

CONSTRUCTIONS OF THREE KINDS OF BIHOM-SUPERALGEBRAS

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ABSTRACT. The purpose of this paper is to study the constructions between Bihom-alternative superalgebras and Bihom-Malcev superalgebras and Bihom-Jordan superalgebras. First, we explain in detail that every regular Bihom-alternative superalgebra could be Bihom-Malcev-admissible superalgebra or Bihom-Jordan-admissible superalgebra. Next, the bimodules and T_θ^* -extensions of Bihom-alternative superalgebras are also discussed as properties of Bihom-alternative superalgebras.

1. Introduction. As for the non-associative algebras that have been gradually developed on the development path of associative rings and associative algebras, non-associative superalgebras are also important for superalgebras that delete the associativity of multiplication on the basis of associative superalgebras, and then satisfy certain identities. The basis of this paper is more typical non-associative superalgebra, namely alternative superalgebras, Malcev superalgebras and Jordan superalgebras.

As we know, Malcev superalgebra is a generalization of Lie superalgebra. Of course, it is not Lie superalgebra. It is a class of non-associative algebra that is very useful in the geometry of smooth rings. Therefore, it has also been widely concerned in recent years [5, 6, 21]. Also Jordan superalgebra has a very important position in quantum mechanics. For interested readers, please refer to the literature [2, 16, 18, 22, 20]. Closely related to Malcev superalgebras and Jordan superalgebras are alternative superalgebras, which is a superalgebra whose associator is a super-alternating function [25]. In particular, all associative superalgebras are alternative. We have known that alternative superalgebras are Malcev-admissible and Jordan-admissible. More related results on alternative, Malcev and Jordan (super)algebra can be found in [7, 11, 14].

In the past few years, many scholars have conducted indepth research on twisted algebras (also known as Hom-algebras or Bihom-algebras) excited by q-deformation of vector field algebras. We can find more related results in [8, 9, 10, 12, 13, 15, 19, 23, 24, 26, 27]. We noticed recently that quadratic Hom-Malcev superalgebras were studied in [17] and Hom-alternative, Hom-Malcev and Hom-Jordan superalgebras were studied in [1]. A Bihom-algebra is an algebra whose identity defining structure

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is twisted by two homomorphisms α and β . This type of algebra is an extension of Hom-algebras introduced through a category approach in [3, 4].

The main purpose of this paper is to study Bihom-type generalizations of alternative superalgebras, Malcev (-admissible) superalgebras and Jordan (-admissible) superalgebras. Outline of the paper is as follows. In Section 2, we give the definition of Bihom-alternative superalgebras, and provide the construction approach to bidirectional between Bihom-alternative superalgebras and alternative superalgebras. In Section 3, we introduce Bihom-Malcev(-admissible) superalgebras and their basic construction methods. Besides that, we show that every regular Bihom-alternative superalgebra gives rise to a Bihom-Malcev superalgebra via the Bihom-supercommutator bracket. In other words, regular Bihom-alternative superalgebras are all Bihom-Malcev admissible superalgebras, generalizing the well-known fact that Hom-alternative superalgebras (resp. alternative superalgebras) are Hom-Malcev admissible (resp. Malcev admissible superalgebras). In the meanwhile, $\mathbb{Z}_2$ -graded Bihom-type Bruck-Kleinfeld function f is super-alternating and has been studied. In Section 4, we study bimodules of Bihom-alternative superalgebras and discuss its T_θ^* -extension. In Section 5, we introduce Bihom-Jordan(-admissible) superalgebras. Then we give a construction result from Jordan-admissible superalgebras to Bihom-Jordan admissible superalgebras. In the end, we present a main theorem, which says that regular Bihom-alternative superalgebras are Bihom-Jordan admissible.

Throughout the paper, all algebraic systems are supposed to be over a field K of characteristic 0. Let A be a linear superspace over K that is a $\mathbb{Z}_2$ -graded linear space with a direct sum $A = A_0 \oplus A_1$. The elements of $A_j, j = 0, 1$, are said to be homogenous and of parity j . The parity of a homogeneous element x is denoted by $|x|$. In the sequel, we will denote by $\mathcal{H}(A)$ the set of all homogeneous elements of A . In this paper, we need to use the elements, all of which are not specified, are homogeneous.

2. Bihom-alternative superalgebras. In this section, we introduce Bihom-alternative superalgebras, study their general properties and provide some construction results. A Bihom-superalgebra is a quadruple (A, μ, α, β) , where $\mu : A \otimes A \rightarrow A$ is an even bilinear map and $\alpha, \beta : A \rightarrow A$ are even linear maps such that $\alpha \circ \mu = \mu \circ (\alpha \otimes \alpha)$ and $\beta \circ \mu = \mu \circ (\beta \otimes \beta)$ (multiplicativity).

Definition 2.1. Let (A, μ, α, β) be a Bihom-superalgebra.

(1) The Bihom-associator of A is even trilinear map $\tilde{a}s_{\alpha, \beta} : A^{\otimes 3} \rightarrow A$ defined by

$$\tilde{a}s_{\alpha, \beta} = \mu \circ (\mu \otimes \beta - \alpha \otimes \mu). \quad (1)$$

For any $x, y, z \in \mathcal{H}(A)$, $\tilde{a}s_{\alpha, \beta}(x, y, z) = \mu(\mu(x, y), \beta(z)) - \mu(\alpha(x), \mu(y, z))$.

(2) The Bihom-super-Jacobiator of A is even trilinear map $\tilde{J}_{\alpha, \beta} : A^{\otimes 3} \rightarrow A$ defined as

$$\tilde{J}_{\alpha, \beta}(x, y, z) = \circlearrowleft_{x, y, z} (-1)^{|x||z|} \mu(\beta^2(x), \mu(\beta(y), \alpha(z))). \quad (2)$$

In particular, when $\alpha = \beta = \text{id}$, as Bihom-superalgebra is to degenerate to the superalgebra, so are Bihom-associator and Bihom-super-Jacobiator degenerate to the original associator and super-Jacobiator.

Definition 2.2. [23] A Bihom-associative superalgebra is a quadruple (A, μ, α, β) , where $\alpha, \beta : A \rightarrow A$ are even linear maps and $\mu : A \times A \rightarrow A$ is an even bilinear map

such that $\alpha\beta = \beta\alpha, \alpha\mu = \mu\alpha^{\otimes 2}, \beta\mu = \mu\beta^{\otimes 2}$ and satisfying the following condition:

$$\tilde{a}s_{\alpha,\beta}(x, y, z) = 0, \text{ for all } x, y, z \in \mathcal{H}(A). \text{ (Bihom-associativity condition)}$$

Clearly, when $\alpha = \beta$, we get a Hom-associative superalgebra.

Definition 2.3. A left Bihom-alternative superalgebra (resp. right Bihom-alternative superalgebra) is a quadruple (A, μ, α, β) where $\alpha, \beta : A \rightarrow A$ are even linear maps and $\mu : A \times A \rightarrow A$ is an even bilinear map such that $\alpha\beta = \beta\alpha, \alpha\mu = \mu\alpha^{\otimes 2}, \beta\mu = \mu\beta^{\otimes 2}$ and satisfying the left Bihom-alternative super-identity, i.e. for all $x, y, z \in \mathcal{H}(A)$,

$$\tilde{a}s_{\alpha,\beta}(\beta(x), \alpha(y), z) + (-1)^{|x||y|}\tilde{a}s_{\alpha,\beta}(\beta(y), \alpha(x), z) = 0, \quad (3)$$

respectively, the right Bihom-alternative super-identity,

$$\tilde{a}s_{\alpha,\beta}(x, \beta(y), \alpha(z)) + (-1)^{|y||z|}\tilde{a}s_{\alpha,\beta}(x, \beta(z), \alpha(y)) = 0. \quad (4)$$

A Bihom-alternative superalgebra is one which is both a left and right Bihom-alternative superalgebra.

Observe that when $\alpha = \beta = \text{id}$ the left Bihom-alternative super-identity (resp. right Bihom-alternative super-identity) reduces to left alternative super-identity (resp. right alternative super-identity).

Definition 2.4. Let (A, μ, α, β) and $(A', \mu', \alpha', \beta')$ be two Bihom-alternative superalgebras. An even linear map $f : A \rightarrow A'$ is said to be a Bihom-alternative superalgebra homomorphism if the following conditions hold

$$f\mu = \mu'(f \otimes f) \text{ (weak morphism condition)}, \quad f\alpha = \alpha'f, \quad f\beta = \beta'f. \quad (5)$$

Definition 2.5. A Bihom-superalgebra (A, μ, α, β) is called

- (1) regular if α and β are algebra automorphisms.
- (2) involutive if α and β are two involutions, that is $\alpha^2 = \beta^2 = \text{id}$.

Proposition 1. Let (A, μ, α, β) be a Bihom-superalgebra such that $\alpha\beta = \beta\alpha, \alpha\mu = \mu\alpha^{\otimes 2}$ and $\beta\mu = \mu\beta^{\otimes 2}$. Then (A, μ, α, β) is a regular Bihom-alternative superalgebra if and only if, for all $x, y, z \in \mathcal{H}(A)$,

$$\tilde{a}s_{\alpha,\beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z)) + (-1)^{|x||y|}\tilde{a}s_{\alpha,\beta}(\beta^2(y), \alpha\beta(x), \alpha^2(z)) = 0 \quad (6)$$

and

$$\tilde{a}s_{\alpha,\beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z)) + (-1)^{|y||z|}\tilde{a}s_{\alpha,\beta}(\beta^2(x), \alpha\beta(z), \alpha^2(y)) = 0. \quad (7)$$

Proof. Straightforward by Definition 2.3. $\square$

Remark 1. Let (A, μ, α, β) be a Bihom-superalgebra. Then A is a regular Bihom-alternative superalgebra if and only if $\tilde{a}s_{\alpha,\beta} \circ (\beta^2 \otimes \alpha\beta \otimes \alpha^2)$ is super-alternating.

Lemma 2.6. Let (A, μ, α, β) be a Bihom-alternative superalgebra. Then for all $x, y, z \in \mathcal{H}(A)$ we have

$$\tilde{a}s_{\alpha,\beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z)) = -(-1)^{|z||x|+|z||y|+|x||y|}\tilde{a}s_{\alpha,\beta}(\beta^2(z), \alpha\beta(y), \alpha^2(x)),$$

$$\tilde{a}s_{\alpha,\beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z)) = (-1)^{|z||x|+|z||y|}\tilde{a}s_{\alpha,\beta}(\beta^2(z), \alpha\beta(x), \alpha^2(y)).$$

Proof.

$$\begin{aligned}\tilde{as}_{\alpha,\beta}(\beta^2(x), \beta\alpha(y), \alpha^2(z)) &= -(-1)^{|z||y|} \tilde{as}_{\alpha,\beta}(\beta^2(x), \beta\alpha(z), \alpha^2(y)) \\ &= (-1)^{|z||y|+|z||x|} \tilde{as}_{\alpha,\beta}(\beta^2(z), \beta\alpha(x), \alpha^2(y)) \\ &= -(-1)^{|z||y|+|z||x|+|x||y|} \tilde{as}_{\alpha,\beta}(\beta^2(z), \beta\alpha(y), \alpha^2(x)).\end{aligned}$$

Another equation is obvious. $\square$

Next, we provide a way to construct Bihom-alternative superalgebras starting from an alternative superalgebra and two commuting alternative superalgebra morphisms. This approach was applied to other algebras such as Bihom-Lie algebras, Bihom-Novikov algebras and so on in [9], [10].

Definition 2.7. Let (A, μ) be a superalgebra and $\alpha, \beta : A \rightarrow A$ be two even algebra morphisms satisfying $\alpha\beta = \beta\alpha$. Define the Bihom-superalgebra induced by α and β as

$$A_{\alpha,\beta} = (A, \mu_{\alpha,\beta} = \mu(\alpha \otimes \beta), \alpha, \beta).$$

Theorem 2.8. (i) Let (A, μ) be a left alternative superalgebra (resp. right alternative superalgebra) and $\alpha, \beta : A \rightarrow A$ be two even left alternative superalgebra morphisms (resp. right alternative superalgebra morphisms) such that $\alpha\beta = \beta\alpha$. Then $A_{\alpha,\beta}$ is a left Bihom-alternative superalgebra (resp. right Bihom-alternative superalgebra).

(ii) Suppose that (A', μ') is another left alternative superalgebra (resp. right alternative superalgebra) and $\alpha', \beta' : A' \rightarrow A'$ be two commuting even left alternative superalgebra morphisms (resp. right alternative superalgebra morphisms). If $f : A \rightarrow A'$ is an even left alternative superalgebra morphism (resp. right alternative superalgebra morphism) that satisfies $f\alpha = \alpha'f$ and $f\beta = \beta'f$, then

$$f : A_{\alpha,\beta} \rightarrow A'_{\alpha',\beta'}$$

is a left Bihom-alternative superalgebra (resp. right Bihom-alternative superalgebra) morphism.

Proof. We show that $A_{\alpha,\beta}$ satisfies the left Bihom-alternative super-identity. For simplicity, we will write $\mu(x, y) = xy$ and $\mu_{\alpha,\beta}(x, y) = x \circ y$. Then for all $x, y, z \in \mathcal{H}(A_{\alpha,\beta})$, we have

$$\begin{aligned}\tilde{as}_{\alpha,\beta}(\beta(x), \alpha(y), z) &= (\beta(x) \circ \alpha\beta(y)) \circ \beta(z) - \alpha\beta(x) \circ (\alpha(y) \circ z) \\ &= (\alpha\beta(x)\alpha\beta(y)) \circ \beta(z) - \alpha\beta(x) \circ (\alpha^2(y)\alpha\beta(z)) \\ &= (\alpha^2\beta(x)\alpha^2\beta(y)) \beta^2(z) - \alpha^2\beta(x) (\alpha^2\beta(y)\beta^2(z)) \\ &= \tilde{as}(\alpha^2\beta(x), \alpha^2\beta(y), \beta^2(z)) \\ &= -(-1)^{|x||y|} \tilde{as}(\alpha^2\beta(y), \alpha^2\beta(x), \beta^2(z)) \\ &= -(-1)^{|x||y|} \tilde{as}_{\alpha,\beta}(\beta(y), \alpha(x), z).\end{aligned}$$

So conclusion (i) is finished. (5) follows from

$$f\mu_{\alpha,\beta} = f\mu(\alpha \otimes \beta) = \mu'(f \otimes f)(\alpha \otimes \beta) = \mu'(\alpha' \otimes \beta')(f \otimes f) = \mu'_{\alpha',\beta'}(f \otimes f).$$

That is, (ii) holds. $\square$

Remark 2. Similar to Theorem 2.8, we may also construct a new Bihom-alternative superalgebra starting with a given Bihom-alternative superalgebra and a pair of commuting Bihom-superalgebra homomorphisms.

On the contrary, we will give an approach to get an alternative superalgebra from Bihom-alternative superalgebra in the following proposition.

Proposition 2. Let (A, μ, α, β) be a regular Bihom-alternative superalgebra. Define a new multiplication on A by $x \diamond y = \mu(\alpha^{-1}(x), \beta^{-1}(y))$. Then $(A, \diamond)$ is an alternative superalgebra.

Proof. Denote $\mu(x, y) = xy$ and $x \diamond y = \alpha^{-1}(x)\beta^{-1}(y)$, for all $x, y \in \mathcal{H}(A)$. We just need to show

$$\tilde{as}(x, y, z) + (-1)^{|x||y|}\tilde{as}(y, x, z) = 0 \text{ (left alternative super-identity)}$$

and

$$\tilde{as}(x, y, z) + (-1)^{|y||z|}\tilde{as}(x, z, y) = 0. \text{ (right alternative super-identity)}$$

Since α and β are bijective and A is a Bihom-alternative superalgebra, we have

$$\begin{aligned} & \tilde{as}(x, y, z) + (-1)^{|x||y|}\tilde{as}(y, x, z) \\ = & (x \diamond y) \diamond z - x \diamond (y \diamond z) + (-1)^{|x||y|}(y \diamond x) \diamond z - (-1)^{|x||y|}y \diamond (x \diamond z) \\ = & (\alpha^{-2}(x)(\alpha\beta)^{-1}(y))\beta^{-1}(z) - \alpha^{-1}(x)((\alpha\beta)^{-1}(y)\beta^{-2}(z)) \\ & + (-1)^{|x||y|}(\alpha^{-2}(y)(\alpha\beta)^{-1}(x))\beta^{-1}(z) - (-1)^{|x||y|}\alpha^{-1}(y)((\alpha\beta)^{-1}(x)\beta^{-2}(z)) \\ = & \tilde{as}_{\alpha, \beta}(\alpha^{-2}(x), (\alpha\beta)^{-1}(y), \beta^{-2}(z)) + (-1)^{|x||y|}\tilde{as}_{\alpha, \beta}(\alpha^{-2}(y), (\alpha\beta)^{-1}(x), \beta^{-2}(z)) \\ = & \tilde{as}_{\alpha, \beta}(\beta(\beta^{-1}\alpha^{-2}(x)), \alpha(\beta^{-1}\alpha^{-2}(y)), \beta^{-2}(z)) \\ & + (-1)^{|x||y|}\tilde{as}_{\alpha, \beta}(\beta(\beta^{-1}\alpha^{-2}(y)), \alpha(\beta^{-1}\alpha^{-2}(x)), \beta^{-2}(z)) \\ = & 0. \end{aligned}$$

Similarly, right alternative super-identity is easy to prove. $\square$

Corollary 1. Let (A, μ, α, β) be an involutive Bihom-alternative superalgebra. Define a new multiplication on A by $x \star y = \mu(\alpha(x), \beta(y))$. Then $(A, \star)$ is an alternative superalgebra.

Proof. We write $\mu(x, y) = xy$ and $x \star y = \alpha(x)\beta(y)$, for all $x, y \in \mathcal{H}(A)$. It is easy to show its left and right alternative super-identities. $\square$

3. Bihom-Malcev superalgebras. In this section, we introduce Bihom-Malcev superalgebras (as a generalization of Bihom-Lie superalgebras) and provide a way to construct Bihom-Malcev superalgebras. Some general properties are studied. In the end, we show that Bihom-Malcev superalgebra could come into being from regular Bihom-alternative superalgebra via the Bihom-supercommutator bracket.

Definition 3.1. [23] A Bihom-Lie superalgebra is a 4-tuple $(A, [\cdot, \cdot], \alpha, \beta)$ consisting of $\alpha, \beta : A \rightarrow A$ are even linear maps and $[\cdot, \cdot] : A \times A \rightarrow A$ is an even bilinear map, which satisfies the following conditions: for all $x, y, z \in \mathcal{H}(A)$,

- (i) $\alpha\beta = \beta\alpha$,
- (ii) $\alpha([x, y]) = [\alpha(x), \alpha(y)]$, $\beta([x, y]) = [\beta(x), \beta(y)]$,
- (iii) $[\beta(x), \alpha(y)] = -(-1)^{|x||y|}[\beta(y), \alpha(x)]$, (Bihom-skew supersymmetry)
- (iv) $\tilde{J}_{\alpha, \beta}^c(x, y, z) = (-1)^{|x||z|}\tilde{J}_{\alpha, \beta}(x, y, z)$. (Bihom-Jacobi superidentity)
- ($\tilde{J}_{\alpha, \beta}(x, y, z) = \circlearrowleft_{x, y, z}(-1)^{|x||z|}[\beta^2(x), [\beta(y), \alpha(z)]] = 0$)

Definition 3.2. A Bihom-Malcev superalgebra is a 4-tuple $(A, [\cdot, \cdot], \alpha, \beta)$ consisting of $\alpha, \beta : A \rightarrow A$ are even linear maps and $[\cdot, \cdot] : A \otimes A \rightarrow A$ is an even bilinear map, which satisfies the following conditions: for all $x, y, z \in \mathcal{H}(A)$,

- (i) $\alpha\beta = \beta\alpha$,
- (ii) $\alpha([x, y]) = [\alpha(x), \alpha(y)], \beta([x, y]) = [\beta(x), \beta(y)],$
- (iii) $[\beta(x), \alpha(y)] = -(-1)^{|x||y|}[\beta(y), \alpha(x)],$ (Bihom-skew supersymmetry)
- (iv) $\begin{aligned} & \tilde{J}_{\alpha, \beta}^c(\alpha\beta(x), \alpha\beta(y), [\beta(t), \alpha(z)]) \\ & + (-1)^{|x||y|+|t|(|x|+|y|)} \tilde{J}_{\alpha, \beta}^c(\alpha\beta(t), \alpha\beta(y), [\beta(x), \alpha(z)]) \\ & = (-1)^{|t||z|} [\tilde{J}_{\alpha, \beta}^c(\beta(x), \beta(y), \beta(z)), \alpha^2\beta^2(t)] \\ & + (-1)^{|x|(|y|+|z|+|t|)+|t||y|} [\tilde{J}_{\alpha, \beta}^c(\beta(t), \beta(y), \beta(z)), \alpha^2\beta^2(x)]. \end{aligned}$
(Bihom-Malcev super-identity)

If $\alpha = \beta = \text{id}$, the Bihom-Malcev super-identity will reduce to the Malcev super-identity or equivalently,

$$\begin{aligned} 2 \left[t, \tilde{J}(x, y, z) \right] &= \tilde{J}(t, x, [y, z]) + (-1)^{|x|(|y|+|z|)} \tilde{J}(t, y, [z, x]) \\ &+ (-1)^{|z|(|x|+|y|)} \tilde{J}(t, z, [x, y]), \end{aligned}$$

for all $x, y, z \in \mathcal{H}(A)$.

Theorem 3.3. Let $(A, [\cdot, \cdot])$ be a Malcev superalgebra and $\alpha, \beta : A \rightarrow A$ be two algebra homomorphisms such that: $\alpha\beta = \beta\alpha$. Then $(A, [\cdot, \cdot]', \alpha, \beta)$ is a Bihom-Malcev superalgebra, where $[\cdot, \cdot]' : A \times A \rightarrow A$ is defined by $[x, y]' = [\alpha(x), \beta(y)]$, for all $x, y \in \mathcal{H}(A)$.

Proof. **Step 1**, for any $x, y \in \mathcal{H}(A)$, we have

$$[\alpha(x), \alpha(y)]' = [\alpha^2(x), \alpha\beta(y)] = \alpha([\alpha(x), \beta(y)]) = \alpha([x, y]').$$

Similarly, we can check that $[\beta(x), \beta(y)]' = \beta([x, y]')$.

Step 2, we are going to show the Bihom-Malcev super-identity. For all $x, y, z \in \mathcal{H}(A)$,

$$\begin{aligned} & \tilde{J}_{\alpha, \beta}^c(\alpha\beta(x), \alpha\beta(y), [\beta(t), \alpha(z)]') \\ &= (-1)^{|x||t|+|z|} \tilde{J}_{\alpha, \beta}^c(\alpha\beta(x), \alpha\beta(y), [\alpha\beta(t), \alpha\beta(z)]) \\ &= (-1)^{|x||t|+|z|} \alpha\beta \left(\odot_{x, y, [t, z]} (-1)^{|x||t|+|z|} [\beta^2(x), [\beta(y)[\alpha(t), \alpha(z)]]'] \right)' \\ &= (-1)^{|x||t|+|z|} \alpha^2\beta^3 \left(\odot_{x, y, [t, z]} (-1)^{|x||t|+|z|} [x, [y, [t, z]]] \right) \\ &= \alpha^2\beta^3(\tilde{J}(x, y, [t, z])). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \left[\tilde{J}_{\alpha, \beta}^c(\beta(x), \beta(y), \beta(z)), \alpha^2\beta^2(t) \right]' \\ &= [(-1)^{|x||z|} (\odot_{x, y, z} (-1)^{|x||z|} [\beta^2(\beta(x)), [\beta(\beta(y)), \alpha(\beta(z))]]')', \alpha^2\beta^2(t)]' \\ &= [(-1)^{|x||z|} (\odot_{x, y, z} (-1)^{|x||z|} \alpha\beta^3[x, [y, z]]), \alpha^2\beta^2(t)]' \\ &= [(-1)^{|x||z|} (\odot_{x, y, z} (-1)^{|x||z|} \alpha^2\beta^3[x, [y, z]]), \alpha^2\beta^3(t)] \\ &= \alpha^2\beta^3[\tilde{J}(x, y, z), t]. \end{aligned}$$

So,

$$\begin{aligned}
& \tilde{J}_{\alpha,\beta}^c(\alpha\beta(x), \alpha\beta(y), [\beta(t), \alpha(z)]') \\
& + (-1)^{|x||y|+|t|(|x|+|y|)} \tilde{J}_{\alpha,\beta}^c(\alpha\beta(t), \alpha\beta(y), [\beta(x), \alpha(z)]') \\
& = \alpha^2\beta^3(\tilde{J}(x, y, [t, z]) + (-1)^{|x||y|+|t|(|x|+|y|)} \tilde{J}(t, y, [x, z])) \\
& = \alpha^2\beta^3((-1)^{|t||z|} [\tilde{J}(x, y, z), t] + (-1)^{|x|(|y|+|z|+|t|)+|t||y|} [\tilde{J}(t, y, z), x]) \\
& = (-1)^{|t||z|} [\tilde{J}_{\alpha,\beta}^c(\beta(x), \beta(y), \beta(z)), \alpha^2\beta^2(t)]' \\
& + (-1)^{|x|(|y|+|z|+|t|)+|t||y|} [\tilde{J}_{\alpha,\beta}^c(\beta(t), \beta(y), \beta(z)), \alpha^2\beta^2(x)]'.
\end{aligned}$$

Thus, $(A, [\cdot, \cdot]', \alpha, \beta)$ is a Bihom-Malcev superalgebra. $\square$

Remark 3. Similar to Theorem 3.3, one may construct a new Bihom-Malcev superalgebra starting with a given Bihom-Malcev superalgebra and a pair of commuting Bihom-superalgebra homomorphisms.

Lemma 3.4. Let $(A, [\cdot, \cdot], \alpha, \beta)$ be a Bihom-superalgebra. Then we have,

$$\tilde{J}_{\alpha,\beta} \circ (\alpha\beta)^{\otimes 3} = \alpha\beta \circ \tilde{J}_{\alpha,\beta}.$$

Proof. Let $x, y, z \in \mathcal{H}(A)$, we have

$$\begin{aligned}
\tilde{J}_{\alpha,\beta}(\alpha\beta(x), \alpha\beta(y), \alpha\beta(z)) & = \circlearrowleft_{x,y,z} (-1)^{|x||z|} [\beta^2(\alpha\beta(x)), [\beta(\alpha\beta(y)), \alpha(\alpha\beta(z))]] \\
& = \circlearrowleft_{x,y,z} (-1)^{|x||z|} \alpha\beta[\beta^2(x), [\beta(y), \alpha(z)]] \\
& = \alpha\beta\tilde{J}_{\alpha,\beta}(x, y, z).
\end{aligned}$$

So the Lemma 3.4 holds. $\square$

Remark 4. Let $(A, [\cdot, \cdot], \alpha, \beta)$ be a Bihom-superalgebra. Then we have

$$\tilde{J}_{\alpha,\beta}^c \circ (\alpha\beta)^{\otimes 3} = \alpha\beta \circ \tilde{J}_{\alpha,\beta}^c.$$

We recall the associative super-commutator in [23] in the following definition.

Definition 3.5. Let (A, μ, α, β) be a regular Bihom-superalgebra. Define the associative super-commutator Bihom-superalgebra as the Bihom-superalgebra

$$A^- = (A, [\cdot, \cdot] = \mu \circ (id - (\alpha^{-1}\beta \otimes \alpha\beta^{-1}) \circ \tau), \alpha, \beta),$$

where $\tau(x, y) = (-1)^{|x||y|}(y, x)$, i.e. for all $x, y \in \mathcal{H}(A)$, we have

$$[x, y] = \mu(x, y) - (-1)^{|x||y|}\mu(\alpha^{-1}\beta(y), \alpha\beta^{-1}(x)).$$

We call a Bihom-superalgebra A is Bihom-Malcev-admissible if A^- is a Bihom-Malcev superalgebra.

Lemma 3.6. Let (A, μ, α, β) be a regular Bihom-superalgebra. Then, in the Bihom-superalgebra A^- , we have

$$\tilde{J}_{\alpha,\beta}^c = \tilde{a}s_{\alpha,\beta}(\alpha^{-1}\beta \otimes \beta \otimes \alpha) \circ (id + \xi + \xi^2) \circ (\delta - id),$$

where $\xi(x \otimes y \otimes z) = (-1)^{|z||x|+|z||y|}\xi(z \otimes x \otimes y)$ and $\delta(x \otimes y \otimes z) = (-1)^{|y||z|}\delta(x \otimes z \otimes y)$.

Proof. For $x, y, z \in \mathcal{H}(A)$,

$$\begin{aligned}
& O\tilde{J}_{\alpha,\beta}^c(x, y, z) \\
&= (-1)^{|x||z|}(\circlearrowleft_{x,y,z} (-1)^{|x||z|}[\beta^2(x), [\beta(y), \alpha(z)]]) \\
&= (-1)^{|x||z|}(\circlearrowleft_{x,y,z} (-1)^{|x||z|}[\beta^2(x), \beta(y)\alpha(z) - (-1)^{|y||z|}\beta(z)\alpha(y)]) \\
&= (-1)^{|x||z|}(\circlearrowleft_{x,y,z} (-1)^{|x||z|}[\beta^2(x), \beta(y)\alpha(z)] \\
&\quad - \circlearrowleft_{x,y,z} (-1)^{|z|(|x|+|y|)}[\beta^2(x), \beta(z)\alpha(y)]) \\
&= (-1)^{|x||z|}(\circlearrowleft_{x,y,z} (-1)^{|x||z|}\beta^2(x)(\beta(y)\alpha(z)) \\
&\quad - \circlearrowleft_{x,y,z} (-1)^{|x||z|}(\alpha^{-1}\beta^2(x)\beta(y))\alpha\beta(z) - \circlearrowleft_{x,y,z} (-1)^{|z|(|x|+|y|)}\beta^2(x)(\beta(z)\alpha(y)) \\
&\quad + \circlearrowleft_{x,y,z} (-1)^{|z|(|x|+|y|)}(\alpha^{-1}\beta^2(x)\beta(z))\alpha\beta(y)) \\
&= -(-1)^{|x||z|}(\circlearrowleft_{x,y,z} (-1)^{|x||z|}\tilde{a}s_{\alpha,\beta}(\alpha^{-1}\beta^2(x), \beta(y), \alpha(z))) \\
&\quad + (-1)^{|x||z|}(\circlearrowleft_{x,y,z} (-1)^{|z|(|x|+|y|)}\tilde{a}s_{\alpha,\beta}(\alpha^{-1}\beta^2(x), \beta(z), \alpha(y))) \\
&= \tilde{a}s_{\alpha,\beta} \circ (\alpha^{-1}\beta^2 \otimes \beta \otimes \alpha) \circ (id + \xi + \xi^2) \circ (\delta - id)(x, y, z).
\end{aligned}$$

□

Proposition 3. Let (A, μ, α, β) be a regular Bihom-alternative superalgebra and A^- be its associative super-commutator Bihom-superalgebra. Then

$$\tilde{J}_{\alpha,\beta}^c(x, y, z) = -6\tilde{a}s_{\alpha,\beta}(\alpha^{-1}\beta^2(x), \beta(y), \alpha(z)), \text{ for all } x, y, z \in \mathcal{H}(A),$$

where the Bihom-associator is taken in A and the Bihom-super-Jacobiator is considered in A^- .

Proof. The result follows immediately from Lemma 3.6. □

Definition 3.7. Let (A, μ, α, β) be a Bihom-superalgebra. Define the function $H : A^{\otimes 4} \rightarrow A$ by:

$$\begin{aligned}
H(w, x, y, z) &= \tilde{a}s_{\alpha,\beta}(\beta^2(w)\alpha\beta(x), \alpha^2\beta(y), \alpha^3(z)) \\
&\quad - (-1)^{|w||x|+|w||y|+|w||z|}\tilde{a}s_{\alpha,\beta}(\beta^2(x)\alpha\beta(y), \alpha^2\beta(z), \alpha^3(w)) \\
&\quad + (-1)^{(|y|+|z|)(|w|+|x|)}\tilde{a}s_{\alpha,\beta}(\beta^2(y)\alpha\beta(z), \alpha^2\beta(w), \alpha^3(x)).
\end{aligned}$$

Lemma 3.8. In a regular Bihom-alternative superalgebra (A, μ, α, β) , the identity $H(w, x, y, z) = \alpha^2\beta^2(w)\tilde{a}s_{\alpha,\beta}(\alpha\beta(x), \alpha^2(y), \alpha^3\beta^{-1}(z)) + \tilde{a}s_{\alpha,\beta}(\beta^2(w), \alpha\beta(x), \alpha^2(y))\alpha^3\beta(z)$ holds, for all $x, y, z, w \in \mathcal{H}(A)$.

Proof. First, since (A, μ, α, β) is a regular Bihom-alternative superalgebra, we have

$$\begin{aligned}
& \tilde{a}s_{\alpha,\beta}(\beta^2(x)\alpha\beta(y), \alpha^2\beta(z), \alpha^3(w)) \\
&= \tilde{a}s_{\alpha,\beta}(\beta^2(x\alpha\beta^{-1}(y)), \alpha\beta(\alpha(z)), \alpha^2(\alpha(w))) \\
&= (-1)^{|w||x|+|w||y|+|w||z|}\tilde{a}s_{\alpha,\beta}(\alpha\beta^2(w), \alpha\beta(x)\alpha^2(y), \alpha^3(z)).
\end{aligned}$$

In addition,

$$\begin{aligned}
& \tilde{a}s_{\alpha,\beta}(\beta^2(y)\alpha\beta(z), \alpha^2\beta(w), \alpha^3(x)) \\
&= \tilde{a}s_{\alpha,\beta}(\beta^2(y\alpha\beta^{-1}(z)), \alpha\beta(\alpha(w)), \alpha^2(\alpha(x))) \\
&= (-1)^{(|w|+|x|)(|y|+|z|)}\tilde{a}s_{\alpha,\beta}(\alpha\beta^2(w), \alpha^2\beta(x), \alpha^2(y)\alpha^3\beta^{-1}(z)).
\end{aligned}$$

So,

$$\begin{aligned} H(w, x, y, z) &= \tilde{a}s_{\alpha, \beta}(\beta^2(w)\alpha\beta(x), \alpha^2\beta(y), \alpha^3(z)) - \tilde{a}s_{\alpha, \beta}(\beta^2(w)\alpha\beta(x), \alpha^2\beta(y), \alpha^3(z)) \\ &\quad + \tilde{a}s_{\alpha, \beta}(\beta^2(w)\alpha\beta(x), \alpha^2\beta(y), \alpha^3(z)) \\ &= \alpha^2\beta^2(w)\tilde{a}s_{\alpha, \beta}(\alpha\beta(x), \alpha^2(y), \alpha^3\beta^{-1}(z)) + \tilde{a}s_{\alpha, \beta}(\beta^2(w), \alpha\beta(x), \alpha^2(y))\alpha^3\beta(z). \end{aligned}$$

The proof is finished. $\square$

Definition 3.9. Let (A, μ, α, β) be a regular Bihom-superalgebra. Define the $\mathbb{Z}_2$ -graded Bihom-Bruck-Kleinfeld function $f : A^{\otimes 4} \rightarrow A$ as the even multilinear map,

$$\begin{aligned} f(w, x, y, z) &= \tilde{a}s_{\alpha, \beta}(\beta^2(w)\alpha\beta(x), \alpha^2\beta(y), \alpha^3(z)) \\ &\quad - (-1)^{|w||x|+|w||y|+|w||z|}\tilde{a}s_{\alpha, \beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z))\alpha^3\beta(w) \\ &\quad - (-1)^{|w||x|}\alpha^2\beta^2(x)\tilde{a}s_{\alpha, \beta}(\alpha\beta(w), \alpha^2(y), \alpha^3\beta^{-1}(z)), \end{aligned} \quad (8)$$

for all $w, x, y, z \in \mathcal{H}(A)$.

Define another even multi-linear map $F : A^{\otimes 4} \rightarrow A$ as

$$F = [\cdot, \cdot] \circ ((\tilde{a}s_{\alpha, \beta} \circ (\beta^2 \otimes \alpha\beta \otimes \alpha^2)) \otimes \alpha^3\beta) \circ (id - \xi + \xi^2 - \xi^3), \quad (9)$$

where $[\cdot, \cdot] = \mu \circ (id - (\alpha^{-1}\beta \otimes \alpha\beta^{-1}) \circ \tau)$ is the super-commutator bracket of μ and ξ is the super cyclic permutation

$$\xi(w \otimes x \otimes y \otimes z) = (-1)^{|z||w|+|z||x|+|z||y|}z \otimes w \otimes x \otimes y,$$

i.e., for all $w, x, y, z \in \mathcal{H}(A)$,

$$\begin{aligned} F(w, x, y, z) &= [\tilde{a}s_{\alpha, \beta}(\beta^2(w), \alpha\beta(x), \alpha^2(y)), \alpha^3\beta(z)] \\ &\quad - (-1)^{|z||w|+|z||x|+|z||y|}[\tilde{a}s_{\alpha, \beta}(\beta^2(z), \alpha\beta(w), \alpha^2(x)), \alpha^3\beta(y)] \\ &\quad + (-1)^{(|y|+|z|)(|w|+|x|)}[\tilde{a}s_{\alpha, \beta}(\beta^2(y), \alpha\beta(z), \alpha^2(w)), \alpha^3\beta(x)] \\ &\quad - (-1)^{|w||x|+|w||y|+|w||z|}[\tilde{a}s_{\alpha, \beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z)), \alpha^3\beta(w)]. \end{aligned}$$

Lemma 3.10. Let (A, μ, α, β) be a Bihom-alternative superalgebra, the F and f described above satisfy the following equation:

$$F = f \circ (id - \rho + \rho^2),$$

where $\rho = \xi^3$ is the super cyclic permutation $\rho(w \otimes x \otimes y \otimes z) = (-1)^{|w||x|+|w||y|+|w||z|}(x \otimes y \otimes z \otimes w)$. Further, for all $w, x, y, z \in \mathcal{H}(A)$,

$$\begin{aligned} F(w, x, y, z) &= f(w, x, y, z) - (-1)^{|w||x|+|w||y|+|w||z|}f(x, y, z, w) + (-1)^{(|y|+|z|)(|w|+|x|)}f(y, z, w, x). \end{aligned}$$

Proof. For each $w, x, y, z \in \mathcal{H}(A)$, we have

$$\begin{aligned} &[\tilde{a}s_{\alpha, \beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z)), \alpha^3\beta(w)] \\ &= \tilde{a}s_{\alpha, \beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z))\alpha^3\beta(w) \\ &\quad - (-1)^{|w||x|+|w||y|+|w||z|}\alpha^2\beta^2(w)\tilde{a}s_{\alpha, \beta}(\alpha\beta(x), \alpha^2(y), \alpha^3\beta^{-1}(z)). \end{aligned}$$

Then, by Lemma 3.8 we get

$$\begin{aligned} &\alpha^2\beta^2(w)\tilde{a}s_{\alpha, \beta}(\alpha\beta(x), \alpha^2(y), \alpha^3\beta^{-1}(z)) + \tilde{a}s_{\alpha, \beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z))\alpha^3\beta(z) \\ &= \tilde{a}s_{\alpha, \beta}(\beta^2(w)\alpha\beta(x), \alpha^2\beta(y), \alpha^3(z)) \\ &\quad - (-1)^{|w||x|+|w||y|+|w||z|}\tilde{a}s_{\alpha, \beta}(\beta^2(x)\alpha\beta(y), \alpha^2\beta(z), \alpha^3(w)) \end{aligned}$$

$$\begin{aligned}
& + (-1)^{(|y|+|z|)(|w|+|x|)} \tilde{a}s_{\alpha,\beta}(\beta^2(y)\alpha\beta(z), \alpha^2\beta(w), \alpha^3(x)) \\
& = f(w, x, y, z) + (-1)^{|w||x|+|w||y|+|w||z|} \tilde{a}s_{\alpha,\beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z))\alpha^3\beta(w) \\
& \quad + (-1)^{|w||x|} \alpha^2\beta^2(x) \tilde{a}s_{\alpha,\beta}(\alpha\beta(w), \alpha^2(y), \alpha^3\beta^{-1}(z)) \\
& \quad - (-1)^{|w||x|+|w||y|+|w||z|} f(x, y, z, w) \\
& \quad - (-1)^{(|w|+|x|)(|y|+|z|)} \tilde{a}s_{\alpha,\beta}(\beta^2(y), \alpha\beta(z), \alpha^2(w))\alpha^3\beta(x) \\
& \quad - (-1)^{|w||x|+|w||y|+|w||z|+|x||y|} \alpha^2\beta^2(y) \tilde{a}s_{\alpha,\beta}(\alpha\beta(x), \alpha^2(z), \alpha^3\beta^{-1}(w)) \\
& \quad + (-1)^{(|y|+|z|)(|w|+|x|)} f(y, z, w, x) \\
& \quad + (-1)^{|z||x|+|z||w|+|z||y|} \tilde{a}s_{\alpha,\beta}(\beta^2(z), \alpha\beta(w), \alpha^2(x))\alpha^3\beta(y) \\
& \quad + (-1)^{(|y|+|z|)(|w|+|x|)} \alpha^2\beta^2(z) \tilde{a}s_{\alpha,\beta}(\alpha\beta(y), \alpha^2(w), \alpha^3\beta^{-1}(x)).
\end{aligned}$$

Moreover, since $\tilde{a}s_{\alpha,\beta} \circ (\beta^2 \otimes \alpha\beta \otimes \alpha^2)$ is super-alternating, we have

$$\begin{aligned}
\tilde{a}s_{\alpha,\beta}(\alpha\beta(y), \alpha^2(w), \alpha^3\beta^{-1}(x)) & = \alpha\beta^{-1}(\tilde{a}s_{\alpha,\beta}(\beta^2(y), \alpha\beta(w), \alpha^2(x))) \\
& = (-1)^{|y||w|+|y||x|} \alpha\beta^{-1}(\tilde{a}s_{\alpha,\beta}(\beta^2(w), \alpha\beta(x), \alpha^2(y))) \\
& = (-1)^{|y||w|+|y||x|} \tilde{a}s_{\alpha,\beta}(\alpha\beta(w), \alpha^2(x), \alpha^3\beta^{-1}(y)),
\end{aligned}$$

Similarly, $\tilde{a}s_{\alpha,\beta}(\alpha\beta(x), \alpha^2(z), \alpha^3\beta^{-1}(w))$ and $\tilde{a}s_{\alpha,\beta}(\alpha\beta(w), \alpha^2(y), \alpha^3\beta^{-1}(z))$ can also be obtained as follows:

$$\begin{aligned}
\tilde{a}s_{\alpha,\beta}(\alpha\beta(x), \alpha^2(z), \alpha^3\beta^{-1}(w)) & = (-1)^{|x||w|+|x||z|} \tilde{a}s_{\alpha,\beta}(\alpha\beta(z), \alpha^2(w), \alpha^3\beta^{-1}(x)), \\
\tilde{a}s_{\alpha,\beta}(\alpha\beta(w), \alpha^2(y), \alpha^3\beta^{-1}(z)) & = (-1)^{|w||y|+|w||z|} \tilde{a}s_{\alpha,\beta}(\alpha\beta(y), \alpha^2(z), \alpha^3\beta^{-1}(w)).
\end{aligned}$$

Then

$$\begin{aligned}
F(w, x, y, z) & = f(w, x, y, z) - (-1)^{|w||x|+|w||y|+|w||z|} f(x, y, z, w) \\
& \quad + (-1)^{(|y|+|z|)(|w|+|x|)} f(y, z, w, x),
\end{aligned}$$

for all $w, x, y, x \in \mathcal{H}(A)$. $\square$

Proposition 4. Let (A, μ, α, β) be a regular Bihom-alternative superalgebra. Then the even Bihom-Bruck-Kleinfeld function f is super-alternating.

Proof. The proof is similar to that Proposition 4.1 in Reference [1], it is easy to check it. $\square$

Next, we will give a relationship between Bihom-alternate superalgebra and Bihom-Malcev superalgebra, and use the above preparation knowledge to prove it.

Theorem 3.11. Every regular Bihom-alternative superalgebra is Bihom-Malcev-admissible.

Proof. Let (A, μ, α, β) be a Bihom-alternative superalgebra and $A^- = (A, [\cdot, \cdot], \alpha, \beta)$ be its super-commutator Bihom-superalgebra. For any $x, y \in \mathcal{H}(A^-)$:

Step 1, it's not difficult to find multiplicativity for α and β in A^- .

Step 2, since the associative super-commutator bracket $[\cdot, \cdot] = \mu \circ (id - (\alpha^{-1}\beta \otimes \alpha\beta^{-1}) \circ \tau)$ is Bihom-skew-supersymmetric, that is to say $[\beta(x), \alpha(y)] = -(-1)^{|x||y|} [\beta(y), \alpha(x)]$.

Step 3, we need to show that the Bihom-Malcev super-identity holds in A^- , i.e.,

$$\tilde{J}_{\alpha,\beta}^c(\alpha\beta(x), \alpha\beta(y), [\beta(t), \alpha(z)]) + (-1)^{|x||y|+|t|(|x|+|y|)} \tilde{J}_{\alpha,\beta}^c(\alpha\beta(t), \alpha\beta(y), [\beta(x), \alpha(z)])$$

$$\begin{aligned}
&= (-1)^{|t||z|} \left[\tilde{J}_{\alpha,\beta}^c(\beta(x), \beta(y), \beta(z)), \alpha^2 \beta^2(t) \right] \\
&\quad + (-1)^{|x|(|y|+|z|+|t|)+|t||y|} \left[\tilde{J}_{\alpha,\beta}^c(\beta(t), \beta(y), \beta(z)), \alpha^2 \beta^2(x) \right].
\end{aligned}$$

Note first that $\tilde{J}_{\alpha,\beta}^c(x, y, z) = -6\tilde{a}s_{\alpha,\beta}(\alpha^{-1}\beta^2(x), \beta(y), \alpha(z))$.

In addition, $[\beta(x), \alpha(z)] = \beta(x)\alpha(z) - (-1)^{|x||z|}\beta(z)\alpha(x)$. So,

$$\begin{aligned}
&\tilde{J}_{\alpha,\beta}^c(\alpha\beta(x), \alpha\beta(y), [\beta(t), \alpha(z)]) + (-1)^{|x||y|+|t|(|x|+|y|)} \tilde{J}_{\alpha,\beta}^c(\alpha\beta(t), \alpha\beta(y), [\beta(x), \alpha(z)]) \\
&= -6\tilde{a}s_{\alpha,\beta}(\beta^3(x), \alpha\beta^2(y), \alpha\beta(t)\alpha^2(z)) + 6(-1)^{|t||z|} \tilde{a}s_{\alpha,\beta}(\beta^3(x), \alpha\beta^2(y), \alpha\beta(z)\alpha^2(t)) \\
&\quad - 6(-1)^{|x||y|+|t|(|x|+|y|)} \tilde{a}s_{\alpha,\beta}(\beta^3(t), \alpha\beta^2(y), \alpha\beta(x)\alpha^2(z)) \\
&\quad + 6(-1)^{|x|(|y|+|z|)+|t|(|x|+|y|)} \tilde{a}s_{\alpha,\beta}(\beta^3(t), \alpha\beta^2(y), \alpha\beta(z)\alpha^2(x)) \\
&= -6(-1)^{(|t|+|z|)(|x|+|y|)} \tilde{a}s_{\alpha,\beta}(\alpha^{-1}\beta^3(t)\beta^2(z), \alpha\beta^2(x), \alpha^2\beta(y)) \\
&\quad + 6(-1)^{|t||z|+|t|(|x|+|z|)(|x|+|y|)} \tilde{a}s_{\alpha,\beta}(\alpha^{-1}\beta^3(z)\beta^2(t), \alpha\beta^2(x), \alpha^2\beta(y)) \\
&\quad - 6(-1)^{|t||y|+|z||t|+|z||y|} \tilde{a}s_{\alpha,\beta}(\alpha^{-1}\beta^3(x)\beta^2(z), \alpha\beta^2(t), \alpha^2\beta(y)) \\
&\quad + 6(-1)^{|x||z|+|t||y|+|z||t|+|z||y|} \tilde{a}s_{\alpha,\beta}(\alpha^{-1}\beta^3(z)\beta^2(x), \alpha\beta^2(t), \alpha^2\beta(y)) \\
&= 6(-1)^{(|t|+|z|)(|x|+|y|)} \alpha^{-1}\beta\{(-1)^{|t|(|x|+|y|+|z|)} [\tilde{a}s_{\alpha,\beta}(\beta^2(z), \alpha\beta(x), \alpha^2(y)), \alpha^3\beta(t)] \\
&\quad - (-1)^{|z|(|x|+|y|)} [\tilde{a}s_{\alpha,\beta}(\beta^2(t), \alpha\beta(x), \alpha^2(y)), \alpha^3\beta(z)]\} \\
&\quad + 6(-1)^{|t||y|+|z||t|+|z||y|} \alpha^{-1}\beta\{(-1)^{|x|(|z|+|t|+|y|)} [\tilde{a}s_{\alpha,\beta}(\beta^2(z), \alpha\beta(t), \alpha^2(y)), \alpha^3\beta(x)] \\
&\quad - (-1)^{|z|(|t|+|y|)} [\tilde{a}s_{\alpha,\beta}(\beta^2(x), \alpha\beta(t), \alpha^2(y)), \alpha^3\beta(z)]\} \\
&= (-1)^{|t||z|} \left[\tilde{J}_{\alpha,\beta}^c(\beta(x), \beta(y), \beta(z)), \alpha^2 \beta^2(t) \right] \\
&\quad + (-1)^{|x|(|y|+|z|+|t|)+|t||y|} \left[\tilde{J}_{\alpha,\beta}^c(\beta(t), \beta(y), \beta(z)), \alpha^2 \beta^2(x) \right].
\end{aligned}$$

Thus, every regular Bihom-alternative superalgebra is Bihom-Malcev-admissible. $\square$

Proposition 5. Let (A, μ, α, β) be a regular Bihom-alternative superalgebra. Then the $\mathbb{Z}_2$ -graded Bihom-Bruck-Kleinfeld function f satisfies

$$f = \frac{1}{3}F = \tilde{a}s_{\alpha,\beta} \circ ([\cdot, \cdot] \circ (\beta^2 \otimes \alpha\beta)) \otimes \alpha^2\beta \otimes \alpha^3 \circ (id + \zeta), \quad (10)$$

where F is defined as before and ζ is the super permutation

$$\zeta(w \otimes x \otimes y \otimes z) = (-1)^{(|y|+|z|)(|w|+|x|)} y \otimes z \otimes w \otimes x.$$

Proof. By Lemma 3.14, we have $F = f \circ (id - \rho + \rho^2) = 3f$, which proves the first equality in (10). It remains to prove that f is equal to the second entry in (10).

Use the definition of f and its super-alternativity,

$$\begin{aligned}
2f(t, x, y, z) &= f(t, x, y, z) - (-1)^{|x||t|} f(x, t, y, z) \\
&= \tilde{a}s_{\alpha,\beta}(\beta^2(t)\alpha\beta(x), \alpha^2\beta(y), \alpha^3(z)) \\
&\quad - (-1)^{|t|(|x|+|y|+|z|)} \tilde{a}s_{\alpha,\beta}(\beta^2(x), \alpha\beta(y), \alpha^2(z))\alpha^3\beta(t) \\
&\quad - (-1)^{|t||x|} \alpha^2\beta^2(x) \tilde{a}s_{\alpha,\beta}(\alpha\beta(t), \alpha^2(y), \alpha^3\beta^{-1}(z)) \\
&\quad - (-1)^{|x||t|} \tilde{a}s_{\alpha,\beta}(\beta^2(x)\alpha\beta(t), \alpha^2\beta(y), \alpha^3(z)) \\
&\quad + (-1)^{|x|(|y|+|z|)} \tilde{a}s_{\alpha,\beta}(\beta^2(t), \alpha\beta(y), \alpha^2(z))\alpha^3\beta(x) \\
&\quad + \alpha^2\beta^2(t) \tilde{a}s_{\alpha,\beta}(\alpha\beta(x), \alpha^2(y), \alpha^3\beta^{-1}(z)),
\end{aligned}$$

which implies that

$$\begin{aligned} & \tilde{a}s_{\alpha,\beta}([\beta^2(t), \alpha\beta(x)], \alpha^2\beta(y), \alpha^3(z)) \\ &= 2f(t, x, y, z) - [\alpha^2\beta^2(t), \tilde{a}s_{\alpha,\beta}(\alpha\beta(x), \alpha^2(y), \alpha^3\beta^{-1}(z))] \\ & \quad + (-1)^{|t|(|x|+|y|+|z|)}[\alpha^2\beta^2(x), \tilde{a}s_{\alpha,\beta}(\alpha\beta(y), \alpha^2(z), \alpha^3\beta^{-1}(t))]. \end{aligned} \quad (11)$$

Interchanging (t, x) with (y, z) in (11) and using the super-alternativity of f , we obtain

$$\begin{aligned} & (-1)^{(|t|+|x|)(|y|+|z|)}\tilde{a}s_{\alpha,\beta}([\beta^2(y), \alpha\beta(z)], \alpha^2\beta(t), \alpha^3(x)) \\ &= (-1)^{|y|(|t|+|x|+|z|)}[\alpha^2\beta^2(z), \tilde{a}s_{\alpha,\beta}(\alpha\beta(t), \alpha^2(x), \alpha^3\beta^{-1}(y))] \\ & \quad - (-1)^{(|t|+|x|)(|y|+|z|)}[\alpha^2\beta^2(y), \tilde{a}s_{\alpha,\beta}(\alpha\beta(z), \alpha^2(t), \alpha^3\beta^{-1}(x))] \\ & \quad + 2(-1)^{(|t|+|x|)(|y|+|z|)}f(y, z, t, x) \\ &= (-1)^{|y|(|t|+|x|+|z|)}[\alpha^2\beta^2(z), \tilde{a}s_{\alpha,\beta}(\alpha\beta(t), \alpha^2(x), \alpha^3\beta^{-1}(y))] \\ & \quad - (-1)^{(|t|+|x|)(|y|+|z|)}[\alpha^2\beta^2(y), \tilde{a}s_{\alpha,\beta}(\alpha\beta(z), \alpha^2(t), \alpha^3\beta^{-1}(x))] \\ & \quad + 2f(t, x, y, z), \end{aligned}$$

so

$$\begin{aligned} & \tilde{a}s_{\alpha,\beta}([\beta^2(t), \alpha\beta(x)], \alpha^2\beta(y), \alpha^3(z)) + (-1)^{(|t|+|x|)(|y|+|z|)}\tilde{a}s_{\alpha,\beta}([\beta^2(y), \alpha\beta(z)], \alpha^2\beta(t), \alpha^3(z)) \\ &= (4f - F)(t, x, y, z) \\ &= f(t, x, y, z) \text{ (since } F = 3f). \end{aligned}$$

Thus f is equal to the second entry in (10). $\square$

4. Bimodules and T_{θ}^* -extension of Bihom-alternative superalgebras. For convenience and without confusion, we denote Bihom-alternative superalgebra (A, μ, α, β) by $(A, \circ)$

Definition 4.1. Let $(A, \circ, \alpha, \beta)$ be a Bihom-alternative superalgebra and V be a $\mathbb{Z}_2$ -graded vector space. Let $L, R : A \rightarrow \text{gl}(V)$ be two even linear maps $(L(x)(u) = xu, R(y)v = (-1)^{|v||y|}vy)$ and $\alpha_V, \beta_V \in \text{gl}(V)$, $\alpha_V\beta_V = \beta_V\alpha_V$. $(V, L, R, \alpha_V, \beta_V)$ is called a representation or a bimodule of A , if for any $x, y, z \in \mathcal{H}(A)$ and $v \in \mathcal{H}(V)$,

$$\alpha_V L(x) = L(\alpha(x))\alpha_V, \quad \alpha_V R(x) = R(\alpha(x))\alpha_V, \quad (12)$$

$$\beta_V L(x) = L(\beta(x))\beta_V, \quad \beta_V R(x) = R(\beta(x))\alpha_V, \quad (13)$$

$$\begin{aligned} & L(\beta(x) \circ \alpha(y))\beta_V(v) - L(\alpha\beta(x))L(\alpha(y))v + (-1)^{|x||y|}L(\beta(y) \circ \alpha(x))\beta_V(v) \\ & - (-1)^{|x||y|}L(\alpha\beta(y))L(\alpha(x))v = 0, \end{aligned} \quad (14)$$

$$\begin{aligned} & (-1)^{|z||y|}R(\alpha\beta(z))R(\beta(y))v - R(\beta(y) \circ \alpha(z))\alpha_V(v) + R(\alpha\beta(y))R(\beta(z))v \\ & - (-1)^{|y||z|}R(\beta(z)\alpha(y))\alpha_V(v) = 0, \end{aligned} \quad (15)$$

$$\begin{aligned} & (-1)^{|x||y|}R(\beta(y))L(\beta(x))\alpha_V(v) - L(\alpha\beta(x))R(y)\alpha_V(v) + (-1)^{|x||y|}R(\beta(y))R(\alpha(x))\beta_V(v) \\ & - R(\alpha(x) \circ y)\alpha_V\beta_V(v) = 0, \end{aligned} \quad (16)$$

$$\begin{aligned} & (-1)^{|x||y|}R(\alpha\beta(y))L(x)\beta_V(v) - L(\alpha(x))R(\alpha(y))\beta_V(v) + L(x \circ \beta(y))\alpha_V\beta_V(v) \\ & - L(\alpha(x))L(\beta(y))\alpha_V(v) = 0. \end{aligned} \quad (17)$$

Let $(A, \circ, \alpha, \beta)$ be a Bihom-alternative superalgebra, $(V, L, R, \alpha_V, \beta_V)$ be its a bimodule and $L^*, R^* : A \rightarrow \text{gl}(V^*)$, $\alpha^*, \beta^* : A^* \rightarrow A^*$, $\alpha_V^*, \beta_V^* : V^* \rightarrow V^*$ be the dual even maps of respectively $\alpha, \beta, \alpha_V, \beta_V$ given by

$$\langle L^*(x)u^*, v \rangle = \langle u^*, L(x)v \rangle, \quad \langle R^*(x)u^*, v \rangle = (-1)^{|u||x|} \langle u^*, R(x)v \rangle,$$

$$\alpha^*(x^*)(y) = x^*(\alpha(y)), \quad \beta^*(x^*)(y) = x^*(\beta(y)),$$

$$\alpha_V^*(u^*)(v) = u^*(\alpha_V(v)), \quad \beta_V^*(u^*)(v) = u^*(\beta_V(v)).$$

Proposition 6. *Let $(V, L, R, \alpha_V, \beta_V)$ be a bimodule of a Bihom-alternative superalgebra $(A, \circ, \alpha, \beta)$. Then $(V^*, L^*, R^*, \alpha_V^*, \beta_V^*)$ is a bimodule of $(A, \circ, \alpha, \beta)$ provided that for any x, y, z, v are homogeneous,*

$$\begin{aligned} & \beta_V L(\beta(x) \circ \alpha(y))(v) - (-1)^{|x||y|} L(\alpha(y)) L(\alpha\beta(x))v + (-1)^{|x||y|} \beta_V L(\beta(y) \circ \alpha(x))(v) \\ & - L(\alpha(x)) L(\alpha\beta(y))v = 0, \end{aligned} \quad (18)$$

$$\begin{aligned} & R(\beta(y)) R(\alpha\beta(z))v - \alpha_V R(\beta(y) \circ \alpha(z))(v) + (-1)^{|z||y|} R(\beta(z)) R(\alpha\beta(y))v \\ & - (-1)^{|y||z|} \alpha_V R(\beta(z) \alpha(y))(v) = 0, \end{aligned} \quad (19)$$

$$\begin{aligned} & \alpha_V L(\beta(x)) R(\beta(y))(v) - (-1)^{|x||y|} \alpha_V R(y) L(\alpha\beta(x))(v) + \beta_V R(\alpha(x)) R(\beta(y))(v) \\ & - \alpha_V \beta_V R(\alpha(x) \circ y)(v) = 0, \end{aligned} \quad (20)$$

$$\begin{aligned} & \beta_V L(x) R(\alpha\beta(y))(v) - (-1)^{|x||y|} \beta_V R(\alpha(y)) L(\alpha(x))(v) + \alpha_V \beta_V L(x \circ \beta(y))(v) \\ & - (-1)^{|x||y|} \alpha_V L(\beta(y)) L(\alpha(x))(v) = 0, \end{aligned} \quad (21)$$

Proof. Straightforward by calculation. $\square$

Theorem 4.2. *Let $(V, L, R, \alpha_V, \beta_V)$ be a bimodule of Bihom-alternative superalgebra $(A, \circ, \alpha, \beta)$, $\theta : A \times A \rightarrow V$ be an even bilinear map. Define a new multiplication on $A \oplus V$ by*

$$(x + u) \star (y + v) = x \circ y + \theta(x, y) + L(x)v + (-1)^{|x||y|} R(y)u, \quad (22)$$

and $\alpha + \alpha_V, \beta + \beta_V \in \text{End}(A \oplus V)$ such that

$$(\alpha + \alpha_V)(x + u) = \alpha(x) + \alpha_V(u), \quad (\beta + \beta_V)(x + u) = \beta(x) + \beta_V(u), \quad (23)$$

for any $x, y \in \mathcal{H}(A)$, $u, v \in \mathcal{H}(V)$, then $(A \oplus V, \star, \alpha + \alpha_V, \beta + \beta_V)$ is a Bihom-alternative superalgebra if and only if θ satisfies the following conditions: for any $x_1, x_2, x_3 \in \mathcal{H}(A)$,

$$\begin{aligned} & \theta(\beta(x_1) \circ \alpha(x_2), \beta(x_3)) + (-1)^{|x_3|(|x_1|+|x_2|)} R(\beta(x_3)) \theta(\beta(x_1), \alpha(x_2)) \\ & - \theta(\alpha\beta(x_1), \alpha(x_2) \circ x_3) - L(\alpha\beta(x_1)) \theta(\alpha(x_2), x_3) + (-1)^{|x_1||x_2|} \theta(\beta(x_2) \circ \alpha(x_1), \beta(x_3)) \\ & + (-1)^{|x_3|(|x_1|+|x_2|)+|x_1||x_2|} R(\beta(x_3)) \theta(\beta(x_2), \alpha(x_1)) - (-1)^{|x_1||x_2|} \theta(\alpha\beta(x_2), \alpha(x_1) \circ x_3) \\ & - (-1)^{|x_1||x_2|} L(\alpha\beta(x_2)) \theta(\alpha(x_1), x_3) = 0, \end{aligned} \quad (24)$$

$$\begin{aligned} & \theta(x_1 \circ \beta(x_2), \alpha\beta(x_3)) + (-1)^{|x_3|(|x_1|+|x_2|)} R(\alpha\beta(x_3)) \theta(x_1, \beta(x_2)) \\ & - \theta(\alpha(x_1), \beta(x_2) \circ \alpha(x_3)) - L(\alpha(x_1)) \theta(\beta(x_2), \alpha(x_3)) + (-1)^{|x_2||x_3|} \theta(x_1 \circ \beta(x_3), \alpha\beta(x_2)) \\ & + (-1)^{|x_1||x_2|} R(\alpha\beta(x_2)) \theta(x_1, \beta(x_3)) - (-1)^{|x_2||x_3|} L(\alpha(x_1)) \theta(\beta(x_3), \alpha(x_2)) \\ & - (-1)^{|x_2||x_3|} \theta(\alpha(x_1), \beta(x_3) \circ \alpha(x_2)) = 0. \end{aligned} \quad (25)$$

Proof. First we prove the commutativity of maps,

$$\begin{aligned}(\alpha + \alpha_V)(\beta + \beta_V)(x + u) &= (\alpha + \alpha_V)(\beta(x) + \beta_V(u)) \\ &= \alpha\beta(x) + \alpha_V\beta_V(u) \\ &= \beta\alpha(x) + \beta_V\alpha_V(u) \\ &= (\beta + \beta_V)(\alpha + \alpha_V)(x + u).\end{aligned}$$

By the fact that $(A \oplus V, \star, \alpha + \alpha_V, \beta + \beta_V)$ is a Bihom-alternative superalgebra. BiHom left super-alternativity holds.

$$\begin{aligned}& \tilde{a}s_{\alpha,\beta}((\beta + \beta_V)(x_1 + v_1), (\alpha + \alpha_V)(x_2 + v_2), x_3 + v_3) \\ & + (-1)^{|x_1||x_2|}\tilde{a}s_{\alpha,\beta}((\beta + \beta_V)(x_2 + v_2), (\alpha + \alpha_V)(x_1 + v_1), x_3 + v_3) \\ & = ((\beta + \beta_V)(x_1 + v_1) \star (\alpha + \alpha_V)(x_2 + v_2)) \star (\beta + \beta_V)(x_3 + v_3) \\ & \quad - (\alpha + \alpha_V)(\beta + \beta_V)(x_1 + v_1) \star ((\alpha + \alpha_V)(x_2 + v_2) \star (x_3 + v_3)) \\ & \quad + (-1)^{|x_1||x_2|}((\beta + \beta_V)(x_2 + v_2) \star (\alpha + \alpha_V)(x_1 + v_1)) \star (\beta + \beta_V)(x_3 + v_3) \\ & \quad - (-1)^{|x_1||x_2|}(\alpha + \alpha_V)(\beta + \beta_V)(x_2 + v_2) \star ((\alpha + \alpha_V)(x_1 + v_1) \star (x_3 + v_3)) \\ & = (\beta(x_1) \circ \alpha(x_2)) \circ \beta(x_3) + \theta(\beta(x_1) \circ \alpha(x_2), \beta(x_3)) + L(\beta(x_1) \circ \alpha(x_2))\beta_V(v_3) \\ & \quad + (-1)^{|x_3|(|x_1|+|x_2|)}R(\beta(x_3))\theta(\beta(x_1), \alpha(x_2)) + (-1)^{|x_3|(|x_1|+|x_2|)}R(\beta(x_3))L(\beta(x_1))\alpha_V(v_2) \\ & \quad + (-1)^{|x_3|(|x_1|+|x_2|)+|x_1||x_2|}R(\beta(x_3))R(\alpha(x_2))\beta_V(v_1) \\ & \quad - \alpha\beta(x_1) \circ (\alpha(x_2) \circ x_3) - \theta(\alpha\beta(x_1), \alpha(x_2) \circ x_3) - L(\alpha\beta(x_1))\theta(\alpha(x_2), x_3) \\ & \quad - L(\alpha\beta(x_1))L(\alpha(x_2))v_3 - (-1)^{|x_2||x_3|}L(\alpha\beta(x_1))R(x_3)\alpha_V(v_2) \\ & \quad - (-1)^{|x_1|(|x_2|+|x_3|)}R(\alpha(x_2) \circ x_3)\alpha\beta(v_1) \\ & \quad + (-1)^{|x_1||x_2|}(\beta(x_2) \circ \alpha(x_1)) \circ \beta(x_3) + (-1)^{|x_1||x_2|}\theta(\beta(x_2) \circ \alpha(x_1), \beta(x_3)) \\ & \quad + (-1)^{|x_1||x_2|}L(\beta(x_2) \circ \alpha(x_1))\beta_V(v_3) + (-1)^{|x_3|(|x_1|+|x_2|)+|x_1||x_2|}R(\beta(x_3))\theta(\beta(x_2), \alpha(x_1)) \\ & \quad + (-1)^{|x_3|(|x_1|+|x_2|)+|x_1||x_2|}R(\beta(x_3))L(\beta(x_2))\alpha_V(v_1) \\ & \quad + (-1)^{|x_3|(|x_1|+|x_2|)}R(\beta(x_3))R(\alpha(x_1))\beta_V(v_2) \\ & \quad - (-1)^{|x_1||x_2|}\alpha\beta(x_2) \circ (\alpha(x_1) \circ x_3) - (-1)^{|x_1||x_2|}\theta(\alpha\beta(x_2), \alpha(x_1) \circ x_3) \\ & \quad - (-1)^{|x_1||x_2|}L(\alpha\beta(x_2))\theta(\alpha(x_1), x_3) - (-1)^{|x_1||x_2|}L(\alpha\beta(x_2))L(\alpha(x_1))v_3 \\ & \quad - (-1)^{|x_1|(|x_2|+|x_3|)}L(\alpha\beta(x_2))R(x_3)\alpha_V(v_1) - (-1)^{|x_2||x_3|}R(\alpha(x_1) \circ x_3)\alpha\beta(v_2).\end{aligned}\tag{26}$$

Since $(V, L, R, \alpha_V, \beta_V)$ is a bimodule of Bihom-alternative superalgebra $(A, \circ, \alpha, \beta)$, by (14), we set $x = x_1, y = x_2, v = v_3$, then we rewrite (14) as (a), by (16), we set $x = x_1, y = x_3, v = v_2$, then we rewrite (16) as (b), and by (16), we set $x = x_2, y = x_3, v = v_1$, then we rewrite (16) as (c). In addition, the fact is that $\tilde{a}s_{\alpha,\beta}(\beta(x_1), \alpha(x_2), x_3) + (-1)^{|x_1||x_2|}\tilde{a}s_{\alpha,\beta}(\beta(x_2), \alpha(x_1), x_3) = 0$.

$$\begin{aligned}(26) &= \tilde{a}s_{\alpha,\beta}(\beta(x_1), \alpha(x_2), x_3) + (-1)^{|x_1||x_2|}\tilde{a}s_{\alpha,\beta}(\beta(x_2), \alpha(x_1), x_3) \\ & \quad + (a) + (-1)^{|x_2||x_3|}(b) + (-1)^{|x_1|(|x_2|+|x_3|)}(c) \\ & \quad + \theta(\beta(x_1) \circ \alpha(x_2), \beta(x_3)) + (-1)^{|x_3|(|x_1|+|x_2|)}R(\beta(x_3))\theta(\beta(x_1), \alpha(x_2)) \\ & \quad - \theta(\alpha\beta(x_1), \alpha(x_2) \circ x_3) - L(\alpha\beta(x_1))\theta(\alpha(x_2), x_3) + (-1)^{|x_1||x_2|}\theta(\beta(x_2) \circ \alpha(x_1), \beta(x_3)) \\ & \quad + (-1)^{|x_3|(|x_1|+|x_2|)+|x_1||x_2|}R(\beta(x_3))\theta(\beta(x_2), \alpha(x_1)) - (-1)^{|x_1||x_2|}\theta(\alpha\beta(x_2), \alpha(x_1) \circ x_3) \\ & \quad - (-1)^{|x_1||x_2|}L(\alpha\beta(x_2))\theta(\alpha(x_1), x_3) \\ &= \theta(\beta(x_1) \circ \alpha(x_2), \beta(x_3)) + (-1)^{|x_3|(|x_1|+|x_2|)}R(\beta(x_3))\theta(\beta(x_1), \alpha(x_2)) \\ & \quad - \theta(\alpha\beta(x_1), \alpha(x_2) \circ x_3) - L(\alpha\beta(x_1))\theta(\alpha(x_2), x_3) + (-1)^{|x_1||x_2|}\theta(\beta(x_2) \circ \alpha(x_1), \beta(x_3)) \\ & \quad + (-1)^{|x_3|(|x_1|+|x_2|)+|x_1||x_2|}R(\beta(x_3))\theta(\beta(x_2), \alpha(x_1)) - (-1)^{|x_1||x_2|}\theta(\alpha\beta(x_2), \alpha(x_1) \circ x_3) \\ & \quad - (-1)^{|x_1||x_2|}L(\alpha\beta(x_2))\theta(\alpha(x_1), x_3) = 0.\end{aligned}$$

So the BiHom left super-alternativity is finished. By the same calculation, we also obtain (25) holds. Thus, $(A \oplus V, \star, \alpha + \alpha_V, \beta + \beta_V)$ is a Bihom-alternative superalgebra if and only if θ satisfies Eqs.(24) and (25). $\square$

Theorem 4.3. *Let $(A, \circ, \alpha, \beta)$ be a Bihom-alternative superalgebra, $\theta : A \times A \rightarrow A^*$, define two new operations on $A \oplus A^*$:*

$$(x + a^*) \star (y + b^*) = x \circ y + \theta(x, y) + (-1)^{|x||y|} R^*(x) b^* + L^*(y) a^*, \quad (27)$$

$$(\alpha + \alpha^*)(x + u) = \alpha(x) + \alpha^*(u), \quad (\beta + \beta^*)(x + u) = \beta(x) + \beta^*(u), \quad (28)$$

for any $x, y \in A; a^*, b^* \in A^*$, then $(A \oplus A^*, \star, \alpha + \alpha^*, \beta + \beta^*)$ is a regular Bihom-alternative algebra if and only if θ satisfies the following conditions:

$$\begin{aligned} & \theta(\beta(x_1) \circ \alpha(x_2), \beta(x_3)) + (-1)^{|x_3|(|x_1|+|x_2|)} L^*(\beta(x_3)) \theta(\beta(x_1), \alpha(x_2)) \\ & - \theta(\alpha\beta(x_1), \alpha(x_2) \circ x_3) - R^*(\alpha\beta(x_1)) \theta(\alpha(x_2), x_3) + (-1)^{|x_1||x_2|} \theta(\beta(x_2) \circ \alpha(x_1), \beta(x_3)) \\ & + (-1)^{|x_3|(|x_1|+|x_2|)+|x_1||x_2|} L^*(\beta(x_3)) \theta(\beta(x_2), \alpha(x_1)) - (-1)^{|x_1||x_2|} \theta(\alpha\beta(x_2), \alpha(x_1) \circ x_3) \\ & - (-1)^{|x_1||x_2|} R^*(\alpha\beta(x_2)) \theta(\alpha(x_1), x_3) = 0, \\ & \theta(x_1 \circ \beta(x_2), \alpha\beta(x_3)) + (-1)^{|x_3|(|x_1|+|x_2|)} L^*(\alpha\beta(x_3)) \theta(x_1, \beta(x_2)) \\ & - \theta(\alpha(x_1), \beta(x_2) \circ \alpha(x_3)) - R^*(\alpha(x_1)) \theta(\beta(x_2), \alpha(x_3)) + (-1)^{|x_2||x_3|} \theta(x_1 \circ \beta(x_3), \alpha\beta(x_2)) \\ & + (-1)^{|x_1||x_2|} L^*(\alpha\beta(x_2)) \theta(x_1, \beta(x_3)) - (-1)^{|x_2||x_3|} R^*(\alpha(x_1)) \theta(\beta(x_3), \alpha(x_2)) \\ & - (-1)^{|x_2||x_3|} \theta(\alpha(x_1), \beta(x_3) \circ \alpha(x_2)) = 0. \end{aligned}$$

For any $x_1, x_2, x_3 \in \mathcal{H}(A)$. Under the circumstances, $A \oplus A^*$ is called T_θ^* -extension of A , write as $T_\theta^* A$.

Proof. By Proposition 6, we know A^* is also a bimodule of A , so it is clear by Theorem 4.2. $\square$

In Section 2, we introduced the construction from alternative superalgebra to Bihom-alternative superalgebra. As an application, next we will give the construction from bimodules of alternative superalgebras to bimodules of Bihom alternative superalgebra.

Proposition 7. *Let (V, L, R) be a bimodule of an alternative superalgebra $(A, \circ)$. Given four even linear maps $\alpha, \beta : A \rightarrow A$ and $\alpha_V, \beta_V : V \rightarrow V$ such that $\alpha\beta = \beta\alpha$, $\alpha_V\beta_V = \beta_V\alpha_V$, $\alpha_V L(x) = L(\alpha(x))\alpha_V$, $\alpha_V R(x) = R(\alpha(x))\alpha_V$, $\beta_V L(x) = L(\beta(x))\beta_V$ and $\beta_V R(x) = R(\beta(x))\beta_V$. Then $(V, \tilde{L}, \tilde{R}, \alpha_V, \beta_V)$ is a bimodule of the Bihom-alternative superalgebra $(A, \mu_{\alpha,\beta}, \alpha, \beta)$, where $\tilde{L} = L(\alpha(x))\beta_V$, $\tilde{R} = R(\beta(x))\alpha_V$ and $\mu_{\alpha,\beta}(x, y) = \alpha(x) \circ \beta(y)$.*

Proof. Let $x, y \in \mathcal{H}(A)$, $v \in \mathcal{H}(V)$ and set $\mu_{\alpha,\beta}(x, y) = x * y$, that is $x * y = \alpha(x) \circ \beta(y)$. Then we have

$$\begin{aligned} & \tilde{L}(\beta(x) * \alpha(y))\beta_V(v) - \tilde{L}(\alpha\beta(x))\tilde{L}(\alpha(y))v + (-1)^{|x||y|} \tilde{L}(\beta(y) * \alpha(x))\beta_V(v) \\ & - (-1)^{|x||y|} \tilde{L}(\alpha\beta(y))\tilde{L}(\alpha(x))v \\ & = \tilde{L}(\alpha\beta(x) \circ \alpha\beta(y))\beta_V(v) - \tilde{L}(\alpha\beta(x))L(\alpha^2(y))\beta_V(v) \\ & + (-1)^{|x||y|} \tilde{L}(\alpha\beta(y) \circ \alpha\beta(x))\beta_V(v) \\ & - (-1)^{|x||y|} \tilde{L}(\alpha\beta(y))L(\alpha^2(x))\beta_V(v) \\ & = L(\alpha^2\beta(x) \circ \alpha^2\beta(y))\beta_V^2(v) - L(\alpha^2\beta(x))L(\alpha^2\beta(y))\beta_V^2(v) \end{aligned}$$

$$\begin{aligned}
& + (-1)^{|x||y|} L(\alpha^2 \beta(y) \circ \alpha^2 \beta(x)) \beta_V^2(v) \\
& - (-1)^{|x||y|} L(\alpha^2 \beta(y)) L(\alpha^2 \beta(x)) \beta_V^2(v) \\
& = 0
\end{aligned}$$

and

$$\begin{aligned}
& (-1)^{|x||y|} \tilde{R}(\beta(y)) \tilde{L}(\beta(x)) \alpha_V(v) - \tilde{L}(\alpha \beta(x)) \tilde{R}(y) \alpha_V(v) \\
& + (-1)^{|x||y|} \tilde{R}(\beta(y)) \tilde{R}(\alpha(x)) \beta_V(v) - \tilde{R}(\alpha(x) * y) \alpha_V \beta_V(v) \\
& = (-1)^{|x||y|} \tilde{R}(\beta(y)) L(\alpha \beta(x)) \alpha_V \beta_V(v) - \tilde{L}(\alpha \beta(x)) R(\beta(y)) \alpha_V^2(v) \\
& + (-1)^{|x||y|} \tilde{R}(\beta(y)) R(\alpha \beta(x)) \alpha_V \beta_V(v) - \tilde{R}(\alpha^2(x) \circ \beta(y)) \alpha_V \beta_V(v) \\
& = (-1)^{|x||y|} R(\beta^2(y)) L(\alpha^2 \beta(x)) \alpha_V^2 \beta_V(v) - L(\alpha^2 \beta(x)) R(\beta^2(y)) \alpha_V^2 \beta_V(v) \\
& + (-1)^{|x||y|} R(\beta^2(y)) R(\alpha^2 \beta(x)) \alpha_V^2 \beta_V(v) - R(\alpha^2 \beta(x) \circ \beta^2(y)) \alpha_V^2 \beta_V(v) \\
& = 0.
\end{aligned}$$

Similarly, other identities can also be obtained. $\square$

5. Bihom-Jordan superalgebras. In this section, We also give constructions for Bihom-Jordan and Bihom-Jordan admissible superalgebras. Additionally, we figure out that regular Bihom-alternative superalgebras are Bihom-Jordan admissible.

Definition 5.1. A BiHom superalgebra (A, μ, α, β) is called a Bihom-Jordan superalgebra if for all $x, y, z, t \in \mathcal{H}(A)$:

- (i) $\alpha\beta = \beta\alpha$,
- (ii) $\mu(\beta(x), \alpha(y)) = (-1)^{|x||y|} \mu(\beta(y), \alpha(x))$,
(Bihom-super commutativity condition)
- (iii) $\circ_{x,y,t} (-1)^{|t|(|x|+|z|)} \tilde{a}_{\alpha,\beta}(\mu(\beta^2(x), \alpha\beta(y)), \alpha^2\beta(z), \alpha^3(t)) = 0$.
(Bihom-Jordan super-identity)

In particular, it is reduced to a Jordan superalgebra when $\alpha = \beta = \text{id}$.

Definition 5.2. Let (A, μ, α, β) be a Bihom-superalgebra such that α and β are bijective maps. Define its plus Bihom-superalgebra as the Bihom-superalgebra $A^+ = (A, *, \alpha, \beta)$, where

$$x * y = \frac{1}{2}(\mu(x, y) + (-1)^{|x||y|} \mu(\alpha^{-1}\beta(y), \alpha\beta^{-1}(x))).$$

Note that product $*$ is Bihom-supercommutative. In fact, for all $x, y \in \mathcal{H}(A)$,

$$\begin{aligned}
\beta(x) * \alpha(y) &= \frac{1}{2}(\beta(x)\alpha(y) + (-1)^{|x||y|} \beta(y)\alpha(x)) \\
&= (-1)^{|x||y|} \frac{1}{2}(\beta(y)\alpha(x) + \beta(x)\alpha(y)) = (-1)^{|x||y|} \beta(y) * \alpha(x).
\end{aligned}$$

Moreover, the plus Bihom-superalgebra $A^+ = (A, *, \alpha, \beta)$ of a Bihom-Jordan-admissible superalgebra is a Bihom-Jordan superalgebra.

Proposition 8. Let (A, μ) be a Jordan-admissible superalgebra and $\alpha, \beta : A \rightarrow A$ be two even superalgebra morphisms satisfying $\alpha\beta = \beta\alpha$ and they are reversible. Then the induced BiHom-superalgebra $A_{\alpha,\beta} = (A, \mu' = \mu(\alpha \otimes \beta), \alpha, \beta)$ is a BiHom-Jordan-admissible superalgebra.

Proof. We need to prove that $(A, *, \alpha, \beta)$ is a BiHom-Jordan superalgebra, where

$$x * y = \frac{1}{2}(\mu'(x, y) + (-1)^{|x||y|}\mu'(\alpha^{-1}\beta(y), \alpha\beta^{-1}(x))).$$

The Jordan superalgebra obtained from Jordan-admissible superalgebra (A, μ) is denoted as $(A, \circ)$, where $\circ = \frac{1}{2}(\mu + \mu^{op})$ and $\mu^{op}(x, y) = (-1)^{|x||y|}\mu(y, x)$.

Also, it is easy to check that $x * y = \alpha(x) \circ \beta(y)$ and $\beta(x) * \alpha(y) = (-1)^{|x||y|}\beta(y) * \alpha(x)$. On the other hand,

$$\tilde{a}s_{\alpha, \beta}(\beta^2(x) * \alpha\beta(y), \alpha^2\beta(z), \alpha^3(t)) = \alpha^3\beta^2\tilde{a}s(x \circ y, z, t).$$

So,

$$\begin{aligned} & \circ_{x, y, t} (-1)^{|t|(|x|+|z|)} \tilde{a}s_{\alpha, \beta}(\beta^2(x) * \alpha\beta(y), \alpha^2\beta(z), \alpha^3(t)) \\ &= \alpha^3\beta^2 \circ_{x, y, t} (-1)^{|t|(|x|+|z|)} \tilde{a}s(x \circ y, z, t) = 0. \end{aligned}$$

Conclusion proof ends. $\square$

Theorem 5.3. *Let (A, μ) be a Jordan superalgebra and $\alpha, \beta : A \rightarrow A$ be two even commuting and reversible superalgebra morphisms. Then $(A, \mu' = \mu \circ (\alpha \otimes \beta), \alpha, \beta)$ is a Bihom-Jordan superalgebra.*

Proof. Firstly, note that μ is super-commutative since A is a Jordan superalgebra. For any $x, y, z, t \in \mathcal{H}(A)$:

Step 1,

$$\mu'(\beta(x), \alpha(y)) = \mu(\alpha\beta(x), \alpha\beta(y)) = (-1)^{|x||y|}\mu(\alpha\beta(y), \alpha\beta(x)) = (-1)^{|x||y|}\mu'(\beta(y), \alpha(x)).$$

Step 2, by direct calculation,

$$\tilde{a}s_{\alpha, \beta}(\mu'(\beta^2(x), \alpha\beta(y)), \alpha^2\beta(z), \alpha^3(t)) = \alpha^3\beta^2\tilde{a}s(\mu(x, y), z, t).$$

further, we have

$$\begin{aligned} & \circ_{x, y, t} (-1)^{|t|(|x|+|z|)} \tilde{a}s_{\alpha, \beta}(\mu'(\beta^2(x), \alpha\beta(y)), \alpha^2\beta(z), \alpha^3(t)) \\ &= \alpha^3\beta^2 \circ_{x, y, t} (-1)^{|t|(|x|+|z|)} \tilde{a}s(\mu(x, y), z, t) = 0. \end{aligned}$$

Thus, $(A, \mu' = \mu \circ (\alpha \otimes \beta), \alpha, \beta)$ is a Bihom-Jordan superalgebra. $\square$

Remark 5. Similar to Theorem 5.3, we may construct a new Bihom-Jordan superalgebra starting with a given Bihom-Jordan superalgebra and a pair of supercommuting Bihom-superalgebra morphisms.

Theorem 5.4. *Every regular BiHom-alternative superalgebra (A, μ, α, β) is BiHom-Jordan-admissible, that is to say $(A^+, *, \alpha, \beta)$ is a Bihom-Jordan superalgebra.*

Proof. As usual we write $\mu(x, y)$ as the juxtaposition xy . we only need to verify BiHom-Jordan super-identity. For all $x, y, z, t \in \mathcal{H}(A)$,

$$\begin{aligned} & \circ_{x, y, t} 8(-1)^{|t|(|x|+|z|)} \tilde{a}s_{\alpha, \beta}(\beta^2(x) * \alpha\beta(y), \alpha^2\beta(z), \alpha^3(t)) \\ &= 8(-1)^{|t|(|x|+|z|)} \underbrace{(((\beta^2(x) * \alpha\beta(y)) * \alpha^2\beta(z)) * \alpha^3\beta(t) - (\alpha\beta^2(x) * \alpha^2\beta(y)) * (\alpha^\beta(z) * \alpha^3(t)))}_{\xi} \\ &+ 8(-1)^{|x|(|y|+|z|)} \underbrace{(((\beta^2(y) * \alpha\beta(t)) * \alpha^2\beta(z)) * \alpha^3\beta(x) - (\alpha\beta^2(y) * \alpha^2\beta(t)) * (\alpha^\beta(z) * \alpha^3(x)))}_{\eta} \\ &+ 8(-1)^{|y|(|t|+|z|)} \underbrace{(((\beta^2(t) * \alpha\beta(x)) * \alpha^2\beta(z)) * \alpha^3\beta(y) - (\alpha\beta^2(t) * \alpha^2\beta(x)) * (\alpha^\beta(z) * \alpha^3(y)))}_{\gamma}, \end{aligned}$$

Further, we have the following via calculations:

$$\begin{aligned}
\xi = & (-1)^{|t|(|x|+|z|)} \{ ((\beta^2(x)\alpha\beta(y))\alpha^2\beta(z))\alpha^3\beta(t) \\
& + (-1)^{|t|(|x|+|y|+|z|)} \alpha^2\beta^2(t)((\alpha\beta(x)\alpha^2(y))\alpha^3(z)) \\
& + (-1)^{|z|(|x|+|y|)} (\alpha\beta^2(z)(\alpha\beta(x)\alpha^2(y)))\alpha^3\beta(t) \\
& + (-1)^{|z|(|x|+|y|)+|t|(|x|+|y|+|z|)} \alpha^2\beta^2(t)(\alpha^2\beta(z)(\alpha^2(x)\alpha^3\beta^{-1}(y))) \\
& + (-1)^{|x||y|} ((\beta^2(y)\alpha\beta(x))\alpha^2\beta(z))\alpha^3\beta(t) \\
& + (-1)^{|t|(|x|+|y|+|z|)+|x||y|} \alpha^2\beta^2(t)((\alpha\beta(y)\alpha^2(x))\alpha^3(z)) \\
& + (-1)^{|z|(|x|+|y|)+|x||y|} (\alpha\beta^2(z)(\alpha\beta(y)\alpha^2(x)))\alpha^3\beta(t) \\
& + (-1)^{|z|(|x|+|y|)+|t|(|x|+|y|+|z|)+|x||y|} \alpha^2\beta^2(t)(\alpha^2\beta(z)(\alpha^2(y)\alpha^3\beta^{-1}(x))) \\
& - (\alpha\beta^2(x)\alpha^2\beta(y))(\alpha^2\beta(z)\alpha^3(t)) - (-1)^{(|z|+|t|)(|x|+|y|)} (\alpha\beta^2(z)\alpha^2\beta(t))(\alpha^2\beta(x)\alpha^3(y)) \\
& - (-1)^{|z||t|} (\alpha\beta^2(x)\alpha^2\beta(y))(\alpha^2\beta(t)\alpha^3(z)) \\
& - (-1)^{(|z|+|t|)(|x|+|y|)+|z||t|} (\alpha\beta^2(t)\alpha^2\beta(z))(\alpha^2\beta(x)\alpha^3(y)) \\
& - (-1)^{|x||y|} (\alpha\beta^2(y)\alpha^2\beta(x))(\alpha^2\beta(z)\alpha^3(t)) \\
& - (-1)^{(|z|+|t|)(|x|+|y|)+|x||y|} (\alpha\beta^2(z)\alpha^2\beta(t))(\alpha^2\beta(y)\alpha^3(x)) \\
& - (-1)^{|x||y|+|z||t|} (\alpha\beta^2(y)\alpha^2\beta(x))(\alpha^2\beta(t)\alpha^3(z)) \\
& - (-1)^{(|z|+|t|)(|x|+|y|)+|x||y|+|z||t|} (\alpha\beta^2(t)\alpha^2\beta(z))(\alpha^2\beta(y)\alpha^3(x)) \},
\end{aligned}$$

analogously, there are concrete equations of η and γ , in this time,

$$\odot_{x,y,t} 8(-1)^{|t|(|x|+|z|)} \widetilde{as}_{\alpha,\beta}(\beta^2(x) * \alpha\beta(y), \alpha^2\beta(z), \alpha^3(z)) = \xi + \eta + \gamma.$$

So,

$$\begin{aligned}
& \odot_{x,y,t} 8(-1)^{|t|(|x|+|z|)} \widetilde{as}_{\alpha,\beta}(\beta^2(x) * \alpha\beta(y), \alpha^2\beta(z), \alpha^3(z)) \\
& = \xi + \eta + \gamma \\
& = \{ (-1)^{|t|(|x|+|z|)} \widetilde{as}_{\alpha,\beta}(\beta^2(x)\alpha\beta(y), \alpha^2\beta(z), \alpha^3(t)) \\
& \quad - (-1)^{|z|(|x|+|y|)+|t||y|} \widetilde{as}_{\alpha,\beta}(\alpha\beta^2(t), \alpha^2\beta(z), \alpha^2(x)\alpha^3\beta^{-1}(y)) \\
& \quad + (-1)^{|x|(|y|+|z|)} \widetilde{as}_{\alpha,\beta}(\beta^2(y)\alpha\beta(t), \alpha^2\beta(z), \alpha^3(x)) \\
& \quad - (-1)^{|z|(|t|+|y|)+|t||x|} \widetilde{as}_{\alpha,\beta}(\alpha\beta^2(x), \alpha^2\beta(z), \alpha^2(y)\alpha^3\beta^{-1}(t)) \\
& \quad + (-1)^{|y|(|t|+|z|)} \widetilde{as}_{\alpha,\beta}(\beta^2(t)\alpha\beta(x), \alpha^2\beta(z), \alpha^3(y)) \\
& \quad - (-1)^{|z|(|t|+|x|)+|y||x|} \widetilde{as}_{\alpha,\beta}(\alpha\beta^2(y), \alpha^2\beta(z), \alpha^2(t)\alpha^3\beta^{-1}(x)) \\
& \quad + \{ (-1)^{|t|(|x|+|z|)+|x||y|} \widetilde{as}_{\alpha,\beta}(\beta^2(y)\alpha\beta(x), \alpha^2\beta(z), \alpha^3(t)) \\
& \quad - (-1)^{|z|(|x|+|y|)+|y|(|x|+|t|)} \widetilde{as}_{\alpha,\beta}(\alpha\beta^2(t), \alpha^2\beta(z), \alpha^2(y)\alpha^3\beta^{-1}(x)) \\
& \quad + (-1)^{|x|(|y|+|z|)+|y||t|} \widetilde{as}_{\alpha,\beta}(\beta^2(t)\alpha\beta(y), \alpha^2\beta(z), \alpha^3(x)) \\
& \quad - (-1)^{|z|(|t|+|y|)+|t|(|x|+|y|)} \widetilde{as}_{\alpha,\beta}(\alpha\beta^2(x), \alpha^2\beta(z), \alpha^2(t)\alpha^3\beta^{-1}(y)) \\
& \quad + (-1)^{|y|(|t|+|z|)+|t||x|} \widetilde{as}_{\alpha,\beta}(\beta^2(x)\alpha\beta(t), \alpha^2\beta(z), \alpha^3(y)) \\
& \quad - (-1)^{|z|(|t|+|x|)+|x|(|t|+|y|)} \widetilde{as}_{\alpha,\beta}(\alpha\beta^2(y), \alpha^2\beta(z), \alpha^2(x)\alpha^3\beta^{-1}(t)) \} \\
& \quad \{ + (-1)^{|t||y|} \alpha^2\beta^2(t)((\alpha\beta(x)\alpha^2(y))\alpha^3(z))
\end{aligned} \tag{29}$$

$$\begin{aligned}
& + (-1)^{|t|(|x|+|z|)+|z|(|x|+|y|)}(\alpha\beta^2(z)(\alpha\beta(x)\alpha^2(y)))\alpha^3\beta(t) \\
& + (-1)^{|y|(|t|+|x|)}\alpha^2\beta^2(t)((\alpha\beta(y)\alpha^2(x))\alpha^3(z)) \\
& + (-1)^{|t|(|x|+|z|)+|z|(|x|+|y|)+|x||y|}(\alpha\beta^2(z)(\alpha\beta(y)\alpha^2(x)))\alpha^3\beta(t) \\
& + (-1)^{|x||t|}\alpha^2\beta^2(x)((\alpha\beta(y)\alpha^2(t))\alpha^3(z)) \\
& + (-1)^{|x|(|y|+|z|)+|z|(|t|+|y|)}(\alpha\beta^2(z)(\alpha\beta(y)\alpha^2(t)))\alpha^3\beta(x) \\
& + (-1)^{|t|(|x|+|y|)}\alpha^2\beta^2(x)((\alpha\beta(t)\alpha^2(y))\alpha^3(z)) \\
& + (-1)^{|x|(|y|+|z|)+|z|(|t|+|y|)+|t||y|}(\alpha\beta^2(z)(\alpha\beta(t)\alpha^2(y)))\alpha^3\beta(x) \\
& + (-1)^{|y||x|}\alpha^2\beta^2(y)((\alpha\beta(t)\alpha^2(x))\alpha^3(z)) \\
& + (-1)^{|y|(|t|+|z|)+|z|(|t|+|x|)}(\alpha\beta^2(z)(\alpha\beta(t)\alpha^2(x)))\alpha^3\beta(y) \\
& + (-1)^{|x|(|y|+|t|)}\alpha^2\beta^2(y)((\alpha\beta(x)\alpha^2(t))\alpha^3(z)) \\
& + (-1)^{|y|(|t|+|z|)+|z|(|t|+|x|)+|t||x|}(\alpha\beta^2(z)(\alpha\beta(x)\alpha^2(t)))\alpha^3\beta(y) \\
& - (-1)^{(|z|+|t|)(|x|+|y|)}(\alpha\beta^2(z)\alpha^2\beta(t))(\alpha^2\beta(x)\alpha^3(y)) \\
& - (-1)^{|z||t|}(\alpha\beta^2(x)\alpha^2\beta(y))(\alpha^2\beta(t)\alpha^3(z)) \\
& - (-1)^{(|z|+|t|)(|x|+|y|)+|x||y|}(\alpha\beta^2(z)\alpha^2\beta(t))(\alpha^2\beta(y)\alpha^3(x)) \\
& - (-1)^{|x||y|+|z||t|}(\alpha\beta^2(y)\alpha^2\beta(x))(\alpha^2\beta(t)\alpha^3(z)) \\
& - (-1)^{(|z|+|x|)(|y|+|t|)}(\alpha\beta^2(z)\alpha^2\beta(x))(\alpha^2\beta(y)\alpha^3(t)) \\
& - (-1)^{|z||x|}(\alpha\beta^2(y)\alpha^2\beta(t))(\alpha^2\beta(x)\alpha^3(z)) \\
& - (-1)^{(|z|+|x|)(|y|+|t|)+|t||y|}(\alpha\beta^2(z)\alpha^2\beta(x))(\alpha^2\beta(t)\alpha^3(y)) \\
& - (-1)^{|t||y|+|z||x|}(\alpha\beta^2(t)\alpha^2\beta(y))(\alpha^2\beta(x)\alpha^3(z)) \\
& - (-1)^{(|z|+|y|)(|x|+|t|)}(\alpha\beta^2(z)\alpha^2\beta(y))(\alpha^2\beta(t)\alpha^3(x)) \\
& - (-1)^{|z||y|}(\alpha\beta^2(t)\alpha^2\beta(x))(\alpha^2\beta(y)\alpha^3(z)) \\
& - (-1)^{(|z|+|y|)(|x|+|t|)+|x||t|}(\alpha\beta^2(z)\alpha^2\beta(y))(\alpha^2\beta(x)\alpha^3(t)) \\
& - (-1)^{|x||t|+|z||y|}(\alpha\beta^2(x)\alpha^2\beta(t))(\alpha^2\beta(y)\alpha^3(z))\}. \tag{30}
\end{aligned}$$

Since A is a Bihom-alternative superalgebra, $\tilde{a}s_{\alpha,\beta}(\beta^2 \otimes \alpha\beta \otimes \alpha^2)$ is super-alternating. (29) equals 0. So

$$\begin{aligned}
& \circ_{x,y,t} 8(-1)^{|t|(|x|+|z|)}\tilde{a}s_{\alpha,\beta}(\beta^2(x) * \alpha\beta(y), \alpha^2\beta(z), \alpha^3(z)) \\
& = (30) \\
& = (-1)^{|t||y|}[\tilde{a}s_{\alpha,\beta}(\beta^2(t), \alpha\beta(x), \alpha^2(y)) + (-1)^{|x||y|}\tilde{a}s_{\alpha,\beta}(\beta^2(t), \alpha\beta(y), \alpha^2(x)), \alpha^3\beta(z)] \\
& \quad + (-1)^{|x||t|}[\tilde{a}s_{\alpha,\beta}(\beta^2(x), \alpha\beta(y), \alpha^2(t)) + (-1)^{|t||y|}\tilde{a}s_{\alpha,\beta}(\beta^2(x), \alpha\beta(t), \alpha^2(y)), \alpha^3\beta(z)] \\
& \quad + (-1)^{|y||x|}[\tilde{a}s_{\alpha,\beta}(\beta^2(y), \alpha\beta(t), \alpha^2(x)) + (-1)^{|x||t|}\tilde{a}s_{\alpha,\beta}(\beta^2(y), \alpha\beta(x), \alpha^2(t)), \alpha^3\beta(z)] \\
& = 0.
\end{aligned}$$

Thus, it satisfies the Bihom-Jordan super-identity for $A^+ = (A, *, \alpha, \beta)$. $\square$

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