

A STABILIZER FREE WG METHOD FOR THE STOKES EQUATIONS WITH ORDER TWO SUPERCONVERGENCE ON POLYTOPAL MESH

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ABSTRACT. A stabilizer free WG method is introduced for the Stokes equations with superconvergence on polytopal mesh in primary velocity-pressure formulation. Convergence rates two order higher than the optimal-order for velocity of the WG approximation is proved in both an energy norm and the L^2 norm. Optimal order error estimate for pressure in the L^2 norm is also established. The numerical examples cover low and high order approximations, and 2D and 3D cases.

1. Introduction. A stabilizing/penalty term is often used in finite element methods with discontinuous approximations to enforce connection of discontinuous functions across element boundaries. Development of stabilizer free discontinuous finite element method is desirable since it simplifies finite element formulation and reduces programming complexity. The stabilizer free WG method and the stabilizer DG method on polytopal mesh were first introduced in [11, 12] for second order elliptic problems. The main idea in [11, 12] is to raise the degree of polynomials used to compute weak gradient ∇_w . In [11, 12], gradient is approximated by a polynomial of order $j = k + n - 1$ where n is the number of sides of polygonal element. This result has been improved in [1, 2] by reducing the degree of polynomial j . Recently, new stabilizer free WG methods have been developed in [13, 14] for second order elliptic equations on polytopal mesh, which have superconvergence. Wachspress coordinates are used to approximate ∇_w in [6, 7] for solving the Stokes equations on polytopal mesh. Wachspress coordinates are usually rational functions, instead of polynomials. The WG methods in [6, 7] are limited to the lowest order WG elements.

In general, discontinuous finite element methods tend to have complex formulations which are often necessary to enforce weak continuity of discontinuous solutions

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across element boundaries. Most of discontinuous finite element methods have one or more stabilizing terms to guarantee stability and convergence of the methods. The stabilizer free WG method has a super clean finite element formulations as (4)-(5) compared to the weak Galerkin method and other DG methods, which reduces programming complexity.

One obvious disadvantage of discontinuous finite element methods is their rather complex formulations which are often necessary to enforce weak continuity of discontinuous solutions across element boundaries. Most of discontinuous finite element methods have one or more stabilizing terms to guarantee stability and convergence of the methods. Existing of stabilizing terms further complicates formulations.

In this paper, we introduce a new stabilizer free WG method of any order to solve the Stokes problem: find unknown functions $\mathbf{u}$ and p such that

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad (2)$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \quad (3)$$

where Ω is a polygonal or polyhedral domain in $\mathbb{R}^d$ ($d = 2, 3$). Our new WG method has the following formulations without any stabilizers: seek $(\mathbf{u}_h, p_h) \in V_h \times W_h$ satisfying the following for all $(\mathbf{v}, w) \in V_h \times W_h$,

$$(\nabla_w \mathbf{u}_h, \nabla_w \mathbf{v}) - (\nabla_w \cdot \mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}), \quad (4)$$

$$(\nabla_w \cdot \mathbf{u}_h, w) = 0. \quad (5)$$

Here ∇_w and $\nabla_w \cdot$ are weak gradient and weak divergence, respectively. In addition, we have proved that the WG approximations have the convergence rates two order higher than the optimal-order for velocity in both an energy norm and the L^2 norm and the optimal convergence rate for pressure in the L^2 norm. Extensive numerical examples are tested for the new WG elements of different degrees k in both two and three dimensional spaces.

Comparing to other weak Galerkin finite element methods, the WG methods without stabilizers leads to a 2-order superconvergent solution with a carefully chose weak Galerkin. So far all weak Galerkin finite element methods with stabilizers do not have superconvergence in both an energy norm and the L^2 norm. A order one superconvergence in an energy norm only is derived in [4, 3]. The new method reduces the computation cost greatly of the other WG methods. The price to pay for removing the stabilizers from the WG methods is that the weak gradient, an intermediate variable, need to be approximated by one degree higher and piecewise polynomials.

2. Preliminary. Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions specified in [10]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ be the set of all interior edges or flat faces. For every element $T \in \mathcal{T}_h$, we denote by h_T its diameter and mesh size $h = \max_{T \in \mathcal{T}_h} h_T$ for $\mathcal{T}_h$. Let $P_k(T)$ consist all the polynomials on T with degree no greater than k .

For $k \geq 0$ and given $\mathcal{T}_h$, define two finite element spaces for velocity

$$V_h = \{ \mathbf{v} = \{ \mathbf{v}_0, \mathbf{v}_b \} : \mathbf{v}_0|_T \in [P_k(T)]^d, \mathbf{v}_b|_e \in [P_{k+1}(e)]^d, e \subset \partial T \}, \quad (6)$$

and for pressure

$$W_h = \{ w \in L_0^2(\Omega) : w|_T \in P_{k+1}(T) \}. \quad (7)$$

Let V_h^0 be a subspace of V_h consisting of functions with vanishing boundary value.

The space $H(\text{div}; \Omega)$ is defined as

$$H(\text{div}; \Omega) = \{\mathbf{v} \in [L^2(\Omega)]^d : \nabla \cdot \mathbf{v} \in L^2(\Omega)\}.$$

For any $T \in \mathcal{T}_h$, it can be divided into a set of disjoint triangles T_i with $T = \cup T_i$. Then we define a space $\Lambda_k(T)$ for the approximation of weak gradient on each element T as

$$\begin{aligned} \Lambda_k(T) = \{ \boldsymbol{\psi} \in [H(\text{div}; T)]^d : \boldsymbol{\psi}|_{T_i} \in [P_{k+1}(T_i)]^{d \times d}, \nabla \cdot \boldsymbol{\psi} \in [P_k(T)]^d, \\ \boldsymbol{\psi} \cdot \mathbf{n}|_e \in [P_{k+1}(e)]^d, e \subset \partial T \}. \end{aligned}$$

For a function $\mathbf{v} \in V_h$, its weak gradient $\nabla_w \mathbf{v}$ is a piecewise polynomial satisfying $\nabla_w \mathbf{v}|_T \in \Lambda_k(T)$ and the following equation,

$$(\nabla_w \mathbf{v}, \boldsymbol{\tau})_T = -(\mathbf{v}_0, \nabla \cdot \boldsymbol{\tau})_T + \langle \mathbf{v}_b, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\partial T} \quad \forall \boldsymbol{\tau} \in \Lambda_k(T). \quad (8)$$

For a function $\mathbf{v} \in V_h$, its weak divergence $\nabla_w \cdot \mathbf{v}$ is a piecewise polynomial satisfying $\nabla_w \cdot \mathbf{v}|_T \in P_{k+1}(T)$ and the following equation,

$$(\nabla_w \cdot \mathbf{v}, w)_T = -(\mathbf{v}_0, \nabla w)_T + \langle \mathbf{v}_b \cdot \mathbf{n}, w \rangle_{\partial T} \quad \forall w \in P_{k+1}(T). \quad (9)$$

The proof of the following lemma can be found in [14].

Lemma 2.1. *For $\boldsymbol{\tau} \in [H(\text{div}; \Omega)]^d$, there exists a projection Π_h with $\Pi_h \boldsymbol{\tau} \in [H(\text{div}; \Omega)]^d$ satisfying $\Pi_h \boldsymbol{\tau}|_T \in \Lambda_k(T)$ and the followings*

$$(\nabla \cdot \boldsymbol{\tau}, \mathbf{q})_T = (\nabla \cdot \Pi_h \boldsymbol{\tau}, \mathbf{q})_T \quad \forall \mathbf{q} \in [P_k(T)]^d, \quad (10)$$

$$-(\nabla \cdot \boldsymbol{\tau}, \mathbf{v}_0) = (\Pi_h \boldsymbol{\tau}, \nabla_w \mathbf{v}) \quad \forall \mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h, \quad (11)$$

$$\|\Pi_h \boldsymbol{\tau} - \boldsymbol{\tau}\| \leq Ch^{k+2} |\boldsymbol{\tau}|_{k+2}. \quad (12)$$

3. Finite element method and its well posedness. We start this section by introducing the following WG finite element scheme without stabilizers.

Weak Galerkin Algorithm 3.1. *A numerical approximation for (1)-(3) is finding $(\mathbf{u}_h, p_h) \in V_h^0 \times W_h$ such that for all $(\mathbf{v}, w) \in V_h^0 \times W_h$,*

$$(\nabla_w \mathbf{u}_h, \nabla_w \mathbf{v}) - (\nabla_w \cdot \mathbf{v}, p_h) = (f, \mathbf{v}), \quad (13)$$

$$(\nabla_w \cdot \mathbf{u}_h, w) = 0. \quad (14)$$

Let Q_0 and Q_b be the two element-wise defined L^2 projections onto $[P_k(T)]^d$ and $[P_{k+1}(e)]^d$ with $e \subset \partial T$ on T respectively. Define $Q_h \mathbf{u} = \{Q_0 \mathbf{u}, Q_b \mathbf{u}\} \in V_h$ for the true solution $\mathbf{u}$. Let $\mathbb{Q}_h$ be the element-wise defined L^2 projection onto $\Lambda_k(T)$ on each element T . Finally denote by $\mathcal{Q}_h$ the element-wise defined L^2 projection onto $P_{k+1}(T)$ on each element T .

Lemma 3.1. *Let $\boldsymbol{\phi} \in [H_0^1(\Omega)]^d$, then on $T \in \mathcal{T}_h$*

$$\nabla_w Q_h \boldsymbol{\phi} = \mathbb{Q}_h \nabla \boldsymbol{\phi}, \quad (15)$$

$$\nabla_w \cdot Q_h \boldsymbol{\phi} = \mathcal{Q}_h \nabla \cdot \boldsymbol{\phi}. \quad (16)$$

Proof. Using (8) and integration by parts, we have that for any $\boldsymbol{\tau} \in \Lambda_k(T)$

$$\begin{aligned} (\nabla_w Q_h \boldsymbol{\phi}, \boldsymbol{\tau})_T &= -(Q_0 \boldsymbol{\phi}, \nabla \cdot \boldsymbol{\tau})_T + \langle Q_b \boldsymbol{\phi}, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\partial T} \\ &= -(\boldsymbol{\phi}, \nabla \cdot \boldsymbol{\tau})_T + \langle \boldsymbol{\phi}, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\nabla \boldsymbol{\phi}, \boldsymbol{\tau})_T = (\mathbb{Q}_h \nabla \boldsymbol{\phi}, \boldsymbol{\tau})_T, \end{aligned}$$

which implies the identity (15).

Using (9) and integration by parts, we have that for any $w \in P_{k+1}(T)$

$$\begin{aligned} (\nabla_w \cdot Q_h \phi, w)_T &= -(Q_0 \phi, \nabla w)_T + \langle Q_b \phi \cdot \mathbf{n}, w \rangle_{\partial T} \\ &= -(\phi, \nabla w)_T + \langle \phi \cdot \mathbf{n}, w \rangle_{\partial T} \\ &= (\nabla \cdot \phi, w)_T = (Q_h \nabla \cdot \phi, w)_T, \end{aligned}$$

which proves (15). $\square$

For any function $\varphi \in H^1(T)$, the following trace inequality holds true (see [10] for details):

$$\|\varphi\|_e^2 \leq C (h_T^{-1} \|\varphi\|_T^2 + h_T \|\nabla \varphi\|_T^2). \quad (17)$$

We introduce two semi-norms $\|\mathbf{v}\|$ and $\|\mathbf{v}\|_{1,h}$ for any $\mathbf{v} \in V_h$ as follows:

$$\|\mathbf{v}\|^2 = \sum_{T \in \mathcal{T}_h} (\nabla_w \mathbf{v}, \nabla_w \mathbf{v})_T, \quad (18)$$

$$\|\mathbf{v}\|_{1,h}^2 = \sum_{T \in \mathcal{T}_h} \|\nabla \mathbf{v}_0\|_T^2 + \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2. \quad (19)$$

It is easy to see that $\|\mathbf{v}\|_{1,h}$ defines a norm in V_h^0 . Next we will show that $\|\cdot\|$ also defines a norm in V_h^0 by proving the equivalence of $\|\cdot\|$ and $\|\cdot\|_{1,h}$ in V_h .

The following norm equivalence has been proved in [14] for each component of $\mathbf{v}$,

$$C_1 \|\mathbf{v}\|_{1,h} \leq \|\mathbf{v}\| \leq C_2 \|\mathbf{v}\|_{1,h} \quad \forall \mathbf{v} \in V_h. \quad (20)$$

Unlike the traditional finite elements [5, 8, 9, 16, 17, 18, 19, 20, 21], the inf-sup condition for the weak Galerkin finite element is easily satisfied due to the large velocity space with independent element boundary degrees of freedom.

Lemma 3.2. *There exists a positive constant β independent of h such that for all $\rho \in W_h$,*

$$\sup_{\mathbf{v} \in V_h} \frac{(\nabla_w \cdot \mathbf{v}, \rho)}{\|\mathbf{v}\|} \geq \beta \|\rho\|. \quad (21)$$

Proof. For any given $\rho \in W_h \subset L_0^2(\Omega)$, there exists a function $\tilde{\mathbf{v}} \in [H_0^1(\Omega)]^d$ such that

$$\frac{(\nabla \cdot \tilde{\mathbf{v}}, \rho)}{\|\tilde{\mathbf{v}}\|_1} \geq C \|\rho\|, \quad (22)$$

where $C > 0$ is a constant independent of h . Let $\mathbf{v} = Q_h \tilde{\mathbf{v}} = \{Q_0 \tilde{\mathbf{v}}, Q_b \tilde{\mathbf{v}}\} \in V_h$. It follows from (20), (17) and $\tilde{\mathbf{v}} \in [H_0^1(\Omega)]^d$,

$$\begin{aligned} \|\mathbf{v}\|^2 &\leq C \|\mathbf{v}\|_{1,h}^2 = C \left(\sum_{T \in \mathcal{T}_h} \|\nabla \mathbf{v}_0\|_T^2 + \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2 \right) \\ &\leq C \left(\sum_{T \in \mathcal{T}_h} \|\nabla Q_0 \tilde{\mathbf{v}}\|_T^2 + \sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_0 \tilde{\mathbf{v}} - Q_b \tilde{\mathbf{v}}\|_{\partial T}^2 \right) \\ &\leq C \left(\sum_{T \in \mathcal{T}_h} \|\nabla Q_0 \tilde{\mathbf{v}}\|_T^2 + \sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_0 \tilde{\mathbf{v}} - \tilde{\mathbf{v}}\|_{\partial T}^2 \right) \\ &\leq C \|\tilde{\mathbf{v}}\|_1^2, \end{aligned}$$

which implies

$$\|\mathbf{v}\| \leq C \|\tilde{\mathbf{v}}\|_1. \quad (23)$$

It follows from (9) that

$$\begin{aligned}
 (\nabla_w \cdot \mathbf{v}, \rho)_{\mathcal{T}_h} &= -(\mathbf{v}_0, \nabla \rho)_{\mathcal{T}_h} + \langle \mathbf{v}_b, \rho \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\
 &= -(Q_0 \tilde{\mathbf{v}}, \nabla \rho)_{\mathcal{T}_h} + \langle Q_b \tilde{\mathbf{v}}, \rho \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\
 &= -(\tilde{\mathbf{v}}, \nabla \rho)_{\mathcal{T}_h} + \langle \tilde{\mathbf{v}}, \rho \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\
 &= (\nabla \cdot \tilde{\mathbf{v}}, \rho)_{\mathcal{T}_h}.
 \end{aligned} \tag{24}$$

Using (24), (23) and (22), we have

$$\frac{(\nabla_w \cdot \mathbf{v}, \rho)}{\|\mathbf{v}\|} = \frac{(\nabla \cdot \tilde{\mathbf{v}}, \rho)}{\|\tilde{\mathbf{v}}\|} \geq \frac{(\nabla \cdot \tilde{\mathbf{v}}, \rho)}{C\|\tilde{\mathbf{v}}\|_1} \geq \beta\|\rho\|,$$

for a positive constant β . This completes the proof of the lemma. $\square$

Lemma 3.3. *The weak Galerkin method (13)-(14) has a unique solution.*

Proof. It suffices to show that zero is the only solution of (13)-(14) if $\mathbf{f} = 0$. To this end, let $\mathbf{f} = 0$ and take $\mathbf{v} = \mathbf{u}_h$ in (13) and $w = p_h$ in (14). By adding the two resulting equations, we obtain

$$(\nabla_w \mathbf{u}_h, \nabla_w \mathbf{u}_h) = 0,$$

which implies that $\nabla_w \mathbf{u}_h = 0$ on each element T . By (20), we have $\|\mathbf{u}_h\|_{1,h} = 0$ which implies that $\mathbf{u}_h = 0$.

Since $\mathbf{u}_h = 0$ and $\mathbf{f} = 0$, the equation (13) becomes $(\nabla \cdot \mathbf{v}, p_h) = 0$ for any $\mathbf{v} \in V_h$. Then the inf-sup condition (21) implies $p_h = 0$. We have proved the lemma. $\square$

4. Error equations. In this section, we derive the equations that the errors satisfy. Let $\mathbf{e}_h = Q_h \mathbf{u} - \mathbf{u}_h$ and $\varepsilon_h = Q_h p - p_h$.

Lemma 4.1. *The following error equations hold true for any $(\mathbf{v}, w) \in V_h^0 \times W_h$,*

$$(\nabla_w \mathbf{e}_h, \nabla_w \mathbf{v}) - (\varepsilon_h, \nabla_w \cdot \mathbf{v}) = \ell_1(\mathbf{u}, \mathbf{v}) + \ell_2(p, \mathbf{v}), \tag{25}$$

$$(\nabla_w \cdot \mathbf{e}_h, w) = 0, \tag{26}$$

where

$$\ell_1(\mathbf{u}, \mathbf{v}) = (Q_h \nabla \mathbf{u} - \Pi_h \nabla \mathbf{u}, \nabla_w \mathbf{v}), \tag{27}$$

$$\ell_2(p, \mathbf{v}) = \langle Q_h p - p, (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h}. \tag{28}$$

Proof. First, we test (1) by $\mathbf{v}_0$ with $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h^0$ to obtain

$$-(\Delta \mathbf{u}, \mathbf{v}_0) + (\nabla p, \mathbf{v}_0) = (\mathbf{f}, \mathbf{v}_0). \tag{29}$$

It follows from (11) and (15)

$$-(\nabla \cdot \nabla \mathbf{u}, \mathbf{v}_0) = (\Pi_h \nabla \mathbf{u}, \nabla_w \mathbf{v}) = (\nabla_w Q_h \mathbf{u}, \nabla_w \mathbf{v}) - \ell_1(\mathbf{u}, \mathbf{v}). \tag{30}$$

Using integration by parts and the fact $\langle p, \mathbf{v}_b \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0$, we have

$$\begin{aligned}
 (\nabla p, \mathbf{v}_0) &= -(p, \nabla \cdot \mathbf{v}_0)_{\mathcal{T}_h} + \langle p, \mathbf{v}_0 \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\
 &= -(Q_h p, \nabla \cdot \mathbf{v}_0)_{\mathcal{T}_h} + \langle p, (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\
 &= (\nabla Q_h p, \mathbf{v}_0)_{\mathcal{T}_h} - \langle Q_h p, \mathbf{v}_0 \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} + \langle p, (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\
 &= -(Q_h p, \nabla_w \cdot \mathbf{v}) - \langle Q_h p, (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} + \langle p, (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\
 &= -(Q_h p, \nabla_w \cdot \mathbf{v}) - \ell_2(p, \mathbf{v}),
 \end{aligned}$$

which implies

$$(\nabla p, \mathbf{v}_0) = -(Q_h p, \nabla_w \cdot \mathbf{v}) - \ell_2(p, \mathbf{v}). \tag{31}$$

Substituting (30) and (31) into (29) gives

$$(\nabla_w Q_h \mathbf{u}, \nabla_w \mathbf{v}) - (\mathcal{Q}_h p, \nabla_w \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}_0) + \ell_1(\mathbf{u}, \mathbf{v}) + \ell_2(p, \mathbf{v}). \quad (32)$$

The difference of (32) and (13) implies

$$(\nabla_w \mathbf{e}_h, \nabla_w \mathbf{v}) - (\varepsilon_h, \nabla_w \cdot \mathbf{v}) = \ell_1(\mathbf{u}, \mathbf{v}) + \ell_2(p, \mathbf{v}) \quad \forall \mathbf{v} \in V_h^0. \quad (33)$$

Testing equation (2) by $w \in W_h$ and using (16) give

$$(\nabla \cdot \mathbf{u}, w) = (\mathcal{Q}_h \nabla \cdot \mathbf{u}, w) = (\nabla_w \cdot Q_h \mathbf{u}, w) = 0. \quad (34)$$

The difference of (34) and (14) implies (26). We have proved the lemma. $\square$

5. Error estimates in energy norm. In this section, we establish order two superconvergence for the velocity approximation $\mathbf{u}_h$ in $\|\cdot\|$ norm and optimal order error estimate for the pressure approximation p_h in the standard L^2 norm.

Lemma 5.1. *Let $\mathbf{u} \in [H^{k+3}(\Omega)]^d$ and $p \in H^{k+2}(\Omega)$ and $\mathbf{v} \in V_h$. Then, the following estimates hold true*

$$|\ell_1(\mathbf{u}, \mathbf{v})| \leq Ch^{k+2} |\mathbf{u}|_{k+3} \|\mathbf{v}\|, \quad (35)$$

$$|\ell_2(p, \mathbf{v})| \leq Ch^{k+2} |p|_{k+2} \|\mathbf{v}\|. \quad (36)$$

Proof. Using the Cauchy-Schwarz inequality and the definitions of $\mathbb{Q}_h$ and Π_h , we have

$$\begin{aligned} |\ell_1(\mathbf{u}, \mathbf{v})| &= |(\mathbb{Q}_h \nabla \mathbf{u} - \Pi_h \nabla \mathbf{u}, \nabla_w \mathbf{v})| \\ &= |(\mathbb{Q}_h \nabla \mathbf{u} - \nabla \mathbf{u} + \nabla \mathbf{u} - \Pi_h \nabla \mathbf{u}, \nabla_w \mathbf{v})| \\ &\leq Ch^{k+2} |\mathbf{u}|_{k+3} \|\mathbf{v}\|. \end{aligned}$$

It follows from (17) and (20)

$$\begin{aligned} |\ell_2(p, \mathbf{v})| &= |\langle \mathcal{Q}_h p - p, (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h}| \\ &\leq C \sum_{T \in \mathcal{T}_h} \|\mathcal{Q}_h p - p\|_{\partial T} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T} \\ &\leq C \left(\sum_{T \in \mathcal{T}_h} h_T \|\mathcal{Q}_h p - p\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{e \in \mathcal{E}_h} h_T^{-1} \|\mathbf{v}_0 - \mathbf{v}_b\|_e^2 \right)^{\frac{1}{2}} \\ &\leq Ch^{k+2} |p|_{k+2} \|\mathbf{v}\|. \end{aligned}$$

We have proved the lemma. $\square$

Theorem 5.2. *Let $(\mathbf{u}_h, p_h) \in V_h^0 \times W_h$ be the solution of (13)-(14). Then, we have*

$$\|Q_h \mathbf{u} - \mathbf{u}_h\| \leq Ch^{k+2} (|\mathbf{u}|_{k+3} + |p|_{k+2}), \quad (37)$$

$$\|\mathcal{Q}_h p - p_h\| \leq Ch^{k+2} (|\mathbf{u}|_{k+3} + |p|_{k+2}). \quad (38)$$

Proof. By letting $\mathbf{v} = \mathbf{e}_h$ in (25) and $w = \varepsilon_h$ in (26) and using the equation (26), we have

$$\|\mathbf{e}_h\|^2 = |\ell_1(\mathbf{u}, \mathbf{e}_h) + \ell_2(p, \mathbf{e}_h)|. \quad (39)$$

It then follows from (35) and (36) that

$$\|\mathbf{e}_h\|^2 \leq Ch^{k+2} (|\mathbf{u}|_{k+3} + |p|_{k+2}) \|\mathbf{e}_h\|, \quad (40)$$

which implies (37). To estimate $\|\varepsilon_h\|$, we have from (25) that

$$(\varepsilon_h, \nabla \cdot \mathbf{v}) = (\nabla_w \mathbf{e}_h, \nabla_w \mathbf{v}) - \ell_1(\mathbf{u}, \mathbf{v}) - \ell_2(p, \mathbf{v}).$$

Using (40), (35) and (36), we arrive at

$$|(\varepsilon_h, \nabla \cdot \mathbf{v})| \leq Ch^{k+2}(|\mathbf{u}|_{k+3} + |p|_{k+2})\|\mathbf{v}\|.$$

Combining the above estimate with the *inf-sup* condition (21) gives

$$\|\varepsilon_h\| \leq Ch^{k+2}(|\mathbf{u}|_{k+3} + |p|_{k+2}),$$

which yields the desired estimate (38). $\square$

6. Error estimates in L2 norm. In this section, order two superconvergence for velocity in the L^2 norm is obtained by duality argument. Recall that $\mathbf{e}_h = \{\mathbf{e}_0, \mathbf{e}_b\} = Q_h \mathbf{u} - \mathbf{u}_h$ and $\epsilon_h = Q_h p - p_h$. Consider the dual problem: seeking $(\boldsymbol{\psi}, \xi)$ satisfying

$$-\Delta \boldsymbol{\psi} + \nabla \xi = \mathbf{e}_0 \quad \text{in } \Omega, \quad (41)$$

$$\nabla \cdot \boldsymbol{\psi} = 0 \quad \text{in } \Omega, \quad (42)$$

$$\boldsymbol{\psi} = 0 \quad \text{on } \partial\Omega. \quad (43)$$

Assume that the dual problem (41)-(43) satisfy the following regularity assumption:

$$\|\boldsymbol{\psi}\|_2 + \|\xi\|_1 \leq C\|\mathbf{e}_0\|. \quad (44)$$

We need the following lemma first.

Lemma 6.1. *For any $\mathbf{v} \in V_h^0$ and $w \in W_h$, the following equations hold true,*

$$(\nabla_w Q_h \boldsymbol{\psi}, \nabla_w \mathbf{v}) - (Q_h \xi, \nabla_w \cdot \mathbf{v}) = (\mathbf{e}_0, \mathbf{v}_0) + \ell_3(\boldsymbol{\psi}, \mathbf{v}) + \ell_2(\xi, \mathbf{v}), \quad (45)$$

$$(\nabla_w \cdot Q_h \boldsymbol{\psi}, w) = 0, \quad (46)$$

where

$$\ell_3(\boldsymbol{\psi}, \mathbf{v}) = \langle (\nabla \boldsymbol{\psi} - Q_h \nabla \boldsymbol{\psi}) \cdot \mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b \rangle_{\partial\mathcal{T}_h},$$

$$\ell_2(\xi, \mathbf{v}) = \langle Q_h \xi - \xi, (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h}.$$

Proof. Testing (41) by $\mathbf{v}_0$ with $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h^0$ gives

$$-(\Delta \boldsymbol{\psi}, \mathbf{v}_0) + (\nabla \xi, \mathbf{v}_0) = (\mathbf{e}_0, \mathbf{v}_0). \quad (47)$$

It follows from integration by parts and the fact $\langle \nabla \boldsymbol{\psi} \cdot \mathbf{n}, \mathbf{v}_b \rangle_{\partial\mathcal{T}_h} = 0$

$$-(\Delta \boldsymbol{\psi}, \mathbf{v}_0) = (\nabla \boldsymbol{\psi}, \nabla \mathbf{v}_0)_{\mathcal{T}_h} - \langle \nabla \boldsymbol{\psi} \cdot \mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b \rangle_{\partial\mathcal{T}_h}. \quad (48)$$

By integration by parts, (8) and (15)

$$\begin{aligned} (\nabla \boldsymbol{\psi}, \nabla \mathbf{v}_0)_{\mathcal{T}_h} &= (Q_h \nabla \boldsymbol{\psi}, \nabla \mathbf{v}_0)_{\mathcal{T}_h} \\ &= -(\mathbf{v}_0, \nabla \cdot (Q_h \nabla \boldsymbol{\psi}))_{\mathcal{T}_h} + \langle \mathbf{v}_0, Q_h \nabla \boldsymbol{\psi} \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h} \\ &= (Q_h \nabla \boldsymbol{\psi}, \nabla_w \mathbf{v}) + \langle \mathbf{v}_0 - \mathbf{v}_b, Q_h \nabla \boldsymbol{\psi} \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h} \\ &= (\nabla_w Q_h \boldsymbol{\psi}, \nabla_w \mathbf{v}) + \langle \mathbf{v}_0 - \mathbf{v}_b, Q_h \nabla \boldsymbol{\psi} \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h}. \end{aligned} \quad (49)$$

Combining (48) and (49) gives

$$-(\Delta \boldsymbol{\psi}, \mathbf{v}_0) = (\nabla_w Q_h \boldsymbol{\psi}, \nabla_w \mathbf{v}) - \ell_3(\boldsymbol{\psi}, \mathbf{v}). \quad (50)$$

Similar to the derivation of (31), we obtain

$$(\nabla \xi, \mathbf{v}_0) = -(Q_h \xi, \nabla_w \cdot \mathbf{v}) - \ell_2(\xi, \mathbf{v}). \quad (51)$$

Combining (50) and (51) with (47) yields (45). Testing equation (42) by $w \in W_h$ and using (16) give

$$(\nabla \cdot \boldsymbol{\psi}, w) = (Q_h \nabla \cdot \boldsymbol{\psi}, w) = (\nabla_w \cdot Q_h \boldsymbol{\psi}, w) = 0, \quad (52)$$

which implies (46) and we have proved the lemma. $\square$

By the same argument as (50), (25) has another form as

$$(\nabla_w \mathbf{e}_h, \nabla_w \mathbf{v}) - (\epsilon_h, \nabla_w \cdot \mathbf{v}) = \ell_3(\mathbf{u}, \mathbf{v}) + \ell_2(p, \mathbf{v}). \quad (53)$$

Theorem 6.2. *Let $(\mathbf{u}_h, p_h) \in V_h^0 \times W_h$ be the solution of (13)-(14). Assume that (44) holds true. Then, we have*

$$\|Q_0 \mathbf{u} - \mathbf{u}_0\| \leq Ch^{k+3}(|\mathbf{u}|_{k+3} + |p|_{k+2}). \quad (54)$$

Proof. Letting $\mathbf{v} = \mathbf{e}_h$ in (45) yields

$$\|\mathbf{e}_0\|^2 = (\nabla_w Q_h \psi, \nabla_w \mathbf{e}_h) - (Q_h \xi, \nabla_w \cdot \mathbf{e}_h) - \ell_3(\psi, \mathbf{e}_h) - \ell_2(\xi, \mathbf{e}_h). \quad (55)$$

Using the fact $(Q_h \xi, \nabla_w \cdot \mathbf{e}_h) = 0$, (55) becomes

$$\|\mathbf{e}_h\|^2 = (\nabla_w Q_h \psi, \nabla_w \mathbf{e}_h) - \ell_3(\psi, \mathbf{e}_h) - \ell_2(\xi, \mathbf{e}_h). \quad (56)$$

With $\mathbf{v} = Q_h \psi$, (53) becomes

$$(\nabla_w \mathbf{e}_h, \nabla_w Q_h \psi) - (\epsilon_h, \nabla_w \cdot Q_h \psi) = \ell_3(\mathbf{u}, Q_h \psi) + \ell_2(p, Q_h \psi). \quad (57)$$

Using (46), we have $(\epsilon_h, \nabla_w \cdot Q_h \psi) = 0$. Then (57) becomes

$$(\nabla_w \mathbf{e}_h, \nabla_w Q_h \psi) = \ell_3(\mathbf{u}, Q_h \psi) + \ell_2(p, Q_h \psi). \quad (58)$$

Combining (56) and (58), we have

$$\|\mathbf{e}_h\|^2 = \ell_3(\mathbf{u}, Q_h \psi) + \ell_2(p, Q_h \psi) - \ell_3(\psi, \mathbf{e}_h) - \ell_2(\xi, \mathbf{e}_h). \quad (59)$$

Using the Cauchy-Schwarz inequality, the trace inequality (17) and the definition of $\mathbb{Q}_h$, we arrive at

$$\begin{aligned} |\ell_3(\mathbf{u}, Q_h \psi)| &\leq |(\nabla \mathbf{u} - \mathbb{Q}_h \nabla \mathbf{u}) \cdot \mathbf{n}, Q_0 \psi - Q_b \psi|_{\partial T_h}| \\ &\leq \left(\sum_{T \in \mathcal{T}_h} \|\nabla \mathbf{u} - \mathbb{Q}_h \nabla \mathbf{u}\|_{\partial T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} \|Q_0 \psi - \psi\|_{\partial T}^2 \right)^{1/2} \\ &\leq Ch^{k+3} |\mathbf{u}|_{k+3} |\psi|_2. \end{aligned} \quad (60)$$

Similarly, we have

$$\begin{aligned} |\ell_2(p, Q_h \psi)| &\leq |(Q_h p - p, (Q_0 \psi - Q_b \psi) \cdot \mathbf{n})_{\partial T_h}| \\ &\leq C \left(\sum_{T \in \mathcal{T}_h} \|Q_h p - p\|_{\partial T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} \|Q_0 \psi - \psi\|_{\partial T}^2 \right)^{1/2} \\ &\leq Ch^{k+3} |p|_{k+2} |\psi|_2. \end{aligned} \quad (61)$$

It follows from the Cauchy-Schwarz inequality, the trace inequality (20) and (37),

$$\begin{aligned} |\ell_3(\psi, \mathbf{e}_h)| &\leq |(\nabla \psi - \mathbb{Q}_h \nabla \psi) \cdot \mathbf{n}, \mathbf{e}_0 - \mathbf{e}_b|_{\partial T_h}| \\ &\leq \left(\sum_{T \in \mathcal{T}_h} h_T \|\nabla \psi - \mathbb{Q}_h \nabla \psi\|_{\partial T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{e}_0 - \mathbf{e}_b\|_{\partial T}^2 \right)^{1/2} \\ &\leq Ch |\psi|_2 \|\mathbf{e}_h\| \\ &\leq Ch^{k+3} (|\mathbf{u}|_{k+3} + |p|_{k+2}) |\psi|_2. \end{aligned} \quad (62)$$

Similarly,

$$\begin{aligned} |\ell_2(\xi, \mathbf{e}_h)| &\leq |\langle \mathcal{Q}_h \xi - \xi, (\mathbf{e}_0 - \mathbf{e}_b) \cdot \mathbf{n} \rangle_{\partial T_h}| \\ &\leq \left(\sum_{T \in \mathcal{T}_h} h_T \|\mathcal{Q}_h \xi - \xi\|_{\partial T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{e}_0 - \mathbf{e}_b\|_{\partial T}^2 \right)^{1/2} \\ &\leq Ch^{k+3}(|\mathbf{u}|_{k+3} + |p|_{k+2})|\xi|_1. \end{aligned} \quad (63)$$

Combining all the estimates above with (59) yields

$$\|\mathbf{e}_h\|^2 \leq Ch^{k+3}(|\mathbf{u}|_{k+3} + |p|_{k+2})(\|\boldsymbol{\psi}\|_2 + \|\xi\|_1).$$

The estimate (54) follows from the above inequality and the regularity assumption (44). We have completed the proof. $\square$

7. Numerical experiments.

7.1. Example 1. We solve the following 2D stationary Stokes equations with domain $\Omega = (0, 1)^2$:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \begin{pmatrix} 192(-x^2 + x)^2(-2y + 1) \\ -192(-2x + 1)(-y^2 + y)^2 + \\ +128(-x^2 + x)(-2y + 1)^2(-2x + 1) - \\ -256(-x^2 + x)(-y^2 + y)(-2x + 1) \end{pmatrix} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The exact solution is

$$\begin{aligned} \mathbf{u} &= \begin{pmatrix} 32(-x^2 + x)^2(-y^2 + y)(-2y + 1) \\ -(32(-x^2 + x)(-y^2 + y)^2(-2x + 1)) \end{pmatrix}, \\ p &= 64(-x^2 + x)(-y^2 + y)(-2x + 1)(-2y + 1). \end{aligned} \quad (64)$$

In this example, we use quadrilateral grids shown in Figure 1.

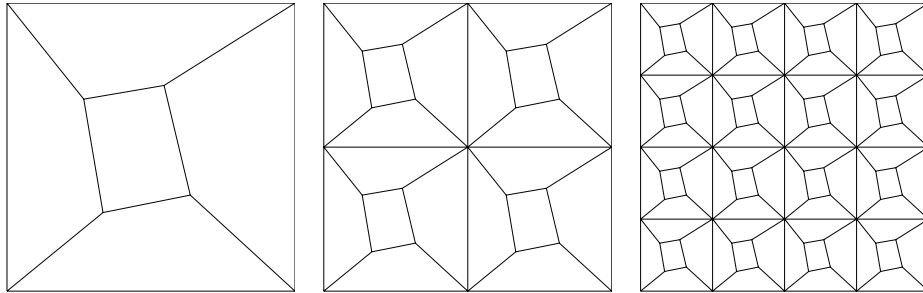


FIGURE 1. The first three quadrilateral grids for the computation of Table 1.

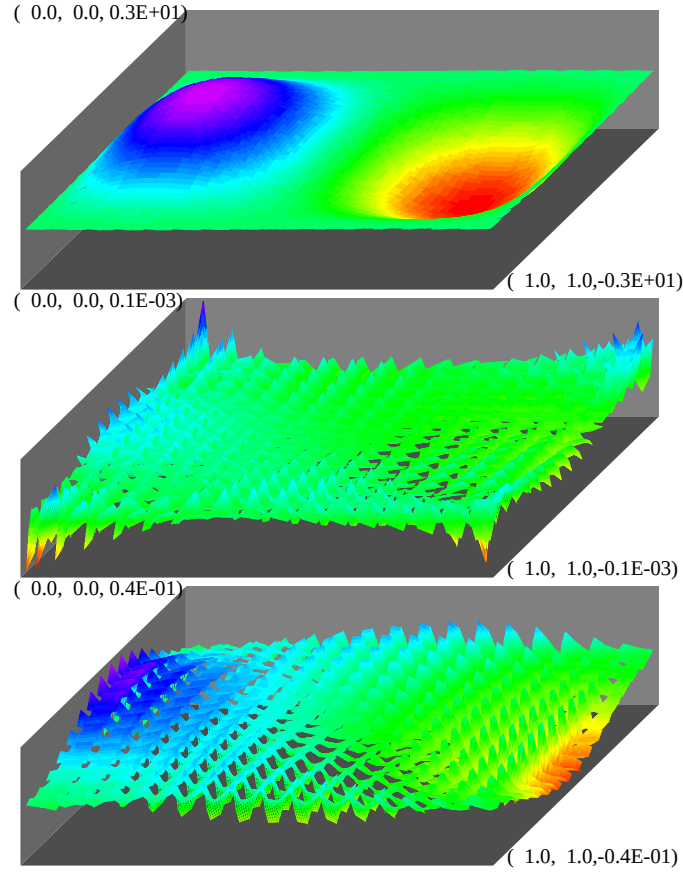


FIGURE 2. The P_1^2 - P_2^2 - P_2 WG finite element (65) solution $(\mathbf{u}_1)_h$ on the fifth grid of Figure 1 (on top), its error (in middle), and the error of the P_1^2 - P_1^2 - P_1 WG finite element (66) solution $(\mathbf{u}_1)_h$ on the fifth grid (at bottom). Both solutions are P_1 polynomials, but the latter error is 1000 times bigger.

We test the newly constructed, two-order superconvergent weak Galerkin finite elements. We compare the results with that of the standard optimal order convergent (no superconvergence) weak Galerkin finite elements. The two-order superconvergent weak Galerkin finite element spaces are

$$\begin{aligned} V_h &= \{ \mathbf{v} = \{ \mathbf{v}_0, \mathbf{v}_b \} : \mathbf{v}_0|_T \in [P_k(T)]^d, \mathbf{v}_b|_e \in [P_{k+1}(e)]^d, e \subset \partial T \}, \\ W_h &= \{ w \in L_0^2(\Omega) : w|_T \in P_{k+1}(T) \}. \end{aligned} \quad (65)$$

The standard weak Galerkin finite element spaces are

$$\begin{aligned} V_h^s &= \{ \mathbf{v} = \{ \mathbf{v}_0, \mathbf{v}_b \} : \mathbf{v}_0|_T \in [P_k(T)]^d, \mathbf{v}_b|_e \in [P_k(e)]^d, e \subset \partial T \}, \\ W_h^s &= \{ w \in L_0^2(\Omega) : w|_T \in P_k(T) \}. \end{aligned} \quad (66)$$

In Table 1, we list the errors and the orders of convergence, for the two types P_1 weak Galerkin finite elements. We can see that two-order superconvergence is achieved for the velocity in L^2 -norm and H^1 -like norm, for the new weak Galerkin finite element (65). The pressure converges at the optimal order in this case. But

TABLE 1. Error profiles and convergence rates for solution (64) on quadrilateral grids shown in Figure 1.

Grid	$\ Q_h \mathbf{u} - \mathbf{u}_h\ _0$	rate	$\ Q_h \mathbf{u} - \mathbf{u}_h\ $	rate	$\ p - p_h\ _0$	rate
by the P_1^2 - P_2^2 - P_2 WG finite element (65)						
4	0.3051E-03	3.95	0.3440E-01	3.02	0.9223E-02	2.95
5	0.1964E-04	3.96	0.4313E-02	3.00	0.1209E-02	2.93
6	0.1248E-05	3.98	0.5421E-03	2.99	0.1555E-03	2.96
by the P_1^2 - P_1^2 - P_1 WG finite element (66)						
4	0.5450E-01	1.88	0.2828E+01	0.94	0.1912E+01	0.89
5	0.1390E-01	1.97	0.1430E+01	0.98	0.9708E+00	0.98
6	0.3492E-02	1.99	0.7171E+00	1.00	0.4873E+00	0.99

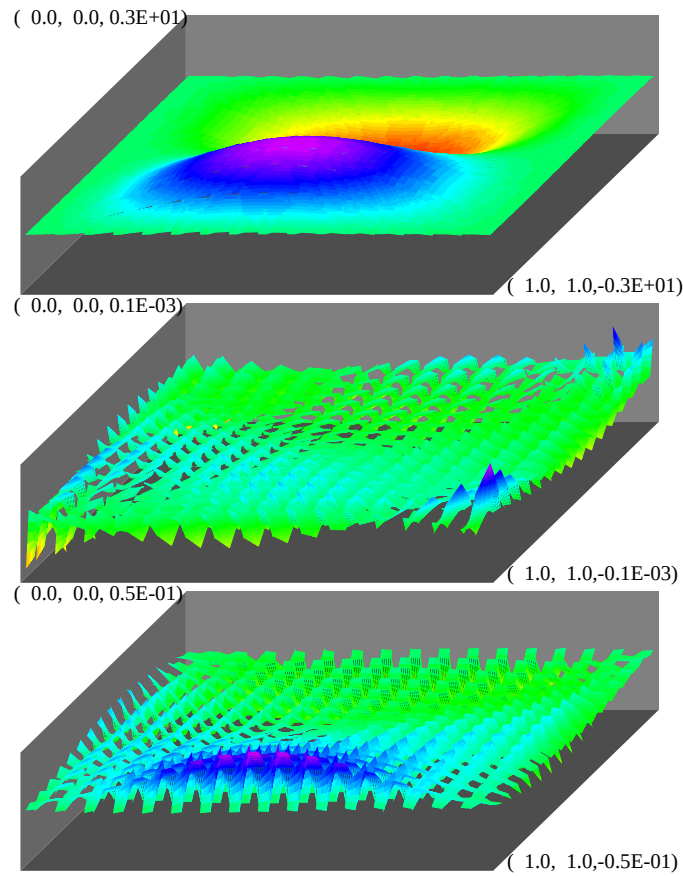


FIGURE 3. The P_1^2 - P_2^2 - P_2 WG finite element (65) solution $(\mathbf{u}_2)_h$ on the fifth grid of Figure 1 (on top), its error (in middle), and the error of the P_1^2 - P_1^2 - P_1 WG finite element (66) solution $(\mathbf{u}_2)_h$ on the fifth grid (at bottom).

the standard weak Galerkin finite element converges only at the optimal order,

with all errors 1000 times bigger than that of the two-order superconvergence weak Galerkin finite element (65).

To see the superconvergence phenomenon, in Figure 2, we plot the first component of $\mathbf{u}_h$ of the P_1^2 - P_2^2 - P_2 WG finite element (65) solution for (64) on the fifth grid of Figure 1 on the top. In middle of Figure 2, we plot its error. At the bottom of Figure 2, we plot the error of the P_1^2 - P_1^2 - P_1 WG finite element (66) solution $(\mathbf{u}_1)_h$ on the fifth grid. We note that both solutions are P_1 polynomials, but the error of latter is 1000 times bigger.

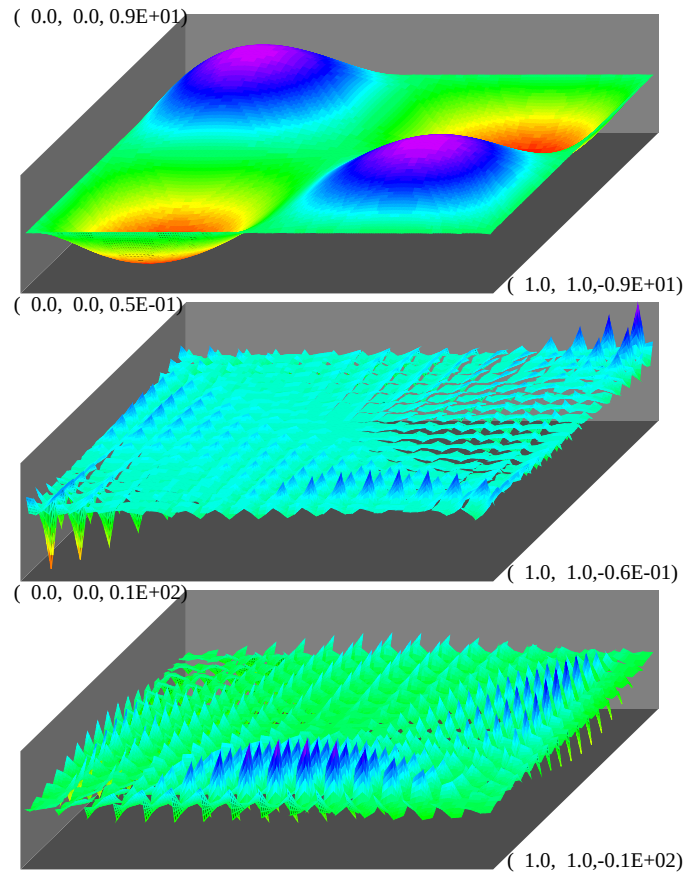


FIGURE 4. The P_1^2 - P_2^2 - P_2 WG finite element (65) solution p_h on the fifth grid of Figure 1 (on top), its error (in middle), and the error of the P_1^2 - P_1^2 - P_1 WG finite element (66) solution p_h on the fifth grid (at bottom).

In Figure 3, we plot the second component of $\mathbf{u}_h$ of the P_1^2 - P_2^2 - P_2 WG finite element (65) solution for (64) on the fifth grid of Figure 1 on the top. In middle of Figure 3, we plot its error. At the bottom of Figure 3, we plot the error of the P_1^2 - P_1^2 - P_1 WG finite element (66) solution $(\mathbf{u}_1)_h$ on the fifth grid. Correspondingly, in Figure 4, we plot the numerical solution for pressure, and the two errors.

In Table 2, we compute the two-order superconvergent P_2 weak Galerkin finite element solutions and the standard P_2 weak Galerkin finite element solutions. We

TABLE 2. Error profiles and convergence rates for solution (64) on quadrilateral grids shown in Figure 1.

Grid	$\ Q_h \mathbf{u} - \mathbf{u}_h\ _0$	rate	$\ Q_h \mathbf{u} - \mathbf{u}_h\ $	rate	$\ p - p_h\ _0$	rate
by the P_2^2 - P_3^2 - P_3 WG finite element (65)						
3	0.8289E-03	5.12	0.8054E-01	4.12	0.5896E-02	4.32
4	0.2507E-04	5.05	0.4871E-02	4.05	0.3609E-03	4.03
5	0.7763E-06	5.01	0.3018E-03	4.01	0.2277E-04	3.99
by the P_2^2 - P_2^2 - P_2 WG finite element (66)						
3	0.8848E-02	3.06	0.4878E+00	2.30	0.3476E+00	1.79
4	0.1128E-02	2.97	0.1213E+00	2.01	0.8938E-01	1.96
5	0.1416E-03	2.99	0.3035E-01	2.00	0.2234E-01	2.00

can see the former has two orders higher convergent rate than the latter. On same grids and with same P_2 polynomials for $\mathbf{u}_0$, the error of former is 1000 times smaller than the latter. In Table 3, both types of P_3 weak Galerkin finite element solutions are listed. They verify the theory.

TABLE 3. Error profiles and convergence rates for solution (64) on quadrilateral grids shown in Figure 1.

Grid	$\ Q_h \mathbf{u} - \mathbf{u}_h\ _0$	rate	$\ Q_h \mathbf{u} - \mathbf{u}_h\ $	rate	$\ p - p_h\ _0$	rate
by the P_3^2 - P_4^2 - P_4 WG finite element (65)						
2	0.6018E-02	6.29	0.3910E+00	5.28	0.1249E-01	5.72
3	0.8806E-04	6.09	0.1146E-01	5.09	0.2933E-03	5.41
4	0.1352E-05	6.03	0.3526E-03	5.02	0.8304E-05	5.14
by the P_3^2 - P_3^2 - P_3 WG finite element (66)						
3	0.1595E-03	5.24	0.1700E-01	4.52	0.1625E-01	0.46
4	0.8572E-05	4.22	0.1641E-02	3.37	0.2105E-02	2.95
5	0.5330E-06	4.01	0.2041E-03	3.01	0.2636E-03	3.00

7.2. Example 2. We solve the problem (64) again, on polygonal grids, consisting of quadrilaterals, pentagons and hexagons, shown in Figure 5. We intentionally perturb the grids to show that the two-order superconvergence is grid independent, i.e., not necessarily on uniform grids. In Table 4, we list the errors and the orders of convergence. The computational results match the theoretic order of convergence, in all cases.

7.3. Example 3. We compute the following 2D driven cavity flow on domain $\Omega = (0, 1)^2$:

$$\begin{aligned}
 -\Delta \mathbf{u} + \nabla p &= \mathbf{0} && \text{in } \Omega, \\
 \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \\
 \mathbf{u} &= \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{on } [0, 1] \times \{1\}, \\ \mathbf{0} & \text{on } \partial\Omega \setminus \{[0, 1] \times \{1\}\}. \end{cases}
 \end{aligned}$$

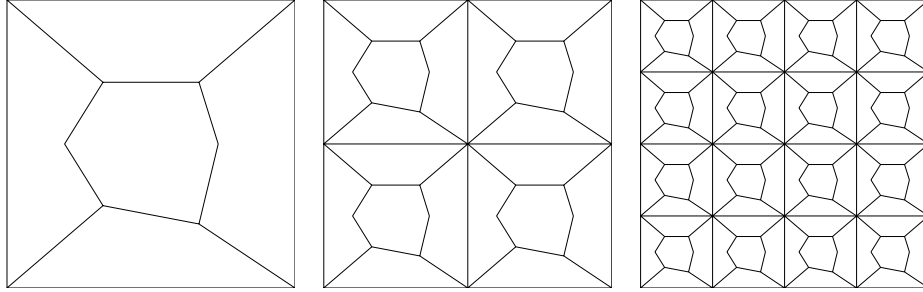


FIGURE 5. The first three polygonal grids for the computation of Table 4.

TABLE 4. Error profiles for solution (64) on polygonal grids shown in Figure 5.

Grid	$\ Q_h \mathbf{u} - \mathbf{u}_h\ _0$	rate	$\ Q_h \mathbf{u} - \mathbf{u}_h\ $	rate	$\ p - p_h\ _0$	rate
by the P_0^2 - P_1^2 - P_1 WG finite element						
4	0.2202E-01	1.80	0.2885E+00	1.91	0.2138E+00	1.97
5	0.5715E-02	1.95	0.7374E-01	1.97	0.5376E-01	1.99
6	0.1442E-02	1.99	0.1861E-01	1.99	0.1351E-01	1.99
by the P_1^2 - P_2^2 - P_2 WG finite element						
4	0.2512E-03	3.86	0.2922E-01	2.94	0.8673E-02	2.93
5	0.1661E-04	3.92	0.3737E-02	2.97	0.1147E-02	2.92
6	0.1066E-05	3.96	0.4735E-03	2.98	0.1481E-03	2.95
by the P_2^2 - P_3^2 - P_3 WG finite element						
3	0.5373E-03	5.10	0.5945E-01	4.16	0.5018E-02	4.11
4	0.1639E-04	5.04	0.3567E-02	4.06	0.3219E-03	3.96
5	0.5101E-06	5.01	0.2208E-03	4.01	0.2063E-04	3.96
by the P_3^2 - P_4^2 - P_4 WG finite element						
2	0.3384E-02	6.26	0.2770E+00	5.30	0.8473E-02	5.54
3	0.4985E-04	6.08	0.8069E-02	5.10	0.2273E-03	5.22
4	0.7855E-06	5.99	0.2525E-03	5.00	0.6876E-05	5.05

The solution is not in $H_0^1(\Omega)$. But the discontinuous $\mathbf{u}_b$ function can handle the jump boundary condition well. Additionally, the discontinuous $\mathbf{u}_b$ function can be defined well on grids with hanging nodes, i.e., non-compatible grids (see Figure 6). In this computation, we use the graded grid shown in Figure 6.

The computed velocity field is plotted in Figure 7.

The extra fine grid near the two top corners gives an accurate solution at these singularity points. The zoom-in plots at the two corners are displayed in Figures 8–9.

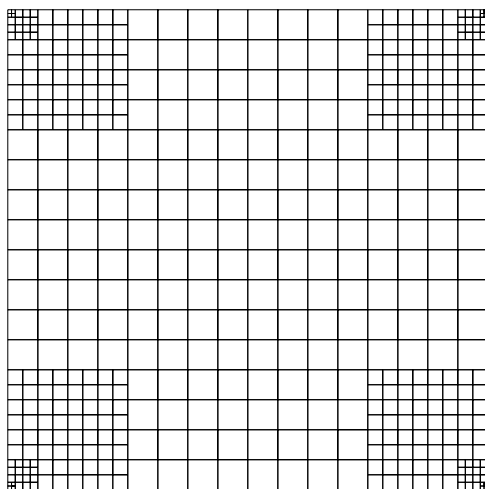


FIGURE 6. The graded grid for the driven cavity computation.

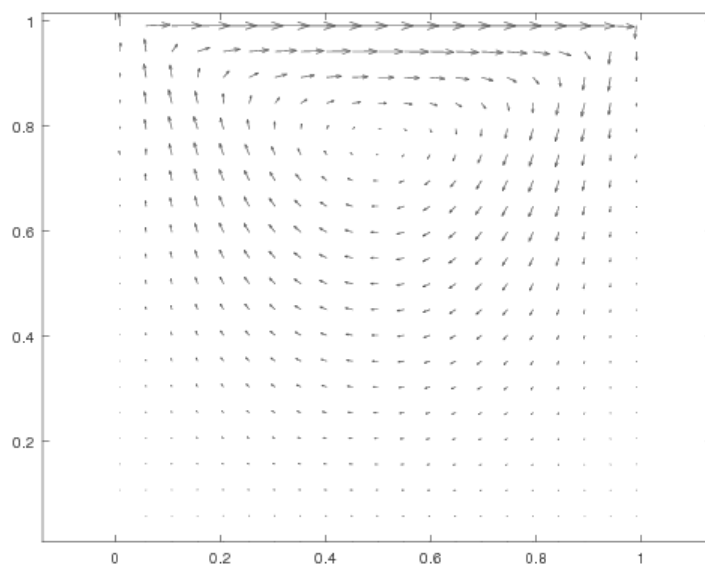


FIGURE 7. The computed driven cavity velocity field on the whole domain.

Because we use smaller square elements at the two bottom corners, we can compute the secondary flow at these two corners well. They are plotted in Figures 10–11.

We zoom-in further the solution at the lower-left corner. We can see in Figure 12 there is another secondary flow generated by the secondary flow in Figure 10. Nevertheless we can see such a computed flow is not mass conservative. We need

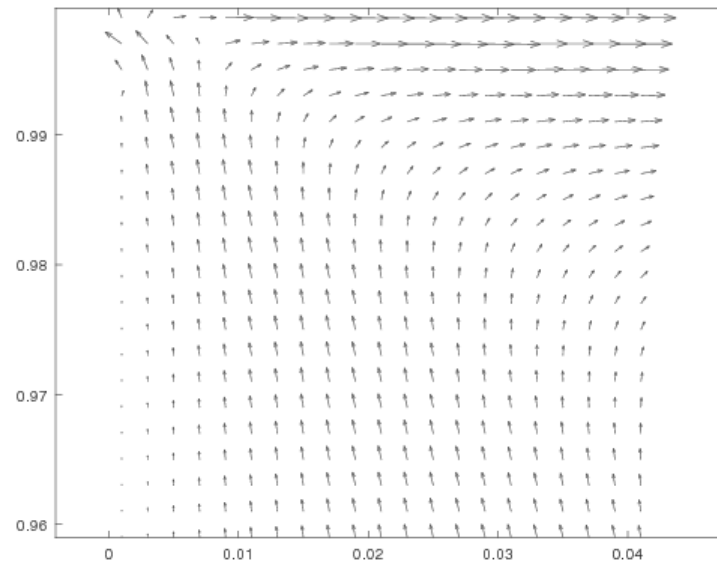


FIGURE 8. The graded grid for the driven cavity computation.

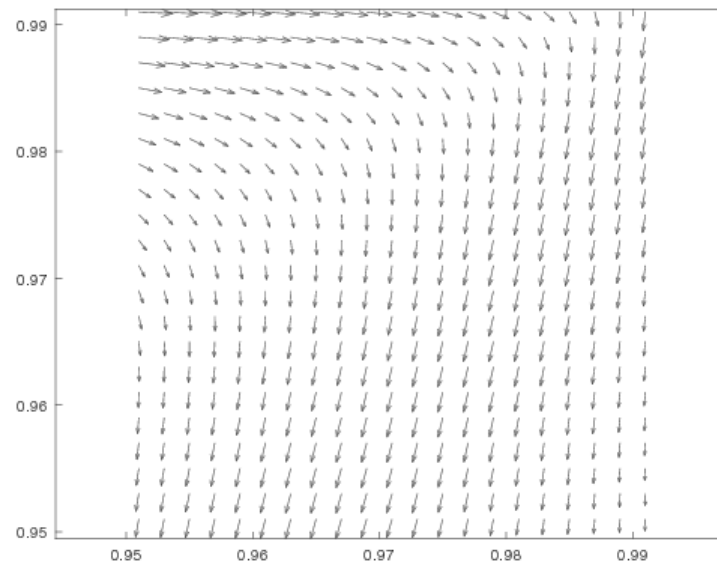


FIGURE 9. The graded grid for the driven cavity computation.

to use $H(\text{div})$ finite elements [15] or divergence-free H^1 finite elements [5, 8, 9, 16, 17, 18, 19, 20, 21] to get a mass conservative solution.

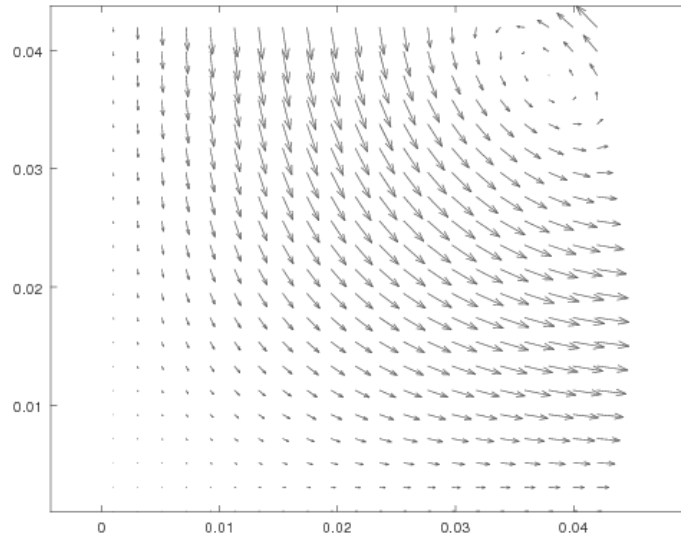


FIGURE 10. The graded grid for the driven cavity computation.

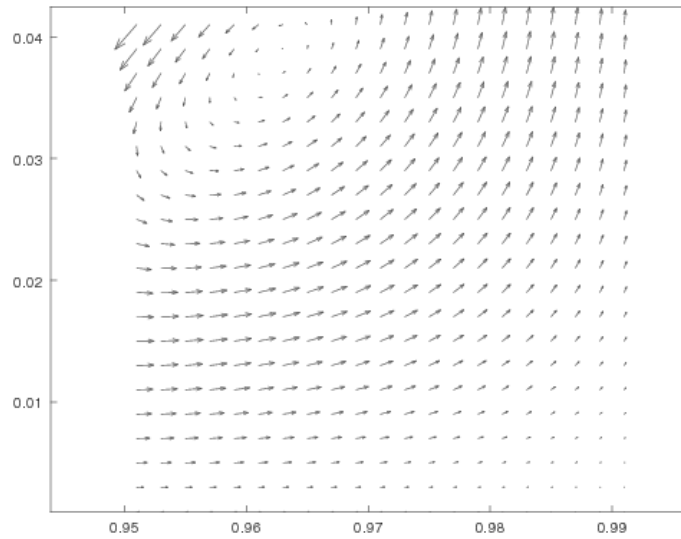


FIGURE 11. The graded grid for the driven cavity computation.

7.4. **Example 4.** We compute a 3D problem (1)–(3) with $\Omega = (0, 1)^3$. The source term and the boundary value $\mathbf{g}$ are chosen so that the exact solution is

$$\mathbf{u} = \begin{pmatrix} -g_y \\ g_x + g_z \\ -g_y \end{pmatrix}, \quad p = g_{yz} \quad \text{where } g = 2^{12}(x - x^2)^2(y - y^2)^2(z - z^2)^2. \quad (67)$$

We use tetrahedral meshes shown in Figure 13. The results of the 3D P_k - P_{k+1} weak Galerkin finite element methods are listed in Table 5. The results show that the method is stable and is of two-order superconvergence (for velocity).

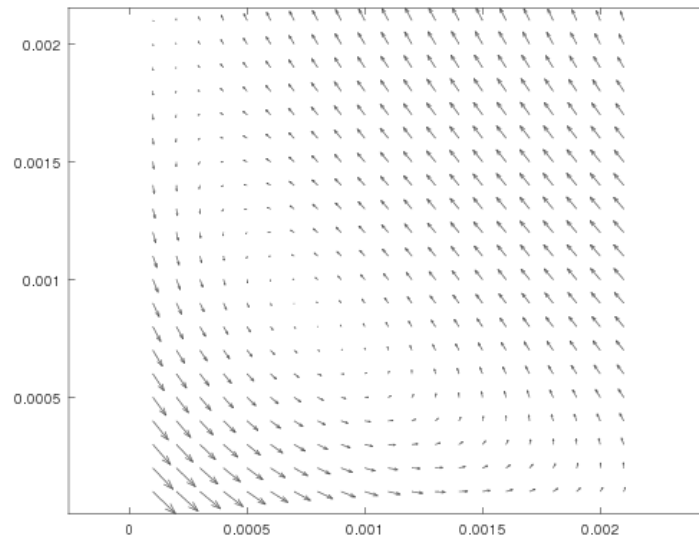


FIGURE 12. The graded grid for the driven cavity computation.

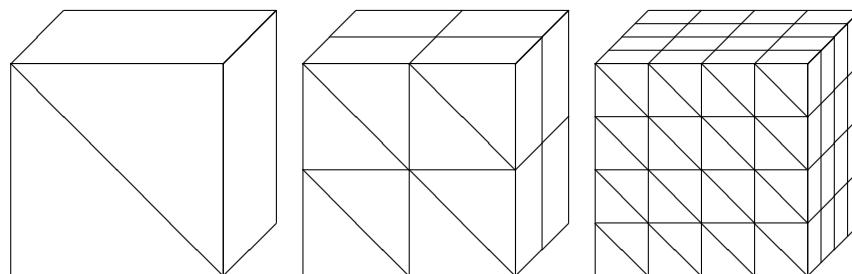


FIGURE 13. The first three levels of wedge grids used in Table 5.

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TABLE 5. Error profiles for solution (67) on wedge grids shown in Figure 13.

Grid	$\ Q_h \mathbf{u} - \mathbf{u}_h\ _0$	rate	$\ Q_h \mathbf{u} - \mathbf{u}_h\ $	rate	$\ p - p_h\ _0$	rate
by the P_0^2 - P_1^2 - P_1 WG finite element						
4	0.8167E-01	1.59	0.1864E+01	1.83	0.5772E+00	1.78
5	0.2228E-01	1.87	0.4851E+00	1.94	0.1575E+00	1.87
6	0.5689E-02	1.97	0.1228E+00	1.98	0.3776E-01	2.06
by the P_1^2 - P_2^2 - P_2 WG finite element						
3	0.6428E-01	3.50	0.4486E+01	2.52	0.6305E+00	3.23
4	0.4636E-02	3.79	0.6105E+00	2.88	0.8163E-01	2.95
5	0.2856E-03	4.02	0.7796E-01	2.97	0.9492E-02	3.10
by the P_2^2 - P_3^2 - P_3 WG finite element						
2	0.7217E+00	3.28	0.3793E+02	1.89	0.2623E+01	5.54
3	0.2563E-01	4.82	0.2898E+01	3.71	0.2215E+00	3.57
4	0.8352E-03	4.94	0.1942E+00	3.90	0.1439E-01	3.94

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