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REGULARITY OF WONG-ZAKAI APPROXIMATION FOR NON-AUTONOMOUS STOCHASTIC QUASI-LINEAR PARABOLIC EQUATION ON $\mathbb{R}^N$

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Abstract. In this paper, we investigate a non-autonomous stochastic quasi-linear parabolic equation driven by multiplicative white noise by a Wong-Zakai approximation technique. The convergence of the solutions of quasi-linear parabolic equations driven by a family of processes with stationary increment to that of stochastic differential equation with white noise is obtained in the topology of $L^2(\mathbb{R}^N)$ space. We establish the Wong-Zakai approximations of solutions in $L^l(\mathbb{R}^N)$ for arbitrary $l \geq q$ in the sense of upper semi-continuity of their random attractors, where q is the growth exponent of the nonlinearity. The L^l-pre-compactness of attractors is proved by using the truncation estimate in L^q and the higher-order bound of solutions.

1. **Introduction.** In this paper, we consider the regularity of Wong-Zakai approximations of a stochastic quasi-linear parabolic equation (with p-Laplacian) with multiplicative white noise on the entire space $\mathbb{R}^N, 1 \leq N \in \mathbb{N}$: For $t > \tau$ and $x \in \mathbb{R}^N$,

$$du = (\mathrm{div}(|\nabla u|^{p-2}\nabla u) - \lambda u + f(t,x,u) + g(t,x))dt + u \circ dW(t), \tag{1}$$

where $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^N))$, $\lambda > 0$ and $p > 2$ are constants, f is a nonlinear function satisfying some dissipative conditions, $W(t)$ is two-sided real-valued Brownian motions on a probability space, $\circ$ signifies the Stratonovich sense of the stochastic term.

We introduce the canonical sample space of Wiener precesses $\Omega := C_0(\mathbb{R})$, the set of continuous functions on $\mathbb{R}$ with 0 at 0 with the compact open topology. $\mathscr{F}$ denotes the Borel σ-algebra of Ω and P is the Wiener measure on $(\Omega, \mathscr{F})$. The Brownian motion $W(t,\omega)$ is identified as $\omega(t)$, *i.e.*, $W(t,\omega) = \omega(t)$. Furthermore, there is a Wiener shift $\{\vartheta\}_{t\in\mathbb{R}}$ on Ω defined by $\vartheta_t\omega(\cdot) = \omega(t+\cdot) - \omega(t)$ for every

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$t \in \mathbb{R}$ and $\omega \in \Omega$. Then $\vartheta : \mathbb{R} \times \Omega \mapsto \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathscr{F}, \mathscr{F})$-measurable with $\vartheta_t P = P$ for all $t \in \mathbb{R}$.

Let the mapping $\mathcal{G}_\delta : \Omega \mapsto \mathbb{R}$ be a random variable such that

$$\mathcal{G}_\delta(\omega) = \frac{\omega(\delta)}{\delta}, \quad \delta \neq 0. \tag{2}$$

By the Wiener shift $\{\vartheta_t\}_{t\in\mathbb{R}}$ on Ω, we get

$$\mathcal{G}_\delta(\vartheta_t\omega) = \frac{\omega(t+\delta) - \omega(t)}{\delta}, \quad t \in \mathbb{R}. \tag{3}$$

By the stationary increment property of $\omega(t)$, we have $\mathcal{G}_\delta(\vartheta_t\omega) \sim N(0, \frac{1}{\delta})$ for every $\delta \neq 0$ and $t \in \mathbb{R}$, and moreover it is easy to check that

$$\mathcal{G}_\delta(\vartheta_{t+r}\omega) - \mathcal{G}_\delta(\vartheta_t\omega) \sim N(0, \frac{2r}{\delta^2}) \text{ for } \delta \geq r,$$

and $\mathcal{G}_\delta(\vartheta_{t+r}\omega) - \mathcal{G}_\delta(\vartheta_t\omega) \sim N(0, \frac{2}{\delta})$ for $\delta < r$. Then the process $\mathcal{G}_\delta(\vartheta_t\omega)$ also possesses stationary increment. Furthermore, $\mathcal{G}_\delta(\vartheta_t\omega)$ may be regarded as an approximation of the white noise in the sense that for every $T > 0$,

$$\lim_{\delta\to 0} \sup_{t\in[0,T]} \Big| \int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr - \omega(t)\Big| = 0; \tag{4}$$

see [21].

Put

$$W_\delta = W_\delta(t,\omega) = \int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr, \quad \forall t \in \mathbb{R}, \omega \in \Omega.$$

In this paper, we study the following point-wise *deterministic* quasi-linear parabolic equations driven by the process W_δ:

$$du_\delta = (\mathrm{div}(|\nabla u_\delta|^{p-2}\nabla u_\delta) - \lambda u_\delta + f(t,x,u_\delta) + g(t,x))dt + u_\delta dW_\delta. \tag{5}$$

Note that Eq.(5) is a random non-autonomous differential equation. Its solutions admit a non-autonomous random dynamical system and therefore one can study its path-wise dynamical properties such as random attractor and its regular properties.

On the other hand, in terms of the convergence property (4), we will show that the limit of solutions of the deterministic differential equations (5) is a solution of stochastic differential equation (1), which is equivalent to the following Itô stochastic differential equations:

$$du = (\mathrm{div}(|\nabla u|^{p-2}\nabla u) - \lambda u + \frac{1}{2}u + f(t,x,u) + g(t,x))dt + udW(t); \tag{6}$$

see Theorem 4.9. The upper semi-continuity of their random attractors in $L^2(\mathbb{R}^N)$ is also proved in Theorem 4.11. We here remark that the random elements (noise) in the sense of Stratonovich or Itô may propose very different stabilization and destabilization effects on the real models; see [6, 5, 23].

Our third work in this paper is to establish the Wong-Zakai approximation in the Banach space $L^q(q \geq p > 2)$, where q is the growth exponent of the nonlinearity. To this end, some further compactness in L^q is needed. This is achieve by a truncation approach, by which we prove that the solution vanishes in $L^q(q \geq p > 2)$ on a domain on which the solution of Eq.(5) diverges to positive and negative infinite. Then by the theorem in [38], we obtain that the random attractor of Eq.(5) converges to that of Eq.(1) in $L^q(\mathbb{R}^N)(q \geq p > 2)$ in the sense of upper semi-continuity; see Theorem 5.4.

Finally, with some small additional assumptions on the coefficients, by using an induction technique we obtain the solution is bounded in $L^l(\mathbb{R}^N)$ for arbitrary $l > q$. By Sobolev interpolation, we establish some compactness of random attractors in $L^l(\mathbb{R}^N)$. Following this, we obtain the Wong-Zakai approximations in this higher-regular space $L^l(\mathbb{R}^N)$ for arbitrary $l > q$; see Theorem 6.4.

We now recollect some literature on the studying of p-Laplacian equation. This equation arises naturally in a boundary value problem of partial differential equations and has been widely used in various fields of science and technology; see [2, 7, 8] and the references therein. In the absence of noise, that is a deterministic p-Laplacian equations, many studies have been done on various aspects of attractors. For the existence of a global attractor in $W^{1,p} \cap L^q$ when the state space is bounded or unbounded, we refer the readers to [11, 12, 14, 32, 33]. For the existence and upper semi-continuity of global attractors of $p(x)$-Laplacian systems is studied in [25]. Recent years, the random dynamics of stochastic p-Laplacian equation with additive or multiplicative noise, has been intensively investigated in several literature. The existence and upper semi-continuity of random attractors in $L^2(\mathbb{R}^N)$ were obtained by Krause et al. [16, 17] by using the the well-established theory of pullback random attractors in Wang [27]. Then the regularity and upper semi-continuity were extensively studied in [18, 19, 20, 34]. By estimating the difference of solutions, Zhao [35, 37] studied the regularity dynamics in higher regular spaces, where the $L^\delta(\mathbb{R}^N)(\forall\delta > 2)$-attracting of the random attractor was proved for arbitrary space dimension $N \geq 1$.

The approximation of stochastic equations by path-wise deterministic equations was initiated by Wong and Zakai [30, 31]. So far, there has been a rich literature about the Wong-Zakai approximations, and we only mention some recent work related to our topic. By means of the Wong-Zakai approximations, Brzeźniak et al.[3] and Manna et al.[22] proved the existence and uniqueness of solution of stochastic Landau-Lifshitz-Gilbert equations with different energy. Lv and Wang et al.[21, 29] and Shen et al.[24] studied the approximations of random attractors and invariant manifolds for stochastic partial differential equations. More recently, Sun et al.[26] studied the upper semi-continuity of attractors for the Wong-Zakai approximation of the fractional stochastic reaction-diffusion equation driven by a white noise in $L^2(\mathbb{R}^N)$. Jiang et al.[13] studied the smooth Wong-Zakai approximations given by a stochastic process via Wiener shift and mollifier of Brownian motions. However, there are few papers to attack the Wong-Zakai approximations in higher regular spaces, except that the recent progress in Zhao and Zhang [40, 39]. Especially, in [39] we require that the nonlinearity exerted on the equations is monotonic.

This paper is organized as follows. In the next section, we introduce some notions on the random dynamical systems. In section 3, we give the conditions for the coefficients of the stochastic p-Laplacian equation, and the related properties of the stationary process $\mathcal{G}_\delta(\vartheta_t\omega)$. Section 4 is devoted to establish the existence of tempered random attractor, the convergence of solutions and the corresponding Wong-Zakai approximation result in $L^2(\mathbb{R}^N)$, which will be used later to prove the upper semi-continuity of attractors in $L^q(\mathbb{R}^N)$. In section 5, we obtain the uniform compactness by a truncation approach, where the high-order Wong-Zakai approximations of the stochastic p-Laplacian equation driven by multiplicative white noise in $L^q(\mathbb{R}^N), \forall q \geq p > 2$ is established. In the final section, we obtain the Wong-Zakai approximation results in $L^l(\mathbb{R}^N)$ for arbitrary $l > q$.

2. **Preliminaries on non-autonomous random attractors.** We present in this section some basic notions about (non-autonomous) random attractor $\mathcal{A}(t,\omega)$ [15, 28], which is a generalization of the (autonomous) random attractor $\mathcal{A}(\omega)$ for random dynamical system; see [10, 9]. For a comprehensive knowledge of random dynamical systems, the reader may refer to Arnold [1]. Let $(H, \|\cdot\|_H)$ be a separable Banach space with σ-algebra $\mathfrak{B}(X)$ and $\vartheta = (\Omega, \mathscr{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ be a metric dynamical system(MDS); see [1]. Denote by $\mathbb{R}^+ = \{s \geq 0 : s \in \mathbb{R}\}$.

Definition 2.1. A mapping $\varphi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \mapsto X$ is called a random cocycle on X over an MDS ϑ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$,
(i) $\varphi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \mapsto X$ is $(\mathfrak{B}(\mathbb{R}^+) \times \mathscr{F} \times \mathfrak{B}(X), \mathfrak{B}(X))$-measurable;
(ii) $\varphi(0, \tau, \omega, \cdot)$ is the identity on X ;
(iii) $\varphi(t+s, \tau, \omega, \cdot) = \varphi(t, \tau+s, \vartheta_s\omega, \varphi(s, \tau, \omega, \cdot))$.
A random cocycle φ is said to be continuous in X if for each $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the mapping $\varphi(t, \tau, \omega, \cdot) : X \mapsto X$ is continuous.

Definition 2.2. A family of sets $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called a random set in X with respect to $\mathscr{F}$ if the mapping $\omega \in \Omega \mapsto \mathrm{dist}_X(\{x\}, K(\tau, \omega))$ is $(\mathscr{F}, \mathcal{B}(\mathbb{R}))$-measurable for every fixed $x \in X$ and $\tau \in \mathbb{R}$, where dist_X is the Hausdorff semi-metric in 2^X, *i.e.*, for two nonempty subsets $A, B \in 2^X$, $\mathrm{dist}_X(A, B) = \sup_{a\in A} \inf_{b\in B} \|a-b\|_X$.

Throughout this paper, we denote by $\mathfrak{D}$ a collection of some families of non-empty subsets of X: $\mathfrak{D} = \{D(\tau, \omega) \subset X : D(\tau, \omega) \neq \emptyset, \tau \in \mathbb{R}, \omega \in \Omega\}$. The elements D_1 and D_2 of $\mathfrak{D}$ is said to be equal if $D_1(\tau, \omega) = D_2(\tau, \omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$. $\mathfrak{D}$ is said to be inclusion closed if for $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ and $D_0(\tau, \omega) \subset D(\tau, \omega)$ for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then $D_0 = \{D_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ (see [28]).

Definition 2.3. Let $\mathfrak{D}$ be a collection of some families of non-empty subsets of X. A family of sets $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called a $\mathfrak{D}$-pullback absorbing set for φ in X if $K \in \mathfrak{D}$ and for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$, there exists a time $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,

$$\varphi(t, \tau - t, \vartheta_{-t}\omega, D(\tau - t, \vartheta_{-t}\omega)) \subseteq K(\tau, \omega).$$

Furthermore, if K is a random set then K is called a $\mathfrak{D}$-pullback random absorbing set for φ.

Definition 2.4. A family of sets $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called a $\mathfrak{D}$-pullback random attractor for φ in X if $\mathcal{A} \in \mathfrak{D}$ is random set and for every $\tau \in \mathbb{R}$ and $\omega \in \mathfrak{D}$, the following three conditions hold:
(i) $\mathcal{A}(\tau, \omega)$ is compact in X;
(ii) $\mathcal{A}(\tau, \omega)$ is invariant, that is, $\varphi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \vartheta_t\omega)$, for arbitrary $t \geq 0$;
(iii) $\mathcal{A}(\tau, \omega)$ is attracting in X, that is, for every element $D \in \mathfrak{D}$,

$$\lim_{t\to+\infty} \mathrm{dist}_X(\varphi(t, \tau - t, \vartheta_{-t}\omega, D(\tau - t, \vartheta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0.$$

Definition 2.5. Let $\mathfrak{D}$ be a collection of some families of non-empty subsets of X. A random cocycle φ is said to be $\mathfrak{D}$-pullback asymptotically compact in X if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $\forall t_n \to +\infty, x_n \in D(\tau - t_n, \vartheta_{-t_n}\omega)$ with $D \in \mathfrak{D}$, the sequence

$$\{\varphi(t_n, \tau - t_n, \vartheta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \text{ has a covergent subsequence in } X.$$

Theorem 2.6 ([28]). *Let $\mathfrak{D}$ be a collection of some families of non-empty subsets of X and φ be a continuous random cocycle on X over an MDS ϑ. Then φ has a unique $\mathfrak{D}$-pullback random attractor $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in X if φ has a closed $\mathfrak{D}$-pullback random absorbing set $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in X and φ is $\mathfrak{D}$-pullback asymptotically compact in X. Furthermore, for all $\tau \in \mathbb{R}, \omega \in \Omega$,*

$$\mathcal{A}(\tau,\omega) = \cap_{s>0}\overline{\cup_{t\geq s}\varphi(t,\tau-t,\vartheta_{-t}\omega,K(\tau-t,\vartheta_{-t}\omega))}^{X},$$

which is the omega-limit set of $K(\tau,\omega)$.

Theorem 2.7 ([19, 38]). *Let I be a metric space, $\delta,\delta_0 \in I$, and φ_δ be a continuous cocycle on X. Suppose that*
(i) For every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\delta_n \to \delta_0$ with $\delta_n \in I$ and $x_n, x \in X$ with $x_n \to x$, there holds

$$\lim_{n\to\infty}\varphi_{\delta_n}(t,\tau,\omega,x_n) = \varphi_{\delta_0}(t,\tau,\omega,x) \text{ in } X;$$

(ii) Let $\mathfrak{D}$ be a collection of some families of nonempty subsets of X, for every $\tau \in \mathbb{R}, \omega \in \Omega$ there exist $\varrho_{\delta_0}(\tau,\omega) > 0$ such that

$$K_{\delta_0}(\tau,\omega) = \{x \in X; \|x\|_X \leq \varrho_{\delta_0}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}.$$

Let $\mathcal{A}_\delta$ be a $\mathfrak{D}$-pullback attractor and K_δ a $\mathfrak{D}$-pullback absorbing set of φ_δ in X, such that for all $\tau \in \mathbb{R}, \omega \in \Omega$,

$$\limsup_{\delta\to\delta_0}\|K_\delta(\tau,\omega)\|_X \leq \varrho_{\delta_0}(\tau,\omega);$$

(iii) For all $\tau \in \mathbb{R}, \omega \in \Omega$, if $\delta_n \to \delta_0$ and $x_n \in \mathcal{A}_{\delta_n}(\tau,\omega)$, then x_n is pre-compact in X.

Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have

$$\lim_{\delta\to\delta_0}\operatorname{dist}_X(\mathcal{A}_\delta(\tau,\omega),\mathcal{A}_{\delta_0}(\tau,\omega)) = 0.$$

In addition, for all $\tau \in \mathbb{R}, \omega \in \Omega$, if $\delta_n \to \delta_0$ and $x_n \in \mathcal{A}_{\delta_n}(\tau,\omega)$, then x_n is pre-compact in Y. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have

$$\lim_{\delta\to\delta_0}\operatorname{dist}_Y(\mathcal{A}_\delta(\tau,\omega),\mathcal{A}_{\delta_0}(\tau,\omega)) = 0,$$

where we assume that $\varphi(t,\tau,\omega,\cdot) : X \mapsto Y$ for every $t > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, and both Y and $X \cap Y$ are Banach spaces.

3. Mathematical settings.

In this article, we denote by $c, c_i > 0 (i = 1,2,...)$ the generic constants which may depend on the τ,ω and T, and may change their values from to line to line. Denote by $\|\cdot\|_p (p > 0)$ the norm in $L^p(\mathbb{R}^N)$ and in particular $\|\cdot\| = \|\cdot\|_2$ for $p = 2$, and $\|\cdot\|_{W^{1,p}}$ the norm in $W^{1,p}(\mathbb{R}^N)$.

In this section, we give some mathematical settings of Eq.(1), including the conditions on the nonlinearity f and some known results in [21, 29]. We assume that the nonlinear function f is continuous on $\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}$ and satisfies the following conditions: for all $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^N$,

$$f(t,x,s)s \leq -\alpha_1|s|^q + \psi_1(t,x), \tag{7}$$

$$|f(t,x,s)| \leq \alpha_2|s|^{q-1} + \psi_2(t,x), \tag{8}$$

$$\frac{\partial}{\partial s}f(t,x,s) \leq \psi_3(t,x), \tag{9}$$

where $2 < p \le q, \alpha_1 > 0, \alpha_2 > 0,\ \psi_1 \in L^1_{loc}(\mathbb{R}; L^1(\mathbb{R}^N)),\ \psi_2 \in L^{q_1}_{loc}(\mathbb{R}; L^{q_1}(\mathbb{R}^N))$ with $\frac{1}{q_1} + \frac{1}{q} = 1$, and $\psi_3 \in L^{\infty}_{loc}(\mathbb{R}; L^{\infty}(\mathbb{R}^N))$.

In addition, the following condition will be needed for the non-autonomous terms g and ψ_1 when deriving uniform estimates of solutions:

$$\int_{-\infty}^{\tau} e^{\lambda s}(\|\psi_1(s)\|_1 + \|g(s)\|^2)ds < +\infty, \forall \tau \in \mathbb{R}, \tag{10}$$

where λ is as in Eq.(1), and especially in order to derive the tempered property of attractor we further assume that for arbitrary $c > 0$,

$$\lim_{t\to-\infty} e^{ct} \int_{-\infty}^{0} e^{\lambda s}(\|\psi_1(s+t)\|_1 + \|g(s+t)\|^2)ds = 0. \tag{11}$$

Let $D = \{D(\tau, \omega) \subset L^2(\mathbb{R}^N) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of $L^2(\mathbb{R}^N)$. Such a family D is called tempered if for every $c > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to-\infty} e^{ct}\|D(\tau + t, \vartheta_t\omega)\|^2 = 0,$$

where $\|D\| = \sup_{u\in D} \|u\|$. In what follows, we always suppose that $\mathfrak{D}$ is a collection of all families of tempered subsets of $L^2(\mathbb{R}^N)$, namely,

$$\mathfrak{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ is tempered in } L^2(\mathbb{R}^N)\}. \tag{12}$$

Then it is obvious that $\mathfrak{D}$ is inclusion closed.

For convenience, let us present the uniform convergence of the integral of stationary process $\mathcal{G}_\delta(\vartheta_t\omega)$ on any finite interval, which is stated in [21].

Lemma 3.1. *Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $T > 0$, then for every $0 < \varepsilon < 1$, there exists $\delta_0 = \delta_0(\varepsilon, \tau, \omega, T) > 0$ such that for all $0 < |\delta| \le \delta_0$,*

$$\sup_{t\in[\tau,\tau+T]} \Big| \int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr - \omega(t)\Big| \le \varepsilon. \tag{13}$$

By the continuity of $\omega(t)$ on $[\tau, \tau + T]$ for any $\tau \in \mathbb{R}$ and $T > 0$, there exists $c = c(\tau, \omega, T)$ such that

$$\sup_{t\in[\tau,\tau+T]} |\omega(t)| \le c. \tag{14}$$

By (13)-(14) there are positive constants $\delta_1 = \delta_1(\tau, \omega, T)$ and $c = c(\tau, \omega, T)$ such that for all $0 < |\delta| \le \delta_1$,

$$\sup_{t\in[\tau,\tau+T]} \Big| \int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr\Big| \le \sup_{t\in[\tau,\tau+T]} \Big| \int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr - \omega(t)\Big| + \sup_{t\in[\tau,\tau+T]} |\omega(t)| \le c. \tag{15}$$

We give a convenient lemma about the property of the stationary process $\mathcal{G}_\delta(\vartheta_t\omega)$, which will be used repeatedly in the subsequential arguments.

Lemma 3.2. *Let $\tau \in \mathbb{R}$, $\omega \in \Omega, \delta \neq 0$ and $\mathcal{G}_\delta(\vartheta_t\omega)$ be defined by (3). Then for any $\gamma > 0$, there exist positive constants $\widetilde{c} = \widetilde{c}(\tau, \omega, \lambda)$ and $\delta_0 = \delta_0(\tau, \omega) < 1$ such that for any $s \le 0$ and $0 < |\delta| \le \delta_0$,*

$$\Big| \int_{-\tau}^{s} \mathcal{G}_\delta(\vartheta_r\omega)dr\Big| \le \widetilde{c} - \min\{\frac{\lambda}{6}, \frac{\gamma}{8}\}s.$$

Proof. By the definition (2) of $\mathcal{G}_\delta$, we have

$$\int_{-\tau}^{s} \mathcal{G}_\delta(\vartheta_r\omega)dr = \frac{1}{\delta}\int_{-\tau+\delta}^{-\tau} \omega(r)dr + \frac{1}{\delta}\int_{s}^{s+\delta} \omega(r)dr. \tag{16}$$

Then according to the integration mean theorem, there exists r_0 between s and $s+\delta$ such that

$$\frac{1}{\delta}\int_{s}^{s+\delta} \omega(r)dr = \omega(r_0). \tag{17}$$

Since the Wiener process $\omega(t)$ satisfies $\lim\limits_{|t|\to\infty} \frac{\omega(t)}{t} = 0$, there exists $T_1 = T_1(\omega,\lambda,\gamma) \le 0$ such that for all $r_0 \le T_1$ we have $|\omega(r_0)| \le -\min\{\frac{\lambda}{6},\frac{\gamma}{8}\}r_0$ for any $\gamma > 0$. Note that $r_0 - s \le |\delta|$. Then if $s \le T_1 - |\delta|$ then $r_0 \le T_1$, whence we know that for all $s \le T_1 - |\delta|$,

$$\Big|\frac{1}{\delta}\int_{s}^{s+\delta} \omega(r)dr\Big| = |\omega(r_0)| \le -\min\{\frac{\lambda}{6},\frac{\gamma}{8}\}r_0. \tag{18}$$

Consider that $s - r_0 \le |\delta|$ also holds true. Then by (18), for all $0 < |\delta| \le 1$ and $s \le T_1 - 1$,

$$\begin{aligned}\Big|\frac{1}{\delta}\int_{s}^{s+\delta} \omega(r)dr\Big| &= |\omega(r_0)| \le -\min\{\frac{\lambda}{6},\frac{\gamma}{8}\}r_0 \le \min\{\frac{\lambda}{6},\frac{\gamma}{8}\}(|\delta| - s)\\ &\le \min\{\frac{\lambda}{6},\frac{\gamma}{8}\}(1-s) \le \frac{\lambda}{6} - \min\{\frac{\lambda}{6},\frac{\gamma}{8}\}s.\end{aligned} \tag{19}$$

On the other hand, we have $T_1 + \delta - 1 \le s + \delta \le \delta$ for $T_1 - 1 \le s \le 0$. Then the number r_0 in (17) may belong to a finite closed interval $[-|T_1| - |\delta| - 1, |T_1| + |\delta| + 1]$ for $T_1 - 1 \le s \le 0$, and therefore $r_0 \in [-|T_1| - 2, |T_1| + 2]$ for $0 < |\delta| \le 1$ and $T_1 - 1 \le s \le 0$. Then by the continuity of ω again, there exists $c_2(\omega)$ such that for all $T_1 - 1 \le s \le 0$ and $0 < |\delta| \le 1$,

$$\Big|\frac{1}{\delta}\int_{s}^{s+\delta} \omega(r)dr\Big| = |\omega(r_0)| \le c_2(\omega). \tag{20}$$

By (19) and (20) we get for all $0 < |\delta| \le 1$ and $s \le 0$,

$$\Big|\frac{1}{\delta}\int_{s}^{s+\delta} \omega(r)dr\Big| \le c_2(\omega) + \frac{\lambda}{6} - \min\{\frac{\lambda}{6},\frac{\gamma}{8}\}s. \tag{21}$$

According to the continuity of $\omega(t)$ in t, we have $\lim\limits_{\delta\to 0} \frac{1}{\delta}\int_{-\tau+\delta}^{-\tau} \omega(r)dr = \omega(-\tau)$, and thereby there exists $\delta_1 = \delta_1(\tau,\omega)$ such that for all $0 < |\delta| \le \delta_1$,

$$\Big|\frac{1}{\delta}\int_{-\tau+\delta}^{-\tau} \omega(r)dr\Big| \le |\omega(-\tau)| + 1. \tag{22}$$

Let $\delta_0 = \min\{\delta_1, 1\}$. Then from (16), (21) and (22) we obtain that, for all $0 < |\delta| \le \delta_0$ and $s \le 0$,

$$\begin{aligned}\Big|\int_{-\tau}^{s} \mathcal{G}_\delta(\vartheta_r\omega)dr\Big| &\le \Big|\frac{1}{\delta}\int_{-\tau+\delta}^{-\tau} \omega(r)dr\Big| + \Big|\frac{1}{\delta}\int_{s}^{s+\delta} \omega(r)dr\Big|\\ &\le |\omega(-\tau)| + 1 + c_2(\omega) + \frac{\lambda}{6} - \min\{\frac{\lambda}{6},\frac{\gamma}{8}\}s,\end{aligned} \tag{23}$$

which completes the proof. ∎

To discuss the random attractors and the Wong-Zakai approximation, we need to transform the stochastic p-Laplacian equation (1) into a path-wise deterministic equation. We first make a transformation for Eq.(1) by means of the geometric Brownian motion $e^{\omega(t)}$. By Itô' formula (see e.g.[4, Theorem 6.2.1]), we have

$$de^{\omega(t)} = e^{\omega(t)}d\omega(t) + \frac{1}{2}e^{\omega(t)}dt.$$

Let $u(t,\tau,\omega,u_\tau) = e^{\omega(t)}v(t,\tau,\omega,v_\tau)$, where u is the solution of Eq.(6). Then we have

$$du = vde^{\omega(t)} + e^{\omega(t)}dv = ud\omega(t) + \frac{1}{2}udt + e^{\omega(t)}dv,$$

in the Itô sense. Therefore v solves the following equation: for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t > \tau$ and $x \in \mathbb{R}^N$,

$$\frac{dv}{dt} = -\lambda v + e^{(p-2)\omega(t)}\text{div}(|\nabla v|^{p-2}\nabla v) + e^{-\omega(t)}f(t,x,u) + e^{-\omega(t)}g(t,x), \tag{24}$$

with the initial condition $v(\tau,x) = v_\tau$.

We also introduce the transformation:

$$v_\delta(t,\tau,\omega,v_{\delta,\tau}) = e^{-\int_0^t \mathcal{G}_\delta(\vartheta_s\omega)ds}u_\delta(t,\tau,\omega,u_{\delta,\tau}), \tag{25}$$

with $v_{\delta,\tau} = e^{-\int_0^\tau \mathcal{G}_\delta(\vartheta_s\omega)ds}u_{\delta,\tau}$. Then we get from Eq.(5) that for any $t > \tau$ and $x \in \mathbb{R}^N$,

$$\begin{aligned}\frac{dv_\delta}{dt} &= -\lambda v_\delta + e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_s\omega)ds}\text{div}(|\nabla v_\delta|^{p-2}\nabla v_\delta) + e^{-\int_0^t \mathcal{G}_\delta(\vartheta_s\omega)ds}f(t,x,u_\delta)\\ &\quad + e^{-\int_0^t \mathcal{G}_\delta(\vartheta_s\omega)ds}g(t,x),\end{aligned} \tag{26}$$

with the initial condition $v_\delta(\tau,x) = v_{\delta,\tau}$.

Give $\omega \in \Omega, \tau \in \mathbb{R}$ and $v_\tau \in L^2(\mathbb{R}^N)$, if f satisfies (7)-(9) then Eq.(24) has a unique solution $v(\cdot,\tau,\omega,v_\tau) \in C([\tau,\infty);L^2(\mathbb{R}^N)) \cap L^q_{loc}((\tau,\infty);L^q(\mathbb{R}^N))$. In addition $v(\cdot,\tau,\omega,v_\tau)$ is continuous in the initial v_τ in $L^2(\mathbb{R}^N)$ and is $(\mathscr{F},\mathcal{B}(L^2(\mathbb{R}^N)))$-measurable in $\omega \in \Omega$, and thus we can define a continuous cocycle $\varphi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ for Eq.(1) by

$$\varphi(t,\tau,\omega,u_\tau) = u(t+\tau,\tau,\vartheta_{-\tau}\omega,u_\tau) = e^{\omega(t)-\omega(-\tau)}v(t+\tau,\tau,\vartheta_{-\tau}\omega,v_\tau), \tag{27}$$

for every $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$. By a similar method, we define the random cocycle for Eq.(5) by

$$\begin{aligned}\varphi_\delta(t,\tau,\omega,u_\tau) &= u_\delta(t+\tau,\tau,\vartheta_{-\tau}\omega,u_{\delta,\tau})\\ &= e^{\int_0^{t+\tau} \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}v_\delta(t+\tau,\tau,\vartheta_{-\tau}\omega,v_{\delta,\tau}).\end{aligned} \tag{28}$$

4. Random attractors and L^2-Wong-Zakai approximations.

In this section, we consider the existence of random attractors and the Wong-Zakai approximations of p-Laplacian equation in $L^2(\mathbb{R}^N)$.

4.1. Existence of random attractors.

In this subsection, we present the existence of random attractors for random cocycles defined by (27) and (28) without detailed proof.

Lemma 4.1. *Suppose (7)-(11) hold and $\mathfrak{D}$ is defined by (12). Then the continuous cocycle φ associated with Eq.(1) admits a closed $\mathfrak{D}$-pullback random absorbing set $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ in $L^2(\mathbb{R}^N)$, which is depicted by, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$K(\tau,\omega) = \{u \in L^2(\mathbb{R}^N) : \|u\|^2 \le \varrho(\tau,\omega)\}, \tag{29}$$

where $\varrho(\tau,\omega)$ is given by

$$\varrho(\tau,\omega) = 4\int_{-\infty}^{0} e^{\frac{4}{3}\lambda s - 2\omega(s)}(\frac{3}{2\lambda}\|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_1)ds. \tag{30}$$

Proof. Taking the inner product of (24) with v in $L^2(\mathbb{R}^N)$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 &+ (-e^{(p-2)\omega(t)}\mathrm{div}(|\nabla v|^{p-2}\nabla v), v) \\ &= (e^{-\omega(t)}f(t,x,u), v) + (e^{-\omega(t)}g(t,x), v).\end{aligned} \tag{31}$$

Using the assumptions (7), we get

$$\begin{aligned}(e^{-\omega(t)}f(t,x,u), v) &= e^{-2\omega(t)}(f(t,x,u), u) \\ &\le -\alpha_1 e^{-2\omega(t)}\|u\|_q^q + e^{-2\omega(t)}\|\psi_1(t)\|_1.\end{aligned} \tag{32}$$

For the p-Laplacian part in (31), we have

$$\begin{aligned}(-e^{(p-2)\omega(t)}\mathrm{div}(|\nabla v|^{p-2}\nabla v), v) &= e^{(p-2)\omega(t)}(|\nabla v|^{p-2}\nabla v, \nabla v) \\ &= e^{-2\omega(t)}\|\nabla u\|_p^p.\end{aligned} \tag{33}$$

The second term on the right hand side of (31) is bounded by

$$(e^{-\omega(t)}g(t,x), v) \le \frac{\lambda}{6}\|v\|^2 + \frac{3}{2\lambda}e^{-2\omega(t)}\|g(t)\|^2. \tag{34}$$

Combine (31)-(34) to find

$$\begin{aligned}\frac{d}{dt}\|v\|^2 + \frac{4\lambda}{3}\|v\|^2 &+ e^{-2\omega(t)}(\|\nabla u\|_p^p + \alpha_1\|u\|_q^q + \frac{\lambda}{3}\|u\|^2) \\ &\le 2e^{-2\omega(t)}(\frac{3}{2\lambda}\|g(t)\|^2 + \|\psi_1(t)\|_1).\end{aligned} \tag{35}$$

Multiplying (35) by $e^{\frac{4\lambda}{3}t}$ and integrating from $\tau - t$ to ξ, along with replacing ω by $\vartheta_{-\tau}\omega$, and then using the formula $\vartheta_{-\tau}\omega(s) = \omega(s-\tau) - \omega(-\tau)$, we find that

$$\begin{aligned}&\|v(\xi,\tau-t,\vartheta_{-\tau}\omega, v_{\tau-t})\|^2 \\ &+ e^{2\omega(-\tau)}\int_{\tau-t}^{\xi} e^{\frac{4\lambda}{3}(s-\xi)-2\omega(s-\tau)}(\|\nabla u(s)\|_p^p + \alpha_1\|u(s)\|_q^q + \frac{\lambda}{3}\|u(s)\|^2)ds \\ &\le e^{\frac{4\lambda}{3}(\tau-t-\xi)}\|v_{\tau-t}\|^2 + 2e^{2\omega(-\tau)}\int_{\tau-t}^{\xi} e^{\frac{4\lambda}{3}(s-\xi)-2\omega(s-\tau)}(\frac{3}{2\lambda}\|g(s)\|^2 + \|\psi_1(s)\|_1)ds.\end{aligned} \tag{36}$$

By the formula $u(\xi,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t}) = e^{\omega(\xi-\tau)-\omega(-\tau)}v(\xi,\tau-t,\vartheta_{-\tau}\omega,v_{\tau-t})$, we have

$$\begin{aligned}
&e^{-2\omega(\xi-\tau)}\|u(\xi,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})\|^2\\
&+\int_{\tau-t}^{\xi} e^{\frac{4\lambda}{3}(s-\xi)-2\omega(s-\tau)}(\|\nabla u(s)\|_p^p+\alpha_1\|u(s)\|_q^q+\frac{\lambda}{3}\|u(s)\|^2)ds\\
&\le e^{\frac{4\lambda}{3}(\tau-t-\xi)-2\omega(-t)}\|u_{\tau-t}\|^2\\
&+2\int_{-\infty}^{\xi-\tau} e^{\frac{4\lambda}{3}(s+\tau-\xi)-2\omega(s)}(\frac{3}{2\lambda}\|g(s+\tau)\|^2+\|\psi_1(s+\tau)\|_1)ds. \qquad (37)
\end{aligned}$$

Let $\xi=\tau$. Then we get

$$\begin{aligned}
\|u(\tau,\tau-t,&\vartheta_{-\tau}\omega,u_{\tau-t})\|^2 \le e^{-\frac{4\lambda}{3}t-2\omega(-t)}\|u_{\tau-t}\|^2\\
&+2\int_{-\infty}^{0} e^{\frac{4\lambda}{3}s-2\omega(s)}(\frac{3}{2\lambda}\|g(s+\tau)\|^2+\|\psi_1(s+\tau)\|_1)ds. \qquad (38)
\end{aligned}$$

Since $u_{\tau-t}\in D(\tau-t,\vartheta_{-t}\omega)$, then there exists a $T=T(\tau,\omega,D)$ such that for all $t\ge T$, $e^{-\frac{4\lambda}{3}t-2\omega(-t)}\|u_{\tau-t}\|^2 \le 2\int_{-\infty}^{0} e^{\frac{4\lambda}{3}s-2\omega(s)}(\frac{3}{2\lambda}\|g(s+\tau)\|^2+\|\psi_1(s+\tau)\|_1)ds$. From (10) we can deduce that $\int_{-\infty}^{0} e^{\frac{4\lambda}{3}s-2\omega(s)}(\frac{3}{2\lambda}\|g(s+\tau)\|^2+\|\psi_1(s+\tau)\|_1)ds < +\infty$. Denote by

$$\varrho(\tau,\omega)=4\int_{-\infty}^{0} e^{\frac{4\lambda}{3}s-2\omega(s)}(\frac{3}{2\lambda}\|g(s+\tau)\|^2+\|\psi_1(s+\tau)\|_1)ds.$$

Then by (11) it is easy to check that $K(\tau,\omega)=\{u\in L^2(\mathbb{R}^N): \|u\|^2\le\varrho(\tau,\omega)\}$ is tempered; that is, $\lim\limits_{t\to-\infty} e^{\gamma t}\|K(\tau+t,\vartheta_t\omega)\|^2 \le \lim\limits_{t\to-\infty} e^{\gamma t}\varrho(\tau+t,\vartheta_t\omega)=0$ for any $\gamma>0$. Furthermore, since for fixed $\tau\in\mathbb{R}$ and for each $\omega\in\Omega$ and $x\in L^2(\mathbb{R}^N)$,

$$\mathrm{dist}(x,K(\tau,\omega))=\begin{cases}0, & x\in K(\tau,\omega);\\ \|x\|-\sqrt{\varrho(\tau,\omega)}, & x\notin K(\tau,\omega),\end{cases}$$

then the mapping $\omega\mapsto dist(x,K(\tau,\omega))$ is $(\mathscr{F},\mathcal{B}(\mathbb{R}))$-measurable for every fixed $x\in X$, and therefore $K(\tau,\omega)=\{u\in L^2(\mathbb{R}^N): \|u\|^2\le\varrho(\tau,\omega)\}$ is a closed random set, which completes the proof. □

To prove the $\mathfrak{D}$-pullback asymptotical compactness in $L^2(\mathbb{R}^N)$ for Eq.(1), we need to prove that the tail of solution is small enough on larger and larger space domains, which is given as below.

Lemma 4.2. *Suppose (7)-(11) hold and $\mathfrak{D}$ is defined by (12). Given $\tau\in\mathbb{R},\omega\in\Omega$, $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathfrak{D}$, then for every $u_{\tau-t}\in D(\tau-t,\vartheta_{-t}\omega)$ and any $\varepsilon>0$, there exist $T=T(\tau,\omega,D,\varepsilon)\ge 1$ and $R=R(\tau,\omega,\varepsilon)>0$ such that for all $\xi\in[\tau-1,\tau]$, the solution u of Eq.(1) satisfies*

$$\sup_{t\ge T}\int_{|x|\ge R}|u(\xi,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})|^2dx\le\varepsilon.$$

Proof. Let ρ be a smooth function defined on $\mathbb{R}^+$ such that $0\le\rho(s)\le 1$ for all $s\in\mathbb{R}^+$,

$$\rho(t)=0 \text{ for } t\in[0,1] \text{ and } \rho(t)=1 \text{ for } t\in[2,+\infty). \qquad (39)$$

Then there exists a positive constant c such that $|\rho'(s)| \leq c_0$ for all $s \in \mathbb{R}^+$. For convenience, we write $\rho(\frac{|x|^2}{k^2}) = \rho_k$. Taking the inner product of Eq.(24) with $\rho_k v$ in $L^2(\mathbb{R}^N)$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\rho_k|v|^2dx + \lambda\int_{\mathbb{R}^N}\rho_k|v|^2dx - \int_{\mathbb{R}^N}e^{(p-2)\omega(t)}\mathrm{div}(|\nabla v|^{p-2}\nabla v)\rho_k v dx \\ \leq \int_{\mathbb{R}^N}e^{-\omega(t)}\rho_k f(t,x,u)vdx + \int_{\mathbb{R}^N}e^{-\omega(t)}\rho_k g(t)vdx. \tag{40}$$

By (7), the first term on the right-hand side of (40) satisfies

$$\int_{\mathbb{R}^N}e^{-\omega(t)}\rho_k f(t,x,u)vdx \leq -\alpha_1 e^{-2\omega(t)}\int_{\mathbb{R}^N}\rho_k|u|^q dx + e^{-2\omega(t)}\int_{\mathbb{R}^N}\rho_k|\psi_1(t,x)|dx. \tag{41}$$

For the forcing term we have

$$\int_{\mathbb{R}^N}e^{-\omega(t)}\rho_k g(t)vdx \leq \frac{\lambda}{3}\int_{\mathbb{R}^N}\rho_k|v|^2dx + \frac{3}{2\lambda}e^{-2\omega(t)}\int_{\mathbb{R}^N}\rho_k|g(t,x)|^2dx. \tag{42}$$

For the p-Lapacian part, by using Young's inequality and Sobolev interpolation inequality $\|u\|_p^p \leq \|u\|_q^q + \|u\|^2 (q \geq p > 2)$, we have

$$\begin{aligned}
-\int_{\mathbb{R}^N}&e^{(p-2)\omega(t)}\mathrm{div}(|\nabla v|^{p-2}\nabla v)\rho_k v dx \\
&= e^{(p-2)\omega(t)}\int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla v(\nabla\rho_k v + \rho_k\nabla v)dx \\
&= e^{(p-2)\omega(t)}\int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla v.\rho_k'\frac{2x}{k^2}vdx + e^{(p-2)\omega(t)}\int_{\mathbb{R}^N}\rho_k|\nabla v|^p dx \\
&\geq -e^{(p-2)\omega(t)}\int_{\mathbb{R}^N}|\nabla v|^{p-1}|\rho_k'|\frac{2|x|}{k^2}|v|dx \\
&\geq -\frac{2\sqrt{2}c_0}{k}e^{-2\omega(t)}(\|\nabla u\|_p^p + \|u\|_p^p) \\
&\geq -\frac{c_1}{k}e^{-2\omega(t)}(\|\nabla u\|_p^p + \|u\|_q^q + \|u\|^2),
\end{aligned} \tag{43}$$

where $c_1 = c_1(p,q,c_0)$. It follows from (40)-(43) that

$$\frac{d}{dt}\int_{\mathbb{R}^N}\rho_k|v|^2dx + \frac{4\lambda}{3}\int_{\mathbb{R}^N}\rho_k|v|^2dx \leq \frac{2c_1}{k}e^{-2\omega(t)}(\|\nabla u\|_p^p + \|u\|_q^q + \|u\|^2) \\ + 2e^{-2\omega(t)}\int_{\mathbb{R}^N}\rho_k|\psi_1(t,x)|dx + \frac{3}{\lambda}e^{-2\omega(t)}\int_{\mathbb{R}^N}\rho_k|g(t,x)|^2dx. \tag{44}$$

Applying Gronwall's lemma and replacing ω by $\vartheta_{-\tau}\omega$, we have for every $\xi \in [\tau-1, \tau]$,

$$\int_{\mathbb{R}^N}\rho_k|v(\xi,\tau-t,\vartheta_{-\tau}\omega,v_{\tau-t})|^2dx \leq e^{\frac{4\lambda}{3}(\tau-t-\xi)}\|v_{\tau-t}\|^2 \\ + e^{2\omega(-\tau)}\frac{2c_1}{k}\int_{\tau-t}^{\xi}e^{\frac{4\lambda}{3}(s-\xi)-2\omega(s-\tau)}(\|\nabla u(s)\|_p^p + \|u(s)\|_q^q + \|u(s)\|^2)ds \\ + e^{2\omega(-\tau)}\int_{-\infty}^{\xi-\tau}e^{\frac{4\lambda}{3}(s+\tau-\xi)-2\omega(s)}\int_{|x|\geq k}(\frac{3}{\lambda}|g(s+\tau,x)|^2 + 2|\psi_1(s+\tau,x)|)dxds. \tag{45}$$

Since $u_{\tau-t} \in D(\tau-t, \vartheta_{-t}\omega)$ and D is tempered, we find that for every $\xi \in [\tau-1, \tau]$,

$$\limsup_{t\to\infty} e^{\frac{4\lambda}{3}(\tau-t-\xi)}\|v_{\tau-t}\|^2 \le e^{2\omega(-\tau)} \limsup_{t\to\infty} e^{-\frac{4\lambda}{3}t-2\omega(-t)}\|u_{\tau-t}\|^2$$
$$\le e^{2\omega(-\tau)} \limsup_{t\to\infty} e^{-\frac{4\lambda}{3}t-2\omega(-t)}\|D(\tau-t, \vartheta_{-t}\omega)\|^2 = 0.$$

As a consequence, there exists a $T_1 = T_1(\tau, \omega, D)$ such that for all $t \ge T_1$,

$$\limsup_{t\to\infty} e^{\frac{4\lambda}{3}(\tau-t-\xi)}\|v_{\tau-t}\|^2 \le \frac{\varepsilon}{3}. \tag{46}$$

Using (10), there exists a radius $R_1 = R_1(\tau, \omega, \varepsilon)$ such that for every $\xi \in [\tau-1, \tau]$ and all $k \ge R_1$,

$$e^{2\omega(-\tau)} \int_{-\infty}^{\xi-\tau} e^{\frac{4\lambda}{3}(s+\tau-\xi)-2\omega(s)} \int_{|x|\ge k} (\frac{3}{\lambda}|g(s+\tau, x)|^2 + 2|\psi_1(s+\tau, x)|)dxds \le \frac{\varepsilon}{3}. \tag{47}$$

From (37) it follows that there exists a $T_2 = T_2(\tau, \omega, D) \ge T_1$ such that for all $t \ge T_2$,

$$\int_{\tau-t}^{\xi} e^{\frac{4\lambda}{3}(s-\xi)-2\omega(s-\tau)}(\|\nabla u(s)\|_p^p + \|u(s)\|_q^q + \|u(s)\|^2)ds \text{ is bounded.}$$

Then there is a radius $R_2 = R_2(\tau, \omega, D, \varepsilon) \ge R_1$ such that for every $\xi \in [\tau-1, \tau]$ and all $t \ge T_2$ and $k \ge R_2$,

$$e^{2\omega(-\tau)}\frac{2c_1}{k} \int_{\tau-t}^{\xi} e^{\frac{4\lambda}{3}(s-\xi)-2\omega(s-\tau)}(\|\nabla u(s)\|_p^p + \|u(s)\|_q^q + \|u(s)\|^2)ds \le \frac{\varepsilon}{3}. \tag{48}$$

Therefore it follows from (45)-(48) that for every $\xi \in [\tau-1, \tau]$ and all $t \ge T_2(\tau, \omega, D)$ and $k \ge R_2$,

$$\int_{|x|\ge\sqrt{2}k} |v(\xi, \tau-t, \vartheta_{-\tau}\omega, v_{\tau-t})|^2 dx$$
$$\le \int_{\mathbb{R}^N} \rho_k |v(\xi, \tau-t, \vartheta_{-\tau}\omega, v_{\tau-t})|^2 dx \le \varepsilon, \tag{49}$$

which along with the formula

$$u(\xi, \tau-t, \vartheta_{-\tau}\omega, u_{\tau-t}) = e^{\omega(\xi-\tau)-\omega(-\tau)} v(\xi, \tau-t, \vartheta_{-\tau}\omega, v_{\tau-t})$$

implies the desired result. □

We now prove the $\mathfrak{D}$-pullback asymptotical compactness of solutions of Eq.(1) in $L^2(\mathbb{R}^N)$.

Lemma 4.3. *Suppose (7)-(11) hold and $\mathfrak{D}$ is defined by (12). Then the continuous cocycle φ defined by (27) is $\mathfrak{D}$-pullback asymptotically compact in $L^2(\mathbb{R}^N)$.*

Proof. This is followed by a same procedure as in [39, Lemm 3.6]. □

By Lemmas 4.1, 4.3 along with Theorem 2.6 we immediately get the following result.

Theorem 4.4. *Suppose (7)-(11) hold and $\mathfrak{D}$ is defined by (12). Then the continuous cocycle φ generated by the solution of Eq.(1) admits a unique tempered*

$\mathfrak{D}$-pullback random attractor $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ in $L^2(\mathbb{R}^N)$ which is structured by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\mathcal{A}(\tau,\omega) = \Omega(K,\tau,\omega) = \cap_{s>0}\overline{\cup_{t\geq s}\varphi(t,\tau-t,\vartheta_{-t}\omega, K(\tau-t,\vartheta_{-t}\omega))}^{L^2(\mathbb{R}^N)},$$

where $K = \{K(\tau,\omega) : \tau \in R, \omega \in \Omega\}$ is given in Lemma 4.1.

Proof. It follows from Lemma 4.1 that there exists $T = T(\tau,\omega,D) > 0$, such that for all $t \geq T$,

$$\varphi(t,\tau-t,\vartheta_{-t}\omega, D(\tau-t,\vartheta_{-t}\omega)) \subseteq K(\tau,\omega),$$

therefore, $K \in \mathfrak{D}$ is a closed $\mathfrak{D}$-pullback random absorbing set of φ, and by Lemma 4.3 φ is $\mathfrak{D}$-pullback asymptotically compact in $L^2(\mathbb{R}^N)$. Then the desired result follows from Theorem 2.6 immediately. □

In what follows, we will prove the existence of $\mathfrak{D}$-pullback random absorbing set K_δ for the cocycle φ_δ in $L^2(\mathbb{R}^N)$.

Lemma 4.5. *Suppose (7)-(11) hold and $\mathfrak{D}$ is defined by (12). Then the continuous cocycle φ_δ associated with Eq.(5) has a closed $\mathfrak{D}$-pullback random absorbing set $K_\delta = \{K_\delta(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$, which is structured by, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$K_\delta(\tau,\omega) = \{u \in L^2(\mathbb{R}^N) : \|u\|^2 \leq \varrho_\delta(\tau,\omega)\}, \tag{50}$$

where $\varrho_\delta(\tau,\omega)$ is given by

$$\varrho_\delta(\tau,\omega) = 4\int_{-\infty}^{0} e^{\frac{4\lambda}{3}s-2\int_0^s \mathcal{G}_\delta(\vartheta_r\omega)dr}(\frac{3}{2\lambda}\|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_1)ds. \tag{51}$$

In addition, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\lim_{\delta\to 0}\varrho_\delta(\tau,\omega) = \varrho(\tau,\omega)$, where $\varrho(\tau,\omega)$ is defined by (30).

Proof. By a similar technique as in the proof of Lemma 4.1 we have

$$\begin{aligned}\frac{d}{dt}\|v_\delta\|^2 + \frac{4\lambda}{3}\|v_\delta\|^2 + e^{-2\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}(\|\nabla u_\delta\|_p^p + \alpha_1\|u_\delta\|_q^q + \frac{\lambda}{3}\|u_\delta\|^2)\\ \leq 2e^{-2\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}(\frac{3}{2\lambda}\|g(t)\|^2 + \|\psi_1(t)\|_1).\end{aligned} \tag{52}$$

In (52), utilizing Gronwall's lemma over the interval $[\tau-t,\xi]$ with $\xi \in [\tau-1,\tau], t \geq 1$, we get

$$\begin{aligned}&\|v_\delta(\xi,\tau-t,\vartheta_{-\tau}\omega, v_{\delta,\tau-t})\|^2\\ &+\int_{\tau-t}^{\xi} e^{\frac{4\lambda}{3}(s-\xi)-2\int_{-\tau}^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\|\nabla u_\delta(s)\|_p^p + \alpha_1\|u_\delta(s)\|_q^q + \frac{\lambda}{3}\|u_\delta(s)\|^2)ds\\ &\leq e^{\frac{4\lambda}{3}(\tau-t-\xi)}\|v_{\delta,\tau-t}\|^2\\ &+2\int_{\tau-t}^{\xi} e^{\frac{4\lambda}{3}(s-\xi)-2\int_{-\tau}^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\frac{3}{2\lambda}\|g(s)\|^2 + \|\psi_1(s)\|_1)ds.\end{aligned} \tag{53}$$

It follows from (53) and the formula

$$v_\delta(\xi,\tau-t,\vartheta_{-\tau}\omega, v_{\delta,\tau}) = e^{-\int_{-\tau}^{\xi-\tau}\mathcal{G}_\delta(\vartheta_s\omega)ds}u_\delta(\xi,\tau-t,\vartheta_{-\tau}\omega, u_{\delta,\tau})$$

that

$$\begin{aligned}
&e^{-2\int_0^{\xi-\tau}\mathcal{G}_\delta(\vartheta_s\omega)ds}\|u_\delta(\xi,\tau-t,\vartheta_{-\tau}\omega,u_{\delta,\tau-t})\|^2\\
&+\int_{\tau-t}^{\xi}e^{\frac{4\lambda}{3}(s-\xi)-2\int_0^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\|\nabla u_\delta(s)\|_p^p+\alpha_1\|u_\delta(s)\|_q^q+\frac{\lambda}{3}\|u_\delta(s)\|^2)ds\\
&\le e^{\frac{4\lambda}{3}(\tau-t-\xi)-2\int_0^{-t}\mathcal{G}_\delta(\vartheta_r\omega)dr}\|u_{\delta,\tau-t}\|^2\\
&+2\int_{\tau-t}^{\xi}e^{\frac{4\lambda}{3}(s-\xi)-2\int_0^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\frac{3}{2\lambda}\|g(s)\|^2+\|\psi_1(s)\|_1)ds\\
&\le e^{\frac{4\lambda}{3}(\tau-t-\xi)-2\int_0^{-t}\mathcal{G}_\delta(\vartheta_r\omega)dr}\|u_{\delta,\tau-t}\|^2\\
&+2\int_{-\infty}^{0}e^{\frac{4\lambda}{3}(s-\tau-\xi)-2\int_0^{s}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\frac{3}{2\lambda}\|g(s+\tau)\|^2+\|\psi_1(s+\tau)\|_1)ds. \qquad (54)
\end{aligned}$$

Let $\xi=\tau$ in (54). Then for every $u_{\delta,\tau-t}\in D(\tau-t,\vartheta_{-t}\omega)$ there exists $T_1=T_1(\tau,\omega,\delta,D)>2$ such that for all $t\ge T_1$,

$$\begin{aligned}
&\|u_\delta(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\delta,\tau-t})\|^2+\int_{\tau-2}^{\tau}e^{\frac{4\lambda}{3}(s-\tau)-2\int_0^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}\|u_\delta(s)\|_q^q ds\\
&\le\varrho_\delta(\tau,\omega):=4\int_{-\infty}^{0}e^{\frac{4\lambda}{3}s-2\int_0^{s}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\frac{3}{2\lambda}\|g(s+\tau)\|^2+\|\psi_1(s+\tau)\|_1)ds. \qquad (55)
\end{aligned}$$

That is, for all $t\ge T_1$,

$$\begin{aligned}
\varphi_\delta(t,\tau-t,\vartheta_{-t}\omega,D(\tau-t,\vartheta_{-t}\omega))&=u_\delta(\tau,\tau-t,\vartheta_{-\tau}\omega,D(\tau-t,\vartheta_{-t}\omega))\\
&\subseteq K_\delta(\tau,\omega). \qquad (56)
\end{aligned}$$

By (55), we can verify that $K_\delta(\tau,\omega)$ is tempered. Indeed, for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$, it follows from (51) that for any $\gamma>0$,

$$\begin{aligned}
&e^{\gamma t}\|K_\delta(\tau+t,\vartheta_t\omega)\|^2\le e^{\gamma t}\varrho_\delta(\tau+t,\vartheta_t\omega)\\
&=4e^{\gamma t}\int_{-\infty}^{0}e^{\frac{4\lambda}{3}s-2\int_0^{s}\mathcal{G}_\delta(\vartheta_{r+t}\omega)dr}(\frac{3}{2\lambda}\|g(s+\tau+t)\|^2+\|\psi_1(s+\tau+t)\|_1)ds. \qquad (57)
\end{aligned}$$

Note that by Lemma 3.2, there exist positive constants $\tilde c=\tilde c(\omega)$ and $\delta_0=\delta_0(\omega)$ such that for every $s\le 0$ and $0<|\delta|\le\delta_0$,

$$|-2\int_0^s\mathcal{G}_\delta(\vartheta_r\omega)dr|\le\tilde c-\min\{\frac{\lambda}{3},\frac{\gamma}{4}\}s. \qquad (58)$$

Then by (58) for every $s\le 0,t\le 0$ and $0<|\delta|\le\delta_0$,

$$\begin{aligned}
\Big|-2\int_0^s\mathcal{G}_\delta(\vartheta_{r+t}\omega)dr\Big|&=\Big|2\int_0^t\mathcal{G}_\delta(\vartheta_r\omega)dr-2\int_0^{s+t}\mathcal{G}_\delta(\vartheta_r\omega)dr\Big|\\
&\le 2\tilde c-\min\{\frac{\lambda}{3},\frac{\gamma}{4}\}t-\min\{\frac{\lambda}{3},\frac{\gamma}{4}\}(s+t)\\
&\le 2\tilde c-\frac{\lambda}{3}s-\frac{\gamma}{2}t. \qquad (59)
\end{aligned}$$

From (57) and (59) it gives that

$$\begin{aligned}
&e^{\gamma t}\|K_\delta(\tau+t,\vartheta_t\omega)\|^2\\
&\quad\le 4e^{\frac{\gamma}{2}t}e^{2\tilde c}\int_{-\infty}^{0}e^{\lambda s}(\frac{3}{2\lambda}\|g(s+\tau+t)\|^2+\|\psi_1(s+\tau+t)\|_1)ds. \qquad (60)
\end{aligned}$$

Consequently, by (11) and (60) we have for any $\gamma > 0$ and $0 < |\delta| \le \delta_0$,

$$\lim_{t\to-\infty} e^{\gamma t}\|K_\delta(\tau+t,\vartheta_t\omega)\|^2 = 0. \tag{61}$$

On the other hand, by Lemma 3.1 and Lebesgue'dominated convergence theorem, we have

$$\lim_{\delta\to 0}\varrho_\delta(\tau,\omega) = \varrho(\tau,\omega)$$

for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$.

Furthermore, since by (58) we get $e^{\frac{4\lambda}{3}(s-\tau)-2\int_0^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr} \ge e^{-\widetilde{c}+\frac{5}{3}\lambda(s-\tau)}$ for every $s\le\tau$ and $0<|\delta|\le\delta_0$, then it follows that for every $0<|\delta|\le\delta_0$, the second term on the left hand side of (55) is bounded by

$$\begin{aligned}
e^{-\widetilde{c}-\frac{10}{3}\lambda}\int_{\tau-2}^{\tau}\|u_\delta(s)\|_q^q ds &\le \int_{\tau-2}^{\tau} e^{-\widetilde{c}+\frac{5}{3}\lambda(s-\tau)}\|u_\delta(s)\|_q^q ds\\
&\le \int_{\tau-2}^{\tau} e^{\frac{4\lambda}{3}(s-\tau)-2\int_0^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}\|u_\delta(s)\|_q^q ds\\
&\le 4e^{\widetilde{c}}\int_{-\infty}^{0} e^{\lambda s}(\|g(s+\tau)\|^2+\|\psi_1(s+\tau)\|_1)ds\\
&< +\infty,
\end{aligned} \tag{62}$$

which concludes the proof. □

Lemma 4.6. *Suppose (7)-(11) hold. Given $\tau\in\mathbb{R},\omega\in\Omega$, then for every $\varepsilon>0$, there exist $\delta_0=\delta_0(\tau,\omega,\varepsilon)>0, T=T(\tau,\omega,\varepsilon)>0$ and $R=R(\tau,\omega,\varepsilon)>0$ such that for all $0<|\delta|\le\delta_0$, the solution of Eq.(5) satisfies*

$$\sup_{t\ge T}\int_{|x|\ge R}|u_\delta(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\delta,\tau-t})|^2dx \le \varepsilon,$$

for all $u_{\delta,\tau-t}\in K_\delta(\tau-t,\vartheta_{-t}\omega)$, where K_δ is given by (50).

Proof. By some similar calculations as in (44), from (26) we get

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^N}\rho_k|v_\delta|^2dx + \frac{4\lambda}{3}\int_{\mathbb{R}^N}\rho_k|v_\delta|^2dx &\le \frac{c_2}{k}e^{-2\int_0^t\mathcal{G}_\delta(\vartheta_r\omega)dr}(\|\nabla u_\delta\|_p^p+\|u_\delta\|_q^q+\|u_\delta\|^2)\\
&+2e^{-2\int_0^t\mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{|x|\ge k}(\frac{3}{2\lambda}|g(t,x)|^2+|\psi_1(t,x)|)dx.
\end{aligned} \tag{63}$$

Multiply (63) by $e^{\frac{4\lambda}{3}t}$ and then integrate over the interval $[\tau-t,\tau]$, along with ω replaced by $\vartheta_{-\tau}\omega$, to find that

$$\begin{aligned}
&e^{-2\int_{-\tau}^0\mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{\mathbb{R}^N}\rho_k|u_\delta(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\delta,\tau-t})|^2dx\\
&\le e^{-\frac{4\lambda}{3}t-2\int_{-\tau}^{-t}\mathcal{G}_\delta(\vartheta_r\omega)dr}\|u_{\delta,\tau-t}\|^2\\
&+\frac{c_2}{k}\int_{\tau-t}^{\tau}e^{\frac{4\lambda}{3}(s-\tau)-2\int_{-\tau}^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\|\nabla u_\delta(s)\|_p^p+\|u_\delta(s)\|_q^q+\|u_\delta(s)\|^2)ds\\
&+2\int_{\tau-t}^{\tau}e^{\frac{4\lambda}{3}(s-\tau)-2\int_{-\tau}^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{|x|\ge k}(\frac{3}{2\lambda}|g(s,x)|^2+|\psi_1(s,x)|)dxds,
\end{aligned}$$

from which it follows that

$$\begin{aligned}&\int_{\mathbb{R}^N}\rho_k|u_\delta(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\delta,\tau-t})|^2dx\leq e^{-\frac{4\lambda}{3}t-2\int_0^{-t}\mathcal{G}_\delta(\vartheta_r\omega)dr}\|u_{\delta,\tau-t}\|^2\\&+\frac{c_2}{k}\int_{\tau-t}^{\tau}e^{\frac{4\lambda}{3}(s-\tau)-2\int_0^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\|\nabla u_\delta(s)\|_p^p+\|u_\delta(s)\|_q^q+\|u_\delta(s)\|^2)ds\\&+2\int_{\tau-t}^{\tau}e^{\frac{4\lambda}{3}(s-\tau)-2\int_0^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{|x|\geq k}(\frac{3}{2\lambda}|g(s,x)|^2+|\psi_1(s,x)|)dxds.\end{aligned}\tag{64}$$

Since $u_{\delta,\tau-t}\in K_\delta(\tau-t,\vartheta_{-t}\omega)$, then by (58) we have for all $0<|\delta|\leq\delta_0$ and $t\geq 0$,

$$e^{-\frac{4\lambda}{3}t-2\int_0^{-t}\mathcal{G}_\delta(\vartheta_r\omega)dr}\|u_{\delta,\tau-t}\|^2\leq e^{\tilde{c}}e^{-\lambda t}\|K_\delta(\tau-t,\vartheta_{-t}\omega)\|^2,$$

from which and (61) that there exists $T_1=T_1(\tau,\omega,\varepsilon)>0$ such that for all $t\geq T_1$ and $0<|\delta|\leq\delta_0$,

$$e^{-\frac{4\lambda}{3}t-2\int_0^{-t}\mathcal{G}_\delta(\vartheta_r\omega)dr}\|u_{\delta,\tau-t}\|^2\leq\frac{\varepsilon}{3}.\tag{65}$$

By (54), (58) and (10), there exists $T_2=T_2(\tau,\omega,\varepsilon)\geq T_1$ such that for all $t\geq T_2$ and $0<|\delta|\leq\delta_0$,

$$\begin{aligned}&\int_{\tau-t}^{\tau}e^{\frac{4\lambda}{3}(s-\tau)-2\int_0^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\|\nabla u_\delta(s)\|_p^p+\alpha_1\|u_\delta(s)\|_q^q+\frac{\lambda}{3}\|u_\delta(s)\|^2)ds\\&\leq 4\int_{-\infty}^{0}e^{\frac{4\lambda}{3}s-2\int_0^{s}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\frac{3}{2\lambda}\|g(s+\tau)\|^2+\|\psi_1(s+\tau)\|_1)ds\\&\leq 4e^{\tilde{c}}\int_{-\infty}^{0}e^{\lambda s}(\frac{3}{2\lambda}\|g(s+\tau)\|^2+\|\psi_1(s+\tau)\|_1)ds<+\infty,\end{aligned}$$

by which it gives that there exists a constant $R_1=R_1(\tau,\omega,\varepsilon)>0$ such that for all $t\geq T_2,k\geq R_1$ and $0<|\delta|\leq\delta_0$,

$$\frac{c_2}{k}\int_{\tau-t}^{\tau}e^{\frac{4\lambda}{3}(s-\tau)-2\int_0^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\|\nabla u_\delta(s)\|_p^p+\|u_\delta(s)\|_q^q+\|u_\delta(s)\|^2)ds\leq\frac{\varepsilon}{3}.\tag{66}$$

By (58) and (10), there exists $R_2=R_2(\tau,\omega,\varepsilon)\geq R_1$ such that for all $k\geq R_2$ and $0<|\delta|\leq\delta_0$,

$$\begin{aligned}&2\int_{\tau-t}^{\tau}e^{\frac{4\lambda}{3}(s-\tau)-2\int_0^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{|x|\geq k}(\frac{3}{2\lambda}|g(s,x)|^2+|\psi_1(s,x)|)dxds\\&\leq 2e^{\tilde{c}}\int_{-\infty}^{0}e^{\lambda s}\int_{|x|\geq k}(\frac{3}{2\lambda}|g(s+\tau,x)|^2+|\psi_1(s+\tau,x)|)dxds\leq\frac{\varepsilon}{3}.\end{aligned}\tag{67}$$

Then combine (64)-(67) to get that for all $k\geq R_2$ and $0<|\delta|\leq\delta_0$,

$$\begin{aligned}&\int_{|x|\geq\sqrt{2}k}|u_\delta(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\delta,\tau-t})|^2dx\\&\qquad\leq\int_{\mathbb{R}^N}\rho_k|u_\delta(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\delta,\tau-t})|^2dx\leq\varepsilon,\end{aligned}\tag{68}$$

which completes the proof. □

Consider that if we let $\sigma(t,x,u)=u$ then f and $\sigma(t,x,u)$ satisfy the total assumptions (8)-(12) in the reference [39]. Thus by Theorem 3.7 in that paper the random cocycle $\varphi_\delta(\delta\neq 0)$ defined by (28) possesses a unique random attractor in $L^2(\mathbb{R}^N)$, which reads as follows.

Theorem 4.7. *Suppose (7)-(11) hold and $\mathfrak{D}$ is defined by (12). Then the continuous cocycle φ_δ defined by (28) admits a unique $\mathfrak{D}$-pullback random attractor in $L^2(\mathbb{R}^N)$, which is pictured by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\mathcal{A}_\delta(\tau,\omega) = \Omega(K_\delta,\tau,\omega) = \cap_{s>0}\overline{\cup_{t\geq s}\varphi(t,\tau-t,\vartheta_{-t}\omega,K_\delta(\tau-t,\vartheta_{-t}\omega))}^{L^2(\mathbb{R}^N)},$$

where $K_\delta = \{K_\delta(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is defined in Lemma 4.5.

4.2. **Wong-Zakai approximation in L^2.** In this subsection, we study the Wong-Zakai approximation of solutions in $L^2(\mathbb{R}^N)$. We first prove the convergence of solutions to Eq.(5).

Lemma 4.8. *Suppose (7)-(11) hold. Given $\tau \in \mathbb{R}, \omega \in \Omega, T > 0$ and $\varepsilon \in (0,1)$, let u and u_δ be the solutions of Eq.(1) and (5) with initial data $u_{\delta,\tau}$ and u_τ, respectively. Then there exist $\delta_0 = \delta_0(\tau,\omega,T,\varepsilon) > 0$ and $c = c(\tau,\omega,T) > 0$ such that*

$$\sup_{0<|\delta|\leq\delta_0}\sup_{t\in[\tau,\tau+T]}\{\|u_\delta(t,\tau,\omega,u_{\delta,\tau}) - u(t,\tau,\omega,u_\tau)\|^2\} \leq c\|u_{\delta,\tau}-u_\tau\|^2$$
$$+ c\varepsilon(\|u_\tau\|^2 + \|u_{\delta,\tau}\|^2 + \int_\tau^{\tau+T}(\|g(s)\|^2 + \|\psi_1(s)\|_1)ds). \qquad (69)$$

Proof. Let $V_\delta(t,\tau,\omega,V_{\delta,\tau}) = v_\delta(t) - v(t) = v_\delta(t,\tau,\omega,v_{\delta,\tau}) - v(t,\tau,\omega,v_\tau)$, where $v(t)$ and $v_\delta(t)$ are the solutions of (24) and (26), respectively. Then V_δ satisfies that for all $t \geq \tau$,

$$\frac{dV_\delta}{dt} = -\lambda V_\delta + e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\mathrm{div}(|\nabla v_\delta|^{p-2}\nabla v_\delta)$$
$$- e^{(p-2)\omega(t)}\mathrm{div}(|\nabla v|^{p-2}\nabla v) + e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}f(t,x,u_\delta) - e^{-\omega(t)}f(t,x,u)$$
$$+ (e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} - e^{-\omega(t)})g(t,x), \qquad (70)$$

with the initial $V_\delta(\tau,x) = V_{\delta,\tau} = v_{\delta,\tau} - v_\tau$. From (70), we have

$$\frac{1}{2}\frac{d}{dt}\|V_\delta\|^2 + \lambda\|V_\delta\|^2 - (e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\mathrm{div}(|\nabla v_\delta|^{p-2}\nabla v_\delta)$$
$$- e^{(p-2)\omega(t)}\mathrm{div}(|\nabla v|^{p-2}\nabla v), V_\delta)$$
$$= \int_{\mathbb{R}^N}(e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}f(t,x,u_\delta) - e^{-\omega(t)}f(t,x,u))V_\delta dx$$
$$+ \int_{\mathbb{R}^N}(e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} - e^{-\omega(t)})g(t,x)V_\delta dx. \qquad (71)$$

Put

$$U_\delta(t) = u_\delta(t) - u(t) = e^{\int_0^t \mathcal{G}_\delta(\vartheta_s\omega)ds}v_\delta(t) - e^{\omega(t)}v(t).$$

Then we have

$$V_\delta(t) = e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}U_\delta - e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}(e^{\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} - e^{\omega(t)})v. \qquad (72)$$

By Lemma 3.1 and (14)-(15), we deduce that for arbitrary $0 < \varepsilon < 1$, there exists $c_0 = c_0(\tau,\omega,T)$ and $\delta_0 = \delta_0(\tau,\omega,T,\varepsilon)$ such that for all $0 < |\delta| \leq \delta_0$ and $t \in [\tau,\tau+T]$,

$$\left|e^{\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} - e^{\omega(t)}\right| \leq \varepsilon; \qquad (73)$$

$$\left|e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} - e^{-\omega(t)}\right| \leq \varepsilon; \qquad (74)$$

$$\left| e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} - e^{(p-2)\omega(t)} \right| \leq \varepsilon, \tag{75}$$

and

$$e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} \leq c_0, \quad e^{-\omega(t)} \leq c_0. \tag{76}$$

We are now ready to estimate the terms in (71). We rewrite the nonlinearity as

$$\begin{aligned}
&(e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} f(t,x,u_\delta) - e^{-\omega(t)} f(t,x,u))V_\delta \\
&\qquad = e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}(f(t,x,u_\delta) - f(t,x,u))V_\delta \\
&\qquad + (e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} - e^{-\omega(t)})f(t,x,u))V_\delta.
\end{aligned} \tag{77}$$

By (72), (8) and (9), the first term on the right hand side of (77) is estimated as

$$\begin{aligned}
&\int_{\mathbb{R}^N} e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}(f(t,x,u_\delta) - f(t,x,u))V_\delta dx \\
&= e^{-2\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} \int_{\mathbb{R}^N} (f(t,x,u_\delta) - f(t,x,u))U_\delta dx \\
&- e^{-2\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}(e^{\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} - e^{\omega(t)}) \int_{\mathbb{R}^N} (f(t,x,u_\delta) - f(t,x,u))v dx \\
&\leq e^{-2\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} \|\psi_3(t)\|_\infty \|U_\delta\|^2 \\
&+ \varepsilon e^{-2\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} \int_{\mathbb{R}^N} |(f(t,x,u_\delta) - f(t,x,u))||v| dx \\
&\leq e^{-2\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} \|\psi_3(t)\|_\infty \|U_\delta\|^2 \\
&+ \varepsilon e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr - \omega(t)} \int_{\mathbb{R}^N} (\alpha_2(|u_\delta|^{q-1} + |u|^{q-1}) + 2\psi_2(t,x))|u| dx \\
&\leq e^{-2\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} \|\psi_3(t)\|_\infty \|U_\delta\|^2 \\
&+ c\varepsilon e^{-2\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr - \omega(t)} (\|u_\delta\|_q^q + \|u\|_q^q + \|\psi_2(t)\|_{q_1}^{q_1}) \\
\text{by } (76) &\leq c\|U_\delta\|^2 + c\varepsilon(\|u_\delta\|_q^q + \|u\|_q^q + \|\psi_2(t)\|_{q_1}^{q_1}),
\end{aligned} \tag{78}$$

where $\frac{1}{q_1} + \frac{1}{q} = 1$, for all $0 < |\delta| \leq \delta_0$ and $t \in [\tau, \tau + T]$, and the second term on the right hand side of (77), using (8) and (74), we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} (e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} - e^{-\omega(t)}) f(t,x,u)V_\delta dx \leq \varepsilon \int_{\mathbb{R}^N} |f(t,x,u)V_\delta| dx \\
&\leq \varepsilon \int_{\mathbb{R}^N} (\alpha_2 |u|^{q-1} + |\psi_2(t,x)|)|V_\delta| dx \leq \varepsilon(\|V_\delta\|_q^q + \|u\|_q^q + \|\psi_2(t)\|_{q_1}^{q_1}) \\
\text{by } (76) &\leq c\varepsilon(\|u_\delta\|_q^q + \|u\|_q^q + \|\psi_2(t)\|_{q_1}^{q_1}),
\end{aligned} \tag{79}$$

for all $0 < |\delta| \leq \delta_0$ and $t \in [\tau, \tau + T]$. By a combination of (77)-(79) we find that for all $0 < |\delta| \leq \delta_0$ and $t \in [\tau, \tau + T]$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} (e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} f(t,x,u_\delta) - e^{-\omega(t)} f(t,x,u))V_\delta dx \\
&\qquad \leq c\varepsilon(\|u_\delta\|_q^q + \|u\|_q^q + \|\psi_2(t)\|_{q_1}^{q_1}) + c\|U_\delta\|^2,
\end{aligned} \tag{80}$$

where $c = c(\tau, \omega, T)$. For the p-Laplacian part we have

$$
\begin{aligned}
&-(e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\operatorname{div}(|\nabla v_\delta|^{p-2}\nabla v_\delta) - e^{(p-2)\omega(t)}\operatorname{div}(|\nabla v|^{p-2}\nabla v), V_\delta)\\
&= -e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{\mathbb{R}^N}(\operatorname{div}(|\nabla v_\delta|^{p-2}\nabla v_\delta) - \operatorname{div}(|\nabla v|^{p-2}\nabla v))V_\delta dx\\
&- \int_{\mathbb{R}^N}(e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} - e^{(p-2)\omega(t)})\operatorname{div}(|\nabla v|^{p-2}\nabla v)V_\delta dx,
\end{aligned}
\tag{81}
$$

where the first term on the right hand side of (81) is estimated as

$$
\begin{aligned}
&- e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{\mathbb{R}^N}(\operatorname{div}(|\nabla v_\delta|^{p-2}\nabla v_\delta) - \operatorname{div}(|\nabla v|^{p-2}\nabla v))V_\delta dx\\
&= e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{\mathbb{R}^N}(|\nabla v_\delta|^{p-2}\nabla v_\delta - |\nabla v|^{p-2}\nabla v)(\nabla v_\delta - \nabla v)dx \geq 0,
\end{aligned}
\tag{82}
$$

since the mapping $s \mapsto |s|^{p-2}s$ for $p \geq 2$ is increasing on $\mathbb{R}$. The second term on the right hand side of (81), by (75) we have

$$
\begin{aligned}
&- \int_{\mathbb{R}^N}(e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} - e^{(p-2)\omega(t)})\operatorname{div}(|\nabla v|^{p-2}\nabla v)V_\delta dx\\
&= \int_{\mathbb{R}^N}(e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} - e^{(p-2)\omega(t)})(|\nabla v|^{p-2}\nabla v)\nabla V_\delta dx\\
&\geq -\varepsilon\int_{\mathbb{R}^N}|\nabla v|^{p-1}(|\nabla v_\delta| + |\nabla v|)dx\\
&= -\varepsilon\int_{\mathbb{R}^N}(|\nabla v|^p + |\nabla v|^{p-1}|\nabla v_\delta|)dx\\
&\geq -c\varepsilon(\|\nabla v_\delta\|_p^p + \|\nabla v\|_p^p).
\end{aligned}
\tag{83}
$$

By a combination of (81)-(83) we get that for all $0 < |\delta| \leq \delta_0$ and $t \in [\tau, \tau + T]$,

$$
\begin{aligned}
&-(e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\operatorname{div}(|\nabla v_\delta|^{p-2}\nabla v_\delta) - e^{(p-2)\omega(t)}\operatorname{div}(|\nabla v|^{p-2}\nabla v), V_\delta)\\
&\geq -c\varepsilon(\|\nabla v_\delta\|_p^p + \|\nabla v\|_p^p).
\end{aligned}
\tag{84}
$$

By (74) and Young's inequality, we have for all $0 < |\delta| \leq \delta_0$ and $t \in [\tau, \tau + T]$,

$$
\int_{\mathbb{R}^N}(e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} - e^{-\omega(t)})g(t,x)V_\delta dx \leq \frac{\lambda}{2}\|V_\delta\|^2 + c\varepsilon\|g(t)\|^2. \tag{85}
$$

Therefore, plugging (80) and (84)-(85) into (71), it follows that for all $0 < |\delta| \leq \delta_0$ and $t \in [\tau, \tau + T]$,

$$
\begin{aligned}
\frac{d}{dt}\|V_\delta\|^2 &\leq c\|U_\delta\|^2 + c\varepsilon(\|u_\delta\|_q^q + \|u\|_q^q + \|u_\delta\|_p^p + \|u\|_p^p + \|g(t)\|^2 + \|\psi_2(t)\|_{q_1}^{q_1})\\
&\leq c\|V_\delta\|^2 + c\varepsilon(\|u\|^2 + \|u_\delta\|_q^q + \|u\|_q^q + \|\nabla u_\delta\|_p^p + \|\nabla u\|_p^p)\\
&+ c\varepsilon(\|g(t)\|^2 + \|\psi_2(t)\|_{q_1}^{q_1}),
\end{aligned}
\tag{86}
$$

where we have used

$$
\|U_\delta(t)\|^2 \leq c\|V_\delta(t)\|^2 + c\varepsilon\|u(t)\|^2. \tag{87}
$$

Apply Gronwall's lemma in (86) over the interval $[\tau, \tau+T]$ to show that for all $0<|\delta|\le\delta_0$ and $t\in[\tau,\tau+T]$,

$$\begin{aligned}\|V_\delta(t)\|^2 &\le e^{c(t-\tau)}\|V_\delta(\tau)\|^2\\ &\quad + c\varepsilon e^{c(t-\tau)}\int_\tau^t (\|u(s)\|^2+\|u_\delta(s)\|_q^q+\|u(s)\|_q^q+\|\nabla u_\delta(s)\|_p^p+\|\nabla u(s)\|_p^p)ds\\ &\quad + c\varepsilon e^{c(t-\tau)}\int_\tau^t(\|g(s)\|^2+\|\psi_2(s)\|_{q_1}^{q_1})ds. \qquad (88)\end{aligned}$$

Integrate (35) from τ to t to yield

$$\begin{aligned}&\int_\tau^t e^{-2\omega(s)}(\|\nabla u(s)\|_p^p+\alpha_1\|u(s)\|_q^q+\frac{\lambda}{3}\|u(s)\|^2)\\ &\qquad\le \|v_\tau\|^2+2\int_\tau^t e^{-2\omega(s)}(\frac{3}{2\lambda}\|g(s)\|^2+\|\psi_1(s)\|_1)ds\\ &\qquad \text{by (76)} \le c(\|u_\tau\|^2+\int_\tau^{\tau+T}(\|g(s)\|^2+\|\psi_1(s)\|_1)ds). \qquad (89)\end{aligned}$$

Similarly, by (52) we have

$$\begin{aligned}&\int_\tau^t e^{-2\int_0^s \mathcal{G}_\delta(\vartheta_r\omega)dr}(\|\nabla u_\delta(s)\|_p^p+\alpha_1\|u_\delta(s)\|_q^q+\frac{\lambda}{3}\|u_\delta(s)\|^2)ds\\ &\qquad\le \|v_{\delta,\tau}\|^2+2\int_\tau^t e^{-2\int_0^s \mathcal{G}_\delta(\vartheta_r\omega)dr}(\frac{3}{2\lambda}\|g(s)\|^2+\|\psi_1(s)\|_1)ds\\ &\qquad\text{by (76)} \le c(\|u_{\delta,\tau}\|^2+\int_\tau^{\tau+T}(\|g(s)\|^2+\|\psi_1(s)\|_1)ds). \qquad (90)\end{aligned}$$

Then (88)-(90) together imply that

$$\begin{aligned}\|V_\delta(t)\|^2 &\le e^{cT}\|V_{\delta,\tau}\|^2+c\varepsilon(\|u_\tau\|^2+\|u_{\tau,\delta}\|^2\\ &\quad+\int_\tau^{\tau+T}(\|g(s)\|^2+\|\psi_1(s)\|_1)ds), \qquad (91)\end{aligned}$$

from which and (87) we get for all $0<|\delta|\le\delta_0$ and $t\in[\tau,\tau+T]$,

$$\|U_\delta(t)\|^2\le c\|U_{\delta,\tau}\|^2+c\varepsilon(\|u_\tau\|^2+\|u_{\delta,\tau}\|^2+\int_\tau^{\tau+T}(\|g(s)\|^2+\|\psi_1(s)\|_1)ds). \qquad (92)$$

This concludes the proof. □

From Lemma 4.8, we immediately get the convergence of solutions in $L^2(\mathbb{R}^N)$ whenever $\delta_n\to 0$ and $\|u_{\delta_n,\tau}-u_\tau\|\to 0$ as $n\to\infty$.

Theorem 4.9. *Given $\tau\in\mathbb{R},\omega\in\Omega$ and $T>0$, if $\delta_n\to 0$ and $\|u_{\delta_n,\tau}-u_\tau\|\to 0$ as $n\to\infty$, then for every $t\in[\tau,\tau+T]$,*

$$u_{\delta_n}(t,\tau,\omega,u_{\delta_n,\tau})\to u(t,\tau,\omega,u_\tau) \quad \text{in } L^2(\mathbb{R}^N) \text{ as } n\to\infty.$$

Next, we derive the compactness result, which is one of the crucial conditions to prove the upper semi-continuity of attractor $\mathcal{A}_\delta=\{\mathcal{A}_\delta(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$.

Lemma 4.10. *Suppose (7)- (11) hold. Then for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$, if $\delta_n\to 0$ as $n\to\infty$ and $u_n\in\mathcal{A}_{\delta_n}(\tau,\omega)$, then $\{u_n\}_{n=1}^\infty$ is pre-compact in $L^2(\mathbb{R}^N)$ for $q\ge p>2$.*

Proof. By Lemma 4.6 and Theorem 4.9, and a similar procedure to prove [39, Lemma 4.11], we can obtain this result and so the detailed proof is omitted. □

We now show that the random attractor of the approximation equation (5) converges to that of the stochastic p-Laplacian equation (1) driven by multiplicative white noise in $L^2(\mathbb{R}^N)$, in the sense of upper semicontinuity of sets. The main result of this section is given below.

Theorem 4.11. *Suppose (7)-(11) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\lim_{\delta\to 0} \operatorname{dist}_{L^2(\mathbb{R}^N)}(\mathcal{A}_\delta(\tau,\omega), \mathcal{A}(\tau,\omega)) = 0,$$

where $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ and $\mathcal{A}_\delta = \{\mathcal{A}_\delta(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ are random attractors of Eqs.(1) and (5), respectively.

Proof. Let $\delta_n \to 0$ and $u_{n,\tau} \to u_\tau$ in $L^2(\mathbb{R}^N)$, then by Lemma 4.9 we have that for all $\tau \in \mathbb{R}, t \geq 0$ and $\omega \in \Omega$, $\varphi_{\delta_n}(t,\tau,\omega,u_{\delta_n,\tau}) \to \varphi(t,\tau,\omega,u_\tau)$ in $L^2(\mathbb{R}^N)$; By Lemma 4.5 we have for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\limsup_{\delta\to 0} \|K_\delta(\tau,\omega)\| \leq \varrho(\tau,\omega)$. Then in conjunction with Lemma 4.10 we show that all conditions in Theorem 2.7 are fulfilled, and whence the proof is concluded. □

5. L^q-Wong-Zakai approximations.

In this section, we discuss the convergence of random attractors in $L^q(\mathbb{R}^N)$, where $q > 2$ is the growth exponent of the nonlinearity. For this purpose, we need to further assume that $\psi_1 \in L^{\frac{q}{2}}_{loc}(\mathbb{R}; L^{\frac{q}{2}}(\mathbb{R}^N))$ such that

$$\int_{-\infty}^{\tau} e^{\lambda s} \|\psi_1(s)\|^{\frac{q}{2}}_{\frac{q}{2}} ds < +\infty. \tag{93}$$

5.1. Uniform truncation estimates of solutions.

We now deal with the uniform L^q-estimate of solution of Eq.(26) by a truncation approach. To this end, we need to prove the bound of solutions in $L^q(\mathbb{R}^N)$.

Lemma 5.1. *Suppose that (7)-(10) and (93) hold and $\mathfrak{D}$ is defined by (12). Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then for every $u_{\delta,\tau-t} \in D(\tau - t, \vartheta_{-t}\omega)$ with $D \in \mathfrak{D}$, there exist positive constants $\delta_0 = \delta_0(\tau,\omega)$, $\widetilde{c}_i = \widetilde{c}_i(\tau,\omega) (i = 1,2)$ and $T = T(\tau,\omega) \geq 2$ such that the solution of Eq.(26) satisfies that*

$$\sup_{t\geq T} \sup_{\xi\in[\tau-1,\tau]} \sup_{0<|\delta|\leq\delta_0} \left\{ \|v_\delta(\xi,\tau - t,\vartheta_{-\tau}\omega, v_{\delta,\tau-t})\|_q^q \right\} \leq \widetilde{c}_1(\tau,\omega), \tag{94}$$

$$\sup_{t\geq T} \sup_{0<|\delta|\leq\delta_0} \left\{ \int_{\tau-1}^{\tau} \|v_\delta(s,\tau - t,\vartheta_{-\tau}\omega, v_{\delta,\tau-t})\|_{2q-2}^{2q-2} ds \right\} \leq \widetilde{c}_2(\tau,\omega), \tag{95}$$

where $v_{\delta,\tau-t} = e^{-\omega(-t)+\omega(-\tau)} u_{\delta,\tau-t}$.

Proof. Taking the inner product of Eq.(26) in $L^2(\mathbb{R}^N)$ with $|v_\delta|^{q-2}v_\delta$, we have

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|v_\delta\|_q^q + \lambda\|v_\delta\|_q^q = &\int_{\mathbb{R}^N} e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} \operatorname{div}(|\nabla v_\delta|^{p-2}\nabla v_\delta)|v_\delta|^{q-2}v_\delta dx \\ &+ e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} \int_{\mathbb{R}^N} f(t,x,u_\delta)|v_\delta|^{q-2}v_\delta dx \\ &+ e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} \int_{\mathbb{R}^N} g(t,x)|v_\delta|^{q-2}v_\delta dx. \end{aligned}\tag{96}$$

The first term on the right hand side of (96), we have

$$\int_{\mathbb{R}^N} e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\operatorname{div}(|\nabla v_\delta|^{p-2}\nabla v_\delta)|v_\delta|^{q-2}v_\delta dx$$
$$= -(q-1)e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{\mathbb{R}^N}|\nabla v_\delta|^p|v_\delta|^{q-2}dx \le 0. \quad (97)$$

By (7), we get

$$\begin{aligned} e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}&\int_{\mathbb{R}^N} f(t,x,u_\delta)|v_\delta|^{q-2}v_\delta dx \\ &\le e^{-2\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{\mathbb{R}^N}(-\alpha_1|u_\delta|^q+\psi_1(t,x))|v_\delta|^{q-2}dx \\ &\le -\alpha_1 e^{(q-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\|v_\delta\|_{2q-2}^{2q-2} \\ &+\frac{q-\frac{4}{3}}{q}\lambda\|v_\delta\|_q^q + ce^{-q\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\|\psi_1(t)\|_{\frac{q}{2}}^{\frac{q}{2}}. \end{aligned} \quad (98)$$

On the other hand, we have

$$\begin{aligned} e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}&\int_{\mathbb{R}^N} g(t,x)|v_\delta|^{q-2}v_\delta dx \\ &\le \alpha_1 e^{(q-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\|v_\delta\|_{2q-2}^{2q-2} + ce^{-q\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\|g(t)\|^2. \end{aligned} \quad (99)$$

By a combination of (96)-(99), we get

$$\begin{aligned} \frac{d}{dt}\|v_\delta\|_q^q + \frac{4\lambda}{3}\|v_\delta\|_q^q + \alpha_1 e^{(q-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\|v_\delta\|_{2q-2}^{2q-2} \\ \le ce^{-q\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}(\|\psi_1(t)\|_{\frac{q}{2}}^{\frac{q}{2}} + \|g(t)\|^2). \end{aligned} \quad (100)$$

We apply [36, Lemma 6.1] in (100) over the interval $[\tau-2,\xi]$ for $\xi\in[\tau-1,\tau]$ and $p>2$, to find that

$$\begin{aligned} \|v_\delta(\xi,&\tau-t,\vartheta_{-\tau}\omega,v_{\delta,\tau-t})\|_q^q + \alpha_1\int_{\tau-1}^{\tau} e^{\frac{4\lambda}{3}(s-\tau)+(q-2)\int_{-\tau}^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}\|v_\delta(s)\|_{2q-2}^{2q-2}ds \\ &\le (e^{\frac{4\lambda}{3}}+1)\int_{\tau-2}^{\tau} e^{\frac{4\lambda}{3}(s-\tau)}\|v_\delta(s,\tau-t,\vartheta_{-\tau}\omega,v_{\delta,\tau-t})\|_q^q ds \\ &+ (e^{\frac{4\lambda}{3}}+2)\int_{\tau-2}^{\tau} e^{\frac{4\lambda}{3}(s-\tau)-q\int_{-\tau}^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\|\psi_1(s)\|_{\frac{q}{2}}^{\frac{q}{2}}+\|g(s)\|^2)ds \\ &\le (e^{\frac{4\lambda}{3}}+1)\int_{\tau-2}^{\tau}\|v_\delta(s,\tau-t,\vartheta_{-\tau}\omega,v_{\delta,\tau-t})\|_q^q ds \\ &+ (e^{\frac{4\lambda}{3}}+2)\int_{-2}^{0} e^{\frac{4\lambda}{3}s-q\int_{-\tau}^{s}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\|\psi_1(s+\tau)\|_{\frac{q}{2}}^{\frac{q}{2}}+\|g(s+\tau)\|^2)ds. \end{aligned} \quad (101)$$

We estimate every term on the right hand side of (101). By (62) we deduce that there exist $T=T(\tau,\omega)>0$ and $\delta_1=\delta_1(\tau,\omega)>0$ such that for all $t\ge T$ and $0<|\delta|\le\delta_1$,

$$\int_{\tau-2}^{\tau}\|v_\delta(s,\tau-t,\vartheta_{-\tau}\omega,v_{\delta,\tau-t})\|_q^q ds < +\infty. \quad (102)$$

By Lemma 3.2, there exists $\delta_2 = \delta_2(\tau, \omega)$ such that for all $0 < |\delta| \le \delta_2$, we have $e^{\frac{4\lambda}{3}s - q\int_{-\tau}^{s} \mathcal{G}_\delta(\vartheta_r\omega)dr} \le e^{\frac{4\lambda}{3}s + qc(\tau,\omega) - \frac{q\lambda}{6}s} \le e^{qc(\tau,\omega)+\frac{q\lambda}{3}}$ for all $s \in [-2, 0]$, and therefore by (10) and (93) it follows that

$$\int_{-2}^{0} e^{\frac{4\lambda}{3}s - q\int_{-\tau}^{s} \mathcal{G}_\delta(\vartheta_r\omega)dr}(\|\psi_1(s+\tau)\|_{\frac{q}{2}}^{\frac{q}{2}} + \|g(s+\tau)\|^2)ds$$
$$\le e^{qc(\tau,\omega)+\frac{q\lambda}{3}} \int_{-2}^{0}(\|\psi_1(s+\tau)\|_{\frac{q}{2}}^{\frac{q}{2}} + \|g(s+\tau)\|^2)ds < +\infty. \tag{103}$$

Let $\delta_0 = \min\{\delta_1, \delta_2\}$. Then (101)-(103) together imply that there exists positive constant $\widetilde{c}_i = \widetilde{c}_i(q, \tau, \omega)(i = 1, 2)$ such that

$$\sup_{t\ge T} \sup_{\xi\in[\tau-1,\tau]} \sup_{0<|\delta|\le\delta_0} \left\{\|v_\delta(\xi, \tau - t, \vartheta_{-\tau}\omega, v_{\delta,\tau-t})\|_q^q\right\} \le \widetilde{c}_1(\tau, \omega),$$

and

$$\sup_{t\ge T} \sup_{0<|\delta|\le\delta_0} \int_{\tau-1}^{\tau} \|v_\delta(s, \tau - t, \vartheta_{-\tau}\omega, v_{\delta,\tau-t})\|_{2q-2}^{2q-2} ds \le \widetilde{c}_2(\tau, \omega).$$

which completes the proof. □

The following lemma is concerned with the truncation approach, by which we show that the solution of Eq.(5) vanishes in L^q with a changing domain on which its value diverges to infinite.

Lemma 5.2. *Suppose that (7)-(10) and (93) hold. Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_{\tau-t} \in K_\delta(\tau - t, \vartheta_{-t}\omega)$. Then for any $\varepsilon > 0$, there exist positive constants $\delta_0 = \delta_0(\tau, \omega, \varepsilon), c = c(\tau, \omega), M_0 = M_0(\tau, \omega, \varepsilon)$ and $T = T(\tau, \omega, \varepsilon)$ such that the solution u_δ of Eq.(1) satisfies that*

$$\sup_{t\ge T} \sup_{0<|\delta|\le\delta_0} \left\{ \int_{(|u_\delta(\tau)|\ge M_0)} |u_\delta(\tau, \tau - t, \vartheta_{-\tau}\omega, u_{\delta,\tau-t})|^q dx\right\} \le c\varepsilon, \tag{104}$$

where $(|u_\delta(\tau)| \ge M_0) = \{x \in \mathbb{R}^{\mathbb{N}} : |u_\delta(\tau, \tau - t, \vartheta_{-\tau}\omega, u_{\delta,\tau-t})| \ge M_0\}$ and K_δ is as in (29).

Proof. Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $s \in [\tau - 1, \tau]$, let $v_\delta(s) = v_\delta(s, \tau - t, \vartheta_{-\tau}\omega, v_{\delta,\tau-t})$ be the solution of Eq.(26) at the sample $\vartheta_{-\tau}\omega$ with the initial value $v_{\delta,\tau-t}$. Put $M = M(\tau, \omega) > 0$. Denote by $(v_\delta(s) - M)_+$ the nonnegative part of $v_\delta(s) - M$. Multiplying (26) by $|(v_\delta(s) - M)_+|^{q-2}(v_\delta(s) - M)_+$ and integrate over $\mathbb{R}^N$, we obtain

$$\frac{1}{p}\frac{d}{ds}\int_{\mathbb{R}^N}(v_\delta(s) - M)_+^q dx + \lambda\int_{\mathbb{R}^N}(v_\delta(s) - M)_+^q dx$$
$$= \int_{\mathbb{R}^N} e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\mathrm{div}(|\nabla v_\delta(s)|^{p-2}\nabla v_\delta(s))|(v_\delta(s) - M)_+|^{q-2}(v_\delta(s) - M)_+ dx$$
$$+ \int_{\mathbb{R}^N} e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} f(t, x, u_\delta(s))|(v_\delta(s) - M)_+|^{q-2}(v_\delta(s) - M)_+ dx$$
$$+ \int_{\mathbb{R}^N} e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr} g(s, x)|(v_\delta(s) - M)_+|^{q-2}(v_\delta(s) - M)_+ dx. \tag{105}$$

The first term on the right hand side of (105) is estimated by

$$\int_{\mathbb{R}^N} e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}\mathrm{div}(|\nabla v_\delta(s)|^{p-2}\nabla v_\delta(s))|(v_\delta(s)-M)_+|^{q-2}(v_\delta(s)-M)_+dx$$
$$= -(q-1)e^{(p-2)\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}\int_{\mathbb{R}^N}|\nabla v_\delta(s)|^p|(v_\delta(s)-M)_+|^{q-2}dx \le 0. \tag{106}$$

Since $v_\delta(s) > M > 0$ for $s \in [\tau-1,\tau]$, we have $u_\delta(s) = e^{\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}v_\delta(s) > 0$. Therefore utilizing (7) the nonlinearity term in (105) is bounded by

$$\begin{aligned} f(s,x,u_\delta(s)) &\le -\alpha_1 e^{(q-1)\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}|v_\delta(s)|^{q-1} + \frac{e^{-\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}\psi_1(s,x)}{v_\delta(s)} \\ &\le -\alpha_1 e^{(q-1)\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}|v_\delta(s)|^{q-1} + \frac{e^{-\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}\psi_1(s,x)}{v_\delta(s)-M}, \end{aligned}$$

from which it follows that for each $s \in [\tau-1,\tau]$,

$$\begin{aligned} &\int_{\mathbb{R}^N} e^{-\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr} f(s,x,u_\delta(s))|(v_\delta(s)-M)_+|^{q-2}(v_\delta(s)-M)_+dx \\ &\le -\alpha_1\int_{\mathbb{R}^N} e^{(q-2)\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}|v_\delta(s)|^{q-1}|(v_\delta(s)-M)_+|^{q-2}(v_\delta(s)-M)_+dx \\ &+ \int_{\mathbb{R}^N} e^{-2\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}\psi_1(s,x)|(v_\delta(s)-M)_+|^{q-2}dx \\ &\le -\alpha_1 e^{(q-2)\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}\int_{\mathbb{R}^N}(\frac{1}{2}M^{q-2}|(v_\delta(s)-M)_+| + \frac{1}{2}|(v_\delta(s)-M)_+|^{q-1}) \\ &|(v_\delta(s)-M)_+|^{q-2}(v_\delta(s)-M)_+dx + ce^{-q\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}\|\psi_1(s)\|_{\frac{q}{2}}^{\frac{q}{2}} \\ &+ \lambda\|(v_\delta(s)-M)_+\|_q^q \\ &\le -\frac{\alpha_1}{2}e^{(q-2)\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}(M^{q-2}\|(v_\delta(s)-M)_+\|_q^q + \|(v_\delta(s)-M)_+\|_{2q-2}^{2q-2}) \\ &+ ce^{-q\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}\|\psi_1(s)\|_{\frac{q}{2}}^{\frac{q}{2}} + \lambda\|(v_\delta(s)-M)_+\|_q^q. \end{aligned} \tag{107}$$

If $v(s) \equiv M$ for $s \in [\tau-1,\tau]$, then (107) also holds true. For the non-autonomous term, using Young's inequality we have

$$\begin{aligned} &\int_{\mathbb{R}^N} e^{-\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}g(s,x)|(v_\delta(s)-M)_+|^{q-2}(v_\delta(s)-M)_+dx \\ &\le \frac{\alpha_1}{2}e^{(q-2)\int_0^t \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}\|(v_\delta(s)-M)_+\|_{2q-2}^{2q-2} + ce^{-q\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}\|g(s)\|^2. \end{aligned} \tag{108}$$

Therefore (107)-(108) together show that

$$\begin{aligned} \frac{d}{ds}\int_{\mathbb{R}^N}(v_\delta(s)-M)_+^q dx + \frac{\alpha_1}{2}e^{(q-2)\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}M^{q-2}\int_{\mathbb{R}^N}(v_\delta(s)-M)_+^q dx \\ \le ce^{-q\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr}(\|\psi_1(s)\|_{\frac{q}{2}}^{\frac{q}{2}} + \|g(s)\|^2). \end{aligned} \tag{109}$$

By Lemma 3.2, there exists a $\delta_1 = \delta_1(\tau,\omega) > 0$ such that for all $0 < |\delta| \le \delta_1$ and $s \in [\tau-1,\tau]$,

$$\Big|\int_0^s \mathcal{G}_\delta(\vartheta_{r-\tau}\omega)dr\Big| \le C_{\tau,\omega}. \tag{110}$$

Then (109) is rewrote as

$$\frac{d}{ds}\int_{\mathbb{R}^N}(v_\delta(s)-M)_+^q dx+\chi(M)\int_{\mathbb{R}^N}(v_\delta(s)-M)_+^q dx \\ \leq ce^{qC_{\tau,\omega}}(\|\psi_1(s)\|_{\frac{q}{2}}^{\frac{q}{2}}+\|g(s)\|^2). \qquad (111)$$

where $\chi(M)=\frac{\alpha_1}{2}e^{-(q-2)C_{\tau,\omega}}M^{q-2}$ and constant $c>0$. Clearly $\lim_{M\to+\infty}\chi(M)\to\infty$ for any $q>2$. We apply [37, Lemma 4.2] to over interval $[\tau-1,\tau]$ to yield

$$\int_{\mathbb{R}^N}(v_\delta(\tau)-M)_+^q dx\leq\int_{\tau-1}^{\tau}e^{\chi(M)(s-\tau)}\|v_\delta(s)\|_q^q ds \\ +ce^{qC_{\tau,\omega}}\int_{\tau-1}^{\tau}e^{\chi(M)(s-\tau)}(\|g(s)\|^2+\|\psi_1(s)\|_{\frac{q}{2}}^{\frac{q}{2}})ds. \qquad (112)$$

In terms of (94), there exist constants $\delta_2=\delta_2(\tau,\omega)>0$, $\widehat{c}=\widehat{c}(\tau,\omega)>0$ and $T_1=T_1(\tau,\omega)$ such that for all $t\geq T_1$ and $0<|\delta|\leq\delta_2$,

$$\int_{\tau-1}^{\tau}e^{\chi(M)(s-\tau)}\|v_\delta(s,\tau-t,\vartheta_{-\tau}\omega,v_{\delta,\tau-t})\|_q^q ds\leq\frac{\widehat{c}}{\chi(M)}\to 0, M\to+\infty. \qquad (113)$$

As for the non-autonomous term of the right in (112), for a large positive number M such that $\chi(M)>\lambda$ and taking $\eta\in(0,1)$, we may derive

$$\int_{\tau-1}^{\tau}e^{\chi(M)(s-\tau)}\|g(s)\|^2ds=\int_{\tau-1}^{\tau-\eta}e^{\chi(M)(s-\tau)}\|g(s)\|^2ds+\int_{\tau-\eta}^{\tau}e^{\chi(M)(s-\tau)}\|g(s)\|^2ds \\ =e^{-\chi(M)\tau}\int_{\tau-1}^{\tau-\eta}e^{(\chi(M)-\lambda)s}e^{\lambda s}\|g(s)\|^2ds+e^{-\chi(M)\tau}\int_{\tau-\eta}^{\tau}e^{\chi(M)s}\|g(s)\|^2ds \\ \leq e^{-\chi(M)\eta}e^{\lambda(\eta-\tau)}\int_{-\infty}^{\tau}e^{\lambda s}\|g(s)\|^2ds+\int_{\tau-\eta}^{\tau}\|g(s)\|^2ds. \qquad (114)$$

for M large enough. Indeed, by (10) the first term on the right hand side of (114) converges to zero when $M\to+\infty$, by $g\in L^2_{loc}(\mathbb{R},L^2(\mathbb{R}^N))$ and η is small enough that the second term is small enough. Analogously, the integral $\int_{\tau-1}^{\tau}e^{\chi(M)(s-\tau)}$ $\|\psi_1(s)\|_{\frac{q}{2}}^{\frac{q}{2}}ds$ also approaches zero when $M\to+\infty$. Put

$$\delta_0=\min\{\delta_1,\delta_2\}.$$

Then from (112)-(114) it follows that there exists M_1 large enough such that

$$\sup_{t\geq T_1}\sup_{0<|\delta|\leq\delta_0}\int_{\mathbb{R}^N}(v_\delta(\tau)-M_1)_+^q dx\leq\varepsilon. \qquad (115)$$

Note that $v_\delta-M_1>\frac{v_\delta}{2}$ for $v_\delta>2M_1$. Then we have the set inclusion $\{x\in\mathbb{R}^N: v_\delta-M_1>\frac{v_\delta}{2}\}\supset\{x\in\mathbb{R}^N: v_\delta\geq 2M_1\}$. So we see from (115) that

$$\sup_{t\geq T_1}\sup_{0<|\delta|\leq\delta_0}\int_{(v_\delta(\tau)\geq 2M_1)}|v_\delta(\tau)|^q dx\leq 2^q\sup_{t\geq T_1}\sup_{0<|\delta|\leq\delta_0}\int_{\mathbb{R}^N}(v_\delta(\tau)-M_1)_+^q dx\leq c\varepsilon,$$

from which and $v_\delta(\tau,\tau-t,\vartheta_{-\tau}\omega,v_{\delta,\tau}) = e^{-\int_{-\tau}^0 \mathcal{G}_\delta(\vartheta_r\omega)dr} u_\delta(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\delta,\tau})$ we have

$$\begin{aligned}&\sup_{t\geq T_1}\sup_{0<|\delta|\leq\delta_0}\int_{(u_\delta(\tau)\geq 2e^{\int_{-\tau}^0 \mathcal{G}_\delta(\vartheta_r\omega)dr}M_1)}|u_\delta(\tau)|^q dx\\ &=\sup_{t\geq T_1}\sup_{0<|\delta|\leq\delta_0} e^{q\int_{-\tau}^0 \mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{(v_\delta(\tau)\geq 2M_1)}|v_\delta(\tau)|^q dx\leq c\varepsilon. \end{aligned}\tag{116}$$

By (110), we get that $u_\delta(\tau)\geq 2e^{\int_{-\tau}^0 \mathcal{G}_\delta(\vartheta_r\omega)dr}M_1$ if $u_\delta(\tau)\geq 2e^{C_{\tau,\omega}}M_1$. Put $\widetilde{M_1} = 2e^{C_{\tau,\omega}}M_1$. Then it follows from (116) that

$$\begin{aligned}&\sup_{t\geq T_1}\sup_{0<|\delta|\leq\delta_0}\int_{(u_\delta(\tau)\geq \widetilde{M_1})}|u_\delta(\tau)|^q dx\\ &\leq\sup_{t\geq T_1}\sup_{0<|\delta|\leq\delta_0}\int_{(u_\delta(\tau)\geq 2e^{\int_{-\tau}^0 \mathcal{G}_\delta(\vartheta_r\omega)dr}M_1)}|u_\delta(\tau)|^q dx\leq c\varepsilon. \end{aligned}\tag{117}$$

Multiplying (26) with $|(v_\delta+M)_-|^{q-2}(v_\delta+M)_-$ and integrate over $\mathbb{R}^N$, where $(v_\delta+M)_-$ is negative part of $v_\delta+M$ for some $M=M(\tau,\omega)>0$, and exerting a similar procedure, we can deduce that there exist positive constants $T_2=T_2(\tau,\omega),\delta_0'=\delta_0'(\tau,\omega)$ and $\widetilde{M_2}=\widetilde{M_2}(\tau,\omega)$ such that

$$\sup_{t\geq T_2}\sup_{0<|\delta|\leq\delta_0'}\int_{(u_\delta(\tau)\leq -\widetilde{M_2})}|u_\delta(\tau)|^q dx\leq c\varepsilon. \tag{118}$$

Then (117) and (118) together imply the desired. □

5.2. **Wong-Zakai approximations in L^q.** By the result in [19, Theorem 3.1], we know that for every $\delta\neq 0$, the $\mathfrak{D}$-pullback random attractor $\mathcal{A}_\delta$ established in $L^2(\mathbb{R}^N)$ is also a unique tempered $\mathfrak{D}$-pullback random attractor in $L^q(\mathbb{R}^N)$ with $q\geq p>2$, that is to say, for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$, $\mathcal{A}_\delta(\tau,\omega)$ is compact and attracting in the topology of L^q.

Proposition 1. *Suppose that (7)-(11) and (93) hold and $\mathfrak{D}$ is defined by (12). Then for every $\delta\neq 0$, the tempered $\mathfrak{D}$-pullback random attractor $\mathcal{A}_\delta$ derived in Theorem 4.7 for the continuous cocycle φ_δ defined by the (28) is also a unique tempered $\mathfrak{D}$-pullback random attractor in $L^q(\mathbb{R}^N)$ with $q\geq p>2$. Moreover, for each $\tau\in\mathbb{R}$ and $\omega\in\Omega$,*

$$\mathcal{A}_\delta(\tau,\omega)=\Omega(K_\delta,\tau,\omega)=\cap_{s>0}\overline{\cup_{t\geq s}\varphi(t,\tau-t,\vartheta_{-t}\omega,K_\delta(\tau-t,\vartheta_{-t}\omega))}^{L^q(\mathbb{R}^N)},$$

where $K_\delta=\{K_\delta(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega)\}$ is given by (50).

We now consider the high-order Wong-Zakai approximation of equation (5) in $L^q(\mathbb{R}^N)$. We first show the pre-compactness of the family of attractors $\mathcal{A}_\delta$ in $L^q(\mathbb{R}^N)$ as $\delta\to 0$.

Lemma 5.3. *Suppose that (7)-(10) and (93) hold. Then for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$, if $\delta_n\to 0$ as $n\to\infty$ and $u_n\in\mathcal{A}_{\delta_n}(\tau,\omega)$, then $\{u_n\}_{n=1}^\infty$ has a convergent subsequence in $L^q(\mathbb{R}^N)$.*

Proof. According to the invariance of $\mathcal{A}_\delta$ and the decomposability of space $\mathbb{R}^N$, along with Lemma 4.10, the proof is quite similar to that of [39, Lemma 5.4]. The details are omitted here. □

We are ready to state the main result of this section in the following theorem, which shows the convergence in $L^q(\mathbb{R}^N)$ for the tempered random attractors $\mathcal{A}_\delta$ and $\mathcal{A}$ for Eqs.(5) and (1).

Theorem 5.4. *Suppose that (7)-(11) and (93) hold. Then for every fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$.*

$$\lim_{\delta\to 0} \mathrm{dist}_{L^q(\mathbb{R}^N)}(\mathcal{A}_\delta(\tau,\omega), \mathcal{A}(\tau,\omega)) = 0,$$

where $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ and $\mathcal{A}_\delta = \{\mathcal{A}_\delta(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ are the tempered pullback random attractors corresponding to Eqs.(1) and (5), respectively.

Proof. By Theorem 4.9 we have that for all $\tau \in \mathbb{R}, t \geq 0$ and $\omega \in \Omega$,

$$\varphi_{\delta_n}(t,\tau,\omega,u_{\delta_n,\tau}) \to \varphi(t,\tau,\omega,u_\tau)$$

in $L^2(\mathbb{R}^N)$ whenever $\delta_n \to 0$ and $u_{n,\tau} \to u_\tau$ in $L^2(\mathbb{R}^N)$. By Lemma 4.5 it follows that $\limsup_{\delta\to 0} \|K_\delta(\tau,\omega)\| \leq \varrho(\tau,\omega)$. Then in conjunction with Lemma 4.10 and Lemma 5.3 we see that all conditions in Theorem 2.7 are fulfilled, which concludes the proof. □

6. L^l-Wong-Zakai approximation ($\forall l > q$).

In this section, we consider the Wong-Zakai approximation in $L^l(\mathbb{R}^N)$ for arbitrary $l > q$, which is an interesting work and not done in any literature. To do this, we need to further assume that

$$g, \psi_1 \in L^\infty_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^N)). \tag{119}$$

By $g \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^N))$ and $\psi_1 \in L^{\frac{q}{2}}_{loc}(\mathbb{R}; L^{\frac{q}{2}}(\mathbb{R}^N))$, we know that

$$g, \psi_1 \in L^l_{loc}(\mathbb{R}; L^l(\mathbb{R}^N))$$

for any $l > q$, under which we can prove that the solution is bounded in $L^l(\mathbb{R}^N)$ for arbitrary $l > q$. This is done by the traditional mathematical induction technique.

Lemma 6.1. *Let $l > q$ be arbitrary. Suppose that (7)-(10) and (119) hold and $\mathfrak{D}$ is defined by (12). Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $k \in \mathbb{N}$, then for every $u_{\delta,\tau-t} \in D(\tau-t, \vartheta_{-t}\omega)$ with $D \in \mathfrak{D}$, there exist positive constants $\delta_0 = \delta_0(\tau,\omega)$, $c = c(\tau,\omega,l)$ and $T = T(\tau,\omega,l) \geq 2$ such that the solution of Eq.(5) satisfies that*

$$\sup_{t\geq T} \sup_{0<|\delta|\leq\delta_0} \Big\{ \|u_\delta(\tau,\tau-t,\vartheta_{-\tau}\omega, u_{\delta,\tau-t})\|_l^l \Big\} \leq c(\tau,\omega,l).$$

Proof. We first prove by induction that there exist positive constants $\delta_0 = \delta_0(\tau,\omega)$, $\widetilde{c}_i^{(k)} = \widetilde{c}_i^{(k)}(\tau,\omega)(i = 1,2)$ and $T_k = T_k(\tau,\omega) \geq 2$ such that the solution v_δ of Eq.(26) satisfies

$$\sup_{t\geq T_k} \sup_{0<|\delta|\leq\delta_0} \Big\{ \|v_\delta(\xi,\tau-t,\vartheta_{-\tau}\omega, v_{\delta,\tau-t})\|_{qa_k}^{qa_k} \Big\} \leq \widetilde{c}_1^{(k)}(\tau,\omega), \quad \xi \in [\tau - \frac{1}{k}, \tau], \tag{120}$$

and

$$\sup_{t\geq T_k} \sup_{0<|\delta|\leq\delta_0} \Big\{ \int_{\tau-\frac{1}{k}}^{\tau} \|v_\delta(s,\tau-t,\vartheta_{-\tau}\omega, v_{\delta,\tau-t})\|_{qa_{k+1}}^{qa_{k+1}} ds \Big\} \leq \widetilde{c}_2^{(k)}(\tau,\omega), \tag{121}$$

where

$$a_1 = 1, \quad a_{k+1} = a_k + \frac{q-2}{q}.$$

If $k = 1$, the result is given in Lemma 5.1. We assume that (120) and (121) hold for k, we then prove that it is true for $k+1$.

Multiplying (26) by $|v_\delta|^{qa_{k+1}-2}v_\delta$ and integrating over $\mathbb{R}^N$, we have

$$\begin{aligned}
&\frac{1}{qa_{k+1}}\frac{d}{dt}\|v_\delta\|_{qa_{k+1}}^{qa_{k+1}}+\lambda\|v_\delta\|_{qa_{k+1}}^{qa_{k+1}}\\
&\qquad=\int_{\mathbb{R}^N} e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\mathrm{div}(|\nabla v_\delta|^{p-2}\nabla v_\delta)|v_\delta|^{qa_{k+1}-2}v_\delta dx\\
&\qquad+\int_{\mathbb{R}^N} e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}f(t,x,u_\delta)|v_\delta|^{qa_{k+1}-2}v_\delta dx\\
&\qquad+e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{\mathbb{R}^N} g(t,x)|v_\delta|^{qa_{k+1}-2}v_\delta dx,
\end{aligned}\tag{122}$$

where we have

$$\int_{\mathbb{R}^N} e^{(p-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\mathrm{div}(|\nabla v_\delta|^{p-2}\nabla v_\delta)|v_\delta|^{qa_{k+1}-2}v_\delta dx\le 0;\tag{123}$$

For the nonlinearity in (122) by using (7) and Young's inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}f(t,x,u_\delta)|v_\delta|^{qa_{k+1}-2}v_\delta dx\\
&\le e^{-2\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{\mathbb{R}^N}(-\alpha_1|u_\delta|^q+\psi_1(t,x))|v_\delta|^{qa_{k+1}-2}dx\\
&=e^{-2\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{\mathbb{R}^N}(-\alpha_1 e^{q\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}|v_\delta|^q+\psi_1(t,x))|v_\delta|^{qa_{k+1}-2}dx\\
&=-\alpha_1 e^{(q-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\|v_\delta\|_{qa_{k+1}+q-2}^{qa_{k+1}+q-2}\\
&+e^{-2\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{\mathbb{R}^N}\psi_1(t,x))|v_\delta|^{qa_{k+1}-2}dx\\
&\le -\alpha_1 e^{(q-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\|v_\delta\|_{qa_{k+2}}^{qa_{k+2}}\\
&+\frac{(qa_{k+1}-\frac{5}{3})\lambda}{qa_{k+1}}\|v_\delta\|_{qa_{k+1}}^{qa_{k+1}}+c_k e^{-qa_{k+1}\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\|\psi_1(t)\|_{\frac{qa_{k+1}}{2}}^{\frac{qa_{k+1}}{2}}.
\end{aligned}\tag{124}$$

And for the last term on the right hand side of (122), by Young's inequality again we get

$$\begin{aligned}
&e^{-\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\int_{\mathbb{R}^N} g(t,x)|v_\delta|^{qa_{k+1}-2}v_\delta dx\\
&\qquad\le\frac{\frac{1}{3}\lambda}{qa_{k+1}}\|v_\delta\|_{qa_{k+1}}^{qa_{k+1}}+c_k e^{-qa_{k+1}\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\|g(t)\|_{qa_{k+1}}^{qa_{k+1}}.
\end{aligned}\tag{125}$$

By (122)-(125), we deduce that

$$\begin{aligned}
&\frac{d}{dt}\|v_\delta\|_{qa_{k+1}}^{qa_{k+1}}+\frac{4\lambda}{3}\|v_\delta\|_{qa_{k+1}}^{qa_{k+1}}+\alpha_1 e^{(q-2)\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}\|v_\delta\|_{qa_{k+2}}^{qa_{k+2}}\\
&\qquad\le c_k e^{-qa_{k+1}\int_0^t \mathcal{G}_\delta(\vartheta_r\omega)dr}(\|g(t)\|_{qa_{k+1}}^{qa_{k+1}}+\|\psi_1(t)\|_{\frac{qa_{k+1}}{2}}^{\frac{qa_{k+1}}{2}}).
\end{aligned}\tag{126}$$

We apply [36, Lemma 6.1] in (126) over the interval $[\tau-\frac{1}{k+1},\tau]$ to find that, for every $\xi\in[\tau-\frac{1}{k+1},\tau]$,

$$\|v_\delta(\xi)\|_{qa_{k+1}}^{qa_{k+1}}+\alpha_1\int_{\tau-\frac{1}{k+1}}^{\tau} e^{\frac{4\lambda}{3}(s-\tau)+(q-2)\int_{-\tau}^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}\|v_\delta(s)\|_{qa_{k+2}}^{qa_{k+2}}ds$$

$$\leq k(k+1)(e^{\frac{4\lambda}{3(k+1)}}+1)\int_{\tau-\frac{1}{k}}^{\tau} e^{\frac{4\lambda}{3}(s-\tau)}\|v_\delta(s)\|_{qa_{k+1}}^{qa_{k+1}}ds$$
$$+c_k(e^{\frac{4\lambda}{3(k+1)}}+2)\int_{\tau-\frac{1}{k}}^{\tau} e^{\frac{4\lambda}{3}(s-\tau)-qa_{k+1}\int_{-\tau}^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\|g(s)\|_{qa_{k+1}}^{qa_{k+1}}$$
$$+\|\psi_1(s)\|_{\frac{qa_{k+1}}{2}}^{\frac{qa_{k+1}}{2}})ds$$
$$\leq k(k+1)(e^{\frac{4\lambda}{3(k+1)}}+1)\int_{\tau-\frac{1}{k}}^{\tau}\|v_\delta(s)\|_{qa_{k+1}}^{qa_{k+1}}ds$$
$$+c_k(e^{\frac{4\lambda}{3(k+1)}}+2)\int_{-\frac{1}{k}}^{0} e^{\frac{4\lambda}{3}s-qa_{k+1}\int_{-\tau}^{s}\mathcal{G}_\delta(\vartheta_r\omega)dr}(\|g(s+\tau)\|_{qa_{k+1}}^{qa_{k+1}}$$
$$+\|\psi_1(s+\tau)\|_{\frac{qa_{k+1}}{2}}^{\frac{qa_{k+1}}{2}})ds. \tag{127}$$

By Lemma 3.2, there exists $\delta_1=\delta_1(\tau,\omega)$ such that for all $0<|\delta|\leq\delta_1$, we have

$$e^{\frac{4\lambda}{3}s-qa_{k+1}\int_{-\tau}^{s}\mathcal{G}_\delta(\vartheta_r\omega)dr}\leq e^{\frac{4\lambda}{3}s+qa_{k+1}c(\tau,\omega)-\frac{q\lambda a_{k+1}}{6}s}\leq e^{qc(\tau,\omega)+\frac{q\lambda a_{k+1}}{6k}}$$

for all $s\in[-\frac{1}{k},0]$. Then by (127) and our induction assumption (121), we get that there exist positive constants $\delta_2=\delta_2(\tau,\omega)\leq\delta_1$, $\widetilde{c}_i^{(k)}=\widetilde{c}_i^{(k)}(\tau,\omega)(i=1,2)$ and $T_k=T_k(\tau,\omega)\geq 2$ such that for every $0<|\delta|\leq\delta_2$, $t\geq T_k$ and $\xi\in[\tau-\frac{1}{k+1},\tau]$,

$$\|v_\delta(\xi)\|_{qa_{k+1}}^{qa_{k+1}}+\alpha_1\int_{\tau-\frac{1}{k+1}}^{\tau} e^{\frac{4\lambda}{3}(s-\tau)+(q-2)\int_{-\tau}^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}\|v_\delta(s)\|_{qa_{k+2}}^{qa_{k+2}}ds$$
$$\leq k(k+1)(e^{\frac{4\lambda}{3(k+1)}}+1)\widetilde{c}_2^{(k)}(\tau,\omega)$$
$$+c_k(e^{\frac{4\lambda}{3(k+1)}}+2)e^{qc(\tau,\omega)+\frac{q\lambda a_{k+1}}{6k}}\int_{-\frac{1}{k}}^{0}(\|g(s+\tau)\|_{qa_{k+1}}^{qa_{k+1}}$$
$$+\|\psi_1(s+\tau)\|_{\frac{qa_{k+1}}{2}}^{\frac{qa_{k+1}}{2}})ds. \tag{128}$$

By Lemma 3.2 again, for every $s\in[\tau-\frac{1}{k+1},\tau]$, there exist constants $C_{\tau,\omega}>0$ and $\delta_3=\delta_3(\tau,\omega)\leq\delta_2$ such that $(q-2)\int_{-\tau}^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr\geq-(q-2)(C_{\tau,\omega}-\frac{\lambda}{6}(s-\tau))$, then we have

$$e^{\frac{4\lambda}{3}(s-\tau)+(q-2)\int_{-\tau}^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}\geq e^{-(q-2)C_{\tau,\omega}+(\frac{4}{3}+\frac{q-2}{6})\lambda(s-\tau)}.$$

Thus $e^{\frac{4\lambda}{3}(s-\tau)+(q-2)\int_{-\tau}^{s-\tau}\mathcal{G}_\delta(\vartheta_r\omega)dr}\geq e^{-(q-2)C_{\tau,\omega}+(\frac{4}{3}+\frac{q-2}{6})\frac{\lambda}{k+1}}$ for every $0<|\delta|\leq\delta_3$. Therefore by (128), we deduce that there exist positive constants $\widetilde{c}_i^{(k+1)}(\tau,\omega)(i=1,2)$ such that for every $0<|\delta|\leq\delta_3$ and $t\geq T_k$,

$$\|v_\delta(\xi,\tau-t,\vartheta_{-\tau}\omega,v_{\delta,\tau-t})\|_{qa_{k+1}}^{qa_{k+1}}\leq\widetilde{c}_1^{(k+1)}(\tau,\omega),\quad \xi\in[\tau-\frac{1}{k+1},\tau]; \tag{129}$$

and

$$\int_{\tau-\frac{1}{k+1}}^{\tau}\|v_\delta(s,\tau-t,\vartheta_{-\tau}\omega,v_{\delta,\tau-t})\|_{qa_{k+2}}^{qa_{k+2}}ds\leq\widetilde{c}_2^{(k+1)}(\tau,\omega). \tag{130}$$

For arbitrary $l>q$, there is a $k_0\in\mathbb{N}$ such that $q<l<qa_{k_0}$, then by the Sobolev interpolation theorem, there exists $\theta\in(0,1)$ with $\frac{1}{l}=\frac{\theta}{q}+\frac{1-\theta}{qa_{k_0}}$ such that

$$\|v_\delta(\tau,\tau-t,\vartheta_{-\tau}\omega,v_{\delta,\tau-t})\|_l^l\leq\|v_\delta(\tau,\tau-t,\vartheta_{-\tau}\omega,v_{\delta,\tau-t})\|_q^{l\theta}$$
$$\times\|v_\delta(\tau,\tau-t,\vartheta_{-\tau}\omega,v_{\delta,\tau-t})\|_{qa_{k_0}}^{l(1-\theta)},$$

which along with (94) and (129) (with $\xi = \tau$), we conclude the proof. □

By the bound in $L^l(\mathbb{R}^N)$ and the Sobolev interpolation, we can obtain the truncation estimate of solution of Eq.(1) in $L^l(\mathbb{R}^N)$ for arbitrary $l > q$.

Lemma 6.2. *Suppose that (7)-(10) and (119) hold and $\mathfrak{D}$ is defined by (12). Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_{\tau-t} \in D(\tau - t, \vartheta_{-t}\omega)$ with $D \in \mathfrak{D}$. Then for any $\varepsilon > 0$, there exist positive constants $\delta_0 = \delta_0(\tau, \omega, \varepsilon), c = c(\tau, \omega, l), M_0 = M_0(\tau, \omega, \varepsilon, l)$ and $T = T(\tau, \omega, \varepsilon, l)$ such that the solution u_δ of Eq.(5) satisfies that*

$$\sup_{t \geq T} \sup_{0<|\delta|\leq\delta_0} \left\{ \int_{(|u_\delta(\tau)|\geq M_0)} |u_\delta(\tau, \tau - t, \vartheta_{-\tau}\omega, u_{\delta,\tau-t})|^l dx \right\} \leq c\varepsilon,$$

where $(|u_\delta(\tau)| \geq M_0) = \{x \in \mathbb{R}^{\mathbb{N}} : |u_\delta(\tau, \tau - t, \vartheta_{-\tau}\omega, u_{\delta,\tau-t})| \geq M_0\}$.

Proof. By Lemma 6.1, there exist positive constants k_0, $\delta_1 = \delta_1(\tau, \omega)$, $c^{(k_0)} = c^{(k_0)}(\tau, \omega)$ and $T_{k_0} = T_{k_0}(\tau, \omega) \geq 2$ such that the solution of Eq.(1) satisfies that

$$\sup_{t\geq T_{k_0}} \sup_{0<|\delta|\leq\delta_1} \left\{ \|u_\delta(\tau, \tau - t, \vartheta_{-\tau}\omega, u_{\delta,\tau-t})\|_{qa_{k_0}}^{qa_{k_0}} \right\} \leq \widetilde{c}^{(k_0)}(\tau, \omega). \tag{131}$$

By Lemma 5.2, for any $\varepsilon > 0$ there exist positive constants $\delta_2 = \delta_2(\tau, \omega, \varepsilon) \geq \delta_0, c = c(\tau, \omega), M_0 = M_0(\tau, \omega, \varepsilon)$ and $T_0 = T_0(\tau, \omega, \varepsilon)$ such that the solution u_δ of Eq.(1) satisfies that

$$\sup_{t\geq T_0} \sup_{0<|\delta|\leq\delta_2} \left\{ \int_{(|u_\delta(\tau)|\geq M_0)} |u_\delta(\tau, \tau - t, \vartheta_{-\tau}\omega, u_{\delta,\tau-t})|^q dx \right\} \leq c\varepsilon. \tag{132}$$

For arbitrary $l > q$, there exists a k_0 such that $q < l < qa_{k_0}$, and then by the Sobolev interpolation we get

$$\begin{aligned} &\int_{(|u_\delta(\tau)|\geq M_0)} |u_\delta(\tau, \tau - t, \vartheta_{-\tau}\omega, u_{\delta,\tau-t})|^l dx \\ &\leq \left(\int_{(|u_\delta(\tau)|\geq M_0)} |u_\delta(\tau, \tau - t, \vartheta_{-\tau}\omega, u_{\delta,\tau-t})|^q dx \right)^{\frac{l\theta}{q}} \\ &\times \left(\int_{(|u_\delta(\tau)|\geq M_0)} |u_\delta(\tau, \tau - t, \vartheta_{-\tau}\omega, u_{\delta,\tau-t})|^{qa_{k_0}} dx \right)^{\frac{l(1-\theta)}{qa_{k_0}}}, \end{aligned} \tag{133}$$

where θ satisfies $\frac{1}{l} = \frac{\theta}{q} + \frac{1-\theta}{a_{k_0}}$. Let $T = \max\{T_0, T_{k_0}\}$ and $\delta_0 = \min\{\delta_1, \delta_2\}$. Then by (131)-(133), it follows that for all $t \geq T$ and $0 < |\delta| \leq \delta_0$,

$$\int_{(|u_\delta(\tau)|\geq M_0)} |u_\delta(\tau, \tau - t, \vartheta_{-\tau}\omega, u_{\delta,\tau-t})|^l dx \leq (c\varepsilon)^{\frac{l\theta}{q}} (\widetilde{c}^{(k_0)}(\tau, \omega))^{\frac{l(1-\theta)}{qa_{k_0}}}.$$

This complete the proof. □

We now present the following compactness of random attractor $\mathcal{A}_\delta$ in $L^l(\mathbb{R}^N)$ as $\delta \to 0$ for arbitrary $l > q$.

Lemma 6.3. *Suppose that (7)-(10), (93) and (119) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, if $\delta_n \to 0$ as $n \to \infty$ and $u_n \in \mathcal{A}_{\delta_n}(\tau, \omega)$, then $\{u_n\}_{n=1}^\infty$ has a convergent subsequence in $L^l(\mathbb{R}^N)$ for arbitrary $l > q$.*

Proof. This is followed by a same procedure as in [39, Lemma 5.4], using the asymptotical compactness in $L^q(\mathbb{R}^N)$ as in Lemma 5.3. The detailed is omitted here. □

We now obtain the upper semi-continuity of random attractors in $L^l(\mathbb{R}^N)$ for any $l > q$.

Theorem 6.4. *Suppose that (7)-(11), (93) and (119) hold. Let $l > q$ be arbitrary. Then for every fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\lim_{\delta \to 0} \operatorname{dist}_{L^l(\mathbb{R}^N)}(\mathcal{A}_\delta(\tau,\omega), \mathcal{A}(\tau,\omega)) = 0,$$

where $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ and $\mathcal{A}_\delta = \{\mathcal{A}_\delta(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ are the tempered pullback random attractors corresponding to Eqs.(1) and (5), respectively.

Proof. This is followed by Theorem 4.9, Lemmas 4.5, 4.10, 6.3 and Theorem 2.7. □

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