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ON BLOWUP OF SECANT VARIETIES OF CURVES

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Abstract. In this paper, we show that for a nonsingular projective curve and a positive integer k, the k-th secant bundle is the blowup of the k-th secant variety along the $(k-1)$-th secant variety. This answers a question raised in the recent paper of the authors on secant varieties of curves.

1. **Introduction.** Throughout the paper, we work over an algebraically closed field $\mathbb{k}$ of characteristic zero. Let C be a nonsingular projective curve of genus $g \geq 0$, and L be a very ample line bundle on C. The complete linear system $|L|$ embeds C into a projective space $\mathbb{P}^r := \mathbb{P}(H^0(C, L))$. For an integer $k \geq 0$, the *k-th secant variety*

$$\Sigma_k = \Sigma_k(C, L) \subseteq \mathbb{P}^r$$

of C in $\mathbb{P}^r$ is the Zariski closure of the union of $(k+1)$-secant k-planes to C.

Assume that $\deg L \geq 2g+2k+1$. Then the k-th secant variety Σ_k can be defined by using the secant sheaf $E_{k+1,L}$ and the secant bundle $B^k(L)$ as follows. Denote by C_m the m-th symmetric product of C. Let

$$\sigma_{k+1} \colon C_k \times C \longrightarrow C_{k+1}$$

be the morphism sending (ξ, x) to $\xi + x$, and $p \colon C_k \times C \to C$ the projection to C. The *secant sheaf* $E_{k+1,L}$ on C_{k+1} associated to L is defined by

$$E_{k+1,L} := \sigma_{k+1,*} p^* L,$$

which is a locally free sheaf of rank $k+1$. Notice that the fiber of $E_{k+1,L}$ over $\xi \in C_{k+1}$ can be identified with $H^0(\xi, L|_\xi)$. The *secant bundle* of k-planes over C_{k+1} is

$$B^k(L) := \mathbb{P}(E_{k+1,L})$$

equipped with the natural projection $\pi_k \colon B^k(L) \to C_{k+1}$. We say that a line bundle $\mathcal{L}$ on a variety X *separates $m+1$ points* if the natural restriction map $H^0(X, \mathcal{L}) \to H^0(\xi, \mathcal{L}|_\xi)$ is surjective for any effective zero-cycle $\xi \subseteq X$ with length$(\xi) = m+1$.

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Notice that a line bundle $\mathcal{L}$ is globally generated if and only if $\mathcal{L}$ separates 1 point, and $\mathcal{L}$ is very ample if and only if $\mathcal{L}$ separates 2 points. Since $\deg L \geq 2g + k$, it follows from Riemann–Roch that L separates $k+1$ points. Then the tautological bundle $\mathscr{O}_{B^k(L)}(1)$ is globally generated. We have natural identifications

$$H^0(B^k(L), \mathscr{O}_{B^k(L)}(1)) = H^0(C_{k+1}, E_{k+1,}) = H^0(C, L),$$

and therefore, the complete linear system $|\mathscr{O}_{B^k(L)}(1)|$ induces a morphism

$$\beta_k \colon B^k(L) \longrightarrow \mathbb{P}^r = \mathbb{P}(H^0(C, L)).$$

The k-th secant variety $\Sigma_k = \Sigma_k(C, L)$ of C in $\mathbb{P}^r$ can be defined to be the image $\beta_k(B^k(L))$. Bertram proved that $\beta_k \colon B^k(L) \to \Sigma_k$ is a resolution of singularities (see [1, Section 1]).

It is clear that there are natural inclusions

$$C = \Sigma_0 \subseteq \Sigma_1 \subseteq \cdots \subseteq \Sigma_{k-1} \subseteq \Sigma_k \subseteq \mathbb{P}^r.$$

The preimage of Σ_{k-1} under the morphism β_k is actually a divisor on $B^k(L)$. Thus there exits a natural morphism from $B^k(L)$ to the blowup of Σ_k along Σ_{k-1}. Vermeire proved that $B^1(L)$ is indeed the blowup of Σ_1 along $\Sigma_0 = C$ ([3, Theorem 3.9]). In the recent work [2], we showed that $B^k(L)$ is the normalization of the blowup of Σ_k along Σ_{k-1} ([2, Proposition 5.13]), and raised the problem asking whether $B^k(L)$ is indeed the blowup itself ([2, Problem 6.1]). The purpose of this paper is to give an affirmative answer to this problem by proving the following:

Theorem 1.1. *Let C be a nonsingular projective curve of genus g, and L be a line bundle on C. If $\deg L \geq 2g + 2k + 1$ for an integer $k \geq 1$, then the morphism $\beta_k \colon B^k(L) \to \Sigma_k(C, L)$ is the blowup of $\Sigma_k(C, L)$ along $\Sigma_{k-1}(C, L)$.*

To prove the theorem, we utilize several line bundles defined on symmetric products of the curve. Let us recall the definitions here and refer the reader to [2] for further details. Let

$$C^{k+1} = \underbrace{C \times \cdots \times C}_{k+1 \text{ times}}$$

be the $(k+1)$-fold ordinary product of the curve C, and $p_i \colon C^{k+1} \to C$ be the projection to the i-th component. The symmetric group $\mathfrak{S}_{k+1}$ acts on $p_1^*L \otimes \cdots \otimes p_{k+1}^*L$ in a natural way: a permutation $\mu \in \mathfrak{S}_k$ sends a local section $s_1 \otimes \cdots \otimes s_{k+1}$ to $s_{\mu(1)} \otimes \cdots \otimes s_{\mu(k+1)}$. Then $p_1^*L \otimes \cdots \otimes p_{k+1}^*L$ is invariant under the action of $\mathfrak{S}_{k+1}$, so it descends to a line bundle $T_{k+1}(L)$ on the symmetric product C_{k+1} via the quotient map $q \colon C^{k+1} \to C_{k+1}$. We have $q^*T_{k+1}(L) = p_1^*L \otimes \cdots \otimes p_{k+1}^*L$. Define a divisor δ_{k+1} on C_{k+1} such that the associated line bundle $\mathscr{O}_{C_{k+1}}(\delta_{k+1}) = \det\big(\sigma_{k+1,*}(\mathscr{O}_{C_k \times C})\big)^*$. Let

$$A_{k+1,L} := T_{k+1}(L)(-2\delta_{k+1})$$

be a line bundle on C_{k+1}. When $k = 0$, we use the convention that $T_1(L) = E_{1,L} = L$ and $\delta_1 = 0$.

The main ingredient in the proof of Theorem 1.1 is to study the positivity of the line bundle $A_{k+1,L}$. Some partial results and their geometric consequences have been discussed in [2, Lemma 5.12 and Proposition 5.13]. Along this direction, we establish the following proposition to give a full picture in a general result describing the positivity of the line bundle $A_{k+1,L}$. This may be of independent interest.

Proposition 1.2. *Let C be a nonsingular projective curve of genus g, and L be a line bundle on C. If $\deg L \geq 2g+2k+\ell$ for integers $k, \ell \geq 0$, then the line bundle $A_{k+1,L}$ on C_{k+1} separates $\ell+1$ points.*

In particular, if $\deg L \geq 2g+2k$, then $A_{k+1,L}$ is globally generated, and if $\deg L \geq 2g+2k+1$, then $A_{k+1,L}$ is very ample.

2. **Proof of the main theorem.**

In this section, we prove Theorem 1.1. We begin with showing Proposition 1.2.

Proof of Proposition 1.2. We proceed by induction on k and ℓ. If $k=0$, then $A_{1,L}=L$ and $\deg L \geq 2g+\ell$. It immediately follows from Riemann–Roch that L separates $\ell+1$ points. If $\ell=0$, then $\deg L \geq 2g+2k$. By [2, Lemma 5.12], $A_{k+1,L}$ separates 1 point.

Assume that $k \geq 1$ and $\ell \geq 1$. Let z be a length $\ell+1$ zero-dimensional subscheme of C_{k+1}. We aim to show that the natural restriction map

$$r_{z,k+1,L}\colon H^0(C_{k+1}, A_{k+1,L}) \longrightarrow H^0(z, A_{k+1,L}|_z)$$

is surjective. We can choose a point $p \in C$ such that X_p contains a point in the support of z, where X_p is the divisor on C_{k+1} defined by the image of the morphism $C_k \to C_{k+1}$ sending ξ to $\xi+p$. Let $y := z \cap X_p$ be the scheme-theoretic intersection, and $\mathscr{I}_x := (\mathscr{I}_z : \mathscr{I}_{X_p})$, which defines a subscheme x of z in C_{k+1}, where $\mathscr{I}_z$ and $\mathscr{I}_{X_p}$ are ideal sheaves of z and X_p in C_{k+1}, respectively. We have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{I}_x(-X_p) & \longrightarrow & \mathscr{I}_z & \longrightarrow & \mathscr{I}_z \cdot \mathscr{O}_{X_p} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{O}_{C_{k+1}}(-X_p) & \longrightarrow & \mathscr{O}_{C_{k+1}} & \longrightarrow & \mathscr{O}_{X_p} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{O}_x(-X_p) & \longrightarrow & \mathscr{O}_z & \longrightarrow & \mathscr{O}_y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where all rows and columns are short exact sequences. By tensoring with $A_{k+1,L}$ and taking the global sections of last two rows, we obtain the commutative diagram with exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(A_{k+1,L}(-X_p)) & \longrightarrow & H^0(A_{k+1,L}) & \longrightarrow & H^0(A_{k+1,L}|_{X_p}) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle r_{x,k+1,L(-p)}} & & \Big\downarrow{\scriptstyle r_{z,k+1,L}} & & \Big\downarrow{\scriptstyle r_{y,k,L(-2p)}} & & \\
0 & \longrightarrow & H^0(A_{k+1,L}(-X_p)|_x) & \longrightarrow & H^0(A_{k+1,L}|_z) & \longrightarrow & H^0(A_{k+1,L}|_y) & \longrightarrow & 0,
\end{array}$$

in which we use the fact that $H^1(A_{k+1,L}(-X_p))=0$ (see the proof of [2, Lemma 5.12]). Note that $A_{k+1,L}(-X_p) = A_{k+1,L(-p)}$ and $A_{k+1,L}|_{X_p} \cong A_{k,L(-2p)}$, where we identify $X_p = C_k$.

Since $\text{length}(y) \leq \text{length}(z) = \ell+1$ and $\deg L(-2p) \geq 2g+2(k-1)+\ell$, the induction hypothesis on k implies that $r_{y,k,L(-2p)}$ is surjective. On the other hand, if $x = \emptyset$, which means that z is a subscheme of X_p, then trivially $r_{x,k+1,L(-p)}$ is surjective. Otherwise, suppose that $x \neq \emptyset$. By the choice of X_p, we know that y is not empty, and therefore, we have $\text{length}(x) \leq \text{length}(z) - 1 = \ell$. Now, $\deg L(-p) \geq 2g + 2k + (\ell - 1)$, so the induction hypothesis on ℓ implies that $L(-p)$ separates ℓ points. In particular, $r_{x,k+1,L(-p)}$ is surjective. Hence $r_{z,k+1,L}$ is surjective as desired. □

Lemma 2.1. *Let $\varphi\colon X \to Y$ be a finite surjective morphism between two varieties. If $\varphi^{-1}(q)$ is scheme theoretically a reduced point for each closed point $q \in Y$, then φ is an isomorphism.*

Proof. Note that φ is proper, injective, and unramifield. Then it is indeed a classical result that φ is an isomorphism. Here we give a short proof for reader's convenience. The problem is local. We may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ for some rings A, B. We may regard A as a subring of B. For any $q \in Y$, let $p := \varphi^{-1}(q) \in X$. It is enough to show that the localizations $A' := A_{\mathfrak{m}_q}$ and $B' := B_{\mathfrak{m}_p}$ are isomorphic. Let $\mathfrak{m}'_q, \mathfrak{m}'_p$ be the maximal ideals of the local rings A', B', respectively. The assumption says that $\mathfrak{m}'_q B' = \mathfrak{m}'_p$. We have

$$B'/A' \otimes_{A'} A'/\mathfrak{m}'_q = B'/(\mathfrak{m}'_q B' + A') = B'/(\mathfrak{m}'_p + A') = 0.$$

By Nakayama lemma, we obtain $B'/A' = 0$. □

We keep using the notations used in the introduction. Recall that C is a nonsingular projective curve of genus $g \geq 0$, and L is a very ample line bundle on C. Consider $\xi_k \in C_k$ and $x \in C$, and let $\xi := \xi_k + x \in C_{k+1}$. The divisor ξ_k spans a k-secant $(k-1)$-plane $\mathbb{P}(H^0(\xi_k, L|_{\xi_k}))$ to C in $\mathbb{P}(H^0(C, L))$, and it is naturally embedded in the $(k+1)$-secant k-plane $\mathbb{P}(H^0(\xi, L|_{\xi}))$ spanned by ξ. This observation naturally induces a morphism

$$\alpha_{k,1}\colon B^{k-1}(L) \times C \longrightarrow B^k(L).$$

To see it in details, we refer to [1, p.432, line –5]. We define the *relative secant variety $Z = Z_{k-1}$ of $(k-1)$-planes* in $B^k(L)$ to be the image of the morphism $\alpha_{k,1}$. The relative secant variety Z is a divisor in the secant bundle $B^k(L)$, and it is the preimage of $(k-1)$-th secant variety Σ_{k-1} under the morphism β_k. It plays the role of transferring the codimension two situation (Σ_k, Σ_{k-1}) into the codimension one situation $(B^k(L), Z)$. We collect several properties of Z.

Proposition 2.2 ([2, Proposition 3.15, Theorem 5.2, and Proposition 5.13])**.** *Recall the situation described in the diagram*

$$\begin{array}{ccccc} Z & \hookrightarrow & B^k(L) & \xrightarrow{\beta_k} & \Sigma_k \subseteq \mathbb{P}^r = \mathbb{P}(H^0(C,L)) \\ & & \big\downarrow{\scriptstyle \pi_k} & & \\ & & C_{k+1}. & & \end{array}$$

Let H be the pull back of a hyperplane divisor of $\mathbb{P}^r$ by β_k, and let $I_{\Sigma_{k-1}|\Sigma_k}$ be the ideal sheaf on Σ_k defining the subvariety Σ_{k-1}. Then one has

1. $\mathscr{O}_{B^k(L)}((k+1)H - Z) = \pi_k^* A_{k+1,L}$.
2. $R^i \beta_{k,*} \mathscr{O}_{B^k(L)}(-Z) = \begin{cases} I_{\Sigma_{k-1}|\Sigma_k} & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$
3. $I_{\Sigma_{k-1}|\Sigma_k} \cdot \mathscr{O}_{B^k(L)} = \mathscr{O}_{B^k(L)}(-Z)$.

As a direct consequence of the above proposition, we have an identification

$$H^0(C_{k+1}, A_{k+1,L}) = H^0(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(k+1)).$$

We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let

$$b\colon \widetilde{\Sigma}_k := \mathrm{Bl}_{\Sigma_{k-1}} \Sigma_k \longrightarrow \Sigma_k$$

be the blowup of Σ_k along Σ_{k-1} with exceptional divisor E. As $I_{\Sigma_{k-1}|\Sigma_k} \cdot \mathscr{O}_{B^k(L)} = \mathscr{O}_{B^k(L)}(-Z)$ (see Proposition 2.2), there exists a morphism α from $B^k(L)$ to the blowup $\widetilde{\Sigma}_k$ fitting into the following commutative diagram

$$\begin{array}{ccc} B^k(L) & \xrightarrow{\alpha} & \widetilde{\Sigma}_k \\ & \searrow^{\beta_k} & \downarrow b \\ & & \Sigma_k. \end{array}$$

We shall show that α is an isomorphism.

Write $V := H^0(\Sigma_k, I_{\Sigma_{k-1}|\Sigma_k}(k+1))$. As proved in [2, Theorem 5.2], $I_{\Sigma_{k-1}|\Sigma_k}(k+1)$ is globally generated by V. This particularly implies that on the blowup $\widetilde{\Sigma}_k$ one has a surjective morphism $V \otimes \mathscr{O}_{\widetilde{\Sigma}_k} \to b^*\mathscr{O}_{\Sigma_k}(k+1)(-E)$, which induces a morphism

$$\gamma\colon \widetilde{\Sigma}_k \longrightarrow \mathbb{P}(V).$$

On the other hand, one has an identification $V = H^0(C_{k+1}, A_{k+1,L})$ by Proposition 2.2. Recall from Proposition 1.2 that $A_{k+1,L}$ is very ample. So the complete linear system $|V| = |A_{k+1,L}|$ on C_{k+1} induces an embedding

$$\psi\colon C_{k+1} \longrightarrow \mathbb{P}(V).$$

Also note that $\alpha^*(b^*\mathscr{O}_{\Sigma_k}(k+1)(-E)) = \beta_k^*\mathscr{O}_{\Sigma_k}(k+1)(-Z) = \pi_k^* A_{k+1,L}$ by Proposition 2.2. Hence we obtain the following commutative diagram

$$\begin{array}{ccc} B^k(L) & \xrightarrow{\alpha} & \widetilde{\Sigma}_k \\ {\scriptstyle \pi_k}\downarrow & & \downarrow{\scriptstyle \gamma} \\ C_{k+1} & \xrightarrow{\psi} & \mathbb{P}(V). \end{array}$$

Take an arbitrary closed point $x \in \widetilde{\Sigma}_k$, and consider its image $x' := b(x)$ on Σ_k. There is a nonnegative integer $m \le k$ such that $x' \in \Sigma_m \setminus \Sigma_{m-1} \subseteq \Sigma_k$. In addition, the point x' uniquely determines a degree $m+1$ divisor $\xi_{m+1,x'}$ on C in such a way that $\xi_{m+1,x'} = \Lambda \cap C$, where Λ is a unique $(m+1)$-secant m-plane to C with $x' \in \Lambda$ (see [2, Definition 3.12]). By [2, Proposition 3.13], $\beta_k^{-1}(x') \cong C_{k-m}$ and $\pi_k(\beta_k^{-1}(x')) = \xi_{m+1,x'} + C_{k-m} \subseteq C_{k+1}$. Consider also $x'' := \gamma(x)$ which lies in the image of ψ. As ψ is an embedding, we may think x'' as a point of C_{k+1}. Now, through forming fiber products, we see scheme-theoretically

$$\alpha^{-1}(x) \subseteq \pi_k^{-1}(x'') \cap \beta_k^{-1}(x').$$

However, the restriction of the morphism π_k on $\beta_k^{-1}(x')$ gives an embedding of C_{k-m} into C_{k+1}. This suggests that $\pi_k^{-1}(x'') \cap \beta_k^{-1}(x')$ is indeed a single reduced point, and so is $\alpha^{-1}(x)$. Finally by Lemma 2.1, α is an isomorphism as desired. □

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