ (maybe $-\infty$). Arguments similar to those in the proof of the case $\alpha \in (4 - N, 2)$ and $l > -2$ imply that (i) and (iii) can not occur. Then, (ii) must hold. It follows from the asymptotic behaviors and $\lim_{s\to 0}(v_1 - v_3)(s) = 0$, we see that, for s near 0,

$$(v_1 - v_3)(s) = O(s^{\mu_{n+1}}). \tag{5.31}$$

Note that

$$\begin{aligned}\mu_{n+1} &= nq + \frac{(N'-4)}{2}[np-(n-1)] - \frac{(N-\alpha)}{2}\\ &= n(q+\frac{(N'-4)}{2}p) - (n-1)\frac{(N'-4)}{2} - \frac{(N-\alpha)}{2}\\ &= \frac{n}{2}(N'+4+2\tau) - (n-1)\frac{(N'-4)}{2} - \frac{(N-\alpha)}{2}\\ &= \alpha + n\tau + 4n - 2.\end{aligned}$$

Substituting (5.31) to

$$(u_1-u_3)(s) = -\int_0^s t^{1-N}\int_0^t \rho^{N-\alpha-1}(v_1-v_3)(\rho)d\rho dt,$$

we have

$$(u_1-u_3)(s) = O(s^{n(4+\tau)}), \tag{5.32}$$

where we use $(u_1-u_3)(0)=0$. Substituting (5.32) to

$$(v_1-v_3)(s) = -\int_0^s t^{1-N}\int_0^t \rho^{N+l-1}(u_1^p-u_3^p)d\rho dt,$$

we obtain

$$(v_1-v_3)(s) = O(s^{2+l+n(4+\tau)}), \tag{5.33}$$

where we use $(v_1-v_3)(s)\to 0$ as $s\to 0$. Note that $(\alpha<2)$,

$$\tau > -\frac{(\alpha+4n-2)}{n} > -\frac{(2+\alpha+4n)}{n+1}$$

and

$$2+l+n(4+\tau) = 2+\alpha+4n+(n+1)\tau > 0.$$

Using $(u_1-u_3)(0)=0$ and substituting (5.33) to

$$(u_1-u_3)(s) = -\int_0^s t^{1-N}\int_0^t \rho^{N-\alpha-1}(v_1-v_3)(\rho)d\rho dt$$

again, we obtain

$$(u_1-u_3)(s) = O(s^{(n+1)(4+\tau)}). \tag{5.34}$$

Repeating this procedure, we obtain

$$(u_1-u_3)(s) \equiv 0 \ \text{ for } s \text{ near } 0. \tag{5.35}$$

This implies $\lambda=1$ and $u_1(r)\equiv u_2(r)$ for $r\in[0,1]$.

The uniqueness result for the case of $\tau = -\frac{(\alpha+4n-2)}{n}$ with any $n\geq 2$ can be obtained similarly. The proof of uniqueness result for $\alpha\in(4-N,2)$ and $l\in(\alpha-4,-2)$ of problem (N) is completed.

We now consider the case $\alpha\in(4-N,2)$ and $l>-2$ for problem (D). Let $u_1,u_2\in D^{2,\alpha}_{0,rad}(B)$ be two positive radial solutions of problem (D) for $1\leq p<p_s$. It follows from Theorem 1.1 that $u_1,u_2\in C^4(B\backslash\{0\})\cap C^0(\overline{B})$, $v_1,v_2\in C^2(B\backslash\{0\})\cap C^0(\overline{B})$.

Setting $\lambda^{\frac{(4+\tau)}{(p-1)}} = \frac{u_1(0)}{u_2(0)}$ and defining the function

$$u_3(s) = \lambda^{\frac{(4+\tau)}{(p-1)}}u_2(\lambda s) \quad s\in[0,\lambda^{-1}],$$

clearly we have u_3 satisfies the problem

$$\begin{cases} (s^{N-1}(s^\alpha\Delta u_3)')' = s^{N+l-1}u_3^p & \text{for } s\in(0,\frac{1}{\lambda}),\\ u_3(\frac{1}{\lambda}) = u_3'(\frac{1}{\lambda}) = 0 \end{cases} \tag{5.36}$$

and

$$u_3(0) = u_1(0). \tag{5.37}$$

Let $(u_1(s), v_1(s))$ and $(u_3(s), v_3(s))$ be defined as in the proof of problem (N). We also know from (2.14) and (2.15) that

$$u_1'(s) < 0, \quad v_1'(s) < 0 \text{ for } s \in (0,1), \quad u_3'(s) < 0, \quad v_3'(s) < 0 \text{ for } s \in (0, \lambda^{-1}),$$

$$u_1(s) > 0 \text{ for } s \in [0,1), \quad u_3(s) > 0 \text{ for } s \in [0, \lambda^{-1}),$$

and

$$u_1(1) = 0, \quad u_3(\lambda^{-1}) = 0.$$

Let $R(\lambda) = \min\{1, \lambda^{-1}\}$. We now show

$$v_1(0) = v_3(0). \tag{5.38}$$

On the contrary, without loss of generality, we assume $v_1(0) > v_3(0)$. Then, arguments similar to those in the proof of (5.11) imply

$$v_1(s) > v_3(s) \quad \text{for } s \in [0, R(\lambda)]. \tag{5.39}$$

From this, $u_1(0) = u_3(0)$ and $-(s^{N-1}(u_1 - u_3)')' = s^{N-\alpha-1}(v_1 - v_3)$, we obtain

$$u_1(s) < u_3(s) \quad \text{for } s \in (0, R(\lambda)], \tag{5.40}$$

and

$$(u_1 - u_3)'(s) < 0 \quad \text{for } s \in (0, R(\lambda)]. \tag{5.41}$$

Note that $s^{N-1}(u_1 - u_3)'(s) \to 0$ as $s \to 0$. Since

$$(u_1 - u_3)(R(\lambda)) = \begin{cases} u_1(\lambda^{-1}) & \text{if } \lambda > 1, \\ 0 & \text{if } \lambda = 1, \\ -u_3(1) & \text{if } \lambda < 1, \end{cases}$$

we deduce that necessarily $\lambda < 1$. On the other hand, for $\lambda < 1$, we have $R(\lambda) = 1$ and

$$(u_1 - u_3)'(1) = -u_3'(1) > 0.$$

This contradicts (5.41) and hence (5.38) holds. Arguments similar to those in the proof of the case $\alpha \in (4-N, 2)$ and $l > -2$ for problem (N) above imply that $\lambda = 1$ and

$$u_1(r) \equiv u_2(r) \quad \text{for } r \in [0,1].$$

The proof of uniqueness result for $\alpha \in (4-N, 2)$ and $l > -2$ of problem (D) is completed.

The proof of uniqueness result for $\alpha \in (4-N, 2)$ and $l \in (\alpha - 4, -2]$ of problem (D) is exactly same as that of the same case of problem (N) above. The proof of uniqueness result for $\alpha \in (4-N, 2)$ and $l \in (\alpha-4, -2]$ of problem (D) is completed. These also complete the proof of Theorem 1.3. □

Remark 2. It is known from Theorem 1.1 that if $\alpha \in [2, N)$ and $\tau > -4$, u is u_N or u_D, then $v_N \in C^2(B\backslash\{0\}) \cap C^0(\overline{B})$ or $v_D \in C^2(B\backslash\{0\}) \cap C^0(\overline{B})$. In order to use arguments similar to those in the proof of Theorem 1.3 to obtain the uniqueness result for this case, we can make scaling on v, i.e., for (u_1, v_1) and (u_2, v_2), we set

$$\lambda^{\frac{(2-\alpha)(p-1)+4+\tau}{(p-1)}} = \frac{v_1(0)}{v_2(0)}$$

and

$$u_3(s) = \lambda^{\frac{(4+\tau)}{p-1}} u_2(\lambda s), \quad v_3(s) = \lambda^{\frac{(2-\alpha)(p-1)+4+\tau}{(p-1)}} v_2(\lambda s),$$

we have that (u_3, v_3) satisfies

$$\begin{cases} -\Delta u_3 = s^{-\alpha} v_3 & \text{for } s \in (0, \lambda^{-1}), \\ -\Delta v_3 = s^l u_3^p & \text{for } s \in (0, \lambda^{-1}), \end{cases}$$

and

$$v_3(0) = v_1(0).$$

Therefore, if we can show that there are sufficiently small positive $\epsilon_1, \epsilon_2, \epsilon_3$ such that one of the three cases occurs: (i) $(u_1 - u_3)(s) > 0$ for $s \in (0, \epsilon_1)$, (ii) $(u_1 - u_3)(s) \equiv 0$ for $s \in (0, \epsilon_2)$, (iii) $(u_1 - u_3)(s) < 0$ for $s \in (0, \epsilon_3)$ (actually we can show that one of the three cases occurs by using the asymptotic expansions near $s = 0$ of $u_1(s)$ and $u_3(s)$, since we can obtain the asymptotic expansions up to arbitrary orders at $s = 0$ of $u_1(s)$ and $u_3(s)$), we see that (ii) must hold and the uniqueness result is obtained.

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