ELECTRONIC RESEARCH ARCHIVE
Volume 29, Number 6, December 2021
doi:10.3934/era.2021057
pp. 3687–3720

SEMILINEAR PSEUDO-PARABOLIC EQUATIONS ON MANIFOLDS WITH CONICAL SINGULARITIES

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(Communicated by Runzhang Xu)

Abstract. This paper studies the well-posedness of the semilinear pseudo-parabolic equations on manifolds with conical degeneration. By employing the Galerkin method and performing energy estimates, we first establish the local-in-time well-posedness of the solution. Moreover, to reveal the relationship between the initial datum and the global-in-time well-posedness of the solution we divide the initial datum into three classes by the potential well depth, i.e., the sub-critical initial energy level, the critical initial energy level and the sup-critical initial energy level (included in the arbitrary high initial energy case), and finally we give an affirmative answer to the question whether the solution exists globally or not. For the sub-critical and critical initial energy, thanks to the potential well theory, we not only obtain the invariant manifolds, global existence and asymptotic behavior of solutions, but also prove the finite time blow up of solutions and estimate the lower bound the of blowup time. For the sup-critical case, we show the assumptions for initial datum which cause the finite time blowup of the solution, realized by introducing a new auxiliary function. Additionally, we also provide some results concerning the estimates of the upper bound of the blowup time in the sup-critical initial energy.

1. **Introduction.** In this paper, we consider the following initial boundary value problem of semilinear pseudo-parabolic equations with conical degeneration

$$u_t - \Delta_{\mathbb{B}} u_t - \Delta_{\mathbb{B}} u = |u|^{p-1}u, \quad (x_b, \widetilde{x}) \in \mathrm{int}\mathbb{B}, t > 0, \tag{1.1}$$

$$u(x_b, \widetilde{x}, 0) = u_0, \quad (x_b, \widetilde{x}) \in \mathrm{int}\mathbb{B}, \tag{1.2}$$

$$u(0, \widetilde{x}, t) = 0, \quad (0, \widetilde{x}) \in \partial\mathbb{B}, t \geq 0, \tag{1.3}$$

where $u_0 \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, $1 < p < \frac{n+2}{n-2}$ and $n = l + 1 \geq 2$ is the dimension of $\mathbb{B}$, $l \in \mathbb{N}$. Here the domain $\mathbb{B} = [0, 1) \times X$ is regarded as the local model near the conical singularity on conical singular manifolds, where $X \subset \mathbb{R}^l$ is a closed compact C^∞

2020 *Mathematics Subject Classification.* 35K20, 35K55, 35A01, 35D30.
Key words and phrases. Conical Laplacian, semilinear pseudo-parabolic, global existence, finite time blow up, blowup time, asymptotic behavior.
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manifold. Denoting the interior of $\mathbb{B}$ by int$\mathbb{B}$ and the boundary of $\mathbb{B}$ by $\partial\mathbb{B} := \{0\} \times X$. We use the coordinates $(x_b, \widetilde{x}) := (x_b, x_1, x_2, ..., x_l) \in \mathbb{B}$ for $0 \leq x_b < 1$, $\widetilde{x} \in X$ near $\partial\mathbb{B}$. The conical Laplacian operator is defined as

$$\Delta_{\mathbb{B}} = \nabla_{\mathbb{B}}^2 = (x_b \partial_{x_b})^2 + \partial_{x_1}^2 + ... + \partial_{x_l}^2,$$

which is the totally characteristic degeneracy operators on a stretched conical manifold, and $\nabla_{\mathbb{B}} = (x_b\partial_{x_b}, \partial_{x_1}, ..., \partial_{x_l})$ denotes the corresponding gradient operator with conical degeneracy on the boundary $\partial\mathbb{B}$. In particular, we intend to investigate problem (1.1)-(1.3) in the weighted Mellin-Sobolev spaces $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, and the definition of such distribution spaces will be introduced in Section 2.

The classical pseudo-parabolic equation

$$u_t - \Delta u_t - \Delta u = |u|^{p-1}u, \ x \in \Omega, t > 0, \tag{1.4}$$

defined on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary appeared in various physical and biological phenomena. For example, taking u as the flow velocity, the homogeneous form of the model equation (1.4) was introduced to study the incompressible simple fluids with fading memory and the non-steady flow with the Rivlin-Ericksen tensors [47, 16]. One can also know more about other applications by referring to [23, 24, 5].

It is well known that equation (1.4) in the domains contained in classical Euclidean space with regular boundary has been well investigated. Cao et al. [9] considered the Cauchy problem of following model

$$\frac{\partial}{\partial t}u - k\frac{\partial \Delta u}{\partial t} = \Delta u + u^p, \ x \in \mathbb{R}^n, t > 0,$$

and obtained the critical global existence exponent and the critical Fujita exponent by integral representation and contraction mapping principle. Subsequently, its uniqueness was proved by Khomrutai in [27] for the case $0 < p < 1$. Furthermore, Li and Du [32] considered the Cauchy problem of

$$u_t - k\Delta u_t = \Delta u + |x|^\sigma u^p, \ x \in \mathbb{R}^n, t > 0,$$

and achieved the global existence and blowup in finite time of solutions with the critical Fujita exponent and the second critical exponent respectively. They also showed that the inhomogeneous term $|x|^\sigma$ affects the decay asymptotic behavior of solutions and accelerates the blowup of solutions. Khomrutai [28] studied the Cauchy problem of sublinear pseudo-parabolic equation

$$\partial_t u - \Delta\partial_t u = \Delta u + V(x,t)u^p, \ x \in \mathbb{R}^n, t > 0,$$

where $V(x,t) \sim \lambda(t)|x|^\sigma$ is a non-autonomous and unbounded potential function with $0 < p < 1$, and established the global existence of solutions by approximation and monotonicity argument. They also derived the precise grow-up rate of solutions and critical growth exponent. In order to figure out the effects of small perturbation on the dynamical of diffusion and reaction, Cao and Yin [8] considered the following Cauchy problem

$$\frac{\partial}{\partial t}u - k\frac{\partial \Delta u}{\partial t} = \Delta u + u^p + f(x), \ x \in \mathbb{R}^n, t > 0,$$

and revealed that small perturbation may develop large variation of solutions as time evolves. We also recommend that the reader refer to [43] to learn more about the effects of the power index of nonlinearity on the dynamical behavior of the solution. Different from above studies that focus on the influence of the nonlinearities especially the power index on the global well-posedness of the solution, [50, 49, 36] comprehensively studied equation (1.4) by considering the influences of the initial data on the global well-posedness and corresponding properties of solution. Depending on the potential well depth, they classified the initial data to subcritical initial energy level $J(u_0) < d$, critical initial energy level $J(u_0) = d$ and supcritical initial energy level $J(u_0) > d$, and proved the global existence, finite time blow up and asymptotic behavior of solutions with $J(u_0) \leq d$. Moreover, thanks to the comparison principle, the global existence and nonexistence of solutions were also obtained at supcritical initial energy level $J(u_0) > d$. When the nonlinear effects are dominated by the logarithmic term, Chen [14] investigated the following nonlinear pseudo-parabolic equation, i.e.,

$$\partial_t u - \Delta u - \Delta \partial_t u = u \ln |u|, \ x \in \Omega, t > 0$$

and proved the global existence and the finite time blow up of solutions under the subcritical and critical initial energy case, respectively. Focusing on the high initial energy level, Xu and Wang et al [51] studied the problem proposed in [50] and gave a sufficient condition on initial data leading to blow up in finite time by the potential well method, at the same time, they also estimated the upper bound of the blowup time. As an important method to reveal the influence of initial data on the dynamical behavior of solutions, the potential well theory can be applied not only to the study of the problem of parabolic equations, but also to the study of the problem for various types of nonlinear evolution equations or systems. Xu and Lian et al [48] investigated the global well-posedness of solutions for coupled parabolic systems in the variational framework, and the initial data leading to the global existence or finite time blow up of the solution are divided. Chen and Xu [15] considered a class of damped fourth-order nonlinear wave equations with logarithmic sources. By examining the effect of weak nonlinear sources on the blow up of the solution, they revealed the confrontation mechanism between the damping structure and the nonlinear source and found the initial data that caused the solution to blow up in infinite time. For related results of polynomial nonlinear sources, we refer to [52]. Furthermore, we suggest the readers refer to [53] for the study of high order nonlinear wave equations, [54, 34] for the study of damped nonlinear wave equation problems using improved potential well methods at high initial energy levels, and so on, which are representative recent results, of course we can not list all of the results obtained by the potential well theory here due to the huge amount.

Actually, geometric singularities have attracted considerable interest and have become the focus of extensive physical and mathematical research in recent years. To find static solutions of Einstein's equations coupled to brane sources, Michele [39] studied the generalizations of the so-called "football" shaped extra dimensions scenario to include two codimension branes, which can be transformed into the mathematical problem of solving the Liouville equation with singularities, where the function space he constructed can be described as a sphere with conical singularities at the brane locations. After that some cone solitons (in the case of compact surfaces) were found by Hamilton [25], where cone singularities also arise naturally on the study of such kind of solitons. Not only being widely applied in cosmology and physics, the cone singular manifold itself also brings a lot of interesting topics to pure mathematics, such as the analytic proof of the cobordism theorem [31]. Conical singularities become a hotspot mainly for reasons of two aspects. Firstly, a manifold with conical singularities is one of the most fundamental stratified spaces and the investigation on it is motivated by the desire of understanding the dynamic behavior of the solution of nonlinear evolution equations on such stratified space. Topologically these spaces are of iterated cone type, in which, due to the conical singularity, the classical differential operator cannot be applied to such manifolds. Secondly, the methods developed for the domains with smooth boundaries cannot be directly applied to domains with singularities. It is a challenge and also an interesting problem in the community to restitute the conclusions established on the smooth domain for the problems defined on the conical space.

Inspired by above, it is natural to bring some ideas and develop techniques to establish a comprehensive understanding of operator theory on the manifolds with conical singularities, which was first explored by Kondrat'ev in [29] by introducing the celebrated Mellin-Sobolev spaces $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ as the work space for the partial differential equations, then the theory on the conically degenerate pseudo-differential operator and the weighted Sobolev space on the conical manifolds were summarized by Schulze and Egorov in [45] and [18]. With the development of the study related to the singularity problem, including the study on the partial differential equation on manifolds with conical singularities, there occurs a large number of related results about various kinds of initial boundary value problems for evolution equations. For example, when using the porous medium equation to describe the flow of a substance in a porous medium material, the medium usually shows various irregular shapes in different regions, among which the most essential case is that the boundary of region includes the conical singularities. In order to discuss such situation for the porous medium equation, Lian and Liu [33] studied the initial-boundary value problem of the porous medium equation

$$u_t = \Delta u^m + V(x)u^p,\ x \in D, t > 0$$

in a cone $D = (0, \infty) \times \mathbb{S}^{n-1}$, and they proved that if the nonlinear power index p belongs to a suitable interval then the problem has no global non-negative solutions for any non-negative initial datum u_0 unless $u_0 \equiv 0$. Beside that, they also showed that this problem has global solutions for some $u_0 \geq 0$ when the nonlinear power index p is out of that interval. Considering the following porous medium equation

$$u_t - \Delta u^m = f(u, t),\ x \in D, t > 0 \tag{1.5}$$

Roidos and Schrohe [44] obtained some results about the existence, uniqueness and maximal L^p-regularity of a short time solution and showed the short time asymptotic behavior of the solution near the conical point. In addition, the behavior for large times of non-negative solutions to the linear Dirichlet problem of equation (1.5) in cone-like domains was obtained by Andreucci [2]. For more related work, we refer to Laptev [19, 30] for the high-order evolution inequalities in cone-like domains and Mazzeo et al [38] for the Ricci flow on asymptotically conical surfaces.

As a differential operator reflecting the diffusion form on the conical singular manifold, the emergence of the cone operator $\Delta_{\mathbb{B}}$ brings about the first problem that needs to be solved urgently is the existence of solution of the differential equation. Unlike the usual smooth domain, the appearance of cone singularities makes the classical embedding theorem fail, which baffles the proof of the existence theorem. Therefore, in order to overcome the difficulty and obtain the existence of the weak solution by variational method in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, Chen et al [12] considered the nonlinear Dirichlet boundary value problems on manifolds with conical singularities

$$-\Delta_{\mathbb{B}} u = |u|^{p-1}u,\ x \in \text{int}\mathbb{B},$$

and obtained the existence of non-trivial weak solution. Moreover, they also established the well-known cone Sobolev inequality and Poincaré inequality in the weighted Sobolev spaces. Subsequently, Chen et al [13] extended this result to the nonlinear elliptic equations with a general nonlinear source and the critical Sobolev exponents respectively. These works describe the mechanism by which cone differential operators act on the regularity of solutions of differential equations. Of course, differential operators do more than affect the regularity of solutions. The influence of degenerate differential operators on the solutions of nonlinear elliptic equations is also reflected in many other aspects such as the eigenvalue problems and the existence of multiple solutions. Far from being complete, we refer the readers to [22, 42, 40, 41, 3] and references therein. It is worth to mention here that the change of the variational structure of equations caused by the non-classical forms of differential operators can bring big challenges to the application of variational techniques [21, 7]. Further more, when utilizing the variational techniques the geometrical feature needs to be taken into consideration [20]. In fact we need to overcome the difficulties mentioned above in the application of variational techniques to the parabolic version.

In order to understand the effect of different initial data belonging to $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ on the well-posedness of the solution, Chen and Liu [11] investigated the following conical degenerate parabolic equation on the conical manifold

$$u_t - \Delta_{\mathbb{B}} u = |u|^{p-1}u, \ x \in \mathrm{int}\mathbb{B}, t > 0,$$

and obtained not only the existence of global solutions with exponential decay, but also the blow up in finite time under low initial energy level and critical initial energy level. Recently, Mohsen and Morteza [1] studied the semilinear conical-degenerate parabolic equation

$$\partial_t u - \Delta_{\mathbb{B}} u + V(x)u = g(x)|u|^{p-1}u, \ x \in \mathrm{int}\mathbb{B}, t > 0,$$

where $V(x) \in L^\infty(\mathrm{int}\mathbb{B}) \cap C(\mathrm{int}\mathbb{B})$ is the positive potential function and $g \in L^\infty(\mathrm{int}\mathbb{B}) \cap C(\mathrm{int}\mathbb{B})$ is a non-negative weighted function. Then they got the results of global solutions with exponential decay and showed the finite time blow up of solutions on manifolds with conical singularities under subcritical initial energy level $J(u_0) < d$.

Our goal is to obtian local and global well-posedness of solutions to problem (1.1)-(1.3). In details, by the potential well method, we classify the initial datum and give a threshold condition, which tells us that as long as the initial datum falls into the specified invariant set and the initial energy satisfies $J(u_0) \leq d$, the solution exists as a global one or blow up in a finite time. Moreover, for global solutions, we give an estimation of the asymptotic behavior of them. For the solution that blows up in finite time, we estimate the lower bound of the blowup time. Different from previous proofs in the potential well framework, we integrate the proofs of the sub-critical and critical initial energy cases as a whole part, which makes the results much more concise. Last but not least, we also investigate the finite time blow up of solutions to problem (1.1)-(1.3) at high initial energy levels. By giving a sufficient condition tied to the initial data in weighted Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, the theorem can not only explain what kind of initial data cause the solution to blow up in finite time, but also get the corresponding upper bound estimate of the blowup time.

The content of this paper is arranged as follows. In Section 2, we give the geometric description of conical singularities, the definitions of the weighted Sobolev spaces and several propositions of the manifold with conical singularities. Then we introduce the potential well structure for problem (1.1)-(1.3) and prove a series of corresponding properties in Section 3. Section 4 is concerned with the local existence and uniqueness theory. In Section 5, we not only prove the invariant manifolds, global existence and decay of solutions to describe the corresponding asymptotic behavior, but also prove the finite time blow up of solutions and estimate the lower bound of blowup time in Theorem 5.2. In Section 6, we give a sufficient condition to obtain the finite time blow up of the solution in Theorem 6.4. In particular, we also estimate the upper bound of the blowup time of the solution. Finally, some remarks and acknowledgements about this paper are given.

2. Manifolds with conical singularities.

In this section, main definitions of the manifold with conical singularities together with a brief description of its properties are given, for more details we refer to [45, 18] and the references therein. Furthermore, we introduce some functional inequalities on the manifold with conical singularity, for more applications of these inequalities one can refer to [12, 13].

2.1. Geometric description of conical singularities.

For $l \geq 2$, let $X \subset \mathbb{S}^l$ be a bounded open set in a unit sphere of $\mathbb{R}^{1+l}_{\hat{x}}$, and define the straight cone X^Δ by

$$X^\Delta = \left\{ \hat{x} \in \mathbb{R}^{1+l} \;\middle|\; \hat{x} = 0 \text{ or } \frac{\hat{x}}{|\hat{x}|} \in X \right\}.$$

The polar coordinates (ρ, θ) gives us a description of $X^\Delta \backslash \{0\}$ in the form $X^\wedge = \mathbb{R}_+ \times X$, which is called the open stretched cone with the base X, and $\{0\} \times X$ is the boundary of $X^\wedge$.

Now we extend it to a more general situation by describing the singular space associated with a manifold with conical singularities. A finite dimensional manifold $\mathcal{B}$ with finite conical singularities $\mathcal{B}_0 = \{b_1, b_2, ..., b_N\}$ has the following two properties:

a) $\mathcal{B} - \mathcal{B}_0$ is a C^∞ manifold.
b) Any $b_i \in \mathcal{B}_0$ $(i = 1, 2, ..., N)$ has an open neighborhood G in $\mathcal{B}$, such that there is a homeomorphism $\chi : G \to X^\Delta$ for some closed compact C^∞ manifold $X = X(b_i)$, and φ restricts a diffeomorphism $\varphi' : G\backslash\{0\} \to X^\wedge$.

By above assumptions we can define the stretched manifold associated with $\mathcal{B}$. Let $\mathbb{B}$ be a C^∞ manifold with compact C^∞ boundary $\partial\mathbb{B} \cong \cup_{b_i \in \mathcal{B}_0} X = X(b_i)$ for which there exists a diffeomorphism

$$\mathcal{B} - \mathcal{B}_0 \cong \mathbb{B} - \partial\mathbb{B} := \text{int}\mathbb{B}.$$

Furthermore, the restriction of this diffeomorphism to $G_i - b_i$ is also a diffeomorphism $G_i - b_i \cong U_i - X(b_i)$, where $G_i \subset \mathcal{B}$ is an open neighborhood near b_i, and $U_i \subset \mathbb{B}$ is a collar neiborhood with $U_i \cong [0, 1) \times X(b_i)$.

2.2. Cone Sobolev spaces $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$.

The typical differential operators on a manifold with conical singularities, i.e., the so-called Fuchsian type operators in a neighborhood of $x_b = 0$, have the following form

$$A := x_b^{-\mu} \sum_{k=0}^{\mu} a_k(x_b) \left(-x_b \frac{\partial}{\partial x_b} \right)^k$$

with $(x_b, \widetilde{x}) \in X^\wedge$ and $a_k(x_b) \in C^\infty(\overline{\mathbb{R}}_+, \text{Diff}^{\mu-k}(X))$[45, 18]. For such singular operators, we introduce the following cone weighted Sobolev space.

Definition 2.1 (The space $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)$)**.** For $(x_b, \widetilde{x}) \in \mathbb{R}_+ \times \mathbb{R}^l := \mathbb{R}_+^n$, $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $1 < p < \infty$, assume $u(x_b, \widetilde{x}) \in \mathcal{D}'(\mathbb{R}_+^n)$, where the dual $(C_0^\infty(\mathbb{R}_+^n))' = \mathcal{D}'(\mathbb{R}_+^n)$ is the space of all distributions in $\mathbb{R}_+^n$. We denote the spaces

$$\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n) := \left\{ u \in \mathcal{D}'(\mathbb{R}_+^n) \;\middle|\; x_b^{\frac{N}{p}-\gamma}(x_b\partial_{x_b})^k \partial_{\widetilde{x}}^\alpha u \in L^p(\mathbb{R}_+^n) \right\}$$

for any $k \in \mathbb{N}$, multi-index $\alpha \in \mathbb{N}^n$ with $k + |\alpha| \le m$.

Therefore, $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)$ is a Banach space with the following norm

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)} = \sum_{k+|\alpha|\le m} \left(\int_{\mathbb{R}_+} \int_{\mathbb{R}^N} x_b^N |x_b^{-\gamma}(x_b\partial_{x_b})^k \partial_{\widetilde{x}}^\alpha u(x_b, \widetilde{x})|^p \frac{\mathrm{d}x_b}{x_b} \mathrm{d}\widetilde{x} \right)^{\frac{1}{p}}.$$

Definition 2.2 ([45] The space $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$)**.** We give the definition of $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$ as follows

(i) Let X be a closed compact C^∞ manifold covered by open neighborhoods $\mathcal{O} = \{O_1, ..., O_N\}$ of the coordinate. Let the subordinate partition of unity $\{\psi_1, ..., \psi_N\}$ be fixed and charts $\chi_j : O_j \to \mathbb{R}^n$, $j = 1, ..., N$. Then we say that $u \in \mathcal{H}_p^{m,\gamma}(X^\wedge)$ if and only if $u \in \mathcal{D}'(X^\wedge)$, whose norm is defined as follows

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(X^\wedge)} = \left(\sum_{j=1}^N \|(1 \times \chi_j^*)^{-1} \psi_j u\|_{\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)}^p \right)^{\frac{1}{p}},$$

where $1 \times \chi_j^* : C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n) \to C_0^\infty(\mathbb{R}_+ \times O_j)$ is the pull-back function with respect to $1 \times \chi_j : \mathbb{R}_+ \times O_j \to \mathbb{R}_+ \times \mathbb{R}^n$. And the closure of $C_0^\infty(X^\wedge)$ in $\mathcal{H}_p^{m,\gamma}(X^\wedge)$ is denoted as $\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge)$.

(ii) Let $\mathbb{B}$ be the stretched cone manifolds. Then $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ denotes the subspace of all $u \in W_{\text{loc}}^{m,p}(\text{int}\mathbb{B})$ such that

$$\mathcal{H}_p^{m,\gamma}(\mathbb{B}) = \left\{ u \in W_{\text{loc}}^{m,p}(\text{int}\mathbb{B}) \;\middle|\; \omega u \in \mathcal{H}_{p,0}^{m,\gamma}(X^\wedge) \right\}$$

for any cut-off function $\omega(x_b)$ supported by a collar neighborhood of $x_b \in (0,1)$. Moreover, the subspace $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$ of $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is defined as follows

$$\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B}) = [\omega]\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge) + [1-\omega]W_0^{m,p}(\text{int}\mathbb{B}),$$

where the classical Sobolev space $W_0^{m,p}(\text{int}\mathbb{B})$ denotes the closure of $C_0^\infty(\text{int}\mathbb{B})$ in $W^{m,p}(\widetilde{\mathbb{B}})$ for $\widetilde{\mathbb{B}}$ as a closed compact C^∞ manifold of dimension n containing $\mathbb{B}$ as a submanifold with boundary.

2.3. **Some inequalities on cone Sobolev spaces.**

Proposition 1 (Cone Sobolev inequality [12])**.** *Assuming* $1 \le p < n$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ *then for all* $u(x_b, \widetilde{x}) \in C_0^\infty(\mathbb{R}_+^n)$ *the following estimate*

$$\|u\|_{L_{p^*}^{\gamma^*}(\mathbb{R}_+^n)} \le c_1 \|(x_b\partial_{x_b})u\|_{L_p^\gamma(\mathbb{R}_+^n)} + (c_1 + c_2) \sum_{i=1}^n \|\partial_{x_i} u\|_{L_p^\gamma(\mathbb{R}_+^n)} + \frac{c_2}{c_3} \|u\|_{L_p^\gamma(\mathbb{R}_+^n)} \tag{2.1}$$

holds, where $\gamma, \gamma^* \in \mathbb{R}$ *are constants with* $\gamma^* = \gamma - 1$, *and*

$$c_1 = \frac{(n-1)p}{n(n-p)},$$

$$c_2 = \frac{(n-1)p\left|(n-1) - \frac{(\gamma-1)(n-1)p}{n-p}\right|^{\frac{1}{n}}}{n(n-p)}$$

and

$$c_3 = \frac{(n-1)p}{n-p}.$$

Moreover, for $u(x_b, \widetilde{x}) \in \mathcal{H}^{1,\gamma}_{p,0}(\mathbb{R}^n_+)$, *there holds*

$$\|u\|_{L^{\gamma^*}_{p^*}(\mathbb{R}^n_+)} \leq C_* \|u\|_{\mathcal{H}^{1,\gamma}_{p,0}(\mathbb{R}^n_+)},$$

where $C_* = c_1 + c_2$.

Proposition 2 (Cone Poincaré inequality [12])**.** *Let* $\mathbb{B} = [0,1) \times X \subset \mathbb{R}^n_+$ *is bounded and* $1 < p < \infty$, γ *is a constant. If* $u(x_b, \widetilde{x}) \in \mathcal{H}^{1,\gamma}_{p,0}(\mathbb{B})$ *then*

$$\|u(x_b, \widetilde{x})\|_{L^\gamma_p(\mathbb{B})} \leq c\|\nabla_{\mathbb{B}} u(x_b, \widetilde{x})\|_{L^\gamma_p(\mathbb{B})},$$

where the optimal constant c *depends only on* $\mathbb{B}$ *and* p.

Proposition 3 (Cone Hölder inequality [13])**.** *If* $u \in L^{\frac{n}{p}}_p(\mathbb{B}), v \in L^{\frac{n}{p'}}_{p'}(\mathbb{B})$ *with* $p, p' \in (1, \infty)$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$, *then we have the following cone type Hölder inequality*

$$\int_{\mathbb{B}} |uv| \frac{\mathrm{d}x_b}{x_b} \mathrm{d}\widetilde{x} \leq \left(\int_{\mathbb{B}} |u|^p \frac{\mathrm{d}x_b}{x_b} \mathrm{d}\widetilde{x}\right)^{\frac{1}{p}} \left(\int_{\mathbb{B}} |v|^{p'} \frac{\mathrm{d}x_b}{x_b} \mathrm{d}\widetilde{x}\right)^{\frac{1}{p'}}.$$

Proposition 4 (Eigenvalue problem [13])**.** *There exist* $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_k \leq ... \to \infty$, *such that for each* $k \geq 1$, *the following Dirichlet problem*

$$\begin{cases} -\Delta_{\mathbb{B}} \psi_k = \lambda_k \psi_k, & (x_b, \widetilde{x}) \in \mathrm{int}\mathbb{B}, \\ \psi_k = 0, & (x_b, \widetilde{x}) \in \partial\mathbb{B}, \end{cases}$$

admits a non-trivial solution in $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. *Moreover,* $\{\psi_k\}_{k\geq 1}$ *constitute an orthonormal basis of the Hilbert space* $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$.

3. **Preliminaries.**

3.1. **Some assumptions, functionals and manifolds.**

In order to state our main results, we shall introduce some definitions and notations as follows. In the sequel, for convenience we denote

$$(u, v)_{\mathbb{B}} = \int_{\mathbb{B}} uv \frac{\mathrm{d}x_b}{x_b} \mathrm{d}\widetilde{x} \quad \text{and} \quad \|u\|_{L^{\frac{n}{p}}_p(\mathbb{B})} = \left(\int_{\mathbb{B}} |u|^p \frac{\mathrm{d}x_b}{x_b} \mathrm{d}\widetilde{x}\right)^{\frac{1}{p}}.$$

Furthermore, we denote $\|\cdot\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} := ((\nabla_{\mathbb{B}}\cdot, \nabla_{\mathbb{B}}\cdot)_{\mathbb{B}} + (\cdot,\cdot)_{\mathbb{B}})^{\frac{1}{2}}$. Of course, the norm $\|\cdot\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}$ is equivalent to the norm $\|\nabla\cdot\|_{L^{\frac{n}{2}}_2(\mathbb{B})}$, which can be stemmed from Proposition 2.

Throughout the paper, C will be used to denote various positive constants, whose value may change from line to line, and its dependence on other variables will be emphasised only if needed.

Definition 3.1. (Weak solution). A function u is called a weak solution to problem (1.1)-(1.3) on $[0,T] \times \mathbb{B}$, if it satisfies

(i) $u \in L^\infty(0,T; \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ and $u_t \in L^2(0,T; \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$;
(ii) $u(0) = u_0$;

(iii) for any $\eta \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, the identity

$$\int_{\mathbb{B}} u_t(t)\eta \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x} + \int_{\mathbb{B}} \nabla_{\mathbb{B}} u_t(t)\nabla_{\mathbb{B}}\eta \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x} + \int_{\mathbb{B}} \nabla_{\mathbb{B}} u(t)\nabla_{\mathbb{B}}\eta \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}$$

$$= \int_{\mathbb{B}} |u(t)|^{p-1}u(t)\eta \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}$$

holds for a.e. $t \in [0, T]$.

Let us introduce the following functionals on the cone Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ as the potential energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x} - \frac{1}{p+1}\int_{\mathbb{B}} |u|^{p+1}\frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x} \tag{3.1}$$

and the so-called Nehari functional

$$I(u) = \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x} - \int_{\mathbb{B}} |u|^{p+1}\frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}. \tag{3.2}$$

Then $J(u)$ and $I(u)$ are well-defined and belong to $C^1(\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B}), \mathbb{R})$.

The weak solution $u(t)$ in Definition 3.1 satisfies the conservation of energy, i.e.,

$$\int_0^t \|u_t(\tau)\|^2_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}\mathrm{d}\tau + J(u(t)) = J(u_0), \quad 0 \le t < T. \tag{3.3}$$

By making use of the functionals above, we define the potential well depth d as follows

$$d = \inf_{u\in\mathcal{N}} J(u) > 0, \tag{3.4}$$

where the Nehari manifold

$$\mathcal{N} = \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \;\middle|\; I(u(t)) = 0, \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u(t)|^2 \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x} \neq 0 \right\} \tag{3.5}$$

separates the whole space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ into the following two unbounded manifolds

$$\mathcal{N}_+ = \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \;\middle|\; I(u(t)) > 0 \right\} \tag{3.6}$$

and

$$\mathcal{N}_- = \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \;\middle|\; I(u(t)) < 0 \right\}. \tag{3.7}$$

Furthermore, we introduce the following potential well

$$\mathcal{W} := \mathcal{N}_+ \cup \{0\} \tag{3.8}$$

and the outside of the corresponding potential well

$$\mathcal{V} := \mathcal{N}_-. \tag{3.9}$$

3.2. **Some lemmas and properties of potential well.** Now, we give some corresponding properties of the potential well as follows.

Lemma 3.2 (The properties of the energy functional $J(u)$)**.** *Assume that* $u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ *and* $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$, *we have*

(i) $\lim_{\lambda\to 0} j(\lambda) = 0$ *and* $\lim_{\lambda\to+\infty} j(\lambda) = -\infty$, *where* $j(\lambda) := J(\lambda u)$;

(ii) there exist a unique $\lambda^* = \lambda^*(u) > 0$, *such that* $j'(\lambda^*) = 0$;

(iii) $j(\lambda)$ *is strictly increasing on* $0 \le \lambda < \lambda^*$, *strictly decreasing on* $\lambda > \lambda^*$ *and takes the maximum at* $\lambda = \lambda^*$;

(iv) $i(\lambda) > 0$ *for* $0 \le \lambda < \lambda^*$, $i(\lambda) < 0$ *for* $\lambda > \lambda^*$ *and* $i(\lambda^*) = 0$, *where* $i(\lambda) := I(\lambda u)$.

Proof. (i) From the definition of $J(u)$, we know that

$$j(\lambda) = J(\lambda u) = \frac{\lambda^2}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_b}{x_b} d\widetilde{x} - \frac{\lambda^{p+1}}{p+1}\int_{\mathbb{B}} |u|^{p+1}\frac{dx_b}{x_b} d\widetilde{x},$$

which gives $\lim_{\lambda\to 0} j(\lambda) = 0$ and $\lim_{\lambda\to+\infty} j(\lambda) = -\infty$.

(ii) An easy calculation shows that

$$j'(\lambda) = \lambda\left(\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_b}{x_b} d\widetilde{x} - \lambda^{p-1}\int_{\mathbb{B}} |u|^{p+1}\frac{dx_b}{x_b} d\widetilde{x}\right). \tag{3.10}$$

Then taking $j'(\lambda) = 0$ we obtain that

$$\lambda^* = \left(\frac{\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_b}{x_b} d\widetilde{x}}{\int_{\mathbb{B}} |u|^{p+1}\frac{dx_b}{x_b} d\widetilde{x}}\right)^{\frac{1}{p-1}} > 0.$$

(iii) By a direct calculation, (3.10) gives $j'(\lambda) > 0$, for $0 < \lambda < \lambda^*$, $j'(\lambda) < 0$ for $\lambda^* < \lambda < \infty$. Hence, the conclusion of (iii) holds.

(iv) The conclusion follows from

$$i(\lambda) = I(\lambda u) = \lambda^2\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_b}{x_b} d\widetilde{x} - \lambda^{p+1}\int_{\mathbb{B}} |u|^{p+1}\frac{dx_b}{x_b} d\widetilde{x} = \lambda j'(\lambda).$$

□

Next we give the relationship between $\|u\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}$ and $I(u)$ in the following lemma.

Lemma 3.3 (The properties of the Nehari functional $I(u)$)**.** *Suppose that* $u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ *and* $r = \left(\frac{1}{(c^2+1)C_*^{p+1}}\right)^{\frac{1}{p-1}}$, *where* c *and* C_* *are the optimal constants of the cone Poincaré inequality and cone Sobolev inequality respectively.*

(i) If $0 < \|u\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} < r$, *then* $I(u) > 0$.

(ii) If $I(u) < 0$, *then* $\|u\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} > r$.

(iii) If $I(u) = 0$, *then* $\|u\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} \ge r$ *or* $\|u\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} = 0$.

Proof. (i) From $0 < \|u\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} < r$, we have

$$\begin{aligned}\int_{\mathbb{B}} |u|^{p+1}\frac{dx_b}{x_b} d\widetilde{x} &\le C_*^{p+1}\|u\|^{p+1}_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} < C_*^{p+1} r^{p-1}\|u\|^2_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} \\ &= \frac{1}{c^2+1}\|u\|^2_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} \le \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}.\end{aligned}$$

Then by the definitions of r and $I(u)$ we obtain that $I(u) > 0$.

(ii) It is easy to see that $\|u\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \neq 0$ by $I(u) < 0$. Combining cone Poincaré inequality and cone Sobolev inequality, it follows that

$$\begin{aligned}\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} &= \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x} + \int_{\mathbb{B}} |u|^2 \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}\\ &\leq (c^2+1)\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}\\ &< (c^2+1)\int_{\mathbb{B}} |u|^{p+1} \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}\\ &\leq (c^2+1)C_*^{p+1}\|u\|^{p+1}_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})},\end{aligned}$$

then we get

$$\|u\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} > \left(\frac{1}{(c^2+1)C_*^{p+1}}\right)^{\frac{1}{p-1}} = r.$$

(iii) If $I(u) = 0$ and $\|u\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \neq 0$, then by

$$\begin{aligned}\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} &= \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x} + \int_{\mathbb{B}} |u|^2 \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}\\ &\leq (c^2+1)\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}\\ &= (c^2+1)\int_{\mathbb{B}} |u|^{p+1} \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}\\ &\leq (c^2+1)C_*^{p+1}\|u\|^{p+1}_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})},\end{aligned}$$

we get $\|u\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \geq \left(\frac{1}{(c^2+1)C_*^{p+1}}\right)^{\frac{1}{p-1}} = r$.

□

The depth d can be estimated as follows.

Lemma 3.4 (The potential well depth)**.** *Suppose that $1 < p < \frac{n+2}{n-2}$ and d is defined as (3.4). Then we have*

$$d = \inf_{u\in\mathcal{N}} J(u) = \frac{p-1}{2(p+1)}\left(\frac{1}{(c^2+1)^{\frac{p+1}{2}}C_*^{p+1}}\right)^{\frac{2}{p-1}}, \tag{3.11}$$

where c is the best coefficient of the cone Poincaré inequality.

Proof. Suppose that $u \in \mathcal{N}$, then $\|u\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \geq r$ from Lemma 3.3. Thus according to the definition of $J(u)$, $I(u)$ and cone Poincaré inequality, we arrive at

$$\begin{aligned}J(u) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x} + \frac{1}{p+1}I(u)\\ &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\frac{1}{c^2+1}\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\\ &\geq \frac{p-1}{2(p+1)(c^2+1)}r^2,\end{aligned}$$

which gives (3.11).

□

Lemma 3.5 (Osgood Lemma,[10]). *Let $\rho : [t_0, T] \to [0, \alpha]$ be a measurable function, γ is a locally integrable, positive function defined on $[t_0, T]$, $\mu : [0, \alpha] \to [0, +\infty)$ is a nondecreasing, continuous function and $\mu(0) = 0$, $a \geq 0$ is a constant. If*

$$\rho(t) \leq a + \int_{t_0}^{t} \gamma(s)\mu(\rho(s))\mathrm{d}s,$$

holds almost everywhere for $t \in [t_0, T]$, then

$$-M(\rho(t)) + M(a) \leq \int_{t_0}^{t} \gamma(s)\mathrm{d}s,$$

is true almost everywhere for $t \in [t_0, T]$ when $a > 0$, where $M(x) = \int_x^\alpha \frac{\mathrm{d}s}{\mu(s)}$. In addition, when $a = 0$ and $M(0) = \infty$, $\rho(t) = 0$ holds almost everywhere for $t \in [t_0, T]$.

4. **Local existence.** In this section, we prove the local existence of the solution of problem (1.1)-(1.3). The local existence theorem is given as follows,

Theorem 4.1 (Local existence). *Suppose that $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$. Then there exist $T > 0$ and a unique weak solution $u \in C([0, T], \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}))$ of problem (1.1)-(1.3) on $[0, T] \times \mathbb{B}$. Moreover, if*

$$T_{\max} = \sup\{T > 0 : u = u(t) \textit{ exists on } [0, T]\} < \infty,$$

then

$$\lim_{t \to T_{\max}} \|u(t)\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} = \infty.$$

Proof. We divide the proof into 2 steps.

Step 1. Local exisence. We prove the local existence of the solution to problem (1.1)-(1.3) by virtue of the Galerkin method and the compactness property [35]. For the initial data $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, let $R^2 := 2\|\nabla_{\mathbb{B}} u_0\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}$. For every $m \geq 1$, let $\Psi_m = \mathrm{Span}\{\psi_1, \psi_2, ..., \psi_m\}$, where $\{\psi_j\}$ is the orthonormal complete system of eigenfuctions of $-\Delta_{\mathbb{B}}$ in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ such that $\|\psi_j\|_{L_2^{\frac{n}{2}}(\mathbb{B})} = 1$ for all j. Then, $\{\psi_j\}$ is orthonormal and complete in $L_2^{\frac{n}{2}}(\mathbb{B})$ and $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ by Proposition 4, and we denote by $\{\lambda_j\}$ the corresponding eigenvalues. Let

$$u_{m0} = \sum_{j=1}^{m} (\nabla_{\mathbb{B}} u_0, \nabla_{\mathbb{B}} \psi_j)_{\mathbb{B}} \psi_j, \ m \in \mathbb{N}^+ \tag{4.1}$$

such that $u_{m0} \in \Psi_m$ and

$$u_{m0} \to u_0 \text{ in } \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \text{ as } m \to \infty. \tag{4.2}$$

For all $m \geq 1$, we seek C^1-continuous functions $g_{m1}(t), ..., g_{mm}(t)$ to form an approximate solution to problem (1.1)-(1.3) of the form

$$u_m(t, x_b, \widetilde{x}) = \sum_{j=1}^{m} g_{mj}(t)\psi_j(x_b, \widetilde{x}), \tag{4.3}$$

which solves the problem

$$\begin{cases}(u_{mt}(t),\psi_j)_{\mathbb{B}}+(\nabla_{\mathbb{B}}u_{mt}(t),\nabla_{\mathbb{B}}\psi_j)_{\mathbb{B}}+(\nabla_{\mathbb{B}}u_m(t),\nabla_{\mathbb{B}}\psi_j)_{\mathbb{B}}=(|u_m(t)|^{p-1}u_m(t),\psi_j)_{\mathbb{B}},\\ u_m(0,x_b,\widetilde{x})=u_{m0},\end{cases} \tag{4.4}$$

for $j=1,2,...,m$ and $t\geq 0$. Problem (4.4) is equivalent to the following systems of ODEs

$$\begin{cases}(1+\lambda_j)g_{mjt}(t)+\lambda_j g_{mj}(t)=(|u_m(t)|^{p-1}u_m(t),\psi_j)_{\mathbb{B}},\\ g_{mj}(0)=(\nabla_{\mathbb{B}}u_0,\nabla_{\mathbb{B}}\psi_j)_{\mathbb{B}},\ j=1,2,...,m.\end{cases} \tag{4.5}$$

One can deduce that for any fixed m there exists a $t_m>0$ and a unique solution $g_{mj}\in C^1[0,t_m]$ of the Cauchy problem (4.5) by the Cauchy-Peano theorem since $f(t):=(|u_m(t)|^{p-1}u_m(t),\psi_j)_{\mathbb{B}}$ is continuous with respect to t. Multiplying (4.4) by $g_{mjt}(t)$ and summing for j, we have

$$\begin{aligned}&(u_{mt}(t),u_{mt}(t))_{\mathbb{B}}+(\nabla_{\mathbb{B}}u_{mt}(t),\nabla_{\mathbb{B}}u_{mt}(t))_{\mathbb{B}}+(\nabla_{\mathbb{B}}u_m(t),\nabla_{\mathbb{B}}u_{mt}(t))_{\mathbb{B}}\\&=(|u_m(t)|^{p-1}u_m(t),u_{mt}(t))_{\mathbb{B}},\end{aligned}$$

which tells us that for all $t\in[0,t_m]$,

$$\begin{aligned}&\|u_{mt}(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}+\|\nabla_{\mathbb{B}}u_{mt}(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_{\mathbb{B}}u_m(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\\&=\int_{\mathbb{B}}|u_m(t)|^{p-1}u_m(t)u_{mt}(t)\frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}.\end{aligned} \tag{4.6}$$

For the last term in equation (4.6), by using the cone Hölder inequality (Proposition 1), cone Sobolev inequality (Proposition 3) and Young's inequality, we deduce

$$\begin{aligned}\int_{\mathbb{B}}|u_m(t)|^{p-1}u_m(t)u_{mt}(t)\frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}\leq&\int_{\mathbb{B}}|u_m(t)|^{p}|u_{mt}(t)|\frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}\\ \leq&\|u_{mt}(t)\|_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}\|u_m(t)\|^p_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}\\ \leq&C_*^{p+1}\|\nabla_{\mathbb{B}}u_{mt}(t)\|_{L_2^{\frac{n}{2}}(\mathbb{B})}\|\nabla_{\mathbb{B}}u_m(t)\|^p_{L_2^{\frac{n}{2}}(\mathbb{B})}\\ \leq&\frac{1}{2}\|\nabla_{\mathbb{B}}u_{mt}(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}+\frac{C_*^{2(p+1)}}{2}\|\nabla_{\mathbb{B}}u_m(t)\|^{2p}_{L_2^{\frac{n}{2}}(\mathbb{B})},\end{aligned} \tag{4.7}$$

which together with (4.6) gives,

$$\|u_{mt}(t)\|^2_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}+\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_{\mathbb{B}}u_m(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\leq C_*^{2(p+1)}\|\nabla_{\mathbb{B}}u_m(t)\|^{2p}_{L_2^{\frac{n}{2}}(\mathbb{B})},t\in[0,t_m]. \tag{4.8}$$

Integrating (4.8) over $[0,t]$, we obtain

$$\begin{aligned}&\|\nabla_{\mathbb{B}}u_m(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}+\int_0^t\|u_{mt}(s)\|^2_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}\mathrm{d}s\\&\leq\|\nabla_{\mathbb{B}}u_m(0)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}+C_*^{2(p+1)}\int_0^t\|\nabla_{\mathbb{B}}u_m(s)\|^{2p}_{L_2^{\frac{n}{2}}(\mathbb{B})}\mathrm{d}s,\end{aligned}$$

which combining the formula (4.2) and the fact $R^2 := 2\|\nabla_{\mathbb{B}} u_0\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}$ shows that there exists some $m_0 \in \mathbb{N}^+$ such that

$$\|\nabla_{\mathbb{B}} u_m(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \int_0^t \|u_{mt}(s)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}s \le R^2 + C_*^{2(p+1)} \int_0^t \|\nabla_{\mathbb{B}} u_m(s)\|^{2p}_{L_2^{\frac{n}{2}}(\mathbb{B})} \mathrm{d}s, \tag{4.9}$$

for $t \in [0, t_m]$ and $m \ge m_0$. In order to estimate the first term in (4.9), one can apply the Osgood lemma [10, Lemma 5.2.1] as follows.

For any fixed $m \ge m_0$, let $\rho(t) := \|\nabla_{\mathbb{B}} u_m(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}$, $t \in [0, t_m]$, since $g_{mj} \in C^1[0, t_m]$, we deduce that $\max\limits_{t\in[0,t_m]} |g_{mj}(t)|$ is bounded, $j = 1, ..., m$, then

$$\rho(t) = \sum_{j=1}^m \lambda_j |g_{mj}(t)|^2 \le m \max_{1\le j\le m} \left(\lambda_j \max_{t\in[0,t_m]} |g_{mj}(t)|^2\right) := \alpha_m < +\infty,$$

which implies that $\rho : [0, t_m] \to [0, \alpha_m]$ is a measurable function. Now we pick

$\gamma(s) \equiv C_*^{2(p+1)} : [0, t_m] \to \mathbb{R}$ is a locally integrable, positive function,

$\mu(s) = s^p : [0, \alpha_m] \to [0, +\infty)$ is a continuous, non-decreasing function,

which satisfies $\mu(0) = 0$, and $a = R^2$, then,

$$M(\nu) = \int_\nu^{\alpha_m} \frac{\mathrm{d}s}{\mu(s)} = \frac{1}{p-1}\left(\nu^{-(p-1)} - \alpha_m^{-(p-1)}\right).$$

Therefore, we have

$$-M(\rho(t)) + M(a) \le \int_0^t \gamma(s)\mathrm{d}s = C_*^{2(p+1)} t,$$

that is,

$$\frac{1}{p-1}\left(\alpha_m^{-(p-1)} - \|\nabla_{\mathbb{B}} u_m(t)\|^{-2(p-1)}_{L_2^{\frac{n}{2}}(\mathbb{B})}\right) + \frac{1}{p-1}\left(R^{-2(p-1)} - \alpha_m^{-(p-1)}\right) \le C_*^{2(p+1)} t.$$

By a simple calculation, we obtain

$$\|\nabla_{\mathbb{B}} u_m(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \le \left[R^{2(1-p)} - (p-1)C_*^{2(p+1)} t\right]^{-\frac{1}{p-1}}, \quad t \in [0, t_m]. \tag{4.10}$$

Taking

$$T = T(R) := \frac{R^{2(1-p)}}{2(p-1)C_*^{2(p+1)}}, \tag{4.11}$$

it follows from estimates (4.9)-(4.11) that

$$\|\nabla_{\mathbb{B}} u_m(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \int_0^t \|u_{mt}(s)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}s \le C(R), \quad t \in [0, T], \ m \ge m_0, \tag{4.12}$$

where

$$C(R) = R^2 \left(1 + \frac{2^{\frac{1}{p-1}}}{p-1}\right).$$

Then combining with (4.12) and Proposition 1 we obtain

$$\||u_m(t)|^{p-1} u_m(t)\|_{L^{\frac{np}{p+1}}_{\frac{p+1}{p}}(\mathbb{B})} = \|u_m(t)\|^p_{L^{\frac{n}{p+1}}_{p+1}(\mathbb{B})} \le C_*^p \|\nabla_{\mathbb{B}} u_m(t)\|^p_{L_2^{\frac{n}{2}}(\mathbb{B})} \le \left(C_* \sqrt{C(R)}\right)^p \tag{4.13}$$

for all $t \in [0,T]$ and $m \geq m_0$, which means

$$\{u_m\} \text{ is bounded in } L^\infty(0,T;\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})),$$
$$\{u_{mt}\} \text{ is bounded in } L^2(0,T;\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})),$$
$$\{|u_m|^{p-1}u_m\} \text{ is bounded in } L^\infty(0,T;L^{\frac{np}{p+1}}_{\frac{p+1}{p}}(\mathbb{B})).$$

Hence, by the Aubin-Lions-Simon Lemma [46] and the weak compactness there exist a $u \in C([0,T];\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ and a subsequence of $\{u_m\}$, which is still denoted by $\{u_m\}$, such that

$$\begin{aligned} &u_m \to u \text{ in } L^\infty(0,T;\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})) \text{ weakly star},\\ &u_m \to u \text{ in } C([0,T];L^{\frac{n}{2}}_2(\mathbb{B})) \text{ and a.e. in } [0,T]\times \mathrm{int}\mathbb{B},\\ &u_{mt} \to u_t \text{ in } L^2(0,T;\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})) \text{ weakly},\\ &|u_m|^{p-1}u_m \to \chi \text{ in } L^\infty(0,T;L^{\frac{np}{p+1}}_{\frac{p+1}{p}}(\mathbb{B})) \text{ weakly star}, \end{aligned} \tag{4.14}$$

which together with the Lions Lemma [35, Chap. 1, p12] deduce that

$$\chi = |u|^{p-1}u.$$

Then for each j fixed, let $m \to \infty$ in (4.4), we have

$$\begin{aligned} &\int_{\mathbb{B}} u_t(t)\psi_j \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x} + \int_{\mathbb{B}} \nabla_{\mathbb{B}}u(t)\nabla_{\mathbb{B}}\psi_j \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x} + \int_{\mathbb{B}} \nabla_{\mathbb{B}}u_t(t)\nabla_{\mathbb{B}}\psi_j \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}\\ &= \int_{\mathbb{B}} |u(t)|^{p-1}u(t)\psi_j \frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x} \end{aligned}$$

for a.e. $t \in [0,T]$ and every $j = 1,2,\cdots$. By the fact that $\{\psi_j\}$ is the complete orthonormal basis in $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, we have

$$(u_t(t),\eta)_{\mathbb{B}} + (\nabla_{\mathbb{B}}u(t),\nabla_{\mathbb{B}}\eta)_{\mathbb{B}} + (\nabla_{\mathbb{B}}u_t(t),\nabla_{\mathbb{B}}\eta)_{\mathbb{B}} = \int_{\mathbb{B}} |u(t)|^{p-1}u(t)\eta\frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}$$

for a.e. $t \in [0,T]$ and every $\eta \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. By the convergence (4.14) and formula (4.12), we have

$$\|\nabla_{\mathbb{B}}u(t)\|^2_{L^{\frac{n}{2}}_2(\mathbb{B})} + \int_0^t \|u_t(s)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}s \leq C(R), \quad t\in[0,T]. \tag{4.15}$$

Moreover, it follows from the fact $u_m \to u$ in $C([0,T];L^{\frac{n}{2}}_2(\mathbb{B}))$ that

$$u_{m0} = u_m(0) \to u(0) \text{ in } L^{\frac{n}{2}}_2(\mathbb{B}),$$

which combining with (4.2) implies that

$$u(0) = u_0 \text{ in } \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}).$$

Thus, $u \in C([0,T];\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ is a weak solution of problem (1.1)-(1.3) on $[0,T]\times\mathbb{B}$.

Step 2. Uniqueness. Suppose that there are two solutions u and v to problem (1.1)-(1.3) corresponding to the initial data $u_0 \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. Then, $w = v - v$ solves the following problem

$$w_t - \Delta_{\mathbb{B}}w_t - \Delta_{\mathbb{B}}w = |u|^{p-1}u - |v|^{p-1}v, \qquad (x_b,\widetilde{x}) \in \mathrm{int}\mathbb{B}, t>0, \tag{4.16}$$

$$w(x_b,\widetilde{x},0) = 0, \qquad (x_b,\widetilde{x}) \in \mathrm{int}\mathbb{B}, \tag{4.17}$$

$$w(0,\widetilde{x},t)=0, \qquad (0,\widetilde{x})\in\partial\mathbb{B}, t\geq 0. \tag{4.18}$$

Since $w\in C([0,T];\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}))$, one can multiply both side of equation (4.16) by $w(t)$ and show that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}+\|\nabla_{\mathbb{B}}w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\right)+2\|\nabla_{\mathbb{B}}w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\\ &=2\int_{\mathbb{B}}\left(|u|^{p-1}u-|v|^{p-1}v\right)w\frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}, \quad t\in[0,T],\end{aligned} \tag{4.19}$$

For the last term in equation (4.19), by using the cone Hölder inequality (Proposition 1), cone Sobolev inequality (Proposition 3) and estimate (4.15), we deduce

$$\begin{aligned}&2\int_{\mathbb{B}}\left(|u|^{p-1}u-|v|^{p-1}v\right)w\frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}\\ &=2p\int_{\mathbb{B}}\int_0^1|\theta u+(1-\theta)v|^{p-1}|w|^2\mathrm{d}\theta\frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}\\ &\leq 2p\int_{\mathbb{B}}\left(|u|^{p-1}+|v|^{p-1}\right)|w|^2\frac{\mathrm{d}x_b}{x_b}\mathrm{d}\widetilde{x}\\ &\leq 2p\left(\|u\|^{p-1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}+\|v\|^{p-1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}\right)\|w\|^2_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}\\ &\leq 2pC_*^{p+1}\left(\|\nabla_{\mathbb{B}}u\|^{p-1}_{L_2^{\frac{n}{2}}(\mathbb{B})}+\|\nabla_{\mathbb{B}}v\|^{p-1}_{L_2^{\frac{n}{2}}(\mathbb{B})}\right)\|\nabla_{\mathbb{B}}w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\\ &\leq C_1(R)\|\nabla_{\mathbb{B}}w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}, \quad t\in[0,T],\end{aligned}$$

with $C_1(R)=4pC_*^{p+1}(C(R))^{\frac{p-1}{2}}$, which together with (4.19) gives,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}+\|\nabla_{\mathbb{B}}w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\right)\leq C_1(R)\left(\|w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}+\|\nabla_{\mathbb{B}}w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\right), \quad t\in[0,T]. \tag{4.20}$$

Applying the Gronwall inequality to (4.20) and making use of (4.17), we have

$$\|w(t)\|^2_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}\leq e^{C_1(R)t}\|w(0)\|^2_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}=0, \quad \forall t\in[0,T],$$

which leads to the uniqueness of weak solution.

Concerning formula (4.11) we observe that the local existence time T merely depends on the norms of the initial data. Therefore, by using the similar idea as shown in [4], the solution can be continued as long as $\|u\|^2_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}$ remains bounded. In details, taking $u(\cdot,T)$ as the initial data and repeating above argument, we know that the problem (1.1)-(1.3) has a unique weak solution on the interval $[0,T_1]$ $(T_1>T)$. After an iteration process, we get a single increasing sequence $\{T_k\}_{k=1}^{\infty}$ such that the problem (1.1)-(1.3) has a unique weak solution on $[0,T_k]$, where T_k has two possibilities for sequence $\{T_k\}_{k=1}^{\infty}$, that either $T_{\max}=\lim\limits_{k\to\infty}T_k$ is finite, or $T_{\max}=\infty$. Moreover, if $T_{\max}=\infty$, then the problem (1.1)-(1.3) possesses a unique global solution. If $T_{\max}<+\infty$, then

$$\limsup_{t\to T_{\max}}\|\nabla_{\mathbb{B}}u(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}=+\infty. \tag{4.21}$$

In fact, if

$$\sup_{t\in[0,T_{\max})}\|\nabla_{\mathbb{B}}u(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\leq M(T_{\max}),$$

then for any $t_0 \in (0, T_{\max})$, by taking $u(\cdot, t_0)$ as the initial data, we know from the above argument that there exists a $T^* > 0$ dependent on M and independent on t_0 such that the problem (1.1)-(1.3) has a unique weak solution on the interval $[t_0, t_0+T^*]$. Due to the arbitrariness of $t_0 \in (0, T_{\max})$, the weak solution of problem (1.1)-(1.3) can be extended to the interval $[0, T_{\max}+\varepsilon]$ with arbitrary positive $\varepsilon > 0$, which contradicts the maximal interval $[0, T_{\max})$. This contradiction shows that (4.21) holds. □

Moreover, we give the following corollary.

Corollary 1 (Blow-up of the weak solution)**.** *Suppose that u is a weak solution of problem (1.1)-(1.3), which can be ensured by Theorem 4.1, if $T_{\max} < \infty$, then we have*

$$\lim_{t\to T_{\max}} \|u(t)\|^2_{L_s^{\frac{n}{s}}(\mathbb{B})} = \infty \quad \textit{for} \quad s \geq \max\left\{1, \frac{n(p-1)}{2}\right\}.$$

Proof. Recalling the definition of energy functional $J(u)$ in (3.1) and the energy relation (3.3), we get

$$\frac{1}{2}\|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \leq \frac{1}{p+1}\|u\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})} + J(u_0) \text{ for all } t \in [0, T_{\max}).$$

Combining with the following Gagliardo-Nirenberg interpolation inequality

$$\|u\|_{L_{B_1}^{\frac{n}{B_1}}(\mathbb{B})} \leq C\|\nabla_{\mathbb{B}} u\|^a_{L_{B_2}^{\frac{n}{B_2}}(\mathbb{B})}\|u\|^{1-a}_{L_{B_3}^{\frac{n}{B_3}}(\mathbb{B})}$$

for $\frac{1}{B_1} = \left(\frac{1}{B_2} - \frac{1}{n}\right)a + \frac{1-a}{B_3}$ and $0 < a < 1$, we have

$$\frac{1}{2}\|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - J(u_0) \leq \frac{1}{p+1}\|u\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})} \leq C\|\nabla_{\mathbb{B}} u\|^{(p+1)a}_{L_2^{\frac{n}{2}}(\mathbb{B})}\|u\|^{(p+1)(1-a)}_{L_s^{\frac{n}{s}}(\mathbb{B})},$$

where $B_1 = p+1$, $B_2 = 2$ and $B_3 = s$ with $\frac{n(p-1)}{2} < s < p+1$ such that $0 < a = \frac{2n(p-1-s)}{(p+1)(sn+2s-ns)} < \frac{2}{p+1}$. Thus we have

$$\frac{1}{2}\|\nabla_{\mathbb{B}} u\|^{2-(p+1)a}_{L_2^{\frac{n}{2}}(\mathbb{B})} - J(u_0)\|\nabla_{\mathbb{B}} u\|^{-(p+1)a}_{L_2^{\frac{n}{2}}(\mathbb{B})} \leq C\|u\|^{(p+1)(1-a)}_{L_s^{\frac{n}{s}}(\mathbb{B})},$$

which implies that $\lim_{t\to T_{\max}} \|u(t)\|^2_{L_s^{\frac{n}{s}}(\mathbb{B})} = \infty$. □

5. **Sub-critical and critical initial energy cases.** In this section, we study the well-posedness of solutions of problem (1.1)-(1.3) in the case of sub-critical and critical initial energy levels. The succeeding result is given to show the invariant sets of the solution for problem (1.1)-(1.3).

Lemma 5.1. *Suppose that $u_0 \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ and $J(u_0) < d$, then*

(i) all weak solutions of problem (1.1)-(1.3) belong to $\mathcal{W}$ provided that $u_0 \in \mathcal{W}$;
(ii) all weak solutions of problem (1.1)-(1.3) belong to $\mathcal{V}$ provided that $u_0 \in \mathcal{V}$.

Proof. (i) Suppose that $J(u_0) < d$, $I(u_0) > 0$ and $u(t)$ is the corresponding solution of problem (1.1)-(1.3). If $u_0 = 0$, then $u(t) = 0$, i.e., $u(t) \in \mathcal{W}$. If $u_0 \in \mathcal{N}_+$, i.e., $I(u_0) > 0$, we claim that $u(t) \in \mathcal{N}_+$ for $0 \leq t < T$. By reduction to absurdity, provided that there exists a $t_0 \in (0, T)$ such that $u(t_0) \in \mathcal{N}$ for the first time and

$u(t) \in \mathcal{N}_+$ for $0 \le t < t_0$, i.e., $I(u(t)) > 0$ for $t \in [0, t_0)$ and $I(u_0) = 0$. Since $u(t_0)$ is a solution of problem (1.1)-(1.3), by the conservation of energy, it follows that

$$\int_0^{t_0} \|u_t(\tau)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau + J(u(t_0)) = J(u_0) < d \tag{5.1}$$

for any $t_0 \in (0, T)$. However, by the definition of d and $I(u(t_0)) = 0$, it implies that $J(u(t_0)) \ge d$, which contradicts (5.1). Thus, $u(t) \in \mathcal{N}_+ \subset \mathcal{W}$ for any $t \in [0, T)$.

(ii) Similar to the proof of (i), we can obtain that $u(t) \in \mathcal{V}$ for $t \in [0, T)$ provided $u_0 \in \mathcal{V}$. □

Theorem 5.2 (Global existence and asymptotic behavior)**.** *Suppose that $u_0 \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ and $J(u_0) \le d$. If $u_0 \in \mathcal{W}$, then problem (1.1)-(1.3) admits a unique global weak solution $u \in L^\infty(0,\infty; \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ with $u_t \in L^2(0,\infty; \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ and satisfies*

$$\|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + \int_0^t \|u_t(\tau)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau < \left(1 + \frac{2(p+1)(c^2+1)}{p-1}\right) d, \quad t \ge 0. \tag{5.2}$$

Moreover, the solution satisfies the estimate

$$\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \le e^{-2\beta t} \|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}, \quad 0 \le t < \infty,$$

where $\beta = 1 - C_*^{p+1} \left(\frac{2(p+1)(c^2+1)}{p-1} J(u_0)\right)^{\frac{p-1}{2}}$.

Proof. We divide the proof into two parts, which are the global existence and the asymptotic behavior.

Part I: Global existence.

First, we give the global existence of the solution of problem (1.1)-(1.3) for $J(u_0) < d$.

Since the conclusion is trivial when $u_0 = 0$, we only consider the case $u_0 \in \mathcal{W}\backslash\{0\}$. From Theorem 4.1, let u be the weak solution of problem (1.1)-(1.3) corresponding to the initial data u_0. It follows from (3.1) and (3.2) that

$$J(u(t)) = \frac{p-1}{2(p+1)} \|\nabla_{\mathbb{B}} u(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{1}{p+1} I(u(t)) \tag{5.3}$$

and

$$\int_0^t \|u_t(\tau)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau + J(u(t)) = J(u_0) < d. \tag{5.4}$$

for all $t \in [0, T_{\max})$. Then by Lemma 5.1 we deduce that

$$u(t) \in \mathcal{W}, \text{ for all } 0 < t < T_{\max},$$

which implies

$$I(u(t)) > 0 \quad \text{for all} \quad 0 < t < T_{\max}. \tag{5.5}$$

Thus, the combination of (5.3)-(5.5) and Proposition 1 shows that

$$\|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + \int_0^t \|u_t(\tau)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau < \left(1 + \frac{2(p+1)(c^2+1)}{p-1}\right) d, \quad 0 < t < T_{\max}. \tag{5.6}$$

Therefore, by virtue of the Continuation Principle, it follows $T_{\max} = \infty$, i.e., $u \in L^\infty(0,\infty; \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ with $u_t \in L^2(0,\infty; \mathcal{H}^{\frac{n}{2}}_{2}(\mathbb{B}))$ is a global weak solution of problem (1.1)-(1.3) corresponding to initial data u_0 and satisfies the estimate (5.2).

For the case $J(u_0) = d$, we apply the idea of scalar transformation to get the corresponding global existence result.

Let $\gamma_k = 1 - \frac{1}{k}$, $k \in \mathbb{N}^+$. Then there exists a sequence $\{u_k(0)\}$ such that $u_k(0) = \gamma_k u_0$. Consider the corresponding initial boundary value problems

$$u_t - \Delta_{\mathbb{B}} u_t - \Delta_{\mathbb{B}} u = |u|^{p-1}u, \qquad (x_b, \widetilde{x}) \in \text{int}\mathbb{B}, t > 0, \tag{5.7}$$

$$u(x_b, \widetilde{x}, 0) = u_k(0), \qquad (x_b, \widetilde{x}) \in \text{int}\mathbb{B}, \tag{5.8}$$

$$u(0, \widetilde{x}, t) = 0, \qquad (0, \widetilde{x}) \in \partial\mathbb{B}, t \geq 0, \tag{5.9}$$

according to Theorem 4.1 and the estimate (5.6), we know that the weak solution u to problem (5.7)-(5.9) corresponding to the initial data $u_k(0)$ satisfies

$$\int_0^t \|u_t(\tau)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau + J(u(t)) = J(u_k(0)), 0 < t < T_{\max}, \tag{5.10}$$

where $T_{\max}$ is the maximal existence time of the solution u.

By $I(u_0) > 0$, it follows from (iii)-(iv) in Lemma 3.2 that $\lambda = \lambda(u_0) \in (0, 1)$ and $\lambda^* > 1$, which implies that $I(u_0) > I(\gamma_k u_0) > 0$ and $J(\gamma_k u_0) < J(u_0) \leq d$. By the continuity of $I(u)$, $J(u)$ with respect to u, we can choose a sufficiently large k such that $\|u_k(0) - \gamma_k u_0\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} < \frac{1}{k}$ with $I(u_k(0)) > 0$ and $J(u_k(0)) < d$, which means $u_k(t) \in \mathcal{W}$ for $[0, T_k)$ by Lemma 5.1.

Since $\lim_{k\to\infty} \gamma_k = 1$, then for $u_k(0) \in \mathcal{W}$ we have

$$u_k(0) \to u_0 \text{ strongly in } \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}), \text{ as } k \to \infty, \tag{5.11}$$

and

$$\int_0^t \|u_{kt}(\tau)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau + J(u_k(t)) = J(u_k(0)) < J(u_0) = d, \quad 0 < t < \tilde{T}, \tag{5.12}$$

then we can obtain the boundness of u_k by the same way in the previous case $J(u_0) < d$, namely (5.6), and the global existence result follows.

Part II: Asymptotic behavior.

In this part, we start with the claim that $u(t) \in \mathcal{W}$ for $t \in [0, \infty)$ when $J(u) \leq d$.

If $J(u_0) < d$, then by the assumption that $I(u_0) > 0$ we can derive immediately that $u(t) \in \mathcal{W}$ for $t > 0$ from Lemma 5.1.

If $J(u_0) = d$, then we conclude that there exists an enough small t_0 such that $I(u(t)) > 0$ for $t \in [0, t_0)$ through the continuity of $I(u(t))$ with respect to t and $I(u_0) > 0$. Next, we assert that $I(u(t)) > 0$ for $t \in [t_0, \infty)$. Arguing by contradiction, assuming that $t_1 \in [t_0, \infty)$ is the first time such that $I(u(t_1)) = 0$, then by the definition of the potential well depth, we have

$$J(u(t_1)) \geq d.$$

In addition, by the energy identity (3.3), we have

$$0 < J(u(t_1)) = J(u_0) - \int_0^{t_1} \|u_t(\tau)\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau \leq d,$$

which tells us that $J(u(t_1)) = d$ and $u_t \equiv 0$ for $0 < t < t_1$. On the other hand, as u is a global weak solution of problem (1.1)-(1.3), we directly obtain the following equation by multiplying (1.1) by u and integrating over $\mathbb{B}$

$$(u_t, u)_{\mathbb{B}} + (\nabla_{\mathbb{B}} u_t, \nabla_{\mathbb{B}} u)_{\mathbb{B}} + (\nabla_{\mathbb{B}} u, \nabla_{\mathbb{B}} u)_{\mathbb{B}} = \int_{\mathbb{B}} |u|^{p+1} \frac{\mathrm{d}x_b}{x_b} \mathrm{d}\widetilde{x}, t \geq 0.$$

Then from the definition of $I(u)$ and $I(u(t)) > 0$ for $t \in [0, t_1)$, it follows that

$$(u_t, u)_{\mathbb{B}} + (\nabla_{\mathbb{B}} u_t, \nabla_{\mathbb{B}} u)_{\mathbb{B}} = -I(u) < 0, t \in [0, t_1), \tag{5.13}$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} = -2I(u) < 0, t \in [0, t_1),$$

which contradicts with $u_t \equiv 0$ for $0 < t < t_1$. Hence, we conclude $I(u(t)) > 0$ for $t \geq 0$.

Since $u(t) \in \mathcal{W}$ for $t \geq 0$ of the case $J(u_0) \leq d$, namely $I(u(t)) > 0$ for $0 \leq t < \infty$, and it is easy to derive that the energy functional $J(u(t))$ is non-increasing from the energy identity (3.3), then we have

$$\begin{aligned} J(u_0) &\geq J(u(t)) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{1}{p+1} I(u) \\ &> \frac{p-1}{2(p+1)(c^2+1)} \|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}, \end{aligned} \tag{5.14}$$

where c is the optimal constant in Proposition 2. Moreover,

$$\begin{aligned} \int_{\mathbb{B}} |u|^{p+1} \frac{\mathrm{d}x_b}{x_b} \mathrm{d}\tilde{x} &\leq C_*^{p+1} \|u\|^{p+1}_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \\ &= C_*^{p+1} \left(\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)^{\frac{p-1}{2}} \|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}. \end{aligned} \tag{5.15}$$

Then from (5.14) we define

$$\begin{aligned} \alpha :=& C_*^{p+1} \left(\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)^{\frac{p-1}{2}} \\ <& C_*^{p+1} \left(\frac{2(p+1))(c^2+1)}{p-1} J(u_0)\right)^{\frac{p-1}{2}} \\ \leq& C_*^{p+1} \left(\frac{2(p+1))(c^2+1)}{p-1} d\right)^{\frac{p-1}{2}} = \frac{1}{c^2+1} < 1. \end{aligned}$$

Hence, taking $\beta := 1 - \alpha > 0$, we obtain from (5.15) that

$$\int_{\mathbb{B}} |u|^{p+1} \frac{\mathrm{d}x_b}{x_b} \mathrm{d}\tilde{x} \leq \alpha \|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} = (1-\beta) \|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})},$$

which gives

$$\beta \|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \leq I(u(t)). \tag{5.16}$$

On the other hand, by the definition of $I(u)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{\mathbb{B}} (|u(t)|^2 + |\nabla_{\mathbb{B}} u(t)|^2) \frac{\mathrm{d}x_b}{x_b} \mathrm{d}\tilde{x}\right) = -2I(u(t)) \leq -2\beta \|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}. \tag{5.17}$$

Then by Gronwall's inequality we can obtain

$$\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \leq e^{-2\beta t} \|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}.$$

□

Theorem 5.3 (Finite time blow up and lower bound estimate of blowup time)**.** *Suppose that* $u_0 \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ *and* $J(u_0) \le d$*. If* $u_0 \in \mathcal{V}$*, then* $u(t)$ *blows up in finite time, i.e., there exists a* $T > 0$ *such that*

$$\lim_{t\to T^-} \int_0^t \|u(\tau)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau = +\infty, \quad 0 < t < T.$$

Moreover, T *is bounded below, which can be estimated by*

$$T \ge \frac{\|u_0\|^{-p+1}_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - r^{-p+1}}{(p-1)C_*^{p+1}}.$$

Proof. Firstly, by Theorem 4.1, we already have the local existence for $u_0 \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, then arguing by contradiction, we suppose that the solution u exists globally in time and here we just consider nontrivial u since the trivial ones do not agree the initial condition, thus are not solutions of problem (1.1)-(1.3). Then we define an auxiliary function as

$$M(t) := \int_0^t \|u(\tau)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau, t \in [0,\infty). \tag{5.18}$$

For $t \in [0,\infty)$ we can compute its derivative as follows

$$M'(t) = \|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} = \|u(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|\nabla_{\mathbb{B}} u(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \tag{5.19}$$

and further by (1.1) and (3.2) we have

$$M''(t) = 2(u_t(t), u(t))_{\mathbb{B}} + 2(\nabla_{\mathbb{B}} u_t(t), \nabla_{\mathbb{B}} u(t))_{\mathbb{B}} = -2I(u(t)). \tag{5.20}$$

Next, we will reveal that the solution actually does not exist globally by showing that $M(t)$ tends to infinity in finite time. However, it is easier to demonstrate that $M^{-\gamma}(t)$ has a zero point, where the exponent $-\gamma$ is a negative constant. Therefore, we take the latter as the proof scheme. Before that, we claim that $M^{-\gamma}(t)$ is concave for sufficiently large t by constructing a differential inequality with $M(t)$. Recalling (3.3) and (5.3), for $t \in [0,\infty)$ we can obtain by (5.19) and (5.20) that

$$\begin{aligned} M''(t) &= -2(p+1)J(u) + (p-1)\|\nabla_{\mathbb{B}} u(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\ &\ge 2(p+1)\left(\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau - J(u_0)\right) + \frac{(p-1)M'(t)}{c^2+1}, \end{aligned} \tag{5.21}$$

where c is the optimal constant of the Cone Poincaré inequality (see Proposition 2). Notice that

$$\begin{aligned} \left(\int_0^t (u,u_t)_{\mathbb{B}} \mathrm{d}\tau\right)^2 &= \left(\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau\right)^2 \\ &= \frac{1}{4}\left(\|u\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - 2\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + \|u_0\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right) \\ &= \frac{1}{4}\left((M'(t))^2 - 2M'(t)\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + \|u_0\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right), \end{aligned}$$

hence

$$(M'(t))^2 = 4\left(\int_0^t (u,u_t)_{\mathbb{B}} \mathrm{d}\tau\right)^2 + 2\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} M'(t) - \|u_0\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}. \tag{5.22}$$

From (5.18), (5.21) and (5.22), making use of the Hölder inequality, we have for $t \in [0, \infty)$

$$
\begin{aligned}
& M(t)M''(t) - \frac{p+1}{2}(M'(t))^2 \\
\geq & \int_0^t \|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau \left(2(p+1) \left(\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau - J(u_0) \right) + \frac{(p-1)M'(t)}{c^2+1} \right) \\
& - \frac{p+1}{2} \left(4 \left(\int_0^t (u, u_t)_{\mathbb{B}} \mathrm{d}\tau \right)^2 + 2M'(t)\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - \|u_0\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \right) \\
= & 2(p+1) \left(\int_0^t \|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau \int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau - \left(\int_0^t (u, u_t)_{\mathbb{B}} \mathrm{d}\tau \right)^2 \right) \\
& - 2(p+1)J(u_0)M(t) + \frac{(p-1)M(t)M'(t)}{c^2+1} \\
& - (p+1)M'(t)\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + \frac{p+1}{2}\|u_0\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \\
\geq & - 2(p+1)J(u_0)M(t) + \frac{(p-1)M(t)M'(t)}{c^2+1} - (p+1)M'(t)\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}.
\end{aligned} \tag{5.23}
$$

To estimate the right hand side of (5.23), we claim first that $u(t) \in \mathcal{V}$ for $t \in [0, \infty)$, and the proof of this claim is similar with the proof in the second step of Theorem 5.2, for which we omit it here. Therefore, we have $I(u(t)) < 0$ for $t \geq 0$, then by the density of the real number we can derive that there exists a $\delta > 0$ such that

$$M''(t) = -2I(u(t)) \geq \delta, t \geq 0, \tag{5.24}$$

Integrating over the both sides of (5.24) from 0 to t, we get

$$M'(t) \geq \delta t + M'(0) \geq \delta t, t \geq 0. \tag{5.25}$$

By applying the same operation that we took for (5.24), we have

$$M(t) \geq \delta t^2 + M(0) \geq \delta t^2, t \geq 0. \tag{5.26}$$

Thus by (5.23), for sufficiently large $t > 0$ we have

$$
\begin{aligned}
& M(t)M''(t) - \frac{p+1}{2}(M'(t))^2 \qquad (5.27) \\
\geq & - 2(p+1)J(u_0)M(t) + \frac{(p-1)M(t)M'(t)}{c^2+1} - (p+1)M'(t)\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \\
= & M(t) \left(\frac{p-1}{2(c^2+1)} M'(t) - 2(p+1)J(u_0) \right) \\
& + M'(t) \left(\frac{p-1}{2(c^2+1)} M(t) - (p+1)\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \right) > 0.
\end{aligned}
$$

By direct computation we can get that

$$(M^{-\gamma}(t))'' = -\gamma M^{-\gamma-2}(t) \left(M(t)M''(t) - (\gamma+1)(M'(t))^2 \right). \tag{5.28}$$

Let $\gamma = \frac{p-1}{2} > 0$ and $N(t) := M^{-\gamma}(t)$, then (5.28) implies that $N''(t) < 0$ for sufficiently large $t > 0$ according to (5.27) and the facts that $\gamma > 0$ and $M(t) \geq 0$, which implies that $N(t)$ is concave for sufficiently large $t > 0$.

Note first that there exists a sufficiently small $\tilde{t} > 0$ such that the non-trivial u exists locally for $t \in (0, \tilde{t}]$ by Theorem 4.1, which claims that $N(t) = M^{-\gamma}(t) > 0$ for $t \in (0, \tilde{t}]$. Then we assert that $N(t)$ is decreasing for $t \in (0, \infty)$. Recalling that $M'(t) > 0$ for $t \in (0, \infty)$, hence

$$N'(t) = -\gamma M^{-\gamma-1} M'(t) < 0, t \in (0, \infty),$$

which makes $N(t)$ keep falling for $t > 0$ and guarantees that the zero point can be reached.

In fact, the case of the asymptote won't occur since $N(t)$ is concave for sufficiently large $t > 0$, which forces the curve of $N(t)$ to hit the t-axis for sufficiently large $t > 0$ and ensures the existence of the zero point. Hence there exists a $0 < T < \infty$ such that

$$N(t) = M(t)^{-\gamma} \to 0, t \to T^-,$$

which means also $M(t) = \int_0^t \|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau \to \infty$ as $t \to T^-$, and contradicts the hypothesis of global existence, thus u blows up in finite time.

Next, we seek the lower bound of the blow up time. By (5.20) we have

$$M''(t) = -2I(u(t)) = -2\|\nabla_{\mathbb{B}} u(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + 2\|u(t)\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}. \tag{5.29}$$

Then Proposition 2, (5.19) and (5.29) imply

$$M''(t) \le 2C_*^{p+1} (M'(t))^{\frac{p+1}{2}}. \tag{5.30}$$

By (5.25) we have already known $M'(t) > 0$ for $t \in [0, T)$, thus we can divide (5.30) by $(M'(t))^{\frac{p+1}{2}}$ and get the following inequality,

$$\frac{M''(t)}{(M'(t))^{\frac{p+1}{2}}} \le 2C_*^{p+1}. \tag{5.31}$$

Integrating the inequality (5.31) from 0 to t, we have

$$(M'(0))^{-\frac{p-1}{2}} - (M'(t))^{-\frac{p-1}{2}} \le (p-1)C_*^{p+1} t. \tag{5.32}$$

Let $t \to T$ in (5.32), since $M'(t) = \|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} > r^2$ by (ii) of Lemma 3.3, we can conclude that

$$T \ge \frac{\|u_0\|^{-p+1}_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - r^{-p+1}}{(p-1)C_*^{p+1}}.$$

□

6. **Sup-critical initial energy case.** In this section we give the finite time blow up result for the sup-critical initial energy case, for which we introduce the following three lemmas first.

Lemma 6.1 ([26]). *Suppose that a positive, twice-differentiable function $\psi(t)$ satisfies*

$$\psi''(t)\psi(t) - (1+\theta)(\psi'(t))^2 \ge 0,\ t > 0, \psi(t) \in C^2, \psi(t) > 0$$

where $\theta > 0$ is a constant. If $\psi(0) > 0$ and $\psi'(0) > 0$, then there exists a t_1 with $0 < t_1 \le \frac{\psi(0)}{\theta\psi'(0)}$ such that $\psi(t)$ tends to infinity as $t \to t_1$.

Lemma 6.2. *Suppose that $J(u_0) > 0$ and $u_0 \in \mathcal{V}$, then the map*

$$t \mapsto \|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}$$

increases strictly while $u(t) \in \mathcal{V}$ for $t \in [0, T_0]$, where T_0 is a positive constant.

Proof. Firstly, an auxiliary function is defined as follows

$$F(t) := \|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}. \tag{6.1}$$

Then it follows from (1.1) that

$$F'(t) = 2(u_t(t), u(t))_{\mathbb{B}} + 2(\nabla_{\mathbb{B}} u_t(t), \nabla_{\mathbb{B}} u(t))_{\mathbb{B}} = -2I(u). \tag{6.2}$$

Hence by $u(t) \in \mathcal{V}$ for $t \in [0, T_0]$ we get

$$F'(t) > 0, t \in [0, T_0], \tag{6.3}$$

which implies that the map

$$t \mapsto \|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}$$

is strictly increasing for $t \in [0, T_0]$. □

Lemma 6.3 (Invariant set $\mathcal{V}$)**.** *Assume that u_0 satisfies*

$$\frac{(p-1)}{2(c^2+1)(p+1)}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} > J(u_0) > 0, \tag{6.4}$$

where c is the optimal constant in Proposition 2, then $u(t) \in \mathcal{V}$ for $t \in [0, T)$, where $T \leq +\infty$ is the maximal existence time of the solution.

Proof. First we claim that $u_0 \in \mathcal{V}$ by (6.4), i.e. $u(t) \in \mathcal{V}$ for $t = 0$. By the definition of $J(u(t))$ (3.1) we have

$$\begin{aligned} J(u_0) =& \frac{1}{2}\|\nabla_{\mathbb{B}} u_0\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{p+1}\|u_0\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})} \\ =& \left(\frac{1}{2} - \frac{1}{p+1}\right)\|\nabla_{\mathbb{B}} u_0\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{1}{p+1}I(u_0) \\ \geq& \frac{(p-1)}{2(c^2+1)(p+1)}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + \frac{1}{p+1}I(u_0), \end{aligned} \tag{6.5}$$

thus by (6.4) and (6.5) we can drive that $I(u_0) < 0$, namely $u_0 \in \mathcal{V}$.

Then we prove that $u(t) \in \mathcal{V}$ for $t \in (0, T)$. Since $I(u_0) < 0$, we have $I(u(t)) < 0$ on $[0, t_0)$ for sufficiently small $t_0 > 0$ according to the continuity of $I(u(t))$ with respect to t. Arguing by contradiction, if $I(u(t))$ doesn't remain negative on $[0, T)$, then there must exist a $t_1 \in [t_0, \infty)$ such that $I(u(t_1)) = 0$ for the first time. It follows from Lemma 6.2 that the map $t \mapsto \|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}$ increases strictly for $t \in [0, t_1)$, by which and (6.4) we get

$$\|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} > \|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} > \frac{2(c^2+1)(p+1)}{(p-1)}J(u_0),\ t \in (0, t_1), \tag{6.6}$$

then from the continuity of $\|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}$ with respect to t, we can obtain

$$\|u(t_1)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} > \|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} > \frac{2(c^2+1)(p+1)}{p-1}J(u_0). \tag{6.7}$$

On the other hand, recalling the definition of $J(u(t))$ and (3.3), we have

$$\begin{aligned} J(u_0) =&J(u(t_1)) + \int_0^{t_1} \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau \\ \geq&\frac{1}{2}\|\nabla_{\mathbb{B}} u(t_1)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{p+1}\|u(t_1)\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})} \\ =&\left(\frac{1}{2} - \frac{1}{p+1}\right)\|\nabla_{\mathbb{B}} u(t_1)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{1}{p+1} I(u(t_1)), \end{aligned} \tag{6.8}$$

then applying the Cone Poincaré inequality (Proposition 2) to (6.8) with $I(u(t_1)) = 0$, we have

$$\begin{aligned} J(u_0) &\geq \frac{p-1}{2(p+1)}\|\nabla_{\mathbb{B}} u(t_1)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\ &\geq \frac{(p-1)}{2(c^2+1)(p+1)}\|u(t_1)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}, \\ &\geq \frac{(p-1)}{2(c^2+1)(p+1)}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}, \end{aligned} \tag{6.9}$$

which contradicts (6.7), thus we can drive that $u(t) \in \mathcal{V}$ for $t \in [0, T)$. $\square$

Next we give the blow up results for arbitrary positive initial energy as follows:

Theorem 6.4 (Finite time Blow up with $J(u_0) > 0$)**.** *Let $u_0 \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{B})$. If u_0 satisfies (6.4), then $u(t)$ blows up in finite time. Furthermore, the upper bound of blowup time can be estimated by*

$$T \leq \frac{4\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}{(p-1)\sigma},$$

where $\sigma := \frac{p-1}{2(c^2+1)(p+1)}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - J(u_0)$.

Proof. Similar as Theorem 5.2, we also divide the proof into two parts, which are the finite time blow up and the estimating of the upper bound of the blow up time.

Part I: Finite time blow up.

Firstly, Theorem 4.1 has asserted the local existence of the solution, then arguing by contradiction, we suppose that $u(t)$ exists globally in time, i.e. the maximal existence time $T = \infty$. Now we take a sufficiently small $\varepsilon > 0$ and a positive constant $c_0 > 0$ such that

$$c_0 > \frac{1}{4}\varepsilon^{-2}\|u_0\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \tag{6.10}$$

and define a new auxiliary function for $t \in [0, \infty)$ as follows

$$P(t) := (M(t))^2 + \varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} M(t) + c_0, \tag{6.11}$$

where $M(t)$ is the auxiliary function defined before in (5.18). Hence for $t \in [0, \infty)$,

$$P'(t) = \left(2M(t) + \varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right) M'(t) \tag{6.12}$$

and

$$P''(t) = \left(2M(t) + \varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right) M''(t) + 2(M'(t))^2.$$

Set $\xi := 4c_0 - \varepsilon^{-2}\|u_0\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}$, then (6.10) indicates $\xi > 0$. Thus for $t \in [0,\infty)$ we have

$$\begin{aligned}(P'(t))^2 &= \left(4(M(t))^2 + 4\varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}M(t) + \varepsilon^{-2}\|u_0\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)(M'(t))^2\\ &= \left(4(M(t))^2 + 4\varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}M(t) + 4c_0 - \xi\right)(M'(t))^2\\ &= (4P(t) - \xi)(M'(t))^2,\end{aligned}$$

which tells

$$4P(t)(M'(t))^2 = (P'(t))^2 + \xi(M'(t))^2, t \in [0,\infty). \tag{6.13}$$

By (6.13), for $t \in [0,\infty)$ we get

$$\begin{aligned}&2P(t)P''(t)\\ =&2\left(\left(2M(t) + \varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)M''(t) + 2(M'(t))^2\right)P(t)\\ =&2\left(2M(t) + \varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)M''(t)P(t) + 4P(t)(M'(t))^2\\ =&2\left(2M(t) + \varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)M''(t)P(t) + (P'(t))^2 + \xi(M'(t))^2.\end{aligned} \tag{6.14}$$

Then from (6.13) and (6.14), for $t \in [0,\infty)$ we can derive that

$$\begin{aligned}&2P(t)P''(t) - (1+\beta)(P'(t))^2\\ =&2\left(2M(t) + \varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)M''(t)P(t) + \xi(M'(t))^2 - \beta(P'(t))^2\\ =&2\left(2M(t) + \varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)M''(t)P(t) + \xi(M'(t))^2 - \beta(4P(t) - \xi)(M'(t))^2\\ =&2\left(2M(t) + \varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)M''(t)P(t) - 4\beta P(t)(M'(t))^2 + \xi(1+\beta)(M'(t))^2\\ >&2\left(2M(t) + \varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)M''(t)P(t) - 4\beta P(t)(M'(t))^2,\end{aligned} \tag{6.15}$$

where $\beta > 0$ is a positive constant that will be determined in the sequel.

Next, we estimate the term $M''(t)$. Testing the both sides of (1.1) by u, for $t \in [0,\infty)$ we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} &= -I(u)\\ &= -\|\nabla_{\mathbb{B}}u\|^2_{L^{\frac{n}{2}}_2(\mathbb{B})} + \|u\|^{p+1}_{L^{\frac{n}{p+1}}_{p+1}(\mathbb{B})}\\ &= -2J(u) + \frac{p-1}{p+1}\|u\|^{p+1}_{L^{\frac{n}{p+1}}_{p+1}(\mathbb{B})}.\end{aligned} \tag{6.16}$$

By (6.4), we can take β such that

$$1 < \beta < \frac{(p-1)\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}{2(c^2+1)(p+1)J(u_0)}. \tag{6.17}$$

Notice by Lemma 6.3 that $I(u(t)) < 0$ for $t \in [0, +\infty)$, then combining (6.16) with (3.3), for $t \in [0, \infty)$ we see

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\\
=&-2J(u) + \frac{p-1}{p+1}\|u\|^{p+1}_{L^{\frac{n}{p+1}}_{p+1}(\mathbb{B})}\\
=&2(\beta-1)J(u) - 2\beta J(u) + \frac{p-1}{p+1}\|u\|^{p+1}_{L^{\frac{n}{p+1}}_{p+1}(\mathbb{B})} \qquad (6.18)\\
\geq&-2\beta J(u_0) + 2\beta\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau + \frac{p-1}{p+1}\|u\|^{p+1}_{L^{\frac{n}{p+1}}_{p+1}(\mathbb{B})}\\
=&-2\beta J(u_0) + 2\beta\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau - \frac{p-1}{p+1}I(u) + \frac{p-1}{p+1}\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\\
>&-2\beta J(u_0) + 2\beta\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau + \frac{p-1}{p+1}\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}.
\end{aligned}$$

The application of Cone Poincaré inequality asserts

$$\frac{p-1}{p+1}\|\nabla_{\mathbb{B}}u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \geq \frac{(p-1)}{(c^2+1)(p+1)}\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}. \qquad (6.19)$$

Putting (6.19) into (6.18), for $t \in [0, \infty)$ we have

$$\begin{aligned}
M''(t) =&\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \qquad (6.20)\\
&> -4\beta J(u_0) + 4\beta\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau + \frac{2(p-1)}{(c^2+1)(p+1)}\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}.
\end{aligned}$$

By (6.16), Hölder and Young's inequalities, for $t \in [0, \infty)$ we estimate the term $(M'(t))^2$ as follows

$$\begin{aligned}
(M'(t))^2 =&\|u\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\\
=&\left(\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + 2\int_0^t (u, u_t)_{\mathbb{B}}\mathrm{d}\tau\right)^2 \qquad (6.21)\\
\leq&\left(\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + 2\left(\int_0^t \|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau\right)^{\frac{1}{2}}\right)^2\\
=&\|u_0\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + 4\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\left(\int_0^t \|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau\right)^{\frac{1}{2}}\\
&+ 4M(t)\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau\\
\leq&\|u_0\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + 2\varepsilon\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}M(t) + 2\varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau\\
&+ 4M(t)\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau.
\end{aligned}$$

Then from (6.20), (6.21) and (6.15), for $t \in [0, \infty)$ we have

$$2P''(t)P(t) - (1+\beta)(P'(t))^2$$

$$
\begin{aligned}
&>2\left(2M(t)+\varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)M''(t)P(t)-4\beta P(t)(M'(t))^2\\
&>I_1I_2-I_3I_4,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1:=&2P(t)\left(2M(t)+\varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right),\\
I_2:=&-4\beta J(u_0)+4\beta\int_0^t\|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau+\frac{2(p-1)}{(c^2+1)(p+1)}\|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})},\\
I_3:=&4\beta P(t),\\
I_4:=&\|u_0\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}+2\varepsilon\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}M(t)+2\varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\int_0^t\|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau\\
&+4M(t)\int_0^t\|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau.
\end{aligned}
$$

Taking $\gamma:=\frac{2(p-1)}{(c^2+1)(p+1)}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}-4\beta J(u_0)$, then (6.17) ensures $\gamma>0$. Choosing ε such that

$$
\varepsilon<\frac{\gamma}{2\beta\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}},
$$

then recalling that $\|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}$ is decreasing with respect to t by Lemma 6.2, for $t\in[0,\infty)$ we obtain

$$
\begin{aligned}
&2P''(t)P(t)-(1+\beta)(P'(t))^2\\
>&I_1I_2-I_3I_4\\
=&I_1\left(4\beta\int_0^t\|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau+\gamma\right)-I_3I_4\\
>&I_1\left(4\beta\int_0^t\|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau+2\beta\varepsilon\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)-I_3I_4\\
=&4\beta P(t)\left(2M(t)+\varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)\left(2\int_0^t\|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau+\varepsilon\|u_0\|^2\right)-I_3I_4\\
=&I_3\left(\left(2M(t)+\varepsilon^{-1}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)\left(2\int_0^t\|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau+\varepsilon\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)-I_4\right)\\
=&0.
\end{aligned}
$$

Thus

$$
P''(t)P(t)-\frac{1+\beta}{2}(P'(t))^2>0, t\in[0,\infty).
$$

Since $P(0)=c_0>\frac{1}{4}\varepsilon^{-2}\|u_0\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}>0$ and $P'(0)=\varepsilon^{-1}\|u_0\|^4_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}>0$, by Lemma 6.1 we can conclude that there exists a $0<T<\infty$ such that

$$
\lim_{t\to T}P(t)=+\infty.
$$

According to the definition of $P(t)$, i.e. (6.11), we can conclude that

$$
\lim_{t\to T}M(t)=+\infty,
$$

which claims the blow up of the solution.

Part II: Upper bound of the blow up time

In Part I we have draw the conclusion that the maximal existence time T is finite, next, we estimate the upper bound of blowup time. For $t \in [0, T)$, we define

$$\psi(t) := \int_0^t \|u(\tau)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau + (T-t)\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + \mu(t+\nu)^2, \tag{6.22}$$

where $\mu > 0$ and $\nu > 0$ are constants, which will be determined later in the process of argumentation. Thus we have

$$\psi'(t) = \|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - \|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + 2\mu(t+\nu), \tag{6.23}$$

then it follows from (3.3), Proposition 2 and (6.4) that

$$\begin{aligned}
\psi''(t) =& -2I(u(t)) + 2\mu \\
\geq& (p-1)\|\nabla_{\mathbb{B}} u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - 2(p+1)J(u) \\
\geq& \frac{p-1}{c^2+1}\|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - 2(p+1)J(u) \\
=& \frac{p-1}{c^2+1}\|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - 2(p+1)J(u_0) + 2(p+1)\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau \\
\geq& 2(p+1)\left(\frac{p-1}{2(c^2+1)(p+1)}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - J(u_0) + \int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau\right) > 0.
\end{aligned} \tag{6.24}$$

Furthermore, we obtain $\psi'(t) \geq \psi'(0) = 2\mu(t+\nu) > 0$ via (3.3), which implies $\psi(t) \geq \psi(0) = T\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + \mu\nu^2 > 0$ for all $t \in [0, T)$.

On the other hand, we can derive that

$$\begin{aligned}
-\frac{1}{4}(\psi'(t))^2 =& -\left(\frac{1}{2}\left(\|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - \|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right) + \mu(t+\nu)\right)^2 \\
=& \left(\int_0^t \|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau + \mu(t+\nu)^2\right)\left(\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau + \mu\right) \\
&- \left(\frac{1}{2}\left(\|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - \|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right) + \mu(t+\nu)\right)^2 \\
&- \left(\psi(t) - (T-t)\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)\left(\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau + \mu\right) \\
=& I_5 - I_6 - \left(\psi(t) - (T-t)\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)\left(\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau + \mu\right),
\end{aligned} \tag{6.25}$$

where

$$I_5 := \left(\int_0^t \|u\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau + \mu(t+\nu)^2\right)\left(\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \mathrm{d}\tau + \mu\right),$$

$$I_6 := \left(\frac{1}{2}\left(\|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - \|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right) + \mu(t+\nu)\right)^2.$$

To estimate (6.25) clearly, we will show that $I_5 - I_6 > 0$,

$$I_5 - I_6 = I_5 - \left(\frac{1}{2}(\|u(t)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - \|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}) + \mu(t+\nu)\right)^2$$

$$
\begin{aligned}
&=I_5-\left(\frac{1}{2}\int_0^t \frac{d}{d\tau}\|u(\tau)\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau+\mu(t+\nu)\right)^2\\
&=I_5-\left(\int_0^t (u,u_t)_{\mathbb{B}}\mathrm{d}\tau+\mu(t+\nu)\right)^2\\
&\geq I_5-\left(\int_0^t \|u(\tau)\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\|u_t(\tau)\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau+\mu(t+\nu)\right)^2,
\end{aligned}
$$

then by the Cauchy-Schwartz inequality we get

$$
\begin{aligned}
I_5-I_6\geq& I_5-\left(\int_0^t \|u(\tau)\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau\int_0^t \|u_t(\tau)\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau+\mu(t+\nu)\right)^2\\
=&I_5-(k_1(t)k_2(t)+\mu(t+\nu))^2\\
=&\left((k_1(t))^2+\mu(t+\nu)^2\right)\left((k_2(t))^2+\mu\right)-(k_1(t)k_2(t)+\mu(t+\nu))^2\\
=&(\sqrt{\mu}k_1(t))^2-2\sqrt{\mu}k_1(t)\sqrt{\mu}(t+\nu)k_2(t)+(\sqrt{\mu}(t+\nu)k_2(t))^2\\
=&(\sqrt{\mu}k_1(t)-\sqrt{\mu}(t+\nu)k_2(t))^2\\
\geq&0,
\end{aligned}
$$

where $k_1(t):=\int_0^t \|u(\tau)\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau$ and $k_2(t):=\int_0^t \|u_t(\tau)\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau$. Hence,

$$
\begin{aligned}
-(\psi'(t))^2\geq& -4\left(\psi(t)-(T-t)\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\right)\left(\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau+\mu\right)\\
\geq& -4\psi(t)\left(\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau+\mu\right).
\end{aligned}
\tag{6.26}
$$

Then by (6.22), (6.24) and (6.26), we achieve

$$
\begin{aligned}
&\psi(t)\psi''(t)-\frac{p+1}{2}(\psi'(t))^2\\
\geq&\psi(t)\left(\psi''(t)-2(p+1)\left(\int_0^t \|u_t\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\mathrm{d}\tau+\mu\right)\right)\\
\geq&2(p+1)\psi(t)\left(\frac{p-1}{2(c^2+1)(p+1)}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}-J(u_0)-\mu\right).
\end{aligned}
\tag{6.27}
$$

Let $\sigma:=\frac{p-1}{2(c^2+1)(p+1)}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}-J(u_0)$, then we can take a sufficiently small $\mu\in(0,\sigma]$ such that

$$
\frac{p-1}{2(c^2+1)(p+1)}\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}-J(u_0)-\mu\geq 0, \tag{6.28}
$$

hence for $t\in[0,\infty)$, from (6.27) and (6.28) we have

$$
\psi(t)\psi''(t)-\frac{p+1}{2}(\psi'(t))^2\geq 0.
$$

It is easy to verify that $\psi(0) = T\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} + \mu\nu^2 > 0$, $\psi'(0) \geq 2\mu\nu > 0$, which implies that the conditions of Lemma 6.1 are satisfied. Then applying Lemma 6.1, we can derive that

$$T \leq \frac{2\psi(0)}{(p-1)\psi'(0)} \leq \frac{\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}{(p-1)\mu\nu} T + \frac{\nu}{p-1}. \tag{6.29}$$

Let ν be large enough such that

$$\nu \in \left(\frac{\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}{(p-1)\mu}, +\infty \right), \tag{6.30}$$

then it follows from (6.29) that

$$T \leq \frac{\mu\nu^2}{(p-1)\mu\nu - \|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}. \tag{6.31}$$

Note that $\mu \in (0, \sigma]$ by (6.28). To estimate the upper bound of the blow up time, we can revise the range of μ according to (6.30) and define the following set to describe the pair (μ, ν)

$$\mathfrak{M} := \left\{ (\nu, \mu) \;\middle|\; \nu \in \left(\frac{\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}{(p-1)\sigma}, +\infty \right), \mu \in \left(\frac{\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}{(p-1)\nu}, \sigma \right] \right\}.$$

From the above discussions, we get

$$T \leq \inf_{(\mu,\nu)\in\mathfrak{M}} \frac{\mu\nu^2}{(p-1)\mu\nu - \|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}. \tag{6.32}$$

Next we define

$$f(\mu, \nu) := \frac{\mu\nu^2}{(p-1)\mu\nu - \|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}},$$

where $(\mu, \nu) \in \mathfrak{M}$. Thus the estimation of the blow-up time turns into seeking the minimal value of $f(\mu, \nu)$. First differentating $f(\mu, \nu)$ with respect to μ, we have

$$\frac{\partial}{\partial\mu} f(\mu, \nu) = \frac{-\nu^2\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}{\left((p-1)\mu\nu - \|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} \right)^2} < 0,$$

which means that $f(\mu, \nu)$ is decreasing with respect to μ, for which we have

$$\inf_{(\mu,\nu)\in\mathfrak{M}} f(\mu, \nu) = \inf_{\nu} f(\sigma, \nu),$$

where $\nu \in \left(\frac{\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}{(p-1)\sigma}, +\infty \right)$. Then we compute the partial derivative of $f(\sigma, \nu)$ with respect to ν and let it equal zero, we can obtain the minimum

$$\nu_{\min} = \frac{2\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}{(p-1)\sigma},$$

thus we have

$$\inf_{(\mu,\nu)\in\mathfrak{M}} f(\mu,\nu) = f(\sigma,\nu_{\min}) = \frac{4\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}{(p-1)\sigma},$$

which means also

$$T \le \frac{4\|u_0\|^2_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}}{(p-1)\sigma}.$$

□

Authors' contribution and acknowledgments. In the first version [37] finished in 2016, XP Liu obtained the threshold conditions of the global existence and nonexistence for solutions together with the asymptotic behavior at both subcritical and critical initial energy levels. But this version [37] did not consider the local existence nor the dynamic behavior at the sup-critical initial energy level. Then, the authors realized that the framework of a family of potential wells brought some problems by unnecessary technical complexities without receiving the benefits expected. After XP Liu left HEU in 2016, YT Wang and YX Chen continued to reconsider problem (1.1)-(1.3) and reconstructed the conclusions again in the framework of a single potential well to replace the family version, and finally completed this second version in 2020. This new version of the present paper not only gives the same results with those in the first version [37] by a single potential well, but also proves the local existence and the finite time blow up at sup-critical initial energy level with the estimation of the blowup time at all initial energy levels.

The authors are grateful to Professor Hua Chen in Wuhan University, who strongly encouraged this work when he visited Harbin Engineering University in 2017 and 2018.

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Received March 2021; revised May 2021; early access August 2021.

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