ELECTRONIC RESEARCH ARCHIVE
Volume **29**, Number **6**, December **2021**

doi:10.3934/era.2021060
pp. **3761–3774**

UNIQUENESS AND NONDEGENERACY OF POSITIVE SOLUTIONS TO AN ELLIPTIC SYSTEM IN ECOLOGY

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Abstract. In this paper, we study the follwing important elliptic system which arises from the Lotka-Volterra ecological model in $\mathbb{R}^N$

$$\begin{cases} -\Delta u + \lambda u = \mu_1 u^2 + \beta uv, & x \in \mathbb{R}^N, \\ -\Delta v + \lambda v = \mu_2 v^2 + \beta uv, & x \in \mathbb{R}^N, \\ u, v > 0, u, v \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \leq 5$, λ, μ_1, μ_2 are positive constants, $\beta \geq 0$ is a coupling constant. Firstly, we prove the uniqueness of positive solutions under general conditions, then we show the nondegeneracy of the positive solution and the degeneracy of semi-trivial solutions. Finally, we give a complete classification of positive solutions when $\mu_1 = \mu_2 = \beta$.

1. Introduction.

We study the following coupled elliptic system:

$$\begin{cases} -\Delta u + \lambda u = \mu_1 u^2 + \beta uv, & x \in \mathbb{R}^N, \\ -\Delta v + \lambda v = \mu_2 v^2 + \beta uv, & x \in \mathbb{R}^N, \\ u, v > 0, u, v \in H^1(\mathbb{R}^N), \end{cases} \tag{1}$$

where $N \leq 5$, λ, μ_1, μ_2 are positive constants, $\beta \geq 0$ is a coupling constant. This system is related to the steady state of the following Lotka-Volterra ecological model (please see [22, 8, 6, 5, 27, 18, 21, 17, 16, 11] and the references therein):

$$\begin{cases} u_t - d_1 \Delta u = u(a - bu + cv), & (x,t) \in \Omega \times (0,T), \\ v_t - d_2 \Delta v = v(d + eu - fv), & (x,t) \in \Omega \times (0,T), \\ u, v > 0, & (x,t) \in \Omega \times (0,T), \end{cases} \tag{2}$$

2020 *Mathematics Subject Classification.* Primary: 35J15, 35J47; Secondary: 35J57.

Key words and phrases. Lotka-Volterra system, positive solution, nondegeneracy.

Dedicated to Professor E. N. Dancer's 75th Birthday. Supported by National Natural Science Foundation of China (11771428,12031015, 12026217).

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with Dirichlet ($u = v = 0, (x,t) \in \partial\Omega \times (0,T)$) or Neumann ($u_\nu = v_\nu = 0, (x,t) \in \partial\Omega \times (0,T)$) boundary condition, and $\Omega \subset \mathbb{R}^N$ is a C^2 domain (possibly unbounded), $u(x,t)$ and $v(x,t)$ represent the population densities of two species respectively,$T \in (0,\infty]$, $d_1, d_2 > 0$,

$$a, b, c, d, e, f \in L^\infty\left(\Omega \times (0,\infty)\right).$$

The coefficients a, d describe their intrinsic growth rates (positive or not), $b, f \geq 0$ represent self-limitation of each species, c, e are the interaction coefficients between the species. In the case $c, e > 0$, the system is referred to as a cooperative model; when $c, e < 0$, it is a competitive system; when $c < 0, e > 0$, it is a predator-prey model. In particular, as $d_1 = d_2 = 1$, we have

$$\begin{cases} u_t - \Delta u = u(a - bu + cv), & (x,t) \in \Omega \times (0,T), \\ v_t - \Delta v = v(d + eu - fv), & (x,t) \in \Omega \times (0,T), \\ u, v > 0, & (x,t) \in \Omega \times (0,T). \end{cases} \tag{3}$$

Dancer-Zhang in [8] and Dang-Wang-Zhang in [6] study the dynamics of the competing system in a bounded domain and prove that the solution converges to a stationary point under strong competition for two species and multiple species respectively. Dancer-Wang-Zhang in [5, 27] and Zhang in [27] study the uniform Hölder continuity for competing species in a bounded domain. In [22] Quittner provides Liouville theorems, universal estimates and periodic solutions for cooperative systems. In [18] Julián López-Gómez et al. describe the coexistence states for the predator-prey model with periodic coefficients and analyzes the dynamics of positive solutions of such models. The steady state system of (3) with Dirichlet boundary condition is

$$\begin{cases} -\Delta u = u(a - bu + cv), & x \in \Omega, \\ -\Delta v = v(d + eu - fv), & x \in \Omega, \\ u, v > 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{4}$$

By rescaling, we can readily assume $b = f = 1$, then

$$\begin{cases} -\Delta u = u(a - u + cv), & x \in \Omega, \\ -\Delta v = v(d + eu - v), & x \in \Omega, \\ u, v > 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{5}$$

When Ω is bounded, this system has been studied intensively in [4, 12, 9, 13, 20] and the references therein. We state some results here. For all cases ($c, e \in \mathbb{R}$), Korman and Leung in [12] give the existence of steady state solutions. For the competing case ($c, e < 0$), Cosner and Lazer in [4] investigate the existence, uniqueness, and stability of coexistence states for competing species. In [9] Gui and Lou prove the existence and uniqueness of positive solutions when $\bar{\lambda}_1 < a = d < \bar{\lambda}_2$ ($\bar{\lambda}_i$ is the i_{th} eigenvalue of $-\Delta$ in $H_0^1(\Omega)$). In [13] Korman and Leung give some nonexistence results. For cooperative system ($c, e > 0$), Korman and Leung in [13] give the necessary and sufficient condition for the existence of a positive solution of (5) when $a > d > \bar{\lambda}_1$, and prove the solution is unique when c or e is small. Lou in [20] shows the nonexistence results of positive solutions when $a < \bar{\lambda}_1,\ d < \bar{\lambda}_1,\ ce \leq 1$ and the existence when $a < \bar{\lambda}_1,\ d < \bar{\lambda}_1,\ N \leq 5,\ ce > 1$, and proves the nonexistence when

$a = d < (N - 6/N)\bar{\lambda}_1,\ N \geq 6$, Ω is a star-shaped domain and $(e+1)u = (c+1)v$ whenever $a = d$.

In [26] Wei and Yao discuss some results about the corresponding Schrödinger equations in $\mathbb{R}^N$, in [28] Zhang and Wang study the structure of positive solutions to the corresponding Schrödinger system in bounded domains. The main difference between system (1) and the Schrödinger system is that (1) does not have a variational structure. We are going to discuss the uniqueness and the nondegeneracy for positive solutions of (1) in $\mathbb{R}^N$. When Ω is bounded, people are able to make use of the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$. However, there is only essential spectrum for $-\Delta$ in $\mathbb{R}^N$, so there are more difficulties for (1) in $\mathbb{R}^N$.

In this paper, we first prove the uniqueness of positive solutions for (1), then we show the nondegeneracy of the positive solution and the degeneracy of semi-trivial solutions. We also give a complete classification of positive solutions as $\mu_1 = \mu_2 = \beta$. By [3], we know any positive solution of (1) is radial and decreases with respect to a point in $\mathbb{R}^N$. With the help of classical bootstrap argument, the solution (u, v) of (1) is in $C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$ and tends to 0 as $|x| \to \infty$.

From now on, we assume the solutions of (1) are radial with respect to the origin. We denote the subspace of radial functions of $H^1(\mathbb{R}^N)$ by $H_r^1(\mathbb{R}^N)$.

Our results are dependent on the solution of the following basic equation

$$\begin{cases} -\Delta w + w = w^2,\ x \in \mathbb{R}^N, 1 \leq N \leq 5, \\ w(0) = \max\limits_{\mathbb{R}^N} w(x),\ w > 0,\ w(x) \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{6}$$

By [14], we know (6) has a unique positive solution w. Besides, w is radial, decreasing and with the decay rate

$$\lim_{|y|\to\infty} w(y)\mathrm{e}^{|y|}\, |y|^{\frac{N-1}{2}} = \alpha_0 > 0, \qquad \lim_{|y|\to\infty} \frac{w'(y)}{w(y)} = -1, \tag{7}$$

for some constant $\alpha_0 > 0$ (see [10]). We note that the restriction $1 \leq N \leq 5$ guarantees the power 2 in (6) is subcritical.

Then for $0 \leq \beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$, the system admits a positive solution of the form

$$(u_0, v_0) = \left(\frac{\lambda(\beta - \mu_2)}{\beta^2 - \mu_1\mu_2} w\left(\sqrt{\lambda}x\right), \frac{\lambda(\beta - \mu_1)}{\beta^2 - \mu_1\mu_2} w\left(\sqrt{\lambda}x\right) \right). \tag{8}$$

If $\mu_1 = \mu_2 =: \mu$, this simplifies to

$$u_0 = v_0 = \frac{\lambda}{\beta + \mu} w\left(\sqrt{\lambda}x\right).$$

In fact, for $N = 1$, the unique positive solution of (6) is explicitly of the form

$$w(x) = \frac{6e^x}{(1 + e^x)^2}. \tag{9}$$

Our first result is the following uniqueness property of (u_0, v_0) in $\mathbb{R}^1$.

Theorem 1.1. *Assume $N = 1$, $0 \leq \beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$, then (u_0, v_0) in (8) with the exact following form*

$$(u_0, v_0) = \left((\beta - \mu_2) \frac{6\lambda \mathrm{e}^{\sqrt{\lambda}x}}{(\beta^2 - \mu_1\mu_2)\left(1 + \mathrm{e}^{\sqrt{\lambda}x}\right)^2}, (\beta - \mu_1) \frac{6\lambda \mathrm{e}^{\sqrt{\lambda}x}}{(\beta^2 - \mu_1\mu_2)\left(1 + \mathrm{e}^{\sqrt{\lambda}x}\right)^2} \right)$$

is the unique positive solution to system (1). Furthermore, both u_0 and v_0 decay like $\mathrm{e}^{-\sqrt{\lambda}|x|}$ *as* $|x| \to \infty$.

Remark 1. In fact, if $\min\{\mu_1, \mu_2\} < \beta < \max\{\mu_1, \mu_2\}$, there is no nontrivial solution for system (1). In fact, we multiply the first equation by v and multiply the second equation by u respectively, then integrate over $\mathbb{R}^N$,

$$\begin{cases} -\int_{\mathbb{R}^N} \nabla u \cdot \nabla v - \int_{\mathbb{R}^N} \lambda uv + \int_{\mathbb{R}^N} \mu_1 u^2 v + \int_{\mathbb{R}^N} \beta uv^2 = 0, \\ -\int_{\mathbb{R}^N} \nabla u \cdot \nabla v - \int_{\mathbb{R}^N} \lambda uv + \int_{\mathbb{R}^N} \mu_2 uv^2 + \int_{\mathbb{R}^N} \beta u^2 v = 0. \end{cases}$$

Taking the difference, we have

$$\int_{\mathbb{R}^N} uv\,[(\mu_1 - \beta)u + (\beta - \mu_2)v] = 0.$$

Then if $\min\{\mu_1, \mu_2\} < \beta < \max\{\mu_1, \mu_2\}$, there is no nontrivial solution to system (1).

The second result is the uniqueness of (u_0, v_0) when $\beta > \max\{\mu_1, \mu_2\}$ in higher dimension.

Theorem 1.2. *Assume $\beta > \max\{\mu_1, \mu_2\}$, $2 \le N \le 5$, then (u_0, v_0) in (8) is the unique positive solution to system (1). In addition, both u_0 and v_0 decay like* $\mathrm{e}^{-\sqrt{\lambda}|x|}\,|x|^{\frac{1-N}{2}}$ *as* $|x| \to \infty$ *by (7).*

Remark 2. By [3, Theorem 1], we know any positive solution of (1) is radial and decreases with respect to a point in $\mathbb{R}^N$ $(N \le 5)$. What is more, the condition of $N \le 5$ is presumed to guarantee the positive solution of (6) exists and is unique by [14]. In fact, the expression in (8) relies on the positive solution of (6).

Definition 1.3. We say that (U, V) is a nondegenerate solution of system (1) if the solution set of the linearized system

$$\begin{cases} \Delta\phi_1 - \lambda\phi_1 + 2\mu_1 U\phi_1 + \beta\phi_1 V + \beta U\phi_2 = 0, \\ \Delta\phi_2 - \lambda\phi_2 + 2\mu_2 V\phi_2 + \beta\phi_1 V + \beta U\phi_2 = 0. \end{cases} \tag{10}$$

is exactly N-dimensional, specifically,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \sum_{j=1}^{N} a_j \begin{pmatrix} \frac{\partial U}{\partial x_j} \\ \frac{\partial V}{\partial x_j} \end{pmatrix}$$

for some constants a_j. If there are other solutions to system (10), we say (U, V) is degenerate.

From the mathematical point of view, the nondegeneracy of positive solutions is an important property for constructing concentrating solutions, that is, we usually use the nondegeneracy to construct single or multiple spike solutions (see [10, 26]).

The following theorem is the nondegeneracy property of (u_0, v_0) for system (1).

Theorem 1.4. *Assume $\beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$, then the solution (u_0, v_0) of system (1) in (8) is nondegenerate.*

Now we consider the degeneracy of semi-trivial solutions. System (1) has semi-trivial solutions $(\bar{u}, 0)$, $(0, \bar{v})$, where $\bar{u} = w_1 := \dfrac{\lambda}{\mu_1} w(\sqrt{\lambda}x)$, $\bar{v} = w_2 := \dfrac{\lambda}{\mu_2} w(\sqrt{\lambda}x)$, w is the unique positive solution of (6) and $\bar{u}, \bar{v}$ satisfy

$$-\Delta\bar{u} + \lambda\bar{u} = \mu_1\bar{u}^2,\ x \in \mathbb{R}^N,$$

$$-\Delta \bar{v} + \lambda \bar{v} = \mu_2 \bar{v}^2, \ x \in \mathbb{R}^N.$$

Define

$$a_0 := \inf_{\phi \in H^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \left(|\nabla \phi|^2 + \lambda \phi^2\right)}{\int_{\mathbb{R}^N} w_1 \phi^2}, \quad b_0 := \inf_{\phi \in H^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \left(|\nabla \phi|^2 + \lambda \phi^2\right)}{\int_{\mathbb{R}^N} w_2 \phi^2}.$$

Proposition 1. *For the solution $(u, v) = (\bar{u}, 0)$ as $\beta = a_0$, or $(u, v) = (0, \bar{v})$ as $\beta = b_0$, then the linearized problem of (1) has exactly an one-dimensional set of solutions in $H_r^1(\mathbb{R}^N)$.*

The next result brings about the complete classification of all positive solutions of system (1) when $\mu_1 = \mu_2 = \beta$.

Theorem 1.5. *Suppose that $N \leq 5$ and $\mu_1 = \mu_2 = \beta > 0$, then all positive solutions of system (1) have the following form*

$$(u(x), v(x)) = \left(\frac{C}{1+C} \frac{\lambda}{\beta} w\left(\sqrt{\lambda} x\right), \frac{1}{1+C} \frac{\lambda}{\beta} w\left(\sqrt{\lambda} x\right) \right), \ C > 0, \tag{11}$$

where w is the unique positive solution of equation (6). Moreover, all positive solutions (u, v) decay like $\mathrm{e}^{-\sqrt{\lambda}|x|} |x|^{\frac{1-N}{2}}$ as $|x| \to \infty$ by (7). In particular, for $\lambda = \mu_1 = \mu_2 = \beta = 1$, system (1) has infinitely many positive solutions

$$(u(x), v(x)) = \left(\frac{C}{1+C} w(x), \frac{1}{1+C} w(x) \right), \ C > 0.$$

2. **Proof of Theorem 1.1.** We prove the uniqueness of positive solution for system (1) in one dimension.

Proof of Theorem 1.1. When $N = 1$, wo only need to consider the interval $[0, +\infty)$ due to the symmetry. The system can be simplified as

$$\begin{cases} u'' - \lambda u + \mu_1 u^2 + \beta uv = 0, \\ v'' - \lambda v + \mu_2 v^2 + \beta uv = 0, \\ u(r), v(r) > 0, \\ u'(0) = v'(0) = 0, u(r), v(r) \to 0 \text{ as } r \to \infty. \end{cases}$$

With the help of variable substitution $(u, v) \to \left(u\left(\frac{x}{\sqrt{\lambda}}\right), v\left(\frac{x}{\sqrt{\lambda}}\right) \right)$, we assume $\lambda = 1$. Suppose $(u, \bar{v})$ is a positive solution, then

$$\begin{cases} u'' - u + \mu_1 u^2 + \beta u\bar{v} = 0, \\ \bar{v}'' - \bar{v} + \mu_2 \bar{v}^2 + \beta u\bar{v} = 0, \\ u(r), \bar{v}(r) > 0, \\ u'(0) = \bar{v}'(0) = 0, u(r), \bar{v}(r) \to 0 \text{ as } r \to \infty. \end{cases}$$

Set $a = \dfrac{\beta - \mu_1}{\beta - \mu_2}$ and let $v = \bar{v}/a$. Then (u, v) solves

$$\begin{cases} u'' - u + \mu_1 u^2 + a\beta uv = 0, \\ v'' - v + \mu_2 a v^2 + \beta uv = 0, \\ u(r), v(r) > 0, \\ u'(0) = v'(0) = 0, u(r), v(r) \to 0 \text{ as } r \to \infty. \end{cases}$$

Step 1. We multiply the equation of u by v, then

$$(u'v)' - u'v' - uv + \mu_1 u^2 v + a\beta uv^2 = 0. \tag{12}$$

In a similar way, we multiply the equation of v by u,

$$(v'u)' - u'v' - uv + \mu_2 auv^2 + \beta u^2 v = 0. \tag{13}$$

Take the difference (12)-(13),

$$(u'v - v'u)' + uv(\mu_1 u + a\beta v - \mu_2 av - \beta u) = 0.$$

Since $a = \dfrac{\beta - \mu_1}{\beta - \mu_2}$, the above identity can be reduced to

$$(u'v - v'u)' + uv(\mu_1 - \beta)(u - v) = 0. \tag{14}$$

Integrate (14) over $[0, +\infty)$ and make use of $u'(0) = 0 = v'(0) = u(+\infty) = v(+\infty)$, we have

$$(\mu_1 - \beta)\int_0^{+\infty} uv(u - v)\mathrm{d}r = 0.$$

When $\mu_1 \neq \beta$, from the above identity, we know that if $u \geq v$ or $u \leq v$, then $u \equiv v$.

Step 2. We intend to prove $u - v$ does not change sign.

If not, then $u - v$ changes sign. Set $f = u - v$, then f satisfies

$$\begin{aligned}
0 &= f'' - f + \mu_1 u^2 - \mu_2 av^2 + a\beta uv - \beta uv \\
&= f'' - f + \mu_1 u(u - v) + \mu_1 uv - \mu_2 av(v - u) - \mu_2 auv + a\beta uv - \beta uv \\
&= f'' - f + f(\mu_1 u + \mu_2 av) + uv(\mu_1 - \mu_2 a + a\beta - \beta) \\
&= f'' - f + f(\mu_1 u + \mu_2 av).
\end{aligned}$$

That is, f satisfies

$$f'' + f(\mu_1 u + \mu_2 av - 1) = 0,$$

which is a linear equation for f. By the isolated properties of the zeroes for nontrivial solutions of linear ordinary differential equations, $u - v$ cannot equal to 0 in any nonempty interval. Therefore, if $u - v$ changes sign infinitely many times, the zeros of f should tend to ∞ (set as $+\infty$ up to a variable change). Assume $x_0 > 0$ is a zero for a nontrivial solution which is large enough , f satisfies

$$\begin{cases}
f'' + f(\mu_1 u + \mu_2 av - 1) = 0, & x \in (x_0, +\infty), \\
f(x_0) = 0, \\
f(x) \to 0 \text{ as } r \to +\infty.
\end{cases}$$

Since $u(r), v(r) \to 0$ as $r \to +\infty$ and x_0 is large, we have $\mu_1 u + \mu_2 av - 1 < 0$, $\forall x \in (x_0, +\infty)$. Hence by strong maximum principle, f is equal to 0 in $(x_0, +\infty)$, which is a contradiction with the isolated property of the zeroes. Consequently $u - v$ changes sign only finite times. Thus there exists r_1 large enough such that

$$u(r_1) - v(r_1) = 0, \quad u(r) - v(r) > 0, \ \forall r > r_1. \tag{15}$$

As a result,

$$u'(r_1) - v'(r_1) \geq 0. \tag{16}$$

Integrating (14) over $(r_1, +\infty)$, we get

$$-\left(u'v - v'u\right)(r_1) + (\mu_1 - \beta)\int_{r_1}^{+\infty} uv(u - v)\mathrm{d}r = 0. \tag{17}$$

When $\beta > \mu_1$, by (15) and (16), we obtain $(\mu_1 - \beta)\int_{r_1}^{+\infty} uv(u-v) < 0$ and $-\left(u'v - v'u\right)(r_1) \le 0$, which contradicts with (17). Hence the assumption fails and $u - v$ does not change sign.

When $\beta < \mu_1$, we have $(\mu_1 - \beta)\int_{r_1}^{+\infty} uv(u-v) > 0$, and thus $\left(u'v - v'u\right)(r_1) > 0$.

We claim: there exists $r_2 > r_1$ such that

$$\left(u'v - v'u\right)(r_2) = 0.$$

If the claim holds, then integrate (14) over $(r_2, +\infty)$ we obtain

$$0 = (\mu_1 - \beta)\int_{r_1}^{+\infty} uv(u-v)\mathrm{d}r > 0,$$

which is a contradiction. Hence the assumption fails, and $u - v$ does not change sign.

Step 3. We prove the above claim.

If not, since $\left(u'v - v'u\right)(r_1) > 0$ and $u'v - v'u$ is continuous, we have

$$\left(u'v - v'u\right)(r) > 0,\ \forall r \in (r_1, +\infty). \tag{18}$$

By multiplying the equation of u by u', we get

$$u''u' - u'u + \mu_1 u^2 u' + \beta a u u' v = 0,$$

thus

$$\frac{1}{2}\left[(u')^2\right]' - \frac{1}{2}\left(u^2\right)' + \frac{\mu_1}{3}\left(u^3\right)' + \frac{\beta a}{3}\left(u^2 v\right)' + \frac{\beta a}{3}u(u'v - v'u) = 0. \tag{19}$$

Similarly, multiply the equation of v by v',

$$v''v' - v'v + \mu_2 a v^2 v' + \beta u v' v = 0,$$

thus

$$\frac{1}{2}\left[(v')^2\right]' - \frac{1}{2}\left(v^2\right)' + \frac{\mu_2 a}{3}\left(v^3\right)' + \frac{\beta}{3}\left(uv^2\right)' + \frac{\beta}{3}v(v'u - u'v) = 0 \tag{20}$$

Take the difference (19)-(20), then

$$\begin{aligned}&\frac{1}{2}\left[(u')^2 - (v')^2\right]' - \frac{1}{2}\left(u^2 - v^2\right)' + \frac{\mu_1}{3}\left(u^3\right)' - \frac{\mu_2 a}{3}\left(v^3\right)'\\ &+ \frac{\beta a}{3}\left(u^2 v\right)' - \frac{\beta}{3}\left(uv^2\right)' + \frac{\beta a}{3}u(u'v - v'u) - \frac{\beta}{3}v(v'u - u'v) = 0.\end{aligned} \tag{21}$$

Integrate (21) over $(r_1, +\infty)$, we have

$$\begin{aligned}&-\frac{1}{2}\left[(u')^2 - (v')^2\right](r_1) + \frac{1}{2}\left(u^2 - v^2\right)(r_1) - \frac{\mu_1}{3}u^3(r_1) + \frac{\mu_2 a}{3}v^3(r_1) - \frac{\beta a}{3}\left(u^2 v\right)(r_1)\\ &+\frac{\beta}{3}\left(uv^2\right)(r_1) + \int_{r_1}^{+\infty}\left[\frac{\beta a}{3}u(u'v - v'u) - \frac{\beta}{3}v(v'u - u'v)\right]\mathrm{d}r = 0.\end{aligned}$$

Since $u(r_1) = v(r_1)$, the identity is reduced to

$$\begin{aligned}&-\frac{1}{2}\left[(u')^2 - (v')^2\right](r_1) + \frac{u^3(r_1)}{3}[-\mu_1 + \mu_2 a - \beta a + \beta]\\ &+\frac{\beta}{3}\int_{r_1}^{+\infty}(au + v)(u'v - v'u)\mathrm{d}r = 0,\end{aligned}$$

thus we have

$$-\frac{1}{2}\left[(u')^2-(v')^2\right](r_1)+\frac{\beta}{3}\int_{r_1}^{+\infty}(au+v)(u'v-v'u)\mathrm{d}r=0. \tag{22}$$

Because $0>u'(r_1)>v'(r_1)$, we have $-\frac{1}{2}\left[(u')^2-(v')^2\right](r_1)>0$, and by (18) we know $\frac{\beta}{3}\int_{r_1}^{+\infty}(au+v)(u'v-v'u)\mathrm{d}r>0$, which contradicts with (22). Therefore, the claim is true.

Finally, we obtain $u=v$. Then u satisfies

$$u''-u+(\mu_1+\beta a)u^2=0.$$

With the substitution $w=(\mu_1+\beta a)u$, then w solves

$$w''-w+w^2=0.$$

Similarly, v solves

$$v''-v+(\mu_2 a+\beta)v^2=0.$$

With the substitution $w=(\mu_2 a+\beta)v$, then w satisfies

$$w''-w+w^2=0.$$

Then by (9), we see w is unique and of the form $w(x)=\frac{6e^x}{(1+e^x)^2}$. Hence the unique positive solution of system (1) is

$$\left((\beta-\mu_2)\frac{6\lambda \mathrm{e}^{\sqrt{\lambda}x}}{(\beta^2-\mu_1\mu_2)\left(1+\mathrm{e}^{\sqrt{\lambda}x}\right)^2},(\beta-\mu_1)\frac{6\lambda \mathrm{e}^{\sqrt{\lambda}x}}{(\beta^2-\mu_1\mu_2)\left(1+\mathrm{e}^{\sqrt{\lambda}x}\right)^2}\right)$$

and the decay rate is $\mathrm{e}^{-\sqrt{\lambda}|x|}$ as $|x|$ goes to infinity. In this way, we have proved the existence and uniqueness of positive solutions. □

3. **Proof of Theorem 1.2.**

Proof of Theorem 1.2. Firstly, by [3, Theorem 1], we know any positive solution of (1) is radial and decreases with respect to a point in $\mathbb{R}^N$ $(N\le 5)$. Moreover, the condition of $N\le 5$ is presumed to guarantee the positive solution of (6) exists and is unique by [14]. In fact, the expression in (8) relies on the positive solution of (6). Also it is straightforward to verify that the (u_0,v_0) in (8) is a positive solution to system (1). From now on, we focus on the uniqueness of (u_0,v_0).

Let (u_1,u_2) be a positive solution of system (1) and presume $\beta>\mu_2\ge\mu_1$. Set $a=\dfrac{\beta-\mu_1}{\beta-\mu_2}\ge 1$ and $\bar{u}_2=u_2/a$, then $(u_1,\bar{u}_2)$ solves

$$\begin{cases}\Delta u_1-\lambda u_1+\mu_1 u_1^2+\beta a u_1\bar{u}_2=0,\\ \Delta\bar{u}_2-\lambda\bar{u}_2+\mu_2 a\bar{u}_2^2+\beta u_1\bar{u}_2=0,\\ u_1,\bar{u}_2>0,\ u_1,\bar{u}_2\in H^1(\mathbb{R}^N)\end{cases}$$

Notice that $\mu_2 a-\mu_1=\beta(a-1)>0$, then

$$\begin{aligned}\Delta(u_1-\bar{u}_2)-\lambda(u_1-\bar{u}_2)&=-\mu_1u_1^2+\mu_2a\bar{u}_2^2-\beta a u_1\bar{u}_2+\beta u_1\bar{u}_2\\&=-\mu_1u_1^2+\left[\mu_1+\beta(a-1)\right]\bar{u}_2^2-\beta(a-1)u_1\bar{u}_2\\&=-\mu_1\left(u_1^2-\bar{u}_2^2\right)-\beta(a-1)\bar{u}_2(u_1-\bar{u}_2).\end{aligned}$$

That is,

$$\Delta(u_1-\bar{u}_2)-\lambda(u_1-\bar{u}_2)=-(u_1-\bar{u}_2)\left[\mu_1 u_1+\mu_2 a\bar{u}_2\right].$$

Define $\Omega_+=\left\{x\in\mathbb{R}^N \mid u_1(x)>\bar{u}_2(x)\right\}$, then Ω_+ is a piece-wise C^1 domain. In fact, since u_1, $\bar{u}_2$ are radially non-increasing functions, $u_1-\bar{u}_2$ is radial and with isolated zeroes from the perspective of ODE.

Now multiplying the equation of u_1 by $\bar{u}_2$ and integrating over Ω_+, we get

$$\int_{\partial\Omega_+}\frac{\partial u_1}{\partial n}\bar{u}_2-\int_{\Omega_+}\nabla u_1\cdot\nabla\bar{u}_2-\int_{\Omega_+}\lambda u_1\bar{u}_2+\int_{\Omega_+}\left(\mu_1 u_1^2\bar{u}_2+\beta a u_1\bar{u}_2^2\right)=0. \quad (23)$$

Multiplying the equation of $\bar{u}_2$ by u_1 and integrating over Ω_+ we obtain

$$\int_{\partial\Omega_+}\frac{\partial\bar{u}_2}{\partial n}u_1-\int_{\Omega_+}\nabla u_1\cdot\nabla\bar{u}_2-\int_{\Omega_+}\lambda u_1\bar{u}_2+\int_{\Omega_+}\left(\mu_2 a u_1\bar{u}_2^2+\beta u_1^2\bar{u}_2\right)=0. \quad (24)$$

With the calculation of (23)-(24),

$$\int_{\partial\Omega_+}\left(\frac{\partial u_1}{\partial n}\bar{u}_2-\frac{\partial\bar{u}_2}{\partial n}u_1\right)+\int_{\Omega_+}u_1\bar{u}_2\left(\mu_1 u_1+\beta a\bar{u}_2-\mu_2 a\bar{u}_2-\beta u_1\right)=0.$$

It can be furthermore reduced to

$$\int_{\partial\Omega_+}\left(\frac{\partial u_1}{\partial n}\bar{u}_2-\frac{\partial\bar{u}_2}{\partial n}u_1\right)+\int_{\Omega_+}u_1\bar{u}_2\left(\mu_1-\beta\right)\left(u_1-\bar{u}_2\right)=0. \quad (25)$$

On the one hand, Since $u_1(x)-\bar{u}_2(x)>0$ in Ω_+ and $u_1(x)-\bar{u}_2(x)=0$ on $\partial\Omega_+$, we know

$$\frac{\partial(u_1-\bar{u}_2)}{\partial n}\Big|_{\partial\Omega_+}\leq 0.$$

Therefore

$$\int_{\partial\Omega_+}\left(\frac{\partial u_1}{\partial n}\bar{u}_2-\frac{\partial\bar{u}_2}{\partial n}u_1\right)\leq 0.$$

On the other hand, because $\mu_1<\beta$ and $u_1(x)-\bar{u}_2(x)>0$ in Ω_+, we have

$$\int_{\Omega_+}u_1\bar{u}_2\left(\mu_1-\beta\right)\left(u_1-\bar{u}_2\right)\leq 0.$$

Compared with (25), we see that $\Omega_+=\emptyset$.

Similarly, we can show $\Omega_-=\left\{x\in\mathbb{R}^N \mid u_1(x)<\bar{u}_2(x)\right\}=\emptyset$. Therefore $u_1=\bar{u}_2$. Then u_1 satisfies

$$\Delta u_1-\lambda u_1+(\mu_1+\beta a)u_1^2=0.$$

Set $\tilde{u}_1:=u_1\left(\frac{x}{\sqrt{\lambda}}\right)$, then $\tilde{u}_1$ solves

$$\Delta\tilde{u}_1-\tilde{u}_1+\frac{\mu_1+\beta a}{\lambda}\tilde{u}_1^2=0.$$

Then $w:=\frac{\mu_1+\beta a}{\lambda}\tilde{u}_1$ solves $\Delta w-w+w^2=0$, which is the unique solution of (6) and decays like $\mathrm{e}^{-|x|}|x|^{\frac{1-N}{2}}$ by (7). Therefore

$$u_1=\frac{\lambda}{\mu_1+\beta a}w\left(\sqrt{\lambda}x\right)=\frac{\lambda(\beta-\mu_2)}{\beta^2-\mu_1\mu_2}w\left(\sqrt{\lambda}x\right),$$

$$u_2=a\bar{u}_2=au_1=\frac{a\lambda}{\mu_1+\beta a}w\left(\sqrt{\lambda}x\right)=\frac{\lambda(\beta-\mu_1)}{\beta^2-\mu_1\mu_2}w\left(\sqrt{\lambda}x\right),$$

which is the same as the (u_0, v_0) in (8). Furthermore, both u_1 and u_2 decay like $e^{-\sqrt{\lambda}|x|}|x|^{\frac{1-N}{2}}$ by the above expression and (7). Consequently we have proved the (u_0, v_0) in (8) is the unique positive solution to system (1). □

Remark 3. The conclusion does not hold for all $\beta > 0$ since there are the examples when $\mu_1 = \mu_2 = \beta$ (see Theorem 1.5).

4. **Proof of Theorem 1.4 and Proposition 1.**

We shall prove the nondegeneracy of the unique solution of system (1).

Proof of Theorem 1.4. The linearized system for system (1) is

$$\begin{cases} \Delta\phi_1 - \lambda\phi_1 + 2\mu_1 u\phi_1 + \beta v\phi_1 + \beta u\phi_2 = 0, \\ \Delta\phi_2 - \lambda\phi_2 + 2\mu_2 v\phi_2 + \beta v\phi_1 + \beta u\phi_2 = 0. \end{cases}$$

Without loss of generality, we assume $\lambda = 1$, then $(u_0, v_0) = (c_1 w, c_2 w) := \left(\frac{\beta-\mu_2}{\beta^2-\mu_1\mu_2}w, \frac{\beta-\mu_1}{\beta^2-\mu_1\mu_2}w\right)$. The linearized system at (u_0, v_0) is reduced to

$$\begin{cases} \Delta\phi_1 - \phi_1 + 2\mu_1 c_1 w\phi_1 + \beta c_2 w\phi_1 + \beta c_1 w\phi_2 = 0, \\ \Delta\phi_2 - \phi_2 + 2\mu_2 c_2 w\phi_2 + \beta c_2 w\phi_1 + \beta c_1 w\phi_2 = 0. \end{cases} \tag{26}$$

That is,

$$\begin{cases} \Delta\phi_1 - \phi_1 + [2\mu_1 c_1 + \beta c_2]w\phi_1 + \beta c_1 w\phi_2 = 0, \\ \Delta\phi_2 - \phi_2 + [2\mu_2 c_2 + \beta c_1]w\phi_2 + \beta c_2 w\phi_1 = 0. \end{cases}$$

With the orthogonal transformation:

$$\begin{cases} \Phi_1 = \phi_1 + \phi_2, \\ \Phi_2 = (\beta-\mu_1)\phi_1 + (\mu_2-\beta)\phi_2, \end{cases}$$

then Φ_1, Φ_2 solve

$$\begin{cases} \Delta\Phi_1 - \Phi_1 + 2w\Phi_1 = 0, \\ \Delta\Phi_2 - \Phi_2 + (\mu_2 c_2 + \mu_1 c_1)w\Phi_2 = 0. \end{cases}$$

By [25, Lemma 4.1], the eigenvalues for

$$\Delta\Phi - \Phi + \alpha w\Phi = 0, \ \Phi \in H^1(\mathbb{R}^N)$$

are

$$\alpha_1 = 1, \ \alpha_2 = \cdots = \alpha_{N+1} = 2, \ \alpha_{N+2} > 2,$$

with eigenspaces

$$V_1 = span\{w\}, \ V_2 = span\left\{\frac{\partial w}{\partial x_j}, j = 1, \cdots, N\right\}.$$

From which we obtain $\Phi_1 \in span\left\{\frac{\partial w}{\partial x_j}, j = 1, \cdots, N\right\}$, therefore

$$\phi_1 + \phi_2 \in span\left\{\frac{\partial w}{\partial x_j}, j = 1, \cdots, N\right\}. \tag{27}$$

Moreover, the condition $\beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$ implies that

$$1 \neq \mu_2 c_2 + \mu_1 c_1 = \frac{\mu_2(\beta-\mu_1) + \mu_1(\beta-\mu_2)}{\beta^2-\mu_1\mu_2} < 2,$$

then $\Phi_2 = 0$, and consequently

$$(\beta - \mu_1)\phi_1 + (\mu_2 - \beta)\phi_2 = 0. \tag{28}$$

Finally, we conclude from (27), (28) and the expression of (u_0, v_0) that the solution set of linearized system (26) is exactly N-dimensional, namely,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \sum_{j=1}^{N} a_j \begin{pmatrix} \frac{\partial u_0}{\partial x_j} \\ \frac{\partial v_0}{\partial x_j} \end{pmatrix} \tag{29}$$

for some constants a_j. Then the positive solution (u_0, v_0) is nondegenerate by Definition 1.3. □

We begin to prove the degeneracy of semi-trivial solutions of system (1).

Proof of Proposition 1. **1)** Assume $\beta = a_0, (u, v) = (\bar{u}, 0) = (w_1, 0)$, the linearized system is

$$\begin{cases} \Delta\phi_1 - \lambda\phi_1 + 2\mu_1\bar{u}\phi_1 + a_0\bar{u}\phi_2 = 0, \\ \Delta\phi_2 - \lambda\phi_2 + a_0\bar{u}\phi_2 = 0. \end{cases} \tag{30}$$

It is easily seen that a_0 can be attained by a radial positive function ψ_0 (we assume $\psi_0(0) = 1$). Then by the definition of a_0, there exists a constant c such that $\phi_2 = c\psi_0$. And via the equation for $\bar{u}$ we know $a_0 \le \mu_1$. With the substitution

$$\Phi = (2\mu_1 - a_0)\phi_1 + a_0\phi_2,$$

then (Φ, ϕ_2) solves

$$\begin{cases} \Delta\Phi - \lambda\Phi + 2\mu_1\bar{u}\Phi = 0, \\ \Delta\phi_2 - \lambda\phi_2 + a_0\bar{u}\phi_2 = 0. \end{cases}$$

Set $\tilde{\Phi}(x) = \Phi\left(\dfrac{x}{\sqrt{\lambda}}\right)$ and make use of $\mu_1\bar{u} = \lambda w(\sqrt{\lambda}x)$, then

$$\begin{cases} \Delta\tilde{\Phi} - \tilde{\Phi} + 2w\tilde{\Phi} = 0, \\ \Delta\phi_2 - \lambda\phi_2 + a_0\bar{u}\phi_2 = 0. \end{cases}$$

Again by [25, Lemma 4.1], we know $\tilde{\Phi} = 0$, which means $\Phi = 0$, i.e. $\phi_1 = -\dfrac{a_0}{2\mu_1 - a_0}\phi_2$. Finally we get the radial solution set of (30) is

$$\left\{ \left(-\frac{a_0}{2\mu_1 - a_0} c\psi_0, c\psi_0 \right), c \in \mathbb{R} \right\}.$$

Which means the radial solution set of linearized system is one-dimensional. That is, the solution set of the linearized system is at least $N+1$ dimensional. Consequently, the semi-trivial solution $(\bar{u}, 0)$ is degenerate by Definition 1.3.

2) Assume $\beta = b_0, (u, v) = (0, \bar{v}) = (0, w_2)$, the linearized system is

$$\begin{cases} \Delta\phi_1 - \lambda\phi_1 + b_0\bar{v}\phi_1 = 0, \\ \Delta\phi_2 - \lambda\phi_2 + 2\mu_2\bar{v}\phi_2 + b_0\bar{v}\phi_1 = 0. \end{cases} \tag{31}$$

Similarly, we know b_0 can be attained at a radial positive function ϕ_0 (assume $\phi_0(0) = 1$), and there exists a constant c such that $\phi_1 = c\phi_0$. Moreover via the equation of $\bar{v}$ we get $b_0 \le \mu_2$. With the substitution

$$\Phi = (2\mu_2 - b_0)\phi_2 + b_0\phi_1,$$

then (ϕ_1, Φ) solves

$$\begin{cases} \Delta\phi_1 - \lambda\phi_1 + b_0\bar{v}\phi_1 = 0, \\ \Delta\Phi - \lambda\Phi + 2\mu_2\bar{v}\Phi = 0. \end{cases}$$

Set $\tilde{\Phi}(x) = \Phi\left(\frac{x}{\sqrt{\lambda}}\right)$ and recall $\mu_2\bar{v} = \lambda w(\sqrt{\lambda}x)$, then

$$\begin{cases} \Delta\phi_1 - \lambda\phi_1 + b_0\bar{v}\phi_1 = 0, \\ \Delta\tilde{\Phi} - \tilde{\Phi} + 2w\tilde{\Phi} = 0. \end{cases}$$

Use [25, Lemma 4.1] one more time we know $\tilde{\Phi} = 0$, so $\Phi = 0$, which means $\phi_2 = -\frac{b_0}{2\mu_2 - b_0}\phi_1$. Finally we get the radial solution set of (31) is

$$\left\{ \left(c\phi_0, -\frac{b_0}{2\mu_2 - b_0}c\phi_0, \right), c \in \mathbb{R} \right\}.$$

Hence the radial solution set of linearized system is one-dimensional. Which means the radial solution set of linearized system is one-dimensional. That is, the solution set of the linearized system is at least $N + 1$ dimensional. Consequently, the semi-trivial solution $(0, \bar{v})$ is degenerate by Definition 1.3. □

5. **Proof of Theorem 1.5.**

Proof of Theorem 1.5. Firstly, we recall that by [3, Theorem 1] any positive solution of (1) is radial and decreases with respect to a point in $\mathbb{R}^N$ $(N \leq 5)$. Moreover, the condition of $N \leq 5$ is presumed to guarantee the positive solution of (6) exists and is unique by [14]. In fact, the expression in (11) relies on the positive solution of (6). Also it is straightforward to verify that any (u, v) in (11) is a positive solution to system (1) with the assumptions of Theorem 1.5. From now on, we concentrate on proving that any positive solution to system (1) with the assumptions of Theorem 1.5 is of the form (11).

In this case, the solution (u, v) is a radial function and satisfies

$$\begin{cases} \Delta u - \lambda u + \beta u^2 + \beta uv = 0, \\ \Delta v - \lambda v + \beta v^2 + \beta uv = 0, \\ u(r), v(r) > 0, \\ u'(0) = v'(0) = 0, u(r), v(r) \to 0 \text{ as } r \to \infty. \end{cases}$$

Via the transformation

$$\tilde{u}(x) = \frac{\beta}{\lambda}u\left(\frac{x}{\sqrt{\lambda}}\right), \ \tilde{v}(x) = \frac{\beta}{\lambda}v\left(\frac{x}{\sqrt{\lambda}}\right),$$

we know that $(\tilde{u}, \tilde{v})$ solves

$$\begin{cases} \Delta\tilde{u} - \tilde{u} + \tilde{u}^2 + \tilde{u}\tilde{v} = 0, \\ \Delta\tilde{v} - \tilde{v} + \tilde{v}^2 + \tilde{u}\tilde{v} = 0, \\ \tilde{u}(r), \tilde{v}(r) > 0, \\ \tilde{u}'(0) = \tilde{v}'(0) = 0, \tilde{u}(r), \tilde{v}(r) \to 0 \text{ as } r \to \infty. \end{cases}$$

After multiplying the equation for $\tilde{u}$ by $\tilde{v}$, we obtain

$$\tilde{v}\Delta\tilde{u} - \tilde{u}\tilde{v} + \tilde{u}^2\tilde{v} + \tilde{u}\tilde{v}^2 = 0. \tag{32}$$

Similarly, we have

$$\tilde{u}\Delta\tilde{v} - \tilde{v}\tilde{u} + \tilde{v}^2\tilde{u} + \tilde{u}^2\tilde{v} = 0. \tag{33}$$

Take the difference (32) − (33), we get

$$\tilde{v}\Delta\tilde{u} - \tilde{u}\Delta\tilde{v} = 0.$$

Integrating over the ball B_r, we see that

$$\begin{aligned} 0 &= \int_{B_r} (\tilde{v}\Delta\tilde{u} - \tilde{u}\Delta\tilde{v}) = \int_{B_r} div\ \ (\tilde{v}\nabla\tilde{u} - \tilde{u}\nabla\tilde{v}) \\ &= \int_{\partial B_r} (\tilde{v}\tilde{u}' - \tilde{u}\tilde{v}') = |\partial B_1| r^{N-1} (\tilde{v}\tilde{u}' - \tilde{u}\tilde{v}')(r), \end{aligned}$$

which implies $\tilde{u}'\tilde{v} - \tilde{v}'\tilde{u} \equiv 0$. It readily follows that

$$\frac{\tilde{u}}{\tilde{v}} \equiv Constant.$$

Then $\tilde{v}$ satisfies

$$\Delta\tilde{v} - \tilde{v} + (C+1)\tilde{v}^2 = 0,$$

for some positive constant C. Consequently, $w := (C+1)\tilde{v}$ satisfies $\Delta w - w + w^2 = 0$, which is the unique solution of (6) and decays like $\mathrm{e}^{-|x|}\,|x|^{\frac{1-N}{2}}$ by (7). Therefore

$$u(x) = \frac{C}{1+C}\frac{\lambda}{\beta} w\left(\sqrt{\lambda}x\right),\ v(x) = \frac{1}{1+C}\frac{\lambda}{\beta} w\left(\sqrt{\lambda}x\right),\ C > 0,$$

which is of the form (11). Further, both u and v decay like $\mathrm{e}^{-\sqrt{\lambda}|x|}\,|x|^{\frac{1-N}{2}}$ by the above expression and (7). Therefore we have finished the classification of all positive solutions to system (1) with the assumptions of Theorem 1.5. □

Remark 4. One may study the following interesting coupled elliptic system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^2 + \beta uv, & x \in \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 v^2 + \beta uv, & x \in \mathbb{R}^N, \\ u, v > 0, u, v \in H^1(\mathbb{R}^N), \end{cases} \tag{34}$$

where $\lambda_1 \neq \lambda_2$, the uniqueness and the nondegeneracy for positive solutions to system (34) are still open, we will continue to do research on these problems.

Acknowledgments. The authors would like to thank the reviewers for the valuable suggestions and comments to improve the manuscript.

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Received March 2021; 1st revision May 2021; 2nd revision June 2021; early access August 2021.

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