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MULTIDIMENSIONAL STABILITY OF PYRAMIDAL TRAVELING FRONTS IN DEGENERATE FISHER-KPP MONOSTABLE AND COMBUSTION EQUATIONS

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Abstract. In this paper, multidimensional stability of pyramidal traveling fronts are studied to the reaction-diffusion equations with degenerate Fisher-KPP monostable and combustion nonlinearities. By constructing supersolutions and subsolutions coupled with the comparison principle, we firstly prove that under any initial perturbation (possibly large) decaying at space infinity, the three-dimensional pyramidal traveling fronts are asymptotically stable in weighted L^∞ spaces on $\mathbb{R}^n$ $(n \geq 4)$. Secondly, we show that under general bounded perturbations (even very small), the pyramidal traveling fronts are not asymptotically stable by constructing a solution which oscillates permanently between two three-dimensional pyramidal traveling fronts on $\mathbb{R}^4$.

1. **Introduction.** In this paper, we investigate the large time behavior of solutions to the following Cauchy problem:

$$\begin{cases} u_t(t,\mathbf{x}) = \Delta u(t,\mathbf{x}) + f\left(u(t,\mathbf{x})\right), & \mathbf{x} \in \mathbb{R}^n,\ t > 0, \\ u(0,\mathbf{x}) = u_0(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $n \in \mathbb{N}$, $u_t = \frac{\partial u}{\partial t}$ and Δ is the standard Laplace operator with respect to the space variables $\mathbf{x} \in \mathbb{R}^n$. For some constants $\varsigma \in [0,1]$ and $\iota \in [0,1]$, the nonlinear reaction term $f \in C^{1+\varsigma}([-\iota, 1+\iota], \mathbb{R})$ satisfies

(H1): $f(0) = f(1) = 0,\ f'(0) \geq 0,\ f'(1) < 0,\ f(u) \geq 0$ for $u \in (0,1)$.

Such equations arise in various phenomena in population dynamics, combustion and chemistry ecology (see [1]), where u typically stands for the concentration of a species or the temperature.

In what follows, we shall study the multidimensional stability of three-dimensional pyramidal traveling fronts to Eq. (1) in $\mathbb{R}^n$ with $n \geq 4$. In order to motivate our study, let us recall some known results in the study of traveling fronts of Eq. (1). In $\mathbb{R}$, traveling fronts are solutions taking the form

$$u(t,\mathbf{x}) = \phi_f(\mathbf{p}),\ \mathbf{p} = \mathbf{x} - c_f t,$$

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where $c_f \geq 0$ is the propagation speed and ϕ_f is the wave profile satisfying

$$\begin{cases} \phi_f''(\mathbf{p}) + c_f\phi_f'(\mathbf{p}) + f(\phi_f(\mathbf{p})) = 0, \quad \phi_f'(\mathbf{p}) < 0, \ \forall\ \mathbf{p} \in \mathbb{R}, \\ \phi_f(+\infty) = 0, \ \phi_f(-\infty) = 1. \end{cases} \tag{2}$$

Such solution $u(t, \mathbf{x}) = \phi_f(\mathbf{x} - c_f t)$ are called the planar traveling front since its level set is a hyperplane. Throughout the paper, we further assume that

(H2): There exists $\phi_f(\mathbf{p}) \in C^2(\mathbb{R})$ with speed $c_f^* > 0$ satisfying (2) and

$$\lim_{\mathbf{p}\to+\infty} \frac{\phi_f'(\mathbf{p})}{\phi_f(\mathbf{p})} = \Lambda < \Lambda_1 \leq 0,$$

where Λ and Λ_1 are two real roots of the equation $\mu^2 + c_f^*\mu + f'(0) = 0$.

The equation (1) with assumptions (H1)-(H2) is called degenerate Fisher-KPP monostable and combustion equation. In fact, it follows from [1, 10] that the assumptions (H1)-(H2) hold with c_f^* being the minimal wave speed and the unique wave speed of planar traveling front ϕ_f when the nonlinear reaction term f is of degenerate Fisher-KPP monostable type and combustion type, respectively. See [2, 19] for more details.

In $\mathbb{R}^n$ with $n \geq 2$, the function $\phi_f(z - c_f t)$ is clearly still the solution of Eq. (1) with $\mathbf{x} = (x, y, z) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}$. A very interesting question is to consider the asymptotic stability of one-dimensional traveling front $\phi_f(z - c_f t)$ in n (≥ 2)-dimensional spaces. For this problem, one can refer to [9, 11, 20, 21, 22] and the references therein to Allen-Cahn equation. It is worth to mention that Matano et al. [14, 13] investigated the asymptotic stability of one-dimensional traveling front under any initial spatial decaying perturbations by using sub-super solutions method combining with the comparison principle. Motivated by [14, 13], Lv and Wang [12] and Bu and Wang [4] established the multidimensional stability of planar traveling fronts to Eq. (1) with Fisher-KPP nonlinearity, non-KPP monostable and combustion nonlinearity, respectively. He and Wu [8] using spectral method studied the stability of traveling front for degenerate Fisher type equations.

However, due to the influence of curvature and spatial dimension, there are other types of traveling fronts in $\mathbb{R}^n$ with $n \geq 2$ which are called non-planar traveling fronts, since their level sets are not hyperplanes anymore. Readers can see for instance Bu and Wang [2, 3], Hamel et. al. [7, 6], Ninomiya and Taniguchi [15], Taniguchi [17, 18] and Wang and Bu [19] for the existence and stability of two-dimensional V-shaped fronts, three-dimensional pyramidal fronts and multidimensional conical shaped fronts. Noting that it is also very interesting to investigate the multidimensional stability of nonplanar traveling fronts. See Sheng et. al. [16] and Cheng and Yuan [5] for the multidimensional stability of two-dimensional V-shaped fronts and three-dimensional pyramidal to Allen-Cahn equation under any spatially decaying initial perturbations, respectively. Recently, Bu and Wang [4] also established the stability of two-dimensional V-shaped fronts in $\mathbb{R}^n$ with $n > 2$ to degenerate Fisher-KPP monostbale and combustion equations.

In $\mathbb{R}^n$ with $n \geq 3$, we write $\mathbf{x} = (x, y, z, s)$ with $x \in \mathbb{R}^{n-3}$ and $(y, z, s) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$. It follows from Wang and Bu [19] that the equation (1) exists a three-dimensional pyramidal fronts with the form $u(t, y, z, s) = V(y, z, \varsigma)$ under the assumptions (H1)-(H2) in $\mathbb{R}^3$, where $\varsigma = s - ct$. For simplicity, we still write $V(y, z, \varsigma)$ as $V(y, z, s)$. Let $l \in \mathbb{N}$ with $l \geq 3$ and $\{\theta_j\}_{1\leq j\leq l}$ satisfy

$$0 \leq \theta_1 < \theta_2 < \cdots < \theta_l < 2\pi \text{ and } \max_{1\leq j\leq l} (\theta_{j+1} - \theta_j) < \pi,$$

where $\theta_{l+1} := \theta_1 + 2\pi$. Let $m_* = \frac{\sqrt{c^2 - c_f^{*2}}}{c_f^*}$ and

$$h(y,z) = \max_{1\le j\le l} h_j(y,z) = \max_{1\le j\le l} m_*(y\cos\theta_j + z\sin\theta_j) \quad \text{for } (y,z)\in\mathbb{R}^2,$$

where $c > c_f^*$. Then $\left\{(y,z,s)\in\mathbb{R}^3 \middle| s = h(y,z)\right\}$ is a 3-dimensional pyramid. Let Γ denote the set of all edges of a pyramid.

Theorem I (see Wang and Bu [19]) Assume that (H1)-(H2) hold. For any $c > c_f^*$, Eq. (1) admits a traveling front of pyramidal shape satisfying

$$V_{yy} + V_{zz} + V_{ss} + cV_s + f(V) = 0, \tag{3}$$

$$\lim_{\gamma\to+\infty} \sup_{(y,z,s)\in D(\gamma)} \frac{\left|V(y,z,s) - \phi_f\left(\frac{c_f^*}{c}(s - h(y,z))\right)\right|}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)} = 0,\ \forall\ \beta\in\left(\frac{\Lambda_1}{\Lambda},1\right),$$

where $D(\gamma) = \left\{(y,z,s)\in\mathbb{R}^3 \middle| \mathrm{dist}((y,z,s),\Gamma) > \gamma\right\}$. Moreover, one has $\frac{\partial}{\partial s}V(y,z,s) < 0$ for $(y,z,s)\in\mathbb{R}^3$ and

$$\phi_f\left(\frac{c_f^*}{c}(s-h(y,z))\right) < V(y,z,s) < 1,\ \forall\ (y,z,s)\in\mathbb{R}^3.$$

It is obvious that the three-dimensional pyramidal front in Theorem I is also the solution to Eq. (1) in $\mathbb{R}^n$ with $n > 3$. The aim of this paper is to study the multidimensional stability of three-dimensional pyramidal fronts $V(y,z,s)$ in $\mathbb{R}^n$ with $n > 3$. Motivated by [14, 4], we mainly use the super-sub solutions method combining with the comparison principle. However, since we are treating degenerate Fisher-KPP monostable and combustion equations in $\mathbb{R}^n$ with $n \ge 4$, many modifications and techniques are needed.

In the following, we use the moving coordinate with speed c toward the s direction. Let $\tilde{s} = s - ct$ and $u(t,x,y,z,s) = \vartheta(t,x,y,z,\tilde{s})$. For simplicity, we still denote $\vartheta(t,x,y,z,\tilde{s})$ by $\vartheta(t,x,y,z,s)$. Then the Eq. (1) can be rewritten as

$$\begin{cases}\vartheta_t - \Delta\vartheta - c\vartheta_s - f(\vartheta) = 0,\ t>0,\ (x,y,z,s)\in\mathbb{R}^{n-3}\times\mathbb{R}^3,\\ \vartheta(0,x,y,z,s) = \vartheta_0(x,y,z,s),\ (x,y,z,s)\in\mathbb{R}^{n-3}\times\mathbb{R}^3.\end{cases} \tag{4}$$

In the sequel, the solution to Eq. (4) is written as $\vartheta(t,x,y,z,s;\vartheta_0)$. The main results in the present paper are as follows.

Theorem 1.1. *Assume that* (H1)-(H2) *hold. Suppose that the initial value $\vartheta_0(x,y,z,s)$ is of class $C(\mathbb{R}^n,[0,1])$ with $n>3$ and satisfies*

$$\lim_{R\to+\infty} \sup_{|x|+|y|+|z|+|s|\ge R} \frac{|\vartheta_0(x,y,z,s) - V(y,z,s)|}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)} = 0$$

for some $\beta\in(\frac{\Lambda_1}{\Lambda},1)$. Then the solution $\vartheta(x,y,z,s;\vartheta_0)$ to Eq. (4) *satisfies*

$$\lim_{t\to+\infty} \sup_{(x,y,z,s)\in\mathbb{R}^n} \frac{|\vartheta(t,x,y,z,s;\vartheta_0) - V(y,z,s)|}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)} = 0. \tag{5}$$

The above theorem shows that under the initial perturbations decaying as $|x| + |y| + |z| + |s| \to +\infty$, the three-dimensional pyramidal traveling fronts are asymptotically stable in weighted L^∞ spaces on $\mathbb{R}^n$ ($n \ge 4$). In particular, when the initial perturbations further belong to L^1 in a certain sense, the convergence rate for (5) is algebraic, see the following theorem for more detail.

Theorem 1.2. *Suppose that* (H1)-(H2) *hold and the initial value* $\vartheta_0(x,y,z,s)$ *to Eq.* (4) *satisfies*

$$V\left(y,z,s-\vartheta_0^-(x)\right) \le \vartheta_0(x,y,z,s) \le V\left(y,z,s-\vartheta_0^+(x)\right) \tag{6}$$

for some smooth functions ϑ_0^-, $\vartheta_0^+ \in L^1(\mathbb{R}^{n-3}) \cap L^\infty(\mathbb{R}^{n-3})$ *with* $n > 3$. *Then for any* $\beta \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$, *the solution* $\vartheta(t,x,y,z,s;\vartheta_0)$ *to Eq.* (4) *satisfies*

$$\sup_{(x,y,z,s)\in\mathbb{R}^n} \frac{|\vartheta(t,x,y,z,s;\vartheta_0) - V(y,z,s)|}{\phi_f^\beta\left(\frac{c_f^*}{c}\left(s-h(y,z)\right)\right)} \le Ct^{-\frac{n-3}{2}}, \ t>0, \tag{7}$$

where $C > 0$ *is a constant depending on* β, f, $||\vartheta_0^-||_{L^1(\mathbb{R}^{n-3})}$, $||\vartheta_0^-||_{L^\infty(\mathbb{R}^{n-3})}$, $||\vartheta_0^+||_{L^1(\mathbb{R}^{n-3})}$ *and* $||\vartheta_0^+||_{L^\infty(\mathbb{R}^{n-3})}$.

If the initial perturbations in Theorem 1.2 keep the sign, then we can obtain that the convergence rate (7) is optimal in some sense.

Proposition 1. *Let* ϑ_0 *be as in* (6) *and assume that either* $\vartheta_0^- \ge 0$, $\vartheta_0^- \not\equiv 0$ *or* $\vartheta_0^+ \le 0$, $\vartheta_0^+ \not\equiv 0$. *Then for any* $\beta \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$, *there exist constants* $D_1 > 0$ *and* $D_2 > 0$ *such that*

$$D_1(1+t)^{-\frac{n-3}{2}} \le \sup_{(x,y,z,s)\in\mathbb{R}^n} \frac{|\vartheta(t,x,y,z,s;\vartheta_0) - V(y,z,s)|}{\phi_f^\beta\left(\frac{c_f^*}{c}\left(s-h(y,z)\right)\right)} \le D_2 t^{-\frac{n-3}{2}}, \ t>0.$$

Finally, by constructing a solution to Eq. (1) which oscillates permanently between two pyramidal traveling fronts, we show that the three-dimensional pyramidal traveling fronts are not asymptotically stable under general bounded perturbations (even very small) on $\mathbb{R}^4$.

Theorem 1.3. *Let* $n = 4$. *Assume that* (H1)-(H2) *hold. Then for any* $\beta \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$ *and* $\bar\delta > 0$, *there exists a bounded function* $\omega(x) \in C(\mathbb{R})$ *with* $||\omega||_{L^\infty(\mathbb{R})} = \bar\delta$ *such that the solution* $u(t,x,y,z,s)$ *to Eq.* (1) *with the initial value* $u_0(x,y,z,s) = V(y,z,s-\omega(x))$ *satisfies*

$$\lim_{m\to+\infty} \sup_{|x|\le m!-1,(y,z,s)\in\mathbb{R}^3} \frac{|u(t_m,x,y,z,s) - V(y,z,s-ct_m+(-1)^m\bar\delta)|}{\phi_f^\beta\left(\frac{c_f^*}{c}\left(s-ct_m-h(y,z)\right)\right)} = 0,$$

where $t_m = \frac{m(m!)^2}{4}$.

Remark 1. From the perspective of dynamical systems, the above result yields that in the weighted $L^\infty_{loc}(\mathbb{R}^4)$, the ω-limit set of the solution u to Eq. (1) contains at least two distinct points. And each of them is a translation of the same three-dimensional pyramidal traveling front.

We organize this paper as follows. In Section 2, we give some preliminaries including the properties of the pyramidal traveling fronts, some known results on the curvature flow problem and a mollified pyramid. In Section 3, we prove that the three-dimensional pyramidal traveling fronts are asymptotically stable in $\mathbb{R}^n$ $(n \ge 4)$ by constructing new types of supersolutions and subsolutions coupled with comparison principle. That is, we prove Theorems 1.1-1.2 and Proposition 1. In Section 4, we prove Theorem 1.3 which states the existence of solution to Eq. (1) which oscillates permanently with non-decaying amplitude.

2. **Preliminaries.** In this section, we state some known results which play an important role in the proving of the main results. Throughout the paper, let

$$\gamma_1 := \sup_{\mathbf{p}\in\mathbb{R}} \left|\frac{\phi'(\mathbf{p})}{\phi(\mathbf{p})}\right|, \quad \gamma_2 := \sup_{\mathbf{p}\in\mathbb{R}} \left|\frac{\phi''(\mathbf{p})}{\phi(\mathbf{p})}\right|, \quad \gamma_3 := \sup_{u\in[-\iota,1+\iota]} |f'(u)|$$

and fix $\iota^* \in \left(0, \frac{\iota}{2}\right)$ such that for any $u \in (1-2\iota^*, 1+\iota^*)$,

$$\frac{3}{2}f'(1) < f'(u) < \frac{1}{2}f'(1).$$

We now recall some known results on the curvature flow problem. See [14] for more details. The mean curvature flow for a graphical surface $w(t,x)$ on $\mathbb{R}^{n-3}$ is given by the following Cauchy problem:

$$\begin{cases} \frac{w_t}{\sqrt{1+|\nabla w|^2}} = \operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right), \quad x\in\mathbb{R}^{n-3},\ t>0, \\ w(0,x) = w_0(x), \quad x\in\mathbb{R}^{n-3}. \end{cases} \tag{8}$$

Assume that on $\mathbb{R}^{n-3}$, the first and second derivatives of w with respect to x are bounded, then by direct calculation, there exists a constant $k>0$ large enough such that

$$\begin{aligned} 0 &= w_t - \sqrt{1+|\nabla w|^2}\cdot \operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) \\ &= w_t - \Delta w + \sum_{i,j=1}^{n-3} \frac{w_{x_i}w_{x_j}w_{x_ix_j}}{1+|\nabla w|^2} \\ &\geq w_t - \Delta w - k|\nabla w|^2. \end{aligned}$$

It is clear that $w(t,x)$ is a subsolution of the following Cauchy problem:

$$\begin{cases} v_t^+ = \Delta v^+ + k|\nabla v^+|^2, \quad x\in\mathbb{R}^{n-3},\ t>0, \\ v^+(0,x) = w_0(x), \quad x\in\mathbb{R}^{n-3}. \end{cases}$$

Taking the Cole-Hopf transformation $w^+(t,x) = \exp(kv^+(t,x))$, we have

$$\begin{cases} w_t^+ = \Delta w^+, \quad x\in\mathbb{R}^{n-3},\ t>0, \\ w^+(0,x) = \exp(kw_0(x)), \quad x\in\mathbb{R}^{n-3}. \end{cases}$$

Thus we can obtain that

$$v^+(t,x) = \frac{1}{k}\log\left|\int_{\mathbb{R}^{n-3}} \Gamma(t,x-\eta)\exp(kw_0(\eta))d\eta\right|, \tag{9}$$

where

$$\Gamma(t,\eta) = \frac{1}{(4\pi t)^{\frac{n-3}{2}}}\exp\left(-\frac{|\eta|^2}{4t}\right).$$

Therefore (9) gives an upper estimate for the solution $w(t,x)$ to the Cauchy problem (8). Similarly, the lower estimate for $w(t,x)$ can be given by the Cauchy problem

$$\begin{cases} v_t^- = \Delta v^- - k|\nabla v^-|^2, \quad x\in\mathbb{R}^{n-3},\ t>0, \\ v^-(0,x) = w_0(x), \quad x\in\mathbb{R}^{n-3}. \end{cases}$$

That is,

$$v^-(t,x) = -\frac{1}{k}\log\left|\int_{\mathbb{R}^{n-3}} \Gamma(t,x-\eta)\exp(-kw_0(\eta))d\eta\right|. \tag{10}$$

Let $k > 0$ be any constant and $v^{\pm}(t,x)$ be solutions to the following Cauchy problems

$$\begin{cases} v_t^{\pm} = \Delta v^{\pm} \pm k|\nabla v^{\pm}|^2, \quad x \in \mathbb{R}^{n-3},\ t > 0, \\ v^{\pm}(0,x) = w_0(x), \quad x \in \mathbb{R}^{n-3}. \end{cases}$$

The following lemma gives the large time behavior of $v^{\pm}(t,x)$, see Lemma 2.4 of [14].

Lemma 2.1. *If the initial value $w_0 \in C(\mathbb{R}^{n-3})$ is bounded and satisfies $\lim_{|x|\to\infty} |w_0(x)| = 0$, then the solutions $v^{\pm}(t,x)$ satisfy*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^{n-3}} |v^{\pm}(t,x)| = 0,$$

respectively. If we further assume that $w_0 \in L^1(\mathbb{R}^{n-3})$, then

$$\sup_{x\in\mathbb{R}^{n-3}} |v^{\pm}(t,x)| \le \frac{1}{k} || \exp(kw_0) - 1||_{L^1(\mathbb{R}^{n-3})} \cdot t^{-\frac{n-3}{2}},\ t > 0.$$

Similar to the proof of Lemma 3.2 of [16] and Lemma 2.2 of [14], we can obtain the following key estimates about the three-dimensional pyramidal fronts and planar fronts, respectively.

Lemma 2.2. *Let $V(y,z,s)$ be a pyramidal front to Eq. (3). Then there exists a positive constant $\tilde{k}_1$ (depending on f) such that*

$$\tilde{k}_1 V_s(y,z,s) \le V_{ss}(y,z,s) \le -\tilde{k}_1 V_s(y,z,s), \quad \forall\ (y,z,s) \in \mathbb{R}^3. \tag{11}$$

Lemma 2.3. *Let $\phi_f(\mathbf{p})$ be a planar front to Eq. (2). There exists a constant $\tilde{k}_2 > 0$ depending only on f such that*

$$\tilde{k}_2 \phi_f'(\mathbf{p}) \le \phi_f''(\mathbf{p}) \le -\tilde{k}_2 \phi_f'(\mathbf{p}), \quad \forall\ \mathbf{p} \in \mathbb{R}.$$

Let $\tilde{k} = \max\{\tilde{k}_1, \tilde{k}_2\}$. Thus we have

$$\begin{cases} \tilde{k} V_s(y,z,s) \le V_{ss}(y,z,s) \le -\tilde{k} V_s(y,z,s), \quad \forall\ (y,z,s) \in \mathbb{R}^3, \\ \tilde{k}\phi_f'(\mathbf{p}) \le \phi_f''(\mathbf{p}) \le -\tilde{k}\phi_f'(\mathbf{p}), \quad \forall\ \mathbf{p} \in \mathbb{R}. \end{cases} \tag{12}$$

The following lemma shows some properties on three-dimensional pyramidal traveling fronts V.

Lemma 2.4. ([19, Lemmas 3.3 and 3.4]) *Let $V(y,z,s)$ be a pyramidal front to Eq. (3). One has*

$$\lim_{R\to+\infty} \sup_{|s-h(y,z)|\ge R} \frac{V_s(y,z,s)}{\phi_f^{\beta}\left(\frac{c_f^*}{c}(s-h(y,z))\right)} = 0, \quad \forall\ \beta \in \left(\frac{\Lambda_1}{\Lambda}, 1\right), \tag{13}$$

and

$$\inf_{\delta\le V(y,z,s)\le 1-\delta} V_s(y,z,s) < 0 \quad \textit{for any } \delta \in (0, \iota^*). \tag{14}$$

Remark 2. Obviously, (13) implies that

$$A_1 := \sup_{(y,z,s)\in\mathbb{R}^3} \frac{V_s(y,z,s)}{\phi_f^{\beta}\left(\frac{c_f^*}{c}(s-h(y,z))\right)} < +\infty,$$

$$A_2 := \sup_{(y,z,s)\in\mathbb{R}^3, \theta\in[0,1]} \frac{\phi_f^{\beta}\left(\frac{c_f^*}{c}(s+\theta-h(y,z))\right)}{\phi_f^{\beta}\left(\frac{c_f^*}{c}(s-h(y,z))\right)} < +\infty.$$

Finally, we show a mollified pyramid, which was constructed by Taniguchi [17]. Let $\widetilde{\rho}(r) \in C^\infty[0,\infty)$ satisfy the following properties:

$$\widetilde{\rho}(r) > 0,\ \widetilde{\rho}_r(r) \le 0 \text{ for } r \ge 0,\ \widetilde{\rho}(r) = 1 \text{ if } r > 0 \text{ is small enough},$$
$$\widetilde{\rho}(r) = e^{-r} \quad \text{if } r > 0 \text{ is large enough, say } r > R_0,$$
$$2\pi \int_0^\infty r\widetilde{\rho}(r)dr = 1.$$

Clearly, the function $\rho(y,z) := \widetilde{\rho}\left(\sqrt{y^2+z^2}\right)$ is of class C^∞ and $\int_{\mathbb{R}^2} \rho(y,z)\,dydz = 1$. Without loss of generality, suppose $R_0 > 1$. For all non-negative integers i_1 and i_2 with $0 \le i_1 + i_2 \le 3$, we have

$$\left|D_y^{i_1} D_z^{i_2} \rho(y,z)\right| \le M_* \rho(y,z), \quad \forall\ (y,z) \in \mathbb{R}^2,$$

where $M_* > 0$ is a constant. Define a mollified pyramid $\left\{(y,z,s) \in \mathbb{R}^3 \mid s = \varphi(y,z)\right\}$ as $\varphi(y,z) := \rho * h$ associated with a pyramid $\left\{(y,z,s) \in \mathbb{R}^3 \mid s = h(y,z)\right\}$. That is,

$$\begin{aligned}\varphi(y,z) &= \int_{\mathbb{R}^2} \rho\left(y-y',z-z'\right) h\left(y',z'\right) dy'dz' \\ &= \int_{\mathbb{R}^2} \rho\left(y',z'\right) h\left(y-y',z-z'\right) dy'dz'. \end{aligned} \tag{15}$$

Let

$$S(y,z) := \frac{c}{\sqrt{1+\left|\nabla\varphi(y,z)\right|^2}} - c_f^*, \tag{16}$$

where $\nabla\varphi(y,z) := (\varphi_y(y,z), \varphi_z(y,z))$ and $|\nabla\varphi(y,z)| = \sqrt{\varphi_y^2(y,z) + \varphi_z^2(y,z)}$. Then we have the following two lemmas, see [17, 18].

Lemma 2.5. *Let φ and S be as in (15) and (16), respectively. For any fixed integers $i_1 \ge 0$ and $i_2 \ge 0$, one has*

$$\sup_{(y,z)\in\mathbb{R}^2} \left|D_y^{i_1} D_z^{i_2} \varphi(y,z)\right| < \mathcal{K}_1 \quad \textit{for some constant } \mathcal{K}_1 > 0,$$
$$h(y,z) < \varphi(y,z) \le h(y,z) + 2\pi m_* \int_0^\infty r^2 \widetilde{\rho}(r)dr,$$
$$|\nabla\varphi(y,z)| < m_*,\ 0 < S(y,z) \le c - c_f^*,\ \forall\ (y,z) \in \mathbb{R}^2$$

and

$$\lim_{\lambda\to\infty} \sup\left\{S(y,z)\middle|(y,z) \in \mathbb{R}^2, \operatorname{dist}\left((y,z),\Gamma\right) \ge \lambda\right\} = 0,$$
$$\lim_{\lambda\to\infty} \sup\left\{\varphi(y,z) - h(y,z)\middle|(y,z) \in \mathbb{R}^2, \operatorname{dist}\left((y,z),\Gamma\right) \ge \lambda\right\} = 0.$$

Lemma 2.6. *There exist two positive constants ν_1 and ν_2 such that*

$$0 < \nu_1 = \inf_{(y,z)\in\mathbb{R}^2} \frac{\varphi(y,z) - h(y,z)}{S(y,z)} \le \sup_{(y,z)\in\mathbb{R}^2} \frac{\varphi(y,z) - h(y,z)}{S(y,z)} = \nu_2 < \infty.$$

In addition, for integers $i_1 \ge 0$ and $i_2 \ge 0$ with $2 \le i_1 + i_2 \le 3$, there exists a constant $\mathcal{K}_2 > 0$ such that

$$\sup_{(y,z)\in\mathbb{R}^2} \left|\frac{D_y^{i_1} D_z^{i_2} \varphi(y,z)}{S(y,z)}\right| < \mathcal{K}_2$$

and

$$|\varphi_{yy}(y,z)|\,,\ |\varphi_{zz}(y,z)| \le m_* M_*, \quad \forall\ (y,z) \in \mathbb{R}^2.$$

3. **Stability under spatially decaying initial perturbations.** In this section, we give the proof of asymptotic stability of three-dimensional pyramidal fronts $V(y,z,s)$ in $\mathbb{R}^n$ with $n \ge 4$ under perturbation that decay at space infinity by constructing supersolutions and subsolutions coupled with comparison principle. That is, we prove Theorems 1.1-1.2 and Proposition 1. In the following, the symbols Δ_x and ∇_x denote the $n-3$-dimensional Laplacian and gradient operators with respect to x, respectively. Let $K(\mu) := \mu^2 + c_f^*\mu + f'(0)$. Clearly, we have $K(\beta\Lambda) < 0$ for any $\beta \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$. Take $\lambda := \min\left\{-\frac{1}{16}K(\beta\Lambda), -\frac{1}{16}f'(1), 1\right\}$.

3.1. **Proof of Theorem 1.1.**

Lemma 3.1. *Let $\tilde{k} > 0$ be defined as in (12). Then for $\beta \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$ and $\upsilon \in (0,1)$ with $\upsilon < \min\left\{\frac{c}{48c_f^*\Lambda m_* M_*}K(\beta\Lambda), 1\right\}$, there exist some constants $\delta_0 > 0$ and $\sigma_0 > 0$ such that for any $\delta \in (0, \delta_0]$ and $\sigma \ge \sigma_0$, and any function $v^+(t,x)$ satisfying*

$$v_t^+ = \Delta_x v^+ + \tilde{k}|\nabla_x v^+|^2, \quad x \in \mathbb{R}^{n-3},\ t > 0, \tag{17}$$

the function defined by

$$\begin{aligned} V^+(t,x,y,z,s) =& V(y,z,s - v^+(t,x) - \sigma\delta(1-e^{-\lambda t})) \\ &+ \delta e^{-\lambda t}\phi_f^\beta\left(\frac{c_f^*}{c}\left(s - v^+(t,x) - \sigma\delta(1-e^{-\lambda t}) - \frac{\varphi(\upsilon y, \upsilon z)}{\upsilon}\right)\right) \end{aligned}$$

is a supersolution to Eq. (4) in $(0,+\infty)\times\mathbb{R}^n$.

Proof. Let $\xi(t,x,y,z,s) := \frac{c_f^*}{c}\left(s - v^+(t,x) - \sigma\delta(1-e^{-\lambda t}) - \frac{\varphi(\upsilon y,\upsilon z)}{\upsilon}\right)$, $Y := \upsilon y$, $Z := \upsilon z$ and $\eta(t,x,s) := s - v^+(t,x) - \sigma\delta(1-e^{-\lambda t})$. Using (3), (12) and (17), the direct calculation implies that

$$\begin{aligned} &\mathcal{H}[V^+] : \\ =& V_t^+ - \Delta V^+ - cV_s^+ - f(V^+) \\ =& -V_\eta v_t^+ - \sigma\delta\lambda e^{-\lambda t}V_\eta - \delta\lambda e^{-\lambda t}\phi_f^\beta(\xi) - \frac{c_f^*}{c}\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)v_t^+ \\ &- \frac{c_f^*}{c}\beta\sigma\delta^2\lambda e^{-2\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi) - V_{\eta\eta}|\nabla_x v^+|^2 + V_\eta\Delta_x v^+ \\ &- \frac{c_f^{*2}}{c^2}\delta\beta(\beta-1)e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2|\nabla_x v^+|^2 - \frac{c_f^{*2}}{c^2}\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f''(\xi)|\nabla_x v^+|^2 \\ &+ \frac{c_f^*}{c}\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\Delta_x v^+ - V_{yy} - \frac{c_f^{*2}}{c^2}\beta(\beta-1)\delta e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2\varphi_Y^2(Y,Z) \\ &- \frac{c_f^{*2}}{c^2}\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f''(\xi)\varphi_Y^2(Y,Z) + \frac{c_f^*}{c}\upsilon\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\varphi_{YY}(Y,Z) \\ &- V_{zz} - \frac{c_f^{*2}}{c^2}\beta(\beta-1)\delta e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)\varphi_Z^2(Y,Z) \\ &- \frac{c_f^{*2}}{c^2}\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f''(\xi)\varphi_Z^2(Y,Z) + \frac{c_f^*}{c}\upsilon\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\varphi_{ZZ}(Y,Z) \end{aligned}$$

$$
\begin{aligned}
&- V_{ss} - \beta(\beta-1)\delta e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2\frac{c_f^{*2}}{c^2} - \beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f''(\xi)\frac{c_f^{*2}}{c^2}\\
&- cV_s - c_f^*\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi) - f\left(V(y,z,\eta)+\delta e^{-\lambda t}\phi_f^{\beta}(\xi)\right)\\
=&- \sigma\delta\lambda e^{-\lambda t}V_\eta - \delta\lambda e^{-\lambda t}\phi_f^{\beta}(\xi) - \frac{c_f^*}{c}\beta\sigma\delta^2\lambda e^{-2\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\\
&+ V_\eta\left(-v_t^+ + \Delta_x v^+\right) - V_{\eta\eta}|\nabla_x v^+|^2\\
&+ \frac{c_f^*}{c}\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\left[(-v_t^+ + \Delta_x v^+)\phi_f'(\xi) - \frac{c_f^*}{c}|\nabla_x v^+|^2\phi_f''(\xi)\right]\\
&- \frac{c_f^{*2}}{c^2}\delta\beta(\beta-1)e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2|\nabla_x v^+|^2\\
&- \frac{c_f^{*2}}{c^2}\beta(\beta-1)\delta e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2\left[1+|\nabla\varphi(Y,Z)|^2\right]\\
&- \frac{c_f^{*2}}{c^2}\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f''(\xi)\left[1+|\nabla\varphi(Y,Z)|^2\right] + \frac{c_f^*}{c}\upsilon\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\Delta\varphi(Y,Z)\\
&- c_f^*\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi) - f\left(V(y,z,\eta)+\delta e^{-\lambda t}\phi_f^{\beta}(\xi)\right) + f\left(V(y,z,\eta)\right)\\
=&- \sigma\delta\lambda e^{-\lambda t}V_\eta - \delta\lambda e^{-\lambda t}\phi_f^{\beta}(\xi) - \frac{c_f^*}{c}\beta\sigma\delta^2\lambda e^{-2\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\\
&+ (-\tilde{k}V_\eta - V_{\eta\eta})|\nabla_x v^+|^2\\
&+ \frac{c_f^*}{c}\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\left[\left(-\tilde{k}\phi_f'(\xi) - \frac{c_f^*}{c}\phi_f''(\xi)\right)|\nabla_x v^+|^2\right]\\
&- \frac{c_f^{*2}}{c^2}\delta\beta(\beta-1)e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2|\nabla_x v^+|^2\\
&- \frac{c_f^{*2}}{c^2}\beta(\beta-1)\delta e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2\left[1+|\nabla\varphi(Y,Z)|^2\right]\\
&- \frac{c_f^{*2}}{c^2}\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f''(\xi)\left[1+|\nabla\varphi(Y,Z)|^2\right] + \frac{c_f^*}{c}\upsilon\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\Delta\varphi(Y,Z)\\
&- c_f^*\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi) - f\left(V(y,z,\eta)+\delta e^{-\lambda t}\phi_f^{\beta}(\xi)\right) + f\left(V(y,z,\eta)\right)\\
\geq&- \sigma\delta\lambda e^{-\lambda t}V_\eta - \delta\lambda e^{-\lambda t}\phi_f^{\beta}(\xi) - \frac{c_f^*}{c}\beta\sigma\delta^2\lambda e^{-2\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\\
&- \frac{c_f^{*2}}{c^2}\beta(\beta-1)\delta e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2\left[1+|\nabla\varphi(Y,Z)|^2\right]\\
&- \frac{c_f^{*2}}{c^2}\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f''(\xi)\left[1+|\nabla\varphi(Y,Z)|^2\right] + \frac{c_f^*}{c}\upsilon\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\Delta\varphi(Y,Z)\\
&- c_f^*\beta\delta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi) - \delta e^{-\lambda t}\phi_f^{\beta}(\xi)f'\left(V(y,z,\eta)+\theta\delta e^{-\lambda t}\phi_f^{\beta}(\xi)\right),
\end{aligned}
$$

where $\xi=\xi(t,x,y,z,s)$, $\eta=\eta(t,x,s)$ and $\theta=\theta(t,x,y,z,s)\in(0,1)$.

Since $\lim\limits_{\mathbf{p}\to+\infty}\frac{\phi_f'(\mathbf{p})}{\phi_f(\mathbf{p})}=\Lambda$, then there exists a constant $R_1>0$ large enough such that

$$
\frac{3}{2}\Lambda \leq \frac{\phi_f'(\mathbf{p})}{\phi_f(\mathbf{p})} \leq \frac{1}{2}\Lambda,\ \left(\beta\frac{\phi_f'(\mathbf{p})}{\phi_f(\mathbf{p})}\right)^2 + c_f^*\beta\frac{\phi_f'(\mathbf{p})}{\phi_f(\mathbf{p})} + f'(0) < \frac{1}{2}K(\beta\Lambda), \quad \forall\, \mathbf{p} > R_1. \tag{18}
$$

By $\lim\limits_{\mathbf{p}\to+\infty}\frac{\phi_f'(\mathbf{p})}{\phi_f(\mathbf{p})}=\Lambda$, $\lim\limits_{\mathbf{p}\to+\infty}\frac{\phi_f''(\mathbf{p})}{\phi_f(\mathbf{p})}=\Lambda^2$ and $|\nabla\varphi(Y,Z)|<m_*$, one has

$$\lim_{\mathbf{p}\to+\infty}\frac{c_f^{*2}}{c^2}\left[\left(\frac{\phi_f'(\mathbf{p})}{\phi_f(\mathbf{p})}\right)^2-\frac{\phi_f''(\mathbf{p})}{\phi_f(\mathbf{p})}\right]\left(1+|\nabla\varphi(Y,Z)|^2\right)=0$$

uniformly in $(Y,Z)\in\mathbb{R}^2$. Thus there exists a constant $R_2>0$ large enough such that

$$\left|\frac{c_f^{*2}}{c^2}\left[\left(\frac{\phi_f'(\mathbf{p})}{\phi_f(\mathbf{p})}\right)^2-\frac{\phi_f''(\mathbf{p})}{\phi_f(\mathbf{p})}\right]\left(1+|\nabla\varphi(Y,Z)|^2\right)\right|<-\frac{1}{16}K(\beta\Lambda), \tag{19}$$

for $\mathbf{p}>R_2$ and $(Y,Z)\in\mathbb{R}^2$. The assumption (H1) implies that there exists a constant $K>0$ such that

$$|f'(u_1)-f'(u_2)|\le K\,|u_1-u_2|^{\varsigma},\quad \forall\ u_1,\ u_2\in[-\iota,1+\iota]. \tag{20}$$

Since $|s-h(y,z)|\to+\infty$ gives $\operatorname{dist}((y,z,s),\Gamma)\to+\infty$, then by Theorem I and the fact that $h(y,z)<\frac{\varphi(\upsilon y,\upsilon z)}{\upsilon}\le h(y,z)+\frac{2\pi m_*\int_0^\infty r^2\tilde{\rho}(r)dr}{\upsilon}$, there exists a constant $R_3>0$ large enough such that for all $\delta<\iota^*$,

$$\left|f'(V(y,z,\eta)+\theta\delta e^{-\lambda t}\phi_f^{\beta}(\xi))-f'(0)\right|\le K\left|V(y,z,\eta)+\theta\delta e^{-\lambda t}\phi_f^{\beta}(\xi)\right|^{\varsigma}<-\frac{1}{16}K(\beta\Lambda) \tag{21}$$

for any $(x,y,z,s)\in\mathbb{R}^n$ and $t\ge0$ with $\xi(t,x,y,z,s)>R_3$.

Since $\lim\limits_{\mathbf{p}\to-\infty}\phi_f(\mathbf{p})=1$, $\lim\limits_{\mathbf{p}\to-\infty}\phi_f'(\mathbf{p})=0$ and $\lim\limits_{\mathbf{p}\to-\infty}\phi_f''(\mathbf{p})=0$, then there exists a constant $R_4>0$ large enough such that

$$\left|\frac{\phi_f'(\mathbf{p})}{\phi_f(\mathbf{p})}\right|<\frac{c|f'(1)|}{16c_f^*m_*M_*}\ \text{and}\ \left|\beta\frac{\phi_f''(\mathbf{p})}{\phi_f(\mathbf{p})}\right|<\frac{|f'(1)|}{8},\quad\forall\ \mathbf{p}<-R_4. \tag{22}$$

Since $|s-h(y,z)|\to+\infty$ gives $\operatorname{dist}((y,z,s),\Gamma)\to+\infty$, then it follows from Theorem I and the fact that $h(y,z)<\frac{\varphi(\upsilon y,\upsilon z)}{\upsilon}\le h(y,z)+\frac{2\pi m_*\int_0^\infty r^2\tilde{\rho}(r)dr}{\upsilon}$ that there exists a $R_5>0$ large enough such that for any $(x,y,z,s)\in\mathbb{R}^n$ and $t>0$ with $\xi(t,x,y,z,s)<-R_5$,

$$V(y,z,\eta(t,x,s))>1-\iota^*. \tag{23}$$

Let $R:=\max\{R_1,R_2,R_3,R_4,R_5\}$ and $\bar{R}:=R+\frac{2\pi m_*\int_0^\infty r^2\tilde{\rho}(r)dr}{\upsilon}$. It follows from (14) that there exists a constant $\beta_1>0$ such that

$$\min_{-\bar{R}\le\xi(t,x,y,z,s)\le\bar{R}}\left(-V_s\left(y,z,\eta(t,x,s)\right)\right)>\beta_1. \tag{24}$$

Let $\delta_0:=\iota^*$. Take $\sigma_0>0$ large enough such that

$$\sigma\lambda\beta_1-\lambda-\gamma_2-2m_*M_*\gamma_1-\gamma_3>0,\ \sigma\ge\sigma_0. \tag{25}$$

Note that $\frac{c_f^{*2}}{c^2}\left(1+|\nabla\varphi(Y,Z)|^2\right)\le1$ for any $(Y,Z)\in\mathbb{R}^2$.

Case 1. For $(t, x, y, z, s) \in [0, +\infty) \times \mathbb{R}^{n-3} \times \mathbb{R}^3$ with $\xi(t, x, y, z, s) > R$. Using $-V_\eta > 0$, $-\phi'_f(\xi) > 0$, (18)–(21) with $\delta \in (0, \delta_0)$ and $\sigma \geq \sigma_0$, we have

$$\begin{aligned}
\mathcal{H}[V^+] :=& V_t^+ - \Delta V^+ - cV_s^+ - f(V^+) \\
\geq& \delta e^{-\lambda t}\phi_f^\beta(\xi)\Bigg[- \lambda + \frac{c_f^{*2}}{c^2}\beta\left(\left(\frac{\phi'_f(\xi)}{\phi_f(\xi)}\right)^2 - \frac{\phi''_f(\xi)}{\phi_f(\xi)}\right)\left(1 + (\nabla\varphi(Y, Z))^2\right) \\
&+ \beta^2\left(\frac{\phi'_f(\xi)}{\phi_f(\xi)}\right)^2\left[1 - \frac{c_f^{*2}}{c^2}\left(1 + (\nabla\varphi(Y, Z))^2\right)\right] \\
&- \beta^2\left(\frac{\phi'_f(\xi)}{\phi_f(\xi)}\right)^2 - c_f^*\beta\frac{\phi'_f(\xi)}{\phi_f(\xi)} - f'(0) + \frac{c_f^*}{c}\upsilon\beta\frac{\phi'_f(\xi)}{\phi_f(\xi)}\Delta\varphi(Y, Z) \\
&- f'\left(V(y, z, \eta) + \theta\delta e^{-\lambda t}\phi_f^\beta(\xi)\right) + f'(0)\Bigg] \\
\geq& \delta e^{-\lambda t}\phi_f^\beta(\xi)\left[-\lambda + \frac{K(\beta\Lambda)}{16} - \frac{1}{2}K(\beta\Lambda) + \frac{1}{16}K(\beta\Lambda) + \frac{1}{16}K(\beta\Lambda)\right] \\
\geq& 0.
\end{aligned}$$

Case 2. For $(t, x, y, z, s) \in [0, +\infty) \times \mathbb{R}^{n-3} \times \mathbb{R}^3$ with $\xi(t, x, y, z, s) < -R$, using $-V_\eta > 0$, $-\phi'_f(\xi) > 0$, (22) and (23) with $\delta \in (0, \delta_0)$ and $\sigma \geq \sigma_0$, we have

$$\begin{aligned}
\mathcal{H}[V^+] :=& V_t^+ - \Delta V^+ - cV_s^+ - f(V^+) \\
\geq& \delta e^{-\lambda t}\phi_f^\beta(\xi)\Bigg[- \lambda - \frac{c_f^{*2}}{c^2}\beta\frac{\phi''_f(\xi)}{\phi_f(\xi)}\left(1 + (\nabla\varphi(Y, Z))^2\right) + \frac{c_f^*}{c}\upsilon\beta\frac{\phi'_f(\xi)}{\phi_f(\xi)}\Delta\varphi(Y, Z) \\
&- f'\left(V(y, z, \eta) + \theta\delta e^{-\lambda t}\phi_f^\beta(\xi)\right)\Bigg] \\
\geq& \delta e^{-\lambda t}\phi_f^\beta(\xi)\left[-\lambda + \frac{f'(1)}{8} + \frac{f'(1)}{8} - \frac{f'(1)}{2}\right] \\
\geq& 0.
\end{aligned}$$

Case 3. For $(t, x, y, z, s) \in [0, +\infty) \times \mathbb{R}^{n-3} \times \mathbb{R}^3$ with $-R \leq \xi(t, x, y, z, s) \leq R$, using $-\phi'_f(\xi) > 0$, (24) and (25) with $\delta \in (0, \delta_0)$ and $\sigma \geq \sigma_0$, we have

$$\begin{aligned}
&\mathcal{H}[V^+] : \\
=& V_t^+ - \Delta V^+ - cV_s^+ - f(V^+) \\
\geq& - \sigma\delta\lambda e^{-\lambda t}V_\eta - \delta\lambda e^{-\lambda t}\phi_f^\beta(\xi) - \frac{c_f^{*2}}{c^2}\beta\delta e^{-\lambda t}\phi_f^\beta(\xi)\frac{\phi''_f(\xi)}{\phi_f(\xi)}\left[1 + (\nabla\varphi(Y, Z))^2\right] \\
&+ \frac{c_f^*}{c}\upsilon\beta\delta e^{-\lambda t}\phi_f^\beta(\xi)\frac{\phi'_f(\xi)}{\phi_f(\xi)}\Delta\varphi(Y, Z) - \delta e^{-\lambda t}\phi_f^\beta(\xi)f'\left(V(y, z, \eta) + \theta\delta e^{-\lambda t}\phi_f^\beta(\xi)\right) \\
\geq& \delta e^{-\lambda t}\Bigg[\sigma\lambda\beta_1 - \lambda - \sup_{\xi\in\mathbb{R}}\frac{|\phi''_f(\xi)|}{|\phi_f(\xi)|} \\
&- \sup_{(Y,Z)\in\mathbb{R}^2}|\Delta\varphi(Y, Z)|\sup_{\xi\in\mathbb{R}}\frac{|\phi'_f(\xi)|}{|\phi_f(\xi)|} - \sup_{u\in[-\iota^*, 1+\iota^*]}|f'(u)|\Bigg] \\
\geq& \delta e^{-\lambda t}\left(-\sigma\lambda\beta_1 - \lambda - \gamma_2 - 2m_*M_*\gamma_1 - \gamma_3\right) \\
\geq& 0.
\end{aligned}$$

By the above argument, we get $\mathcal{H}[V^+] \geq 0$ for $(t,x,y,z,s) \in [0,+\infty)\times\mathbb{R}^{n-3}\times\mathbb{R}^3$. Namely, the function $V^+(t,x,y,z,s)$ is a supersolution to Eq. (4) in $[0,+\infty)\times\mathbb{R}^n$. □

Lemma 3.2. *Let $\tilde{k} > 0$ be defined as in (12). Then for $\beta \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$ and $\upsilon \in (0,1)$ with $\upsilon < \min\left\{\frac{cK(\beta\Lambda)}{48\Lambda c_f^* m_* M_*}, 1\right\}$, there exist some constants $\delta_1 > 0$ and $\sigma_1 > 0$ such that, for any $\delta \in (0, \delta_1]$ and $\sigma \geq \sigma_1$, and any bounded functions $v^-(t,x)$ satisfying*

$$v_t^- = \Delta_x v^- - \tilde{k}|\nabla_x v^-|^2, \quad x \in \mathbb{R}^{n-3},\ t > 0, \tag{26}$$

the function defined by

$$\begin{aligned} V^-(t,x,y,z,s) =& V(y,z,s-v^-(t,x)+\sigma\delta(1-e^{-\lambda t})) \\ & - \delta e^{-\lambda t}\phi_f^\beta\left(\frac{c_f^*}{c}\left(s+v^-(t,x)+\sigma\delta(1-e^{-\lambda t})-\frac{\varphi(\upsilon y,\upsilon z)}{\upsilon}\right)\right) \end{aligned}$$

is a subsolution to Eq. (4) in $(0,+\infty)\times\mathbb{R}^n$.

Proof. Let $\xi(t,x,y,z,s) = s+v^-(t,x)+\sigma\delta(1-e^{-\lambda t})-\frac{\varphi(\upsilon y,\upsilon z)}{\upsilon}$, $\eta(t,x,s) = s - v^-(t,x)+\sigma\delta(1-e^{-\lambda t})$ and $Y=\upsilon y,\ Z=\upsilon z$. Using (3), (12) and (26), the direct calculation yields that

$$\begin{aligned}
&\mathcal{H}[V^-] \\
=&V_t^- - \Delta V^- - cV_s^- - f(V^-) \\
=&-V_\eta v_t^- + \sigma\delta\lambda e^{-\lambda t}V_\eta + \delta\lambda e^{-\lambda t}\phi_f^\beta(\xi) - \frac{c_f^*}{c}\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)v_t^- \\
&-\sigma\delta^2\frac{c_f^*}{c}e^{-2\lambda t}\lambda\beta\phi_f^{\beta-1}(\xi)\phi_f'(\xi) - V_{\eta\eta}|\nabla_x v^-|^2 + V_\eta\Delta_x v^- \\
&+\frac{c_f^{*2}}{c^2}\delta\beta(\beta-1)e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2|\nabla_x v^-|^2 + \frac{c_f^{*2}}{c^2}\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f''(\xi)|\nabla_x v^-|^2 \\
&+\frac{c_f^*}{c}\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\Delta_x v^- - V_{yy} - V_{zz} \\
&+\frac{c_f^{*2}}{c^2}\delta\beta(\beta-1)e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2\,|\nabla\varphi(Y,Z)|^2 \\
&+\frac{c_f^{*2}}{c^2}\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f''(\xi)\,|\nabla\varphi(Y,Z)|^2 - \frac{c_f^*}{c}\delta\beta\upsilon e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\Delta\varphi(Y,Z) \\
&-V_{\eta\eta} + \frac{c_f^{*2}}{c^2}\delta\beta(\beta-1)e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2 + \frac{c_f^{*2}}{c^2}\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f''(\xi) \\
&-cV_\eta + c_f^*\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi) - f\left(V(y,z,s,\eta) - \delta e^{-\lambda t}\phi_f^\beta(\xi)\right) \\
=&-\sigma\delta\lambda e^{-\lambda t}V_\eta + \delta\lambda e^{-\lambda t}\phi_f^\beta(\xi) - \frac{c_f^*}{c}\beta\sigma\delta^2\lambda e^{-2\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi) \\
&+V_\eta\left(-v_t^- + \Delta_x v^-\right) - V_{\eta\eta}|\nabla_x v^-|^2 \\
&+\frac{c_f^*}{c}\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\left[(-v_t^- + \Delta_x v^-)\phi_f'(\xi) + \frac{c_f^*}{c}|\nabla_x v^-|^2\phi_f''(\xi)\right] \\
&-V_{yy} - V_{zz} - V_{\eta\eta} - cV_\eta - f(V) \\
&+\frac{c_f^{*2}}{c^2}\delta\beta(\beta-1)e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2\left[1+|\nabla\varphi(Y,Z)|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{c_f^{*2}}{c^2}\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f''(\xi)\left[1+|\nabla\varphi(Y,Z)|^2\right]\\
&+\frac{c_f^{*2}}{c^2}\delta\beta(\beta-1)e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2|\nabla_x v^-|^2\\
&-\frac{c_f^*}{c}\delta\beta\upsilon e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\Delta\varphi(Y,Z)+c_f^*\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\\
&-f\left(V(y,z,\eta)-\delta e^{-\lambda t}\phi_f^{\beta}(\xi)\right)+f\left(V(y,z,\eta)\right)\\
\leq&\sigma\delta\lambda e^{-\lambda t}V_\eta+\delta\lambda e^{-\lambda t}\phi_f^{\beta}(\xi)-\frac{c_f^*}{c}\beta\sigma\delta^2\lambda e^{-2\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\\
&+\frac{c_f^{*2}}{c^2}\delta\beta(\beta-1)e^{-\lambda t}\phi_f^{\beta-2}(\xi)\phi_f'(\xi)^2\left[1+|\nabla\varphi(Y,Z)|^2\right]\\
&+\frac{c_f^{*2}}{c^2}\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f''(\xi)\left[1+|\nabla\varphi(Y,Z)|^2\right]\\
&-\frac{c_f^*}{c}\delta\beta\upsilon e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\Delta\varphi(Y,Z)+c_f^*\delta\beta e^{-\lambda t}\phi_f^{\beta-1}(\xi)\phi_f'(\xi)\\
&+\delta e^{-\lambda t}\phi_f^{\beta}(\xi)f'\left(V(y,z,\eta)-\theta\delta e^{-\lambda t}\phi_f^{\beta}(\xi)\right),
\end{aligned}
$$

where $\xi=\xi(t,x,y,z,s)$, $\eta=\eta(t,x,s)$ and $\theta=\theta(t,x,y,z,s)\in(0,1)$.

It follows from the boundedness of functions $v^-(t,x)$ that there exists a constant $\bar{M}>0$ such that $||v^-||_{L^\infty((0,+\infty)\times\mathbb{R}^{n-3})}\leq\bar{M}$. And thus, Theorem I, (20) and $h(y,z)\leq\frac{\varphi(\upsilon y,\upsilon z)}{\upsilon}\leq h(y,z)+\frac{2\pi m_*\int_0^{+\infty}r^2\tilde{\rho}(r)dr}{\upsilon}$ imply that there exists $\tilde{R}_3>0$ large enough such that for all $\delta<\iota^*$,

$$
\left|f'\left(V(y,z,\eta)-\theta\delta e^{-\lambda t}\phi_f^{\beta}(\xi)\right)-f'(0)\right|\leq K\left|V(y,z,\eta)-\theta\delta e^{-\lambda t}\phi_f^{\beta}(\xi)\right|^{\varsigma}<-\frac{1}{16}k(\beta\Lambda),\tag{27}
$$

for $(t,x,y,z,s)\in[0,+\infty)\times\mathbb{R}^{n-3}\times\mathbb{R}^3$ with $\xi(t,x,y,z,s)>\tilde{R}_3+2\bar{M}$. Theorem I and the fact that $h(y,z)\leq\frac{\varphi(\upsilon y,\upsilon z)}{\upsilon}\leq h(y,z)+\frac{2\pi m_*\int_0^{+\infty}r^2\tilde{\rho}(r)dr}{\upsilon}$ imply that there exists $\tilde{R}_4>0$ large enough such that

$$
V(y,z,\eta)>1-\iota,\tag{28}
$$

for $(t,x,y,z,s)\in[0,+\infty)\times\mathbb{R}^{n-3}\times\mathbb{R}^3$ with $\xi(t,x,y,z,s)<-\tilde{R}_4-2\bar{M}$.

Let R_1,R_2 and R_4 be defined as in (18), (19) and (22). Put $\hat{R}:=\max\{R_1,R_2,R_4,\tilde{R}_3+2\bar{M},\tilde{R}_4+2\bar{M}\}$ and $\hat{R}_*:=\hat{R}+\frac{2\pi m_*\int_0^{+\infty}r^2\tilde{\rho}(r)dr}{\upsilon}$, then (14) yields that there exists a constant $\beta_1'>0$ such that

$$
\min_{-\hat{R}_*\leq\xi(t,x,y,z,s)\leq\hat{R}_*}-V_\eta(y,z,\eta)\geq\beta_1'>0.\tag{29}
$$

Take $\sigma_1>0$ large enough such that

$$
-\frac{\sigma\lambda\beta_1'}{2}+\lambda+\gamma_2+2\gamma_1 m_* M_*+\gamma_3<0,\quad\text{for }\sigma\geq\sigma_1.\tag{30}
$$

Let $\delta_1:=\min\left\{\iota^*,\frac{\beta_1'}{2\gamma_1},-\frac{k(\beta\Lambda)}{16\sigma\lambda\gamma_1},\frac{|f'(1)|}{8\sigma\lambda\gamma_1}\right\}$ for $\sigma\geq\sigma_1$.

Similar to the proof of Lemma 3.1, we can get $\mathcal{H}[V^-]\leq 0$ on $(t,x,y,z,s)\in[0,+\infty)\times\mathbb{R}^{n-3}\times\mathbb{R}^3$. That is, the function $V^-(t,x,y,z,s)$ is a subsolution of Eq. (4) on $(t,x,y,z,s)\in[0,+\infty)\times\mathbb{R}^{n-3}\times\mathbb{R}^3$. □

Lemma 3.3. *Assume that the initial value* $\vartheta_0 \in C(\mathbb{R}^n, [0,1])$ *with* $n > 3$ *satisfies*

$$\lim_{R\to+\infty} \sup_{|x|+|y|+|z|+|s|\geq R} \frac{|\vartheta_0(x,y,z,s) - V(y,z,s)|}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)} = 0,$$

for some $\beta \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$*. Then for any fixed* $T > 0$*, we have*

$$\lim_{R\to+\infty} \sup_{|x|+|y|+|z|+|s|\geq R} \frac{|\vartheta(T,x,y,z,s;\vartheta_0) - V(y,z,s)|}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)} = 0.$$

Proof. The proof of Lemma 3.3 is similar to that of Lemma 3.8 in [4], so we omit it here. □

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. We only show the lower estimate, as the upper estimate can be given in a similar way. We denote $\vartheta(t,x,y,z,s;\vartheta_0)$ by $\vartheta(t,x,y,z,s)$ for simplicity. Take constants $\tilde{k} > 0$ as in (12), $\delta_* := \min\{\delta_0, \delta_1\}$ and $\sigma \geq \max\{\sigma_0, \sigma_1, 1\}$, where σ_0, δ_0 and σ_1, δ_1 are defined as in Lemmas 3.1 and 3.2, respectively. For any $\varepsilon > 0$, set $\hat{\varepsilon} = \min\left\{\frac{\log 2}{2\gamma_1}, \frac{\varepsilon}{8A_1}, \frac{\epsilon}{2\sqrt{2}e^{\gamma_1 D/2\upsilon}}\right\}$ such that $\frac{\hat{\varepsilon}}{\sigma} \in (0, \delta_*)$, where A_1 is defined as in Remark 2, $D := 2\pi m_* \int_0^{+\infty} r^2 \tilde{\rho}(r) dr$ and υ defined as in Lemmas 3.1 and 3.2. Since the initial value $\upsilon_0 \in C(\mathbb{R}^n, [0,1])$ with $n > 3$ satisfies

$$\lim_{R\to+\infty} \sup_{|x|+|y|+|z|+|s|\geq R} \frac{|\vartheta_0(x,y,z,s) - V(y,z,s)|}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)} = 0$$

for some $\beta \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$, then the strong maximum principle yields that

$$0 < \vartheta(t,x,y,z,s) < 1, \quad \forall\ (x,y,z,s) \in \mathbb{R}^{n-3} \times \mathbb{R}^3 \text{ and } t > 0.$$

From Lemma 3.3, it follows that for any fixed $T > 0$, there exists a constant $R^1 > 0$ large enough that

$$\sup_{|x|+|y|+|z|+|s|\geq R^1} \frac{|\vartheta(T,x,y,z,s) - V(y,z,s)|}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)} \leq \frac{\hat{\varepsilon}}{\sigma}.$$

Thus we can choose a continuous function $w_0(x) \leq 0$ satisfying $\lim\limits_{|x|\to\infty} w_0(x) = 0$ and

$$\vartheta(T,x,y,z,s) \geq V(y,z,s-w_0(x)) - \frac{\hat{\varepsilon}}{\sigma}\phi_f^\beta\left(\frac{c_f^*}{c}(s+w_0(x)-h(y,z))\right)$$

for any $(x,y,z,s) \in \mathbb{R}^{n-3} \times \mathbb{R}^3$, and hence,

$$\vartheta(T,x,y,z,s) \geq V(y,z,s-w_0(x)) - \frac{\hat{\varepsilon}}{\sigma}\phi_f^\beta\left(\frac{c_f^*}{c}\left(s+w_0(x)-\frac{\varphi(\upsilon y,\upsilon z)}{\upsilon}\right)\right)$$

for any $(x,y,z,s) \in \mathbb{R}^{n-3} \times \mathbb{R}^3$ from $h(y,z) \leq \varphi(y,z)$ and $-\phi_f'(\mathbf{p}) > 0$, where υ is defined as in Lemma 3.2. Let $v^-(t,x)$ be the solution of Cauchy problem

$$\begin{cases} v_t^- = \Delta_x v^- - \tilde{k}|\nabla v^-|^2, & x \in \mathbb{R}^{n-3},\ t > 0, \\ v^-(0,x) = w_0(x), & x \in \mathbb{R}^{n-3}. \end{cases}$$

Then it follows from Lemma 2.1 that there exists $T_1 > 0$ large enough such that

$$-\hat{\varepsilon} \leq v^-(t,x) \leq 0 \quad \text{for all } x \in \mathbb{R}^{n-3} \text{ and } t \geq T_1.$$

Therefore, the comparison principle together with the subsolution constructed in Lemma 3.2 yields

$$\begin{aligned}
&\vartheta(t,x,y,z,s)\\
\geq &V\left(y,z,s-v^-(t-T,x)+\hat{\varepsilon}(1-e^{-\lambda(t-T)})\right)\\
&-\frac{\hat{\varepsilon}}{\sigma}e^{-\lambda(t-T)}\phi_f^\beta\left(\frac{c_f^*}{c}\left(s+v^-(t-T,x)+\hat{\varepsilon}(1-e^{-\lambda(t-T)})-\frac{\varphi(\upsilon y,\upsilon z)}{\upsilon}\right)\right)\\
\geq &V\left(y,z,s-v^-(t-T,x)+\hat{\varepsilon}(1-e^{-\lambda(t-T)})\right)\\
&-\frac{\hat{\varepsilon}}{\sigma}e^{-\lambda(t-T)}\phi_f^\beta\left(\frac{c_f^*}{c}\left(s+v^-(t-T,x)-\frac{\varphi(\upsilon y,\upsilon z)}{\upsilon}\right)\right)
\end{aligned}$$

for $t \geq T+T_1$ and $(x,y,z,s)\in\mathbb{R}^{n-3}\times\mathbb{R}^3$. Thus

$$\begin{aligned}
\frac{\vartheta(t,x,y,z,s)-V(y,z,s)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\geq &\frac{V(y,z,s-v^-(t-T,x)+\hat{\varepsilon}(1-e^{-\lambda(t-T)}))-V(y,z,s)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\\
&-\frac{\hat{\varepsilon}}{\sigma}e^{-\lambda(t-T)}\frac{\phi_f^\beta\left(\frac{c_f^*}{c}\left(s+v^-(t-T,x)-\frac{\varphi(\upsilon y,\upsilon z)}{\upsilon}\right)\right)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}.
\end{aligned}$$

Let $\underline{v}^-(t,x):=v^-(t-T,x)-\hat{\varepsilon}(1-e^{-\lambda(t-T)})$. One has that $-2\hat{\varepsilon}\leq \underline{v}^-(t,x)\leq 0$ for $x\in\mathbb{R}^{n-3}$ and $t\geq T+T_1$. $\gamma_1 := \sup_{\mathbf{p}\in\mathbb{R}}\left|\frac{\phi_f'(\mathbf{p})}{\phi_f(\mathbf{p})}\right|$ yields that $\phi_f(\mathbf{p}+\tilde{\mathbf{p}})e^{\gamma_1\mathbf{p}}$ is increasing in $\tilde{\mathbf{p}}\in\mathbb{R}$ for each $\mathbf{p}\in\mathbb{R}$. Thus we have

$$\begin{aligned}
&\frac{V(y,z,s-\underline{v}^-(t,x))-V(y,z,s)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}=\frac{-V_s(y,z,s-\theta\underline{v}^-(t,x))}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\underline{v}^-(t,x)\\
=&\frac{-V_s(y,z,s-\theta\underline{v}^-(t,x))}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-\theta\underline{v}^-(t,x)-h(y,z))\right)}\frac{\phi_f^\beta\left(\frac{c_f^*}{c}(s-\theta\underline{v}^-(t,x)-h(y,z))\right)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\underline{v}^-(t,x)\\
\geq&-2\hat{\varepsilon}\sup_{(y,z,s)\in\mathbb{R}^3}\frac{-V_s(y,z,s)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\sup_{(t,x,y,z,s)\in\Omega}\frac{\phi_f^\beta\left(\frac{c_f^*}{c}(s-\theta\underline{v}^-(t,x)-h(y,z))\right)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\\
\geq&-2\hat{\varepsilon}A_1e^{2\gamma_1\hat{\varepsilon}},
\end{aligned}$$

where $\theta\in(0,1)$ and $\Omega:=[T+T_1,+\infty)\times\mathbb{R}^n$. From the fact that $h(y,z)\leq\varphi(y,z)\leq h(y,z)+D$ for all $(y,z)\in\mathbb{R}^2$, it follows that

$$\frac{\phi_f^\beta\left(\frac{c_f^*}{c}(s+v^-(t-T,x)-\frac{\varphi(\upsilon y,\upsilon z)}{\upsilon})\right)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\leq\frac{\phi_f^\beta\left(\frac{c_f^*}{c}(s-\hat{\varepsilon}-h(y,z)-\frac{D}{\upsilon})\right)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\leq e^{\gamma_1\frac{D}{2\upsilon}}e^{\gamma_1\hat{\varepsilon}}.$$

Combining the above argument, we have

$$\frac{\vartheta(t,x,y,z,s)-V(y,z,s)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\geq-2\hat{\varepsilon}A_1e^{2\gamma_1\hat{\varepsilon}}-\hat{\varepsilon}e^{\gamma_1\frac{D}{2\upsilon}}e^{\gamma_1\hat{\varepsilon}}>-\frac{\varepsilon}{2}-\frac{\varepsilon}{2}>-\varepsilon,$$

for $t\geq T+T_1$ and $(x,y,z,s)\in\mathbb{R}^{n-3}\times\mathbb{R}^3$. This completes the proof of Theorem 1.1. $\square$

3.2. **Proofs of Theorem 1.2 and Proposition 1.** In this subsection, we give the proofs of Theorem 1.2 and Proposition 1 motivated by [14, 4]. We firstly construct a pair of subsolution and supersolution to Eq. (4). Let $v^{\pm}(t,x)$ be the solutions to the following Cauchy problem:

$$\begin{cases} v_t^{\pm}(t,x) = \Delta_x v^{\pm}(t,x) \pm \tilde{k}_1 |\nabla_x v^{\pm}(t,x)|^2, & x \in \mathbb{R}^{n-3},\ t>0, \\ v^{\pm}(0,x) = v_0^{\pm}(x), & x \in \mathbb{R}^{n-3}, \end{cases} \tag{31}$$

where $\tilde{k}_1$ is the positive constant defined as in Lemma 2.2.

Lemma 3.4. *Let $V(y,z,s)$ be a pyramidal front to Eq. (3) and $\vartheta(t,x,y,z,s;\vartheta_0)$ be the solution to Eq. (4). Assume that the initial value $\vartheta_0(x,y,z,s)$ satisfies*

$$V(y,z,s-v_0^-(x)) \le \vartheta_0(x,y,z,s) \le V(y,z,s-v_0^+(x)), \quad \forall\ (x,y,z,s) \in \mathbb{R}^{n-3}\times\mathbb{R}^3.$$

Then we have

$$V(y,z,s-v^-(t,x)) \le \vartheta(t,x,y,z,s) \le V(y,z,s-v^+(t,x)) \tag{32}$$

for all $t \ge 0$ and $(x,y,z,s) \in \mathbb{R}^{n-3}\times\mathbb{R}^3$.

Proof. Let $w^+(t,x,y,z,s) = V(y,z,s-v^+(t,x))$. Now we show that the function $w^+(t,x,y,z,s)$ satisfies

$$\mathcal{H}[w^+] := w_t^+ - \Delta w^+ - c w_s^+ - f(w^+) \ge 0,$$

which yields that $w^+(t,x,y,z,s)$ is a supersolution to Eq. (4).

Using (3), Lemma 2.2 and (31), the direct calculation yields that

$$\begin{aligned}\mathcal{H}[w^+] =& - v_t^+ V_s - \Sigma_{j=1}^{n-3}\left[-v_{x_jx_j}^+ V_s + (v_{x_j}^+)^2 V_{ss}\right] - V_{yy} - V_{zz} - V_{ss} - cV_s - f(V) \\ =& - v_t^+ V_s + \Delta_x v^+ V_s - |\nabla_x v^+| V_{ss} \\ =& |\nabla_x v^+|(-\tilde{k}_1 V_s - V_{ss}) \ge 0.\end{aligned}$$

Similarly, we can show that the function $V(y,z,s-v^-(t,x))$ is a subsolution to Eq. (4). Therefore it follows from the assumptions on initial value $\vartheta_0(x,y,z,s)$ and the comparison principle that the inequalities (32) hold. □

Proof of Theorem 1.2. Denote $\vartheta(t,x,y,z,s;\vartheta_0)$ by $\vartheta(t,x,y,z,s)$ for simplicity. For any $t \ge 0$ and $(x,y,z,s) \in \mathbb{R}^{n-2}\times\mathbb{R}^3$, it follows from Lemma 3.4 that we have

$$V(y,z,s-v^-(t,x)) \le \vartheta(t,x,y,z,s) \le V(y,z,s-v^+(t,x))$$

with $v_0^+(x) = \vartheta_0^+(x)$ and $v_0^-(x) = \vartheta_0^-(x)$ in (31). Thus for any $\beta \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$, we have

$$\begin{aligned}\frac{\vartheta(t,x,y,z,s) - V(y,z,s)}{\phi_f^{\beta}\left(\frac{c_f^*}{c}(s-h(y,z))\right)} \le& \frac{V(y,z,s-v^+(t,x)) - V(y,z,s)}{\phi_f^{\beta}\left(\frac{c_f^*}{c}(s-h(y,z))\right)} \\ =& \frac{-V_s(y,z,s-\theta v^+(t,x))}{\phi_f^{\beta}\left(\frac{c_f^*}{c}(s-h(y,z))\right)} v^+(t,x),\end{aligned}$$

where $\theta \in (0,1)$. Since the smooth functions $\vartheta_0^+ \in L^1(\mathbb{R}^{n-3}) \cap L^\infty(\mathbb{R}^{n-3})$ with $n>3$, then (9) implies that $v^+(t,x)$ are bounded uniformly in $t \ge 0$ and $(x,y,z,s) \in$

$\mathbb{R}^{n-2}\times\mathbb{R}^3$. From (15), it follows that there exists a positive constant D_* depending on β such that

$$\begin{aligned}
&\frac{-V_s(y,z,s-\theta v^+(t,x))}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\\
=&\frac{-V_s(y,z,s-\theta v^+(t,x))}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-\theta v^+(t,x)-h(y,z))\right)}\frac{\phi_f^\beta\left(\frac{c_f^*}{c}(s-\theta v^+(t,x)-h(y,z))\right)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\\
\le&\sup_{(y,z,s)\in\mathbb{R}^3}\frac{-V_s(y,z,s)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\sup_{(t,x,y,z,s)\in[0,+\infty)\times\mathbb{R}^n}\frac{\phi_f^\beta\left(\frac{c_f^*}{c}(s-\theta v^+(t,x)-h(y,z))\right)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\\
\le&D_*.
\end{aligned}$$

Lemma 2.1 implies that there exists a constant $D^+>0$ (depending on β, f, $||\vartheta_0^+||_{L^1}$ and $||\vartheta_0^+||_{L^\infty}$) such that

$$\frac{\vartheta(t,x,y,z,s)-V(y,z,s)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\le D_*\sup_{x\in\mathbb{R}^{n-3}}|\vartheta_0^+(t,x)|\le D^+t^{-\frac{n-3}{2}}.$$

In a similar way, we can obtain that there exists a constant $D^->0$ (depending on β, f, $||\vartheta_0^-||_{L^1}$ and $||\vartheta_0^-||_{L^\infty}$) such that

$$\frac{\vartheta(t,x,y,z,s)-V(y,z,s)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\ge D^-t^{-\frac{n-3}{2}}.$$

Let $C:=\max\{D^-,D^+\}$. We complete the proof. □

Proof of Proposition 1. From Theorem 1.2 and (32), it suffices to prove that the solutions $v^\pm(t,x)$ to Eq. (31) with $v_0^\pm(x)=\vartheta_0^\pm(x)$ satisfy $v^+(t,0)\le D_2t^{-\frac{n-3}{2}}$ and $v^-(t,0)\ge D_1(1+t)^{-\frac{n-3}{2}}$ for some constants $D_i>0,\ i=1,2$. In fact, from the first inequality of (32), it follows that for $t\ge0$,

$$\begin{aligned}
\frac{\vartheta(t,\mathbf{0},0,0,0;\vartheta_0)-V(0,0,0)}{\phi_f^\beta(0)}\ge&\frac{V(0,0,-v^-(t,0))-V(0,0,0)}{\phi_f^\beta(0)}\\
\ge&\min_{s\in[-||v^-||_{L^\infty},0]}\left|\frac{V_s(0,0,s)}{\phi_f^\beta(0)}\cdot v^-(t,0)\right|\\
\ge&D_1(1+t)^{-\frac{n-3}{2}},
\end{aligned}$$

where $\mathbf{0}=\underbrace{(0,\cdots,0)}_{n-3}$. From (10), the explicit expression of the solution $v^-(t,x)$ to (31) is given by

$$v^-(t,x)=-\frac{1}{\tilde{k}_1}\log\left(\int_{\mathbb{R}^{n-3}}\Gamma(t,x-\eta)\exp(-\tilde{k}_1\vartheta_0^-(\eta))d\eta\right).$$

Since $\vartheta_0^- \geq 0$ and $\vartheta_0^- \not\equiv 0$, then there exist a positive constant $\varrho > 0$ and a open set $\Theta \neq \emptyset$ in $\mathbb{R}^{n-3}$ such that $\vartheta_0^- \geq \varrho$ in Θ. Therefore

$$\begin{aligned} v^-(t,x) \geq & -\frac{1}{\tilde{k}_1}\log\left(1 - \int_\Theta \Gamma(t, x-\eta)(1-\exp(-\tilde{k}_1\varrho))d\eta\right) \\ \geq & -\frac{1}{\tilde{k}_1}\log\left(1 - |\Theta|(1-\exp(-\tilde{k}_1\varrho))\min_{\eta\in\Theta}\Gamma(t,x-\eta)\right) \\ \geq & \frac{|\Theta|}{\tilde{k}_1}(1-\exp(-\tilde{k}_1\varrho))\min_{\eta\in\Theta}\Gamma(t,x-\eta). \end{aligned}$$

And hence, $v^-(t,0) \geq D_1(1+t)^{-\frac{n-3}{2}}$ for $t \geq 0$. Similarly, we can prove that $v^+(t,0) \leq D_2 t^{-\frac{n-3}{2}}$ for $t \geq 0$. We complete the proof. □

4. **Existence of oscillating solution.** In this section, we show Theorem 1.3. That is, we prove the existence of solution to Eq. (1) which oscillates permanently with non-decaying amplitude. To prove our main result, we need construct a sequence of subsolutions and supersolutions pushing the solution forth and back in the s-direction by combining Lemma 3.4 and the following auxiliary lemma.

Lemma 4.1. [Lemmas 3.1 and 3.2 of [14]] *Let $\tilde{k}_1$ be defined as in Lemma 2.2 and $v^\pm(t,x)$ be solutions to the Eq. (31) with $n=4$. Suppose that initial values $v_0^\pm(x)$ are all bounded on $\mathbb{R}$ and satisfy*

$$\begin{cases} v_0^+(x) \leq \bar{\delta}, & x \in \mathbb{R}, \\ v_0^+(x) \leq -\bar{\delta}, & |x| \in [m!+1, (m+1)!-1] \end{cases}$$

and

$$\begin{cases} v_0^-(x) \geq -\bar{\delta}, & x \in \mathbb{R}, \\ v_0^-(x) \geq \bar{\delta}, & |x| \in [m!+1, (m+1)!-1] \end{cases}$$

for some constant $\bar{\delta} > 0$ and some integer $m \geq 2$, respectively. Then there exists a constant $B > 0$ which depends on $\bar{\delta}$ and $\tilde{k}_1$ such that

$$\sup_{|x|\leq m!-1} v^+(T,x) \leq -\bar{\delta} + B\int_{|\zeta|\in[0,2/\sqrt{m}]\cup[\sqrt{m},+\infty)} e^{-\zeta^2}d\zeta$$

and

$$\sup_{|x|\leq m!-1} v^-(T,x) \geq \bar{\delta} - B\int_{|\zeta|\in[0,2/\sqrt{m}]\cup[\sqrt{m},+\infty)} e^{-\zeta^2}d\zeta,$$

respectively, where $T = \frac{m(m!)^2}{4}$.

Proof of Theorem 1.3. Let

$$\Omega_m := [m!+1, (m+1)!-1], \quad \widehat{\Omega}_m := [0, m!] \cup [(m+1)!, +\infty).$$

Let $\{v_j^\pm(t,x)\}_{j=1,2,\cdots}$ be two sequences of solutions to the Cauchy problem (31) with smooth initial values $v_{0,j}^\pm(x)$ satisfying $|v_{0,j}^\pm(x)| \leq \bar{\delta}$ in $x \in \mathbb{R}$,

$$v_{0,j}^+(x) = \begin{cases} -\bar{\delta}, & |x| \in \Omega_{2j}, \\ \bar{\delta}, & |x| \in \widehat{\Omega}_{2j} \end{cases} \qquad \text{and} \qquad v_{0,j}^-(x) = \begin{cases} \bar{\delta}, & |x| \in \Omega_{2j+1}, \\ -\bar{\delta}, & |x| \in \widehat{\Omega}_{2j+1}, \end{cases}$$

respectively. By the above definitions of $v_{0,j}^{\pm}(x)$, we can choose a function $\omega \in C^\infty(\mathbb{R})$ such that

$$v_{0,j}^{-}(x) \le \omega(x) \le v_{0,j}^{+}(x) \quad \text{for all } j \ge 1 \text{ and } x \in \mathbb{R}.$$

Let $\vartheta(t,x,y,z,s)$ be the solution to Eq. (4) with initial value $\vartheta(0,x,y,z,s) = V(y,z,s-\omega(x))$. Then we have

$$V(y,z,s+\bar\delta) \le V(y,z,s-v_j^-(t,x)) \le \vartheta(t,x,y,z,s) \le V(y,z,s-v_j^+(t,x)) \le V(y,z,s-\bar\delta)$$

from the definition of $\widehat{\vartheta}(x)$ and Lemma 3.4. Thus it follows from Lemma 4.1 that we have

$$\begin{aligned}
&\sup_{|x|\le(2j+1)!-1} \frac{\vartheta(t_{2j+1},x,y,z,s)-V(y,z,s-\bar\delta)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\\
\ge &\sup_{|x|\le(2j+1)!-1} \frac{V(y,z,s-v_j^-(t_{2j+1},x))-V(y,z,s-\bar\delta)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\\
\ge -&\sup_{(y,z,s)\in\mathbb{R}^3} \frac{\left|V_s\left(y,z,s-\bar\delta+\theta_1 B\int_{|\zeta|\in[0,2/\sqrt{2j+1}]\cup[\sqrt{2j+1},+\infty)} e^{-\zeta^2}d\zeta\right)\right|}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\\
&\times B\int_{|\zeta|\in[0,2/\sqrt{2j+1}]\cup[\sqrt{2j+1},+\infty)} e^{-\zeta^2}d\zeta,
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{|x|\le(2j)!-1} \frac{\vartheta(t_{2j},x,y,z,s)-V(y,z,s+\bar\delta)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\\
\le &\sup_{|x|\le(2j)!-1} \frac{V(y,z,s-v_j^+(t_{2j},x))-V(y,z,s+\bar\delta)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\\
\le &\sup_{(y,z,s)\in\mathbb{R}^3} \frac{\left|V_s\left(y,z,s+\bar\delta+\theta_2 B\int_{|\zeta|\in[0,2/\sqrt{2j}]\cup[\sqrt{2j},+\infty)} e^{-\zeta^2}d\zeta\right)\right|}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)}\\
&\times B\int_{|\zeta|\in[0,2/\sqrt{2j}]\cup[\sqrt{2j},+\infty)} e^{-\zeta^2}d\zeta,
\end{aligned}$$

where $\theta_1,\ \theta_2 \in (0,1)$, $t_{2j} = \frac{(2j)((2j)!)^2}{4}$ and $t_{2j+1} = \frac{(2j+1)((2j+1)!)^2}{4}$. By (13), the above two inequalities yield that

$$\lim_{j\to+\infty} \sup_{(y,z,s)\in\mathbb{R}^3} \sup_{|x|\le(2j+1)!-1} \frac{\vartheta(t_{2j+1},x,y,z,s)-V(y,z,s-\bar\delta)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)} = 0$$

and

$$\lim_{j\to+\infty} \sup_{(y,z,s)\in\mathbb{R}^3} \sup_{|x|\le(2j)!-1} \frac{\vartheta(t_{2j},x,y,z,s)-V(y,z,s+\bar\delta)}{\phi_f^\beta\left(\frac{c_f^*}{c}(s-h(y,z))\right)} = 0.$$

Then the conclusion of Theorem 1.3 can be obtained from the above two limits. We complete the proof. □

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