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WELL-POSEDNESS RESULTS AND BLOW-UP FOR A SEMI-LINEAR TIME FRACTIONAL DIFFUSION EQUATION WITH VARIABLE COEFFICIENTS

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Abstract. The semi-linear problem of a fractional diffusion equation with the Caputo-like counterpart of a hyper-Bessel differential is considered. The results on existence, uniqueness and regularity estimates (local well-posedness) of the solutions are established in the case of linear source and the source functions that satisfy the globally Lipschitz conditions. Moreover, we prove that the problem exists a unique positive solution. In addition, the unique continuation of solutions and a finite-time blow-up are proposed with the reaction terms are logarithmic functions.

1. Introduction.

In this paper, we consider a semi-linear time-fractional diffusion equation with time-varying coefficients:

$$\begin{cases} {}^{\mathrm{C}}\left(t^{\sigma}\frac{\partial}{\partial t}\right)^{\alpha} u + \mathbb{L}u = f(u), \quad x\in\Omega,\ t>0, \\ u(x,t) = 0, \quad x\in\partial\Omega,\ t>0, \\ u(x,0) = u_0(x), \quad x\in\Omega, \end{cases} \tag{$\mathbb{P}$}$$

where $\Omega \subset \mathbb{R}^d, (d \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and the initial data u_0 at $t = 0$ is given. The operator $\mathbb{L} : \mathcal{D}(\mathbb{L}) \in L^2(\Omega) \to L^2(\Omega)$ be a positive, self-adjoint operator. In ($\mathbb{P}$), ${}^{\mathrm{C}}\left(t^{\sigma}\frac{\partial}{\partial t}\right)^{\alpha}$ will denote a Caputo-like counterpart (C-LC) to hyper-Bessel operator (H-BO) of order $\alpha \in (0, 1)$ and the parameter $0 < \sigma < 1$ (see formula (3)).

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Fractional calculus is a subject of a long history and has gained great interest in different fields of applied science: mathematics [1–3,9,21,26,31], physics [18,28,30], including stochastic processes [6,25,29], mechanics [13], chemistry and biology [16] and some references therein. It is well known that the original definition of the H-BO differential of higher (integer) order $m \geq 1$ was first introduced by Dimovski [11]:

$$\mathcal{B} = t^{\alpha_0} \frac{\mathrm{d}}{\mathrm{d}t} t^{\alpha_1} \frac{\mathrm{d}}{\mathrm{d}t} \cdots t^{\alpha_{m-1}} \frac{\mathrm{d}}{\mathrm{d}t} t^{\alpha_m},$$

where $\alpha_0, \alpha_1, ..., \alpha_m$ are arbitrary parameters that satisfy $\sum_{i=0}^{m} \alpha_i < m$. The fractional power $\mathcal{B}^\alpha$ of the H-BO (as convolutional products in the sense of operational calculus) is developed by himself in [10] and Kiryakova [19,20], Lamb and McBride [21], McBride [26], and references therein. In probability theory, it is useful to illustrate heat diffusion of the fractional Brownian motion [6,25] and the study of Tricomi-type [43] or Keldysh-type equations [42].

For the general case, Garra et al. [12] consider fractional differential equation with time-varying coefficient of the form

$$\left(t^\sigma \frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha u(t) = -\lambda u(t), \quad \alpha \in (0,1),\ \sigma \in \mathbb{R},\ \lambda > 0,\ t \geq 0. \tag{1}$$

The authors provided the solutions of the relaxation-type equation (1) obtained by the operator $\left(t^\sigma \frac{\partial}{\partial t}\right)^\alpha$ that is related to the Erdélyi-Kober integrals [12]:

$$\left(t^\sigma \frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha g(t) = \begin{cases} (1-\sigma)^\alpha t^{-(1-\sigma)\alpha} I_{1-\sigma}^{0,-\alpha} g(t), & \text{if } \sigma < 1, \\ (\sigma-1)^\alpha I_{1-\sigma}^{-1,-\alpha} t^{(1-\sigma)\alpha} g(t), & \text{if } \sigma > 1, \\ \text{“considered separately”}, & \text{if } \sigma = 1. \end{cases} \tag{2}$$

Note that as $\sigma = 0$, this operator coincides with the Riemann-Liouville fractional derivative, and for $\alpha = 1$ expressions (2) include the conventional first-order derivative. In analogy with the classical theory of fractional calculus operators, we are led to a generalization to describe the regularized C-LC of the H-BO for $\sigma < 1$ in terms of Erdélyi-Kober fractional-order operator (see e.g. [5,12]):

$$\begin{aligned} {}^{\mathrm{C}}\left(t^\sigma \frac{\partial}{\partial t}\right)^\alpha g(t) &:= (1-\sigma)^\alpha t^{-\alpha(1-\sigma)} I_{1-\sigma}^{0,-\alpha} (g(t) - g(0)) \\ &= \left(t^\sigma \frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha g(t) - \frac{g(0)}{(1-\sigma)^{-\alpha}} \frac{t^{-\alpha(1-\sigma)}}{\Gamma(1-\sigma)}. \end{aligned} \tag{3}$$

Recently, Al-Musalhi et al. [5] used the C-LC of H-BO (3) to study both a direct problem and an inverse source problem:

$${}^{\mathrm{C}}\left(t^\sigma \frac{\partial}{\partial t}\right)^\alpha u(x,t) - u_{xx}(x,t) = f(x,t), \quad x \in (0,\pi),\ t \in (0,T).$$

Basing on the appropriate eigenfunction expansions the authors have constructed the solutions and the properties for existence and uniqueness are also presented. In [34], Tuan et al consider a problem of recovering the initial data for a time-fractional diffusion equation with a regularized hyper-Bessel differential. The solution to this problem exists but isn't stable, so, the authors use the fractional Tikhonov method to construct a regularized solution. Also, they also provide the error estimates between the regularized solution and the exact solution. Some recent studies on the behavior of solutions can be listed as [22–24,32,35,36,39,40] and the references therein. All works mentioned above give us a great motivation to study the well-posed behavior of mild solutions to Problem ($\mathbb{P}$).

In this paper, for Problem ($\mathbb{P}$), we study two cases of the source functions:

- the linear source functions;
- the nonlinear source functions.

For the linear case, $f = f(x,t)$, we obtain local well-posedness properties (existence and regularity of (unique) weak solutions). In the case of nonlinear source functions, we consider both the globally and locally Lipschitz cases. The existence of a local solution and the regularity of the solution is established with the source function $f = f(u)$ satisfying the globally Lipschitz condition. Moreover, we prove that the Problem ($\mathbb{P}$) exists a unique positive solution. With the locally Lipschitz source function $f = f_q(u) = |u|^{q-1}\log|u|^q,\ q > 1$, we consider the extension of the solution to larger time periods and the blow-up of the solution. In [7], for the fractional-in time wave equations, the authors also consider the blow-up of solutions with derivatives considered in Caputo sense, and the source function has a general form that satisfies the given conditions. Hence, for this logarithmic type function as above, this is almost the first work to study the blow-up of the solutions to the Problem ($\mathbb{P}$).

The rest of the paper is organized as follows. Section 2, we first present some relevant notations, and secondly, the definition of the Mittag-Leffler functions is given and its useful properties for use throughout the paper. In Section 3, we consider the linear problem, some regularity estimates of weak solutions are obtained. In Section 4, we consider the semi-linear Problem ($\mathbb{P}$). The results of local well-posedness (local existence of solutions, uniqueness, regularity) are established when the source function is global Lipschitz. Furthermore, we prove the problem has a unique positive solution. For the nonlinearity source of the form as logarithmic functions $f_q(u) = |u|^{q-1}\log|u|^q,\ q > 1$ (locally Lipschitz function type), the uniqueness continuation of solutions and a finite-time blow-up are proposed. The conclusion is stated in Section 5.

2. **Preliminaries.**

2.1. **Relevant notations.** Let us recall that the spectral problem

$$\begin{cases} \mathbb{L}\mathrm{e}_p(x) = \lambda_p \mathrm{e}_p(x), & x \in \Omega, \\ \mathrm{e}_p(x) = 0, & x \in \partial\Omega, \end{cases}$$

admits a family of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le ... \le \lambda_p \le ... \nearrow \infty.$$

The notation $\|\cdot\|_B$ stands for the norm in the Banach space B. We denote by $L^q(0,T;B), 1 \le q \le \infty,\ T > 0$, the Banach space of real-valued measurable functions $v : (0;T) \to B$ with norm

$$\|v\|_{L^q(0,T;B)} = \left(\int_0^T \|v(t)\|_B^q \mathrm{d}t \right)^{\frac{1}{q}} < \infty, \quad \text{for } 1 \le q < \infty,$$

$$\|v\|_{L^q(0,T;B)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|v(t)\|_B < \infty, \quad \text{for } q = \infty.$$

The norm of the function space $C^k([0,T];B), 0 \le k \le \infty$ is denoted by

$$\|v\|_{C^k([0,T];B)} = \sum_{i=0}^{k} \sup_{t\in[0,T]} \|v^{(i)}(t)\|_B < \infty.$$

For any $\zeta \ge 0$, we define the Hilbert scale space

$$\mathcal{D}(\mathbb{L}^\zeta) = \left\{ v = \sum_{p=1}^{\infty} (v, \mathrm{e}_p)\mathrm{e}_p(x) \in L^2(\Omega) : \sum_{p=1}^{\infty} (v, \mathrm{e}_p)^2 \lambda_p^{2\zeta} < \infty \right\},$$

is equipped with norm

$$\|v\|_{\mathcal{D}(\mathbb{L}^\zeta)} = \left(\sum_{p=1}^{\infty} (v, \mathrm{e}_p)^2 \lambda_p^{2\zeta} \right)^{\frac{1}{2}}.$$

Obviously, we have $\mathcal{D}(\mathbb{L}^0) = L^2(\Omega)$ if $\zeta = 0$. We denote by $\mathcal{D}(\mathbb{L}^{-\zeta})$ the dual space of $\mathcal{D}(\mathbb{L}^{-\zeta})$ provided that the dual space of $L^2(\Omega)$ is identified with itself, e.g. see [27]. The space $\mathcal{D}(\mathbb{L}^{-\zeta})$ is a Hilbert space with respect to the norm

$$\|v\|_{\mathcal{D}(\mathbb{L}^{-\zeta})} = \left(\sum_{p=1}^{\infty} (v, \mathrm{e}_p)^2_{-\zeta,\zeta} \lambda_p^{-2\zeta} \right)^{\frac{1}{2}},$$

for $w \in \mathcal{D}(\mathbb{L}^{-\zeta})$ where $(\cdot,\cdot)_{-\zeta,\zeta}$ is the dual product between $\mathcal{D}(\mathbb{L}^{-\zeta})$ and $\mathcal{D}(\mathbb{L}^{\zeta})$. We note that

$$(v_1, v_2)_{-\zeta,\zeta} = (v_1, v_2), \quad \text{for } v_1 \in L^2(\Omega), v_2 \in \mathcal{D}(\mathbb{L}^\zeta).$$

Remark 1. From the definitions of the spaces $\mathcal{D}(\mathbb{L}^\zeta)$ and $\mathcal{D}(\mathbb{L}^{-\zeta})$, we observe that

$$\|v\|_{\mathcal{D}(\mathbb{L}^{-\zeta})} \le C_\zeta \|v\|_{L^2}, \quad \text{and} \quad \|v\|_{L^2} \le C_\zeta \|v\|_{\mathcal{D}(\mathbb{L}^\zeta)}, \quad \text{for } C_\zeta > 0.$$

Given a Banach space B, let $C((0,T];B)$ be the set of all continuous functions which map $(0,T]$ into B. For $\mu > 0$, we define the following Banach space

$$X^\mu((0,T];B) = \left\{ v \in C((0,T];B) : \sup_{t\in(0,T]} t^\mu \|v(t)\|_B < \infty \right\},$$

with the norm $\|v\|_{X^\mu((0,T];B)} = \sup_{t\in(0,T]} t^\mu \|v(t)\|_B < \infty$, see [33].

2.2. **Properties of Mittag-Leffler functions and some related results.** The Mittag-Leffler function is defined by (see [14, 15])

$$E_{\alpha,\beta}(y) = \sum_{m=0}^{\infty} \frac{y^m}{\Gamma(\alpha m + \beta)}, \quad y \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants, Γ is the usual Gamma function.

Next, we give some properties of the Mittag-Leffler function. Let $\beta \in \mathbb{R}$, and $\alpha \in (0,2)$, we have:

$$|E_{\alpha,\beta}(-y)| \le \frac{C}{1+|y|}, \quad \mu \le \arg(y) \le \pi,$$

where $C > 0$ depends on α, β, μ and $\frac{\pi\beta}{2} < \mu < \min\{\pi, \pi\beta\}$ (see e.g. [14, 30]).

Lemma 2.1. *For $0 < \alpha_1 < \alpha_2 < 1$ and $\alpha \in [\alpha_1, \alpha_2]$, then there exist the positive constants $C^-, C^+, \bar{C}$ such that*

a) $E_{\alpha,1}(-y) > 0$, for any $y > 0$;

b) $\dfrac{C^-}{1+y} \le E_{\alpha,1}(-y) \le \dfrac{C^+}{1+y}$, *and* $E_{\alpha,\beta}(-y) \le \dfrac{\bar{C}}{1+y}$, *for* $\beta \in \mathbb{R},\ y > 0$.

Lemma 2.2. *Let* $\alpha > 0, \lambda > 0, t > 0, n \in \mathbb{N}$, *we have*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[E_{\alpha,1}(-\lambda t^\alpha)\right] = -\lambda t^{\alpha-n} E_{\alpha,\alpha-n+1}(-\lambda t^\alpha); \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\right] = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha). \tag{5}$$

Lemma 2.3. *The following equality holds (for proof, see [44])*

$$E_{\alpha,1}(-y) = \int_0^\infty W_\alpha(z) e^{-yz} \mathrm{d}z, \quad \text{for } y \in \mathbb{C},$$

where we recall the definition of the Wright type function (see [15], Formula (28))

$$W_\alpha(z) := \sum_{n=0}^\infty \frac{(z)^n}{\Gamma(-\alpha n + 1 - \alpha)}, \quad 0 < \alpha < 1.$$

Moreover, $W_\alpha(z)$ *is a probability density function, that is,*

$$W_\alpha(z) \ge 0, \quad \text{for } z > 0; \quad \text{and} \quad \int_0^\infty W_\alpha(z) \mathrm{d}z = 1.$$

Lemma 2.4 (*Weakly singular Grönwall's inequalities*)**.** *Let* A, B, σ_1, σ_2 *be non-negative constants. For the continuous functions* $u : [0, T] \to [0, \infty)$, *assume that*

$$u(t) \le A t^{-\sigma_1} + B \int_0^t (t-\tau)^{-\sigma_2} u(\tau) \mathrm{d}\tau, \quad \text{for all} \quad t \in [0, T].$$

Then, there exists a positive constant $C(B, \sigma_2, T)$ *such that*

$$u(t) \le \frac{C(B, \sigma_2, T)}{1 - \sigma_1} A t^{-\sigma_1}, \quad \text{for a.e. } t \in (0, T].$$

Proof. See [38], Theorem 1.2, page 2. □

The main result of the paper is based on two main goals, that is, consider problem ($\mathbb{P}$) with linear and nonlinear source functions. We get into the results presented in the next section with the source function $f = f(x, t)$ (depending on the space variable $x \in \Omega$ and the time $t \in [0, T]$).

3. **The linear problem.**

For the source function to be linear, we consider the properties of existence, uniqueness and new regularity estimates. Indeed, we consider the linear problem

$$\begin{cases} {}^{\mathrm{C}}\left(t^\sigma \dfrac{\partial}{\partial t}\right)^\alpha u(x,t) + (\mathbb{L}u)(x,t) = f(x,t), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{6}$$

The function u is a mild solution of (6) if $u \in C([0,T]; L^2(\Omega))$ and satisfies the following integral equation

$$u(t) = E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right) u_0 + \frac{1}{s^\alpha} \int_0^t (t^s - \tau^s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right) f(\tau) \mathrm{d}(\tau^s), \tag{7}$$

where $t < T,\ s = 1 - \sigma,\ \alpha \in (0,1)$.

Theorem 3.1. *For $\sigma \in (\frac{1}{2}, 1)$ and $s = 1-\sigma$, let us choose α such that it satisfies* $\max\left\{\frac{1}{2}; \frac{2s-1}{s}\right\} < \alpha < 1$. *Let $u_0 \in \mathcal{D}(\mathbb{L}^\zeta) \cap \mathcal{D}(\mathbb{L}^{1-\zeta})$, with $\zeta = \frac{1}{\alpha}$ and*

$$f \in C([0,T]; \mathcal{D}(\mathbb{L}^{-\zeta})) \cap L^{2m}(0,T; L^2(\Omega) \cap \mathcal{D}(\mathbb{L}^\zeta) \cap \mathcal{D}(\mathbb{L}^{-\zeta})),$$

for $\frac{1}{m} + \frac{1}{n} = 1$, and $1 \leq n < \frac{1}{2-2\alpha}$. Then the Problem (6) has a unique weak solution $u \in C([0,T]; \mathcal{D}(\mathbb{L}^\zeta))$ given by (7) and $\partial_t u \in L^{\overline{m}}(0,T; \mathcal{D}(\mathbb{L}^{-\zeta}))$, for $1 < \overline{m} < \frac{1}{1-\alpha s}$. Moreover, there exists the constant $C > 0$ such that for all $t \in (0,T]$

$$\begin{aligned} \|u(t)\|_{\mathcal{D}(\mathbb{L}^\zeta)} + \|\partial_t u(t)\|_{\mathcal{D}(\mathbb{L}^{-\zeta})} &\leq C\left(\|u_0\|_{\mathcal{D}(\mathbb{L}^\zeta)} + t^{\alpha s-1}\|u_0\|_{\mathcal{D}(\mathbb{L}^{1-\zeta})}\right) \\ &+ C\left(t^{s(\alpha-1)+\frac{s}{2n}+\frac{1}{2}}\|f\|_{L^{2m}(0,T;\mathcal{D}(\mathbb{L}^\zeta))} + t^{\alpha s-2s+1}\|f\|_{L^\infty(0,T;\mathcal{D}(\mathbb{L}^{1-\zeta}))}\right). \end{aligned}$$

We also have ${}^{\mathrm{C}}\left(t^\sigma \frac{\partial}{\partial t}\right)^\alpha u \in C([0,T]; \mathcal{D}(\mathbb{L}^{-\zeta}))$ *and*

$$\begin{aligned} &\left\| {}^{\mathrm{C}}\left(t^\sigma \frac{\partial}{\partial t}\right)^\alpha u(t)\right\|_{\mathcal{D}(\mathbb{L}^{-\zeta})} \\ &\leq C\left(\|u_0\|_{\mathcal{D}(\mathbb{L}^{1-\zeta})} + t^{s(\alpha-1)+\frac{s}{2n}+\frac{1}{2}}\|f\|_{L^{2m}(0,T;\mathcal{D}(\mathbb{L}^{1-\zeta}))}\right) + \|f\|_{C([0,T];\mathcal{D}(\mathbb{L}^{-\zeta}))}. \end{aligned}$$

In addition, we have

$$\lim_{t\to 0^+}\|u(t)-u_0\|_{\mathcal{D}(\mathbb{L}^\nu)} = 0, \quad \textit{for some } 0 \leq \nu < \zeta = \frac{1}{\alpha}. \tag{8}$$

Proof. The proof is divided into four steps.

Step 1. We prove that $u \in C([0,T]; \mathcal{D}(\mathbb{L}^\zeta))$. For $t \in [0,T]$, we have

$$\|u(t)\|_{\mathcal{D}(\mathbb{L}^\zeta)} \leq \|I_1(t)u_0\|_{\mathcal{D}(\mathbb{L}^\zeta)} + \|I_2(t)f\|_{\mathcal{D}(\mathbb{L}^\zeta)}, \tag{9}$$

where, we set

$$I_1(t)u_0 := E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)u_0, \tag{10}$$

$$I_2(t)f := \frac{1}{s^\alpha}\int_0^t (t^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)f(\tau)\mathrm{d}(\tau^s). \tag{11}$$

Using Lemma 2.1b), we imply that there exists the constant $C > 0$ such that

$$\begin{aligned} \|I_1(t)u_0\|^2_{\mathcal{D}(\mathbb{L}^\zeta)} &= \left\|E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)u_0\right\|^2_{\mathcal{D}(\mathbb{L}^\zeta)} \\ &= \sum_{p=1}^\infty (u_0, \mathrm{e}_p)^2 E^2_{\alpha,1}\left(-\frac{t^{\alpha s}}{s^\alpha}\lambda_p\right)\lambda_p^{2\zeta} \\ &\leq \sum_{p=1}^\infty (u_0, \mathrm{e}_p)^2\left(\frac{C^+}{1+\frac{t^{\alpha s}}{s^\alpha}\lambda_p}\right)^2\lambda_p^{2\zeta} \leq C\|u_0\|^2_{\mathcal{D}(\mathbb{L}^\zeta)}. \end{aligned} \tag{12}$$

Using Lemma 2.1b) and Höder inequality, we infer that there exists the constant $C > 0$ such that

$$\begin{aligned} &\|I_2(t)f\|^2_{\mathcal{D}(\mathbb{L}^\zeta)} \\ &= \sum_{p=1}^\infty\left[\frac{1}{s^\alpha}\int_0^t (f(\tau), \mathrm{e}_p)(t^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\lambda_p\right)\mathrm{d}(\tau^s)\right]^2\lambda_p^{2\zeta} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{t}{s^{2\alpha}} \sum_{p=1}^{\infty} \int_0^t \left(f(\tau), \mathrm{e}_p\right)^2 (t^s - \tau^s)^{2\alpha-2} E^2_{\alpha,\alpha}\left(-\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\lambda_p\right) \mathrm{d}(\tau^s) \lambda_p^{2\zeta} \\
&\leq \frac{(\bar{C})^2 t}{s^{2\alpha}} \int_0^t (t^s - \tau^s)^{2\alpha-2} \sum_{p=1}^{\infty} \left(f(\tau), \mathrm{e}_p\right)^2 \lambda_p^{2\zeta} \mathrm{d}(\tau^s) \\
&\leq \frac{(\bar{C})^2 t}{s^{2\alpha}} \int_0^t (t^s - \tau^s)^{2\alpha-2} \|f(\tau)\|^2_{\mathcal{D}(\mathbb{L}^\zeta)} \mathrm{d}(\tau^s) \\
&\leq \frac{(\bar{C})^2 t}{s^{2\alpha}} \left(\int_0^t (t^s - \tau^s)^{2(\alpha-1)n} \mathrm{d}(\tau^s)\right)^{\frac{1}{n}} \left(\int_0^T \|f(\eta)\|^{2m}_{\mathcal{D}(\mathbb{L}^\zeta)} \mathrm{d}\eta\right)^{\frac{1}{m}}, \ t^s \leq T \\
&\leq C t^{2s(\alpha-1)+\frac{s}{n}+1} \|f\|^2_{L^{2m}(0,T;\mathcal{D}(\mathbb{L}^\zeta))}, \qquad (13)
\end{aligned}
$$

where $m, n > 0 : \frac{1}{m} + \frac{1}{n} = 1$. By the assumption on $1 \leq n < \frac{1}{2-2\alpha}$, we have $s + n + 2s(\alpha-1)n > 0$ and from (9), (12) and (13), we get that the operartors (10)-(11) converges in $\mathcal{D}(\mathbb{L}^\zeta)$ uniformly for all $t \in [0, T]$. Thus, we have proved that $u \in C([0, T]; \mathcal{D}(\mathbb{L}^\zeta))$. We also get from (12) and (13) that there exists the positive constant C holds

$$
\|u(t)\|_{\mathcal{D}(\mathbb{L}^\zeta)} \leq C\left(\|u_0\|_{\mathcal{D}(\mathbb{L}^\zeta)} + t^{s(\alpha-1)+\frac{s}{2n}+\frac{1}{2}} \|f\|_{L^{2m}(0,T;\mathcal{D}(\mathbb{L}^\zeta))}\right), \quad \forall t \in [0, T].
$$

Step 2. We shall prove that $\partial_t u \in L^{\overline{m}}(0, T; \mathcal{D}(\mathbb{L}^{-\zeta}))$, for $1 < \overline{m} < \frac{1}{1-\alpha s}$. For $t \in (0, T]$, we have

$$
\begin{aligned}
\partial_t u(t) = &\sum_{p=1}^{\infty} \left[\left(u_0, \mathrm{e}_p\right) \lambda_p t^{\alpha s-1} E_{\alpha,\alpha}\left(-\frac{t^{\alpha s}}{s^\alpha}\lambda_p\right)\right] \mathrm{e}_p \\
&+ \sum_{p=1}^{\infty} \left[\frac{1}{s^\alpha} \int_0^t \left(f(\tau), \mathrm{e}_p\right) \lambda_p (t^s - \tau^s)^{\alpha-2} E_{\alpha,\alpha-1}\left(-\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\lambda_p\right) \mathrm{d}(\tau^s)\right] \mathrm{e}_p \\
:= &I_3(t) u_0 + I_4(t) f, \qquad \text{(respectively)}. \qquad (14)
\end{aligned}
$$

We proceed as in Step 1, from Lemma 2.1b) and Parseval's relation, one obtains

$$
\begin{aligned}
\|I_3(t) u_0\|^2_{\mathcal{D}(\mathbb{L}^{-\zeta})} &= \sum_{p=1}^{\infty} \left[\left(u_0, \mathrm{e}_p\right) \lambda_p^{1-\zeta} t^{\alpha s-1} E_{\alpha,\alpha}\left(-\frac{t^{\alpha s}}{s^\alpha}\lambda_p\right)\right]^2 \\
&\leq C t^{2\alpha s-2} \|u_0\|^2_{\mathcal{D}(\mathbb{L}^{1-\zeta})}, \quad \forall t \in (0, T],
\end{aligned}
$$

and we have

$$
\begin{aligned}
&\|I_4(t) f\|^2_{\mathcal{D}(\mathbb{L}^{-\zeta})} \\
&= \sum_{p=1}^{\infty} \frac{1}{\lambda_p^{-2\zeta}} \left|\frac{1}{s^\alpha} \int_0^t \left(f(\cdot, \tau), \mathrm{e}_p\right) \lambda_p (t^s - \tau^s)^{\alpha-2} E_{\alpha,\alpha-1}\left(-\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\lambda_p\right) \mathrm{d}(\tau^s)\right|^2 \\
&\leq C \sum_{p=1}^{\infty} \sup_{0 \leq \tau \leq T} \left|\left(f(\cdot, \tau), \mathrm{e}_p\right) \lambda_p^{1-\zeta}\right|^2 \left|\int_0^t (\tau^s)^{\alpha-2} E_{\alpha,\alpha-1}\left(-\frac{(\tau^s)^\alpha}{s^\alpha}\lambda_p\right) \mathrm{d}(\tau^s)\right|^2 \\
&\leq C t^{2\alpha s-4s+2} \|f\|^2_{L^\infty(0,T;\mathcal{D}(\mathbb{L}^{1-\zeta}))}, \quad \text{for } \alpha > \frac{2s-1}{s} \quad \forall t \in [0, T]. \qquad (15)
\end{aligned}
$$

It follows readily from these estimates that there exists the constant $C>0$ such that for all $t\in(0,T]$, $\alpha\ge\frac{2s-1}{s}$,

$$\|\partial_t u(\cdot,t)\|_{\mathcal{D}(\mathbb{L}^{-\varsigma})}\le C\left(t^{\alpha s-1}\|u_0\|_{\mathcal{D}(\mathbb{L}^{1-\varsigma})}+t^{\alpha s-2s+1}\,\|f\|_{L^\infty(0,T;\mathcal{D}(\mathbb{L}^{1-\varsigma}))}\right),$$

this implies $\partial_t u\in L^{\overline{m}}(0,T;\mathcal{D}(\mathbb{L}^{-\varsigma}))$, for $1<\overline{m}<\frac{1}{1-\alpha s}$.

Step 3. Next, show that ${}^{\mathrm{C}}\left(t^\sigma\frac{\partial}{\partial t}\right)^\alpha u\in C([0,T];\mathcal{D}(\mathbb{L}^{-\varsigma}))$. From (6), one has

$$\left\|{}^{\mathrm{C}}\left(t^\sigma\frac{\partial}{\partial t}\right)^\alpha u(t)\right\|_{\mathcal{D}(\mathbb{L}^{-\varsigma})}\le\|\mathbb{L}u(\cdot,t)\|_{\mathcal{D}(\mathbb{L}^{-\varsigma})}+\|f(\cdot,t)\|_{\mathcal{D}(\mathbb{L}^{-\varsigma})}.$$

From (7), we get the following estimates

$$\begin{aligned}&\|\mathbb{L}u(\cdot,t)\|^2_{\mathcal{D}(\mathbb{L}^{-\varsigma})}\\&=2\sum_{p=1}^\infty\left[(u_0,\mathrm{e}_p)\,E_{\alpha,1}\left(-\frac{t^{\alpha s}}{s^\alpha}\lambda_p\right)\right]^2\lambda_p^{2-2\zeta}\\&+2\sum_{p=1}^\infty\left[\frac{1}{s^\alpha}\int_0^t(f(\cdot,\tau),\mathrm{e}_p)\,(t^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\lambda_p\right)\mathrm{d}(\tau^s)\right]^2\lambda_p^{2-2\zeta}\\&\le C\,\|u_0\|^2_{\mathcal{D}(\mathbb{L}^{1-\varsigma})}+Ct^{2s(\alpha-1)+\frac{s}{n}+1}\,\|f\|^2_{L^{2m}(0,T;\mathcal{D}(\mathbb{L}^{1-\varsigma}))}.\end{aligned}$$

By an argument analogous to the previous one. We get for every $t\in[0,T]$,

$$\begin{aligned}\left\|{}^{\mathrm{C}}\left(t^\sigma\frac{\partial}{\partial t}\right)^\alpha u(t)\right\|_{\mathcal{D}(\mathbb{L}^{-\varsigma})}&\le C\,\|u_0\|_{\mathcal{D}(\mathbb{L}^{1-\varsigma})}+\|f\|_{C([0,T];\mathcal{D}(\mathbb{L}^{-\varsigma}))}\\&\quad+Ct^{s(\alpha-1)+\frac{s}{2n}+\frac12}\,\|f\|_{L^{2m}(0,T;\mathcal{D}(\mathbb{L}^{1-\varsigma}))}.\end{aligned}$$

From $n\le\frac{1}{2-2\alpha}$, we imply that $2sn(\alpha-1)+s+n>0$, this implies that ${}^{\mathrm{C}}\left(t^\sigma\frac{\partial}{\partial t}\right)^\alpha u\in C([0,T];\mathcal{D}(\mathbb{L}^{-\varsigma}))$.

Step 4. Next, we shall be proving (8). One has

$$\|u(t)-u_0\|_{\mathcal{D}(\mathbb{L}^\nu)}\le\left\|\left(E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)-1\right)u_0\right\|_{\mathcal{D}(\mathbb{L}^\nu)}+\|I_2(t)f\|_{\mathcal{D}(\mathbb{L}^\nu)}.\tag{16}$$

For $I_2(t)f$ defined as in (11), and since $\frac{1}{\alpha}=\zeta>\nu\ge0$, we deduce that $\mathcal{D}(\mathbb{L}^\zeta)\hookrightarrow\mathcal{D}(\mathbb{L}^\nu)$, and we obtain

$$\|I_2(t)f\|_{\mathcal{D}(\mathbb{L}^\nu)}\le\|I_2(t)f\|_{\mathcal{D}(\mathbb{L}^\zeta)}\le Ct^{s(\alpha-1)+\frac{s}{2n}+1/2}\,\|f\|_{L^{2m}(0,T;\mathcal{D}(\mathbb{L}^\zeta))}\to0,\quad\text{as }t\to0^+.$$

Using Lemma 2.1b), we also have

$$\begin{aligned}\left\|\left(E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)-1\right)u_0\right\|^2_{\mathcal{D}(\mathbb{L}^\nu)}&=\sum_{p=1}^\infty(u_0,\mathrm{e}_p)^2\left(E_{\alpha,1}\left(-\frac{t^{\alpha s}}{s^\alpha}\lambda_p\right)-1\right)^2\lambda_p^{2\nu}\\&\le(C^++1)^2\sum_{p=1}^\infty(u_0,\mathrm{e}_p)^2\lambda_p^{2\nu}\\&\le(C^++1)^2\,\|u_0\|^2_{\mathcal{D}(\mathbb{L}^\nu)}<\infty,\ \forall t\in[0,T].\end{aligned}$$

From the properties of function $E_{\alpha,1}\left(-\frac{t^{\alpha s}}{s^\alpha}\lambda_p\right)\to1$ as $t\to0^+$, then

$$\lim_{t\to0^+}\left(E_{\alpha,1}\left(-\frac{t^{\alpha s}}{s^\alpha}\lambda_p\right)-1\right)=0,\ \forall p\in\mathbb{N}^*.$$

We invoke the Lebesgue's Dominated Convergence Theorem that

$$\lim_{t\to 0^+}\left\|\left(E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)-1\right)u_0\right\|_{\mathcal{D}(\mathbb{L}^\nu)}=0. \tag{17}$$

From (16)-(17) that (8) is satisfied. The proof of the theorem is complete. □

To get the next interesting result, we need to build a complementary lemma. From the properties of Mittag-Leffler functions, we have the following lemma:

Lemma 3.2. *For $\alpha\in(0,1), s=1-\sigma, \rho>0$ and for $0<t\le T$, we have the following:*

$$a)\quad \left\|E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)v\right\|_{L^2(\Omega)}\le C^+s^\alpha t^{-\alpha s}\|v\|_{\mathcal{D}(\mathbb{L}^{-1})}. \tag{18a}$$

$$b)\quad \left\|E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)v\right\|_{\mathcal{D}(\mathbb{L}^\rho)}\le C^+s^\alpha t^{-\alpha s}\|v\|_{\mathcal{D}(\mathbb{L}^{\rho-1})}. \tag{18b}$$

$$c)\quad \left\|E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)v\right\|_{L^2(\Omega)}\le \bar{C}s^\alpha\left(t^s-\tau^s\right)^{-\alpha}\|v\|_{\mathcal{D}(\mathbb{L}^{-1})}. \tag{18c}$$

$$d)\quad \left\|E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)v\right\|_{\mathcal{D}(\mathbb{L}^\rho)}\le \bar{C}s^\alpha\left(t^s-\tau^s\right)^{-\alpha}\|v\|_{\mathcal{D}(\mathbb{L}^{\rho-1})}. \tag{18d}$$

Proof. a) For $v\in\mathcal{D}(\mathbb{L}^{-1})$, using the Lemma 2.1, one obtains

$$\begin{aligned}
&\left\|E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)v\right\|^2_{L^2(\Omega)}\\
&=\sum_{p=1}^\infty E^2_{\alpha,1}\left(-\frac{t^{\alpha s}}{s^\alpha}\lambda_p\right)(v,\mathrm{e}_p)^2\le\sum_{p=1}^\infty\left(\frac{C^+}{1+\frac{t^{\alpha s}}{s^\alpha}\lambda_p}\right)^2(v,\mathrm{e}_p)^2\\
&\le (C^+)^2s^{2\alpha}t^{-2\alpha s}\sum_{p=1}^\infty(v,\mathrm{e}_p)^2\lambda_p^{-2}\le (C^+)^2s^{2\alpha}t^{-2\alpha s}\|v\|^2_{\mathcal{D}(\mathbb{L}^{-1})}.
\end{aligned}$$

Taking the square root, we imply (18a).
b) For $v\in\mathcal{D}(\mathbb{L}^{\rho-1})$, One has similar to the above, we get

$$\begin{aligned}
\left\|E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)v\right\|^2_{\mathcal{D}(\mathbb{L}^\rho)}&=\sum_{p=1}^\infty E^2_{\alpha,1}\left(-\frac{t^{\alpha s}}{s^\alpha}\lambda_p\right)(v,\mathrm{e}_p)^2\lambda_p^{2\rho}\\
&\le\sum_{p=1}^\infty\left(\frac{C^+}{1+\frac{t^{\alpha s}}{s^\alpha}\lambda_p}\right)^2(v,\mathrm{e}_p)^2\lambda_p^{2\rho}\\
&\le (C^+)^2s^{2\alpha}t^{-2\alpha s}\sum_{p=1}^\infty(v,\mathrm{e}_p)^2\lambda_p^{2\rho-2}\\
&\le (C^+)^2s^{2\alpha}t^{-2\alpha s}\|v\|^2_{\mathcal{D}(\mathbb{L}^{\rho-1})},
\end{aligned}$$

which implies (18b).
c) For $v\in\mathcal{D}(\mathbb{L}^{-1})$, one obtains

$$\left\|E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)v\right\|_{L^2(\Omega)}=\sum_{p=1}^\infty E^2_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)(v,\mathrm{e}_p)^2$$

$$\leq \sum_{p=1}^{\infty}\left(\frac{\bar{C}}{1+\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\lambda_p}\right)^2 (v,\mathrm{e}_p)^2$$
$$\leq (\bar{C})^2 s^{2\alpha}\left(t^s-\tau^s\right)^{-2\alpha}\sum_{p=1}^{\infty}(v,\mathrm{e}_p)^2\lambda_p^{-2}$$
$$\leq (\bar{C})^2 s^{2\alpha}\left(t^s-\tau^s\right)^{-2\alpha}\|v\|^2_{\mathcal{D}(\mathbb{L}^{-1})},$$

taking the square root, we obtain (18c). In the same way as in the one above, we obtain (18d). The proof of the lemma is complete. □

Based on the Lemma above, we proceed now to establish the next results.

Theorem 3.3. *For the constants σ, s, α, ζ as given in Theorem 3.1. Let $m, n \in \mathbb{N}^*$ such that $\frac{1}{m}+\frac{1}{n}=1$, and $1 \leq n < 1+\alpha$. We have the following two results:*

- *If $u_0 \in \mathcal{D}(\mathbb{L}^{-1}), f \in L^m(0,T;\mathcal{D}(\mathbb{L}^{-1}))$, then the Problem (6) has a unique weak solution $u \in X^{\alpha s}(0,T;L^2(\Omega))$. Moreover, there exists a positive constant C satisfying the following estimate for all $t \in (0,T]$,*
$$\|u\|_{X^{\alpha s}((0,T];L^2(\Omega))} \leq C\left(\|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})}+T^{s+\alpha s-ns}\|f\|_{L^m(0,T;\mathcal{D}(\mathbb{L}^{-1}))}\right).$$
- *If $u_0 \in \mathcal{D}(\mathbb{L}^{\zeta-1}), f \in L^{2m}(0,T;\mathcal{D}(\mathbb{L}^{\zeta}))$, then $\partial_t u \in X^1((0,T];\mathcal{D}(\mathbb{L}^{\zeta}))$ and we also get that for the constant $C>0$*
$$\|\partial_t u\|_{X^1((0,T];\mathcal{D}(\mathbb{L}^\zeta))} \leq C\left(\|u_0\|_{\mathcal{D}(\mathbb{L}^{\zeta-1})}+T^{s(\alpha-2)+\frac{s}{2n}+\frac{3}{2}}\|f\|_{L^{2m}(0,T;L^2(\Omega))}\right).$$

Proof. The proof is divided into two steps.

Step 1. <u>We show that $u \in X^{\alpha s}((0,T];L^2(\Omega))$.</u> For $t \in [0,T]$, we have

$$\|u(t)\|_{L^2(\Omega)} \leq \|I_1(t)u_0\|_{L^2(\Omega)}+\|I_2(t)f\|_{L^2(\Omega)},$$

where, we set $I_1(t)u_0, I_2(t)f$ are defined as in (10) and (11). Using Lemma 3.2a), we imply that there exists the constant $C>0$ such that

$$\|I_1(t)u_0\|_{L^2(\Omega)} = \left\|E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)u_0\right\|_{L^2(\Omega)} \leq Ct^{-\alpha s}\|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})}.$$

Using Lemma 2.1b) and Höder's inequality, we infer that there is a constant $C>0$ such that

$$\|I_2(t)f\|_{L^2(\Omega)} \leq \frac{1}{s^\alpha}\int_0^t (t^s-\tau^s)^{\alpha-1}\left\|E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)f(\tau)\right\|_{L^2(\Omega)}\mathrm{d}(\tau^s)$$
$$\leq C\int_0^t (t^s-\tau^s)^{\alpha-1}\|f(\tau)\|_{L^2(\Omega)}\mathrm{d}(\tau^s)$$
$$\leq C\left(\int_0^t (t^s-\tau^s)^{(\alpha-1)(m-1)}\mathrm{d}(\tau^s)\right)^{\frac{1}{m-1}}\left(\int_0^T\|f(\eta)\|^{\frac{m-1}{m-2}}_{L^2(\Omega)}\mathrm{d}\eta\right)^{\frac{m-2}{m-1}}$$
$$\leq Ct^{\frac{s(\alpha-1)(m-1)+s}{m-1}}\|f\|_{L^{\frac{m-1}{m-2}}(0,T;L^2(\Omega))},$$

where $2 \leq m < \frac{2-\alpha}{1-\alpha}$. From this condition on m, we have $s[(m-1)(2\alpha-1)+1]>0$. From these inequalities above, we deduce that

$$t^{\alpha s}\|u(t)\|_{L^2(\Omega)} \leq C\left(\|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})}+T^{\frac{s[(m-1)(2\alpha-1)+1]}{m-1}}\|f\|_{L^{\frac{m-1}{m-2}}(0,T;L^2(\Omega))}\right),$$

so we get

$$\|u\|_{X^{\alpha s}((0,T];L^2(\Omega))} \leq C\left(\|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})} + T^{\frac{s[(m-1)(2\alpha-1)+1]}{m-1}}\|f\|_{L^{\frac{m-1}{m-2}}(0,T;L^2(\Omega))}\right). \quad (19)$$

Thus, we have shown that $u \in X^{s\alpha}((0,T];L^2(\Omega))$ and satisfies the estimate (19).

Step 2. Next, we show that $\partial_t u \in X^1((0,T];\mathcal{D}(\mathbb{L}^\zeta))$. For $t \in [0,T]$, from the operators $I_3(t)$ and $I_4(t)$ are defined as in (14), and by an argument similar to Lemma 3.2d) and Parseval's relation, one obtains

$$\begin{aligned}\|I_3(t)u_0\|_{\mathcal{D}(\mathbb{L}^\zeta)} &= \left\|\sum_{p=1}^{\infty}\left[(u_0,\mathrm{e}_p)\,t^{\alpha s-1}E_{\alpha,\alpha}\left(-\frac{t^{\alpha s}}{s^\alpha}\lambda_p\right)\right]\mathrm{e}_p\right\|_{\mathcal{D}(\mathbb{L}^\zeta)}\\ &\leq Ct^{-1}\|u_0\|_{\mathcal{D}(\mathbb{L}^{\zeta-1})}, \quad \forall t\in(0,T],\end{aligned}$$

and the same way as in (15), one obtains

$$\|I_4(t)f\|_{\mathcal{D}(\mathbb{L}^\zeta)} \leq Ct^{s(\alpha-2)+\frac{s}{2n}+\frac{1}{2}}\|f\|_{L^{2m}(0,T;\mathcal{D}(\mathbb{L}^\zeta))}, \quad \forall t\in[0,T].$$

By choosing of n satisfies $1 \leq n < 1+\alpha$, then $s + (2\alpha-4)sn + 3n > 0$. It follows readily from the above inequalities that there exists the positive constant C satisfying the following estimate for all $t \in [0,T]$,

$$t\|\partial_t u(t)\|_{\mathcal{D}(\mathbb{L}^\zeta)} \leq C\left(\|u_0\|_{\mathcal{D}(\mathbb{L}^{\zeta-1})} + T^{s(\alpha-2)+\frac{s}{2n}+\frac{3}{2}}\|f\|_{L^{2m}(0,T;L^2(\Omega))}\right).$$

This implies that $\partial_t u \in X^1((0,T];\mathcal{D}(\mathbb{L}^\zeta))$. This concludes the proof. □

4. **The semi-linear problem.** For the nonlinear source function $f = f(u)$, we consider the following two cases:

- $f(u)$ satisfies the globally Lipschitz condition: local well-posedness and regularity estimates and the problem exists a unique positive solution.
- $f(u)$ satisfies the locally Lipschitz condition: large existence times, continuation and finite-time blow-up.

We consider the semi-linear problem

$$\begin{cases} {}^{\mathrm{C}}\left(t^\sigma\dfrac{\partial}{\partial t}\right)^\alpha u(x,t) + (\mathbb{L}u)(x,t) = f(u), & x\in\Omega,\ t>0,\\ u(x,t) = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega.\end{cases} \quad (20)$$

The mild solution of Problem (20) is represented by the following integral equation

$$u(t) = E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)u_0 + \frac{1}{s^\alpha}\int_0^t (t^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)f(u)\mathrm{d}(\tau^s), \quad (21)$$

where $t < T,\ s = 1-\sigma,\ \alpha\in(0,1)$.

4.1. **The local well-posedness results for the source term f is globally Lipchitz.** In this subsection, we prove that the Problem (20) is a local well-posed. First, prove that for the Problem (20) exists a unique mild solution, then the regularity of the solution is established. Moreover, we prove that the problem exists a unique positive solution.

We shall begin with introducing the following two standing hypotheses for the globally Lispchitz source term:

- Assume that f satisfies the global Lipschitz condition:
$$\|f(v_1)-f(v_2)\|_{L^2(\Omega)} \le K\,\|v_1-v_2\|_{L^2(\Omega)}\,, \tag{$Hyp1$}$$
with $K>0$ independent of v_1, v_2.
- Suppose that $f(0)=0$, and
$$\|f(v)\|_{L^2(\Omega)} \le K\,\|v\|_{L^2(\Omega)}\,. \tag{$Hyp2$}$$

For $\mathcal{E}>0$, denote by $C_{\mathcal{E}}([0,T];B)$ is the function space $C([0,T];B)$ equipped with the following weighted norm:
$$\|v\|_{\mathcal{E},B} = \max_{0\le t\le T} \|\exp(-\mathcal{E}t)\,v(t)\|_B\,, \quad w\in C([0,T];B).$$

The main results of this section are the following theorems.

Theorem 4.1 (*Existence*)**.** *Assume that f satisfies ($Hyp1$). Then, the integral equation (21) has a unique mild solution $u\in C_{\mathcal{E}}([0,T];L^2(\Omega))$ for the constant $\mathcal{E}$ is large enough.*

Proof. For $v\in C_{\mathcal{E}}([0,T];L^2(\Omega))$, we consider the following function
$$\mathbf{J}v(t) = E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)u_0 + \frac{1}{s^\alpha}\int_0^t (t^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)f(v(\tau))\mathrm{d}(\tau^s), \tag{22}$$
for $t\in(0,T]$, and we aim to show that the map $\mathbf{J}: C_{\mathcal{E}}([0,T];L^2(\Omega)) \to C_{\mathcal{E}}([0,T];L^2(\Omega))$, for $\mathcal{E}>0$ has a unique fixed point u then we imply u is a solution of (21). In fact, we will prove that for every $v_1,v_2\in C_{\mathcal{E}}([0,T];L^2(\Omega))$, using Lemma 2.1b) and ($Hyp1$), we have
$$\begin{aligned}
&\|\exp(-\mathcal{E}t^s)(\mathbf{J}v_1(t)-\mathbf{J}v_2(t))\|_{L^2(\Omega)}\\
&= \left\|\frac{1}{s^\alpha}\int_0^t \frac{\exp(-\mathcal{E}t^s)}{(t^s-\tau^s)^{1-\alpha}}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)(f(v_1(\tau))-f(v_2(\tau)))\,\mathrm{d}(\tau^s)\right\|_{L^2(\Omega)}\\
&\le \frac{1}{s^\alpha}\int_0^t \frac{\exp(-\mathcal{E}t^s)}{(t^s-\tau^s)^{1-\alpha}}\left\|E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)(f(v_1(\tau))-f(v_2(\tau)))\right\|_{L^2(\Omega)}\mathrm{d}(\tau^s)\\
&\le \frac{\bar{C}}{s^\alpha}\int_0^t \frac{\exp(-\mathcal{E}t^s)}{(t^s-\tau^s)^{1-\alpha}}\|f(v_1(\tau))-f(v_2(\tau))\|_{L^2(\Omega)}\,\mathrm{d}(\tau^s)\\
&\le KC\int_0^t \frac{\exp(-\mathcal{E}(t^s-\tau^s))}{(t^s-\tau^s)^{1-\alpha}}\|\exp(-\mathcal{E}\tau^s)(v_1(\cdot,\tau)-v_2(\cdot,\tau))\|_{L^2(\Omega)}\,\mathrm{d}(\tau^s)\\
&\le KC\int_0^t (t^s-\tau^s)^{\alpha-1}\exp(-\mathcal{E}(t^s-\tau^s))\,\mathrm{d}(\tau^s)\,\|v_1(\cdot,\tau)-v_2(\cdot,\tau)\|_{\mathcal{E},L^2(\Omega)}\,.
\end{aligned} \tag{23}$$
Since the right hand side of (23), we can see that when τ is close to t and α is less than 1, the integral will be singular. Therefore, using the Höder's inequality we deduce for $m>\frac{1}{\alpha}$
$$\begin{aligned}
&\int_0^t (t^s-\tau^s)^{\alpha-1}\exp(-\mathcal{E}(t^s-\tau^s))\,\mathrm{d}(\tau^s)\\
&\le \left(\int_0^t (t^s-\tau^s)^{\frac{m(\alpha-1)}{m-1}}\mathrm{d}(\tau^s)\right)^{\frac{m-1}{m}}\left(\int_0^t \exp(-m\mathcal{E}(t^s-\tau^s))\,\mathrm{d}(\tau^s)\right)^{\frac{1}{m}}\\
&\le \left(\frac{m-1}{m\alpha-1}\right)^{\frac{m-1}{m}} t^{\frac{sm\alpha-s}{m}}\left(\frac{1}{m\mathcal{E}}\right)^{\frac{1}{m}}(1-\exp(-m\mathcal{E}t^s))^{\frac{1}{m}} \le \frac{C}{(\mathcal{E})^{\frac{1}{m}}}.
\end{aligned}$$

Then we get that

$$\|\mathbf{J}v_1 - \mathbf{J}v_2\|_{\mathcal{E},L^2(\Omega)} \leq \frac{C}{(\mathcal{E})^{\frac{1}{m}}} \|v_1 - v_2\|_{\mathcal{E},L^2(\Omega)}, \quad \text{for } m > \frac{1}{\alpha}.$$

By choosing the constant $\mathcal{E}$ large enough, we claim that the mapping $\mathbf{J}$ of the space $C_{\mathcal{E}}([0,T];L^2(\Omega))$ into itsel defined by (22) is a contraction. We conclude that the integral equation (22) has a unique solution $u \in C_{\mathcal{E}}([0,T];L^2(\Omega))$. □

Theorem 4.2 (*Regularity*)**.** *For $\alpha \in (0,1)$, let $0 < \zeta < 1$ and assume that f satisfies (Hyp2). Then we have the following:*
a) If $u_0 \in \mathcal{D}(\mathbb{L}^{-1})$ and u is the solution of (21), then there exists positive constants K, C independent of variable t, with

$$\|u\|_{X^{\alpha s}((0,T];L^2(\Omega))} \lesssim \|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})}. \tag{24}$$

b) If $u_0 \in \mathcal{D}(\mathbb{L}^{\zeta-1})$ and u is the solution of (21), then there exists positive constants K, C independent of t, with

$$\|u\|_{X^{\alpha s}((0,T];\mathcal{D}(\mathbb{L}^{\zeta}))} \lesssim \|u_0\|_{\mathcal{D}(\mathbb{L}^{\zeta-1})}. \tag{25}$$

Proof. • *Proof a.* First, from (21), one has

$$\begin{aligned}\|u(\cdot,t)\|_{L^2(\Omega)} \leq{} & \left\| E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right) u_0 \right\|_{L^2(\Omega)} \\ & + \frac{1}{s^\alpha}\int_0^t (t^s - \tau^s)^{\alpha-1} \left\| E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right) f(u) \right\|_{L^2(\Omega)} \mathrm{d}(\tau^s).\end{aligned} \tag{26}$$

From (18a), we infer that $C > 0$

$$\left\| E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right) u_0 \right\|_{L^2(\Omega)} \leq C^+ s^\alpha t^{-\alpha s} \|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})} \leq C t^{-\alpha s} \|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})}. \tag{27}$$

Estimating the second term of (26). Based on (18c), (*Hyp*2), and Hölder inequality, we get that

$$\begin{aligned}&\frac{1}{s^\alpha}\int_0^t (t^s - \tau^s)^{\alpha-1} \left\| E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right) f(u) \right\|_{L^2(\Omega)} \mathrm{d}(\tau^s) \\ &\quad \leq C\int_0^t (t^s-\tau^s)^{-1} \|f(u)\|_{\mathcal{D}(\mathbb{L}^{-1})} \mathrm{d}(\tau^s) \\ &\quad \leq C\int_0^t (t^s-\tau^s)^{-1} \|f(u)\|_{L^2(\Omega)} \mathrm{d}(\tau^s) \\ &\quad \leq KC\int_0^t (t^s-\tau^s)^{-1} \|u(\cdot,\tau)\|_{L^2(\Omega)} \mathrm{d}(\tau^s).\end{aligned} \tag{28}$$

From (26), (27) and (28), we deduce that

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq C t^{-\alpha s}\|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})} + KC\int_0^{t^s} (t^s - \eta)^{-1} \|u(\cdot,\eta)\|_{L^2(\Omega)} \mathrm{d}\eta.$$

Thanks to Lemma 2.4 gives

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq \frac{C(K,T)}{1-\alpha s} t^{-\alpha s}\|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})} \leq C t^{-\alpha s}\|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})}. \tag{29}$$

Multiplying by $t^{\alpha s}$ on both sides of (29) and we get

$$t^{\alpha s}\|u(\cdot,t)\|_{L^2} \leq C\,\|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})},$$

which implies (24).

• *Proof b.* From (18b), one obtains

$$\left\| E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)u_0 \right\|_{\mathcal{D}(\mathbb{L}^\zeta)} \leq C^+ s^\alpha t^{-\alpha s}\,\|u_0\|_{\mathcal{D}(\mathbb{L}^{\zeta-1})} \leq Ct^{-\alpha s}\,\|u_0\|_{\mathcal{D}(\mathbb{L}^{\zeta-1})}. \tag{30}$$

Using (18d), Remark 1 and hypothesis (*Hyp*2), we get that for $\zeta < 1$

$$\begin{aligned}
&\frac{1}{s^\alpha}\int_0^t (t^s-\tau^s)^{\alpha-1}\left\| E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)f(u)\right\|_{\mathcal{D}(\mathbb{L}^\zeta)} \mathrm{d}(\tau^s)\\
&\leq C\int_0^t (t^s-\tau^s)^{-1}\,\|f(u)\|_{\mathcal{D}(\mathbb{L}^{\zeta-1})}\,\mathrm{d}(\tau^s)\\
&\leq CC_\zeta\int_0^t (t^s-\tau^s)^{-1}\,\|f(u)\|_{L^2(\Omega)}\,\mathrm{d}(\tau^s)\\
&\leq KCC_\zeta\int_0^t (t^s-\tau^s)^{-1}\,\|u(\cdot,\tau)\|_{L^2(\Omega)}\,\mathrm{d}(\tau^s)\\
&\leq KCC_\zeta^2\int_0^t (t^s-\tau^s)^{-1}\,\|u(\cdot,\tau)\|_{\mathcal{D}(\mathbb{L}^\zeta)}\,\mathrm{d}(\tau^s).
\end{aligned} \tag{31}$$

From (30) and (31), there exists the positive constant C such that

$$\|u(\cdot,t)\|_{\mathcal{D}(\mathbb{L}^\zeta)} \leq Ct^{-\alpha s}\,\|u_0\|_{\mathcal{D}(\mathbb{L}^{\zeta-1})} + KC\int_0^t (t^s-\tau^s)^{-1}\,\|u(\cdot,\tau)\|_{\mathcal{D}(\mathbb{L}^\zeta)}\,\mathrm{d}(\tau^s).$$

From Lemma 2.4 (Grönwall's inequality), one obtains

$$\|u(\cdot,t)\|_{\mathcal{D}(\mathbb{L}^\zeta)} \leq Ct^{-\alpha s}\,\|u_0\|_{\mathcal{D}(\mathbb{L}^{\zeta-1})}.$$

An argument analogous to the previous one yields (25). This complete the proof of the theorem. □

Theorem 4.3 (*Stability*)**.** *For $0<\zeta<1$, and $\alpha\in(0,1)$, assume that f satisfies (Hyp1). For $u_0\in\mathcal{D}(\mathbb{L}^{\zeta-1})$ for $\zeta\in(0,1)$. The solution u depends continuously on the initial data in the following sense. If $u_{0,j}\to u_0$ in $\mathcal{D}(\mathbb{L}^{\zeta-1})$ and if u_j is the corresponding maximal solution with initial data $u_{0,j}$, then $u_j\to u$ in $L^\infty(0,T;\mathcal{D}(\mathbb{L}^\zeta))$ for every interval $(0,T]$.*

Proof. Let $u_0\in\mathcal{D}(\mathbb{L}^{\zeta-1})$ and consider $\{u_{0,j}\}_{j\in\mathbb{N}^*}\subset\mathcal{D}(\mathbb{L}^{\zeta-1})$ such that

$$u_{0,j}\to u_0, \text{ as } j\to\infty.$$

For j sufficiently large we have that

$$\begin{aligned}
&\|u(\cdot,t)-u_j(\cdot,t)\|_{\mathcal{D}(\mathbb{L}^\zeta)}\\
&\leq \left\| E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)(u_0-u_{0,j})\right\|_{\mathcal{D}(\mathbb{L}^\zeta)}\\
&\quad+\frac{1}{s^\alpha}\int_0^t (t^s-\tau^s)^{\alpha-1}\left\| E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)(f(u)-f(u_j))\right\|_{\mathcal{D}(\mathbb{L}^\zeta)}\mathrm{d}(\tau^s).
\end{aligned} \tag{32}$$

Using (18b), we get

$$\left\|E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)(u_0-u_{0,j})\right\|_{\mathcal{D}(\mathbb{L}^\zeta)} \le Ct^{-\alpha s}\left\|u_0-u_{0,j}\right\|_{\mathcal{D}(\mathbb{L}^{\zeta-1})}, \quad \forall t\in(0,T].$$

From (18d), ($Hyp1$) and Remark 1, we have

$$\begin{aligned}\frac{1}{s^\alpha}\int_0^t (t^s-\tau^s)^{\alpha-1}&\left\|E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)(f(u)-f(u_j))\right\|_{\mathcal{D}(\mathbb{L}^\zeta)}\mathrm{d}(\tau^s)\\ &\le \bar{C}\int_0^t (t^s-\tau^s)^{-1}\left\|f(u)-f(u_j)\right\|_{\mathcal{D}(\mathbb{L}^{\zeta-1})}\mathrm{d}(\tau^s)\\ &\le \bar{C}C_\zeta\int_0^t (t^s-\tau^s)^{-1}\left\|f(u)-f(u_j)\right\|_{L^2(\Omega)}\mathrm{d}(\tau^s)\\ &\le K\bar{C}C_\zeta^2\int_0^t (t^s-\tau^s)^{-1}\left\|u(\cdot,\tau)-u_j(\cdot,\tau)\right\|_{\mathcal{D}(\mathbb{L}^\zeta)}\mathrm{d}(\tau^s). \quad (33)\end{aligned}$$

Combining (32)-(33), we deduce that

$$\begin{aligned}\left\|u(\cdot,t)-u_j(\cdot,t)\right\|_{\mathcal{D}(\mathbb{L}^\zeta)} &\le Ct^{-\alpha s}\left\|u_0-u_{0,j}\right\|_{\mathcal{D}(\mathbb{L}^{\zeta-1})}\\ &\quad + KC\int_0^t (t^s-\tau^s)^{-1}\left\|u(\cdot,\tau)-u_j(\cdot,\tau)\right\|_{\mathcal{D}(\mathbb{L}^\zeta)}\mathrm{d}(\tau^s),\end{aligned}$$

for all $t\in(0,T]$. From Lemma 2.4 (Grönwall's inequality) we see that

$$t^{\alpha s}\big\|u(\cdot,t)-u_j(\cdot,t)\big\|_{\mathcal{D}(\mathbb{L}^\zeta)} \le C\left\|u_0-u_{0,j}\right\|_{\mathcal{D}(\mathbb{L}^{\zeta-1})}.$$

Let $j\to\infty$ and we have $u_{0,j}\to u_0$ so $u_j\to u$ in $\mathcal{D}(\mathbb{L}^\zeta)$, for all $t\in(0,T]$, which finishes the proof. □

Remark 2. If $u_0(x)>0$ a.e. $x\in\Omega$ and the continuous function $f(u)$ is nonnegative, then the explicit solution of Problem (20) presented in (21) is positive.

Lemma 4.4. *(see [4]) Let H be a Hausdorff locally convex linear topological space, Q be a convex subset of H, V be an open subset of Q, and $R\in V$. Suppose that $\mathcal{M}:\overline{V}\to Q$ is a continuous, compact map. Then, either*

(i) The map $\mathcal{M}$ has a fixed point in $\overline{V}$; or
(ii) there are $u\in\partial V$ (the boundary of V in Q) and $\xi\in(0,1)$ with $u=\xi\mathcal{M}u+(1-\xi)R$.

Theorem 4.5 (Existence-uniqueness a positive solution)**.** *Assume that nonnegative and continuous function f satisfies the hypotheses ($Hyp1$) and ($Hyp2$), then, there exists a unique positive solution $u\in C([0,+\infty);L^2(\Omega))$ of Problem (20).*

Proof. **Step 1.** Existence a positive solution $u\in C([0,+\infty);L^2(\Omega))$. For $B>0$, let us set

$$Q=\left\{v\in C([0,T];L^2(\Omega))\ \middle|\ u(\cdot,t)\ge B,\ \text{a.e}\ (x,t)\in\Omega\times[0,T],\ \text{for}\ T\in(0,\infty)\right\}.$$

Consider the operator $\mathcal{M}:Q\to Q$ defined by

$$\begin{aligned}&\mathcal{M}v(t)\\ &=E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)u_0+\frac{1}{s^\alpha}\int_0^t (t^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)f(v(\tau))\mathrm{d}(\tau^s).\end{aligned} \quad (34)$$

From Remark 2, since $u_0 > 0$ a.e. in Ω and $f(u)$ is nonnegative, then $\mathcal{M}u$ is nonnegative. By Theorem 4.1, we know that the operator $\mathcal{M}$ has a unique fixed point. Let

$$V := \left\{ u : [0,T] \to L^2(\Omega) \quad \middle| \quad \begin{array}{l} \diamond\ u(x,0) = u_0(x) \\ \diamond\ u \text{ is continous on } (0,T] \\ \diamond\ \|u(\cdot,t)\|_{L^2(\Omega)} \leq t^{-\alpha s} \|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})}, \forall t > 0 \end{array} \right\}.$$

Then, we can show that $\mathcal{M} : \overline{V} \to Q$ is continuous and compact by the usual techniques (see e.g. [41]). Moreover, for $\xi \in (0,1)$, if $u \in Q$ is any solution of the equation (34) then we get

$$\begin{aligned} u(t) &= \xi \mathcal{M}u(t) + (1-\xi)E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)u_0 \\ &= E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)u_0 + \frac{\xi}{s^\alpha}\int_0^t (t^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right) f(v(\tau))\mathrm{d}(\tau^s). \end{aligned}$$

By an argument analogous to that used for the proof of Theorem 4.2a, one obtains

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq Ct^{-\alpha s}\|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})}.$$

We invoke Lemma 4.4 to deduce that $\mathcal{M}$ has a fixed point in $\overline{V}$. Then, this fixed point is a positive solution of Problem (20). Since the arbitrariness of $T \in (0,+\infty)$, then we claim that there exists a positive solution $u \in C([0,+\infty); L^2(\Omega))$ of Problem (20).

Step 2. Uniqueness positive solution in $C([0,+\infty); L^2(\Omega))$. By Step 1, we suppose that $u, v \in C([0,+\infty); L^2(\Omega))$ are two positive solutions of equation (20). Then, we conclude that

$$\begin{aligned} &\|u(\cdot,t) - v(\cdot,t)\|_{L^2(\Omega)} \\ &\leq \frac{1}{s^\alpha}\int_0^t (t^s-\tau^s)^{\alpha-1}\left\|E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)(f(u)-f(v))\right\|_{L^2(\Omega)}\mathrm{d}(\tau^s) \\ &\leq KC\int_0^t (t^s-\tau^s)^{-1}\|u(\tau)-v(\tau)\|_{L^2(\Omega)}\mathrm{d}(\tau^s). \end{aligned}$$

Applying Lemma 2.4 (Grönwall inequality), we derive $u = v$. This implies the uniqueness of the solution. □

4.2. **Large existence times, continuation and finite-time blow-up.** In this subsection, we consider the source term $f(u) = f_q(u) = |u|^{q-1}\log|u|^q$ for $q > 1$, (which is locally Lipschitz).

Lemma 4.6. *For every $\varepsilon > 0$, there exists $A > 0$, such that the real function*

$$h(y) = |y|^a \log|y|, \quad \textit{for } a \geq 0,$$

satisfies

$$h(y) \leq A + |y|^{a+\varepsilon}.$$

Proof. Since $\lim_{|y|\to+\infty}\left(\frac{\log|y|}{|y|^\varepsilon}\right) = 0$, then there exists $y > 0$ such that

$$\frac{\log|y|}{|y|^\varepsilon} < 1, \quad \text{for all } |y| > y_0.$$

So,

$$h(y) \le |y|^{a+\varepsilon}, \quad \text{for all } |y| > y_0.$$

Since $\log x$ is an increasing function, we can conclude for any $a \ge 0$ that $|h(y)| \le A$, for some $A > 0$ and for all $|y| \le y_0$. Thus,

$$h(y) \le A + |y|^{a+\varepsilon}.$$

The proof is complete. □

Lemma 4.7. *(See [17]) For $\Omega \subset \mathbb{R}^d$, it is well-known that we obtain the following Sobolev embeddings*

$$\begin{cases} L^q(\Omega) \hookrightarrow \mathcal{D}(\mathbb{L}^\zeta), & \text{if} \quad \frac{-d}{4} < \zeta \le 0, \quad q \ge \frac{2d}{d-4\zeta}, \\ \mathcal{D}(\mathbb{L}^\zeta) \hookrightarrow L^q(\Omega), & \text{if} \quad 0 \le \zeta < \frac{d}{4}, \quad q \le \frac{2d}{d-4\zeta}. \end{cases}$$

Then, we have more next results on local existence.

Theorem 4.8 (Local-in-time existence)**.** *Let $u_0 \in \mathcal{D}(\mathbb{L}^{-1})$, and assume that $\alpha \in \left(\frac{1}{2}, 1\right)$, $s = 1 - \sigma$. For the nonlinearity source as logarithmic function type $f_q(u) = |u|^{q-1} \log |u|^q$, with $\max\left\{\frac{1}{\alpha}; \frac{3}{2}\right\} < q \le \min\left\{\frac{1}{\alpha s}; 2\right\}$, then there is a time constant $T > 0$ (depending only on u_0) such that the Problem ($\mathbb{P}$) has a unique mild solution on $(0, T]$.*

Proof. Let $T > 0$ and $M > 0$ to be chosen later, we consider the following space

$$\mathbb{S} := \left\{ u \in X^{\alpha s}((0,T]; L^2(\Omega)) : \left\| u \right\|_{X^{\alpha s}((0,T]; L^2(\Omega))} \le M \right\},$$

for $0 < \alpha, s < 1$, and we define the mapping $\mathbf{H}$ on $\mathbb{S}$ by

$$\mathbf{H}u(t) = E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right) u_0 + \overbrace{\frac{1}{s^\alpha} \int_0^t (t^s - \tau^s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right) f_q(u) \mathrm{d}(\tau^s)}^{=:\mathcal{H}(u)(t)}. \tag{35}$$

We show that $\mathbf{H}$ is invariant in $\mathbb{S}$ and $\mathbf{H}$ is a contraction.

Claim 1. <u>If $u_0 \in \mathcal{D}(\mathbb{L}^{-1})$, then $\mathbf{H}$ is $\mathbb{S}$-invariant.</u> In fact, from (18a), we have

$$\left\| E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right) u_0 \right\|_{X^{\alpha s}(0,T; L^2(\Omega))} \le C \left\| u_0 \right\|_{\mathcal{D}(\mathbb{L}^{-1})}, \quad \forall t \in (0, T]. \tag{36}$$

Using Lemma 2.1b) we have for $t \in (0, T]$

$$\begin{aligned} \left\| \mathcal{H}(u)(t) \right\|_{L^2(\Omega)} &= \left\| \frac{1}{s^\alpha} \int_0^t (t^s - \tau^s)^{\alpha-1} E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right) f_q(u) \mathrm{d}(\tau^s) \right\|_{L^2(\Omega)} \\ &\le \frac{1}{s^\alpha} \int_0^t (t^s - \tau^s)^{\alpha-1} \left\| E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right) f_q(u) \right\|_{L^2(\Omega)} \mathrm{d}(\tau^s) \\ &\le \frac{\bar{C}}{s^\alpha} \int_0^t (t^s - \tau^s)^{\alpha-1} \left\| f_q(u) \right\|_{L^2(\Omega)} \mathrm{d}(\tau^s). \end{aligned} \tag{37}$$

From Lemma 4.6, for the constants $A, \varepsilon > 0$, we conclude that for $t \in (0, T]$,

$$\int_\Omega \left| f_q(u) \right|^2 \mathrm{d}x = \int_\Omega \left(\left| u(x,t) \right|^{q-1} \log \left| u(x,t) \right|^q \right)^2 \mathrm{d}x$$

$$\leq 2q\int_\Omega \left(A^2+|u|^{2q-2+2\varepsilon}\right)\mathrm{d}x \leq 2q\left(A^2|\Omega|+\|u(\cdot,t)\|^{2(q-1+\varepsilon)}_{L^{2(q-1+\varepsilon)}(\Omega)}\right). \tag{38}$$

From (37) and (38) and Hölder's inequality for $\frac{1}{\alpha}<q<2$, put $q^*=q-1+\varepsilon$, for $q>1$, choose ε satisfies $\frac{3-2q}{2}\leq\varepsilon\leq 2-q$, one obtains $1\leq 2q^*\leq 2$, we used the Sobolev embedding $L^2(\Omega)\hookrightarrow L^{2q^*}(\Omega)$, we have that

$$\begin{aligned}
&\|\mathcal{H}(u)(t)\|_{L^2(\Omega)}\\
&\leq C\int_0^t (t^s-\tau^s)^{\alpha-1}\left(A|\Omega|^{\frac12}+\|u(\cdot,\tau)\|^{q^*}_{L^{2q^*}(\Omega)}\right)\mathrm{d}(\tau^s)\\
&\leq C\left(\int_0^t (t^s-\tau^s)^{\frac{q(\alpha-1)}{q-1}}\mathrm{d}(\tau^s)\right)^{\frac{q-1}{q}}\left[\int_0^t\left(A|\Omega|^{\frac12}+\|u(\cdot,\tau)\|^{q^*}_{L^{2q^*}(\Omega)}\right)^q\mathrm{d}(\tau^s)\right]^{\frac1q}\\
&\leq C\left(\int_0^t (t^s-\tau^s)^{\frac{q(\alpha-1)}{q-1}}\mathrm{d}(\tau^s)\right)^{\frac{q-1}{q}}\left[\int_0^t\left(A|\Omega|^{\frac12}+\|u(\cdot,\tau)\|^{q^*}_{L^{2}(\Omega)}\right)^q\mathrm{d}(\tau^s)\right]^{\frac1q}\\
&\leq C\left(\int_0^t (t^s-\tau^s)^{\frac{q(\alpha-1)}{q-1}}\mathrm{d}(\tau^s)\right)^{\frac{q-1}{q}}\left[\int_0^t\left(A|\Omega|^{\frac12}+\tau^{-\alpha sq^*}\|u\|^{q^*}_{X^{\alpha s}(0,T;L^2(\Omega))}\right)^q\mathrm{d}(\tau^s)\right]^{\frac1q}\\
&\leq C\left(\int_0^t (t^s-\tau^s)^{\frac{q(\alpha-1)}{q-1}}\mathrm{d}(\tau^s)\right)^{\frac{q-1}{q}}\left[\int_0^t\left(A|\Omega|^{\frac12}+\tau^{-\alpha sq^*}M^{q^*}\right)^q\mathrm{d}(\tau^s)\right]^{\frac1q}\\
&\leq Ct^{s\alpha-s+\frac{s(q-1)}{q}}\left(t^{\frac{s}{q}}A|\Omega|^{\frac12}+t^{\frac{s}{q}-\alpha sq^*}M^{q^*}\right),
\end{aligned} \tag{39}$$

where by choosing a positive number ε is appropriate to $q^*<\frac{1}{q\alpha}$. For $q>\frac{1}{\alpha}$, then $2s\alpha-s+\frac{s(q-1)}{q}>0$, we get that

$$\|\mathcal{H}(u)\|_{X^{\alpha s}(0,T;L^2(\Omega))}\leq CT^{2s\alpha-s+\frac{s(q-1)}{q}}\left(T^{\frac{s}{q}}A|\Omega|^{\frac12}+T^{\frac{s}{q}-\alpha sq^*}M^{q^*}\right). \tag{40}$$

Hence, from (36) and (40), for every $t\in(0,T]$,

$$\begin{aligned}
&\|\mathbf{H}u\|_{X^{\alpha s}(0,T;L^2(\Omega))}\\
&\leq \left\|E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)u_0\right\|_{X^{\alpha s}(0,T;L^2(\Omega))}+\|\mathcal{H}(u)\|_{X^{\alpha s}(0,T;L^2(\Omega))}\\
&\leq C\left(\|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})}+T^{2s\alpha-s+\frac{s(q-1)}{q}}\left(T^{\frac{s}{q}}A|\Omega|^{\frac12}+T^{\frac{s}{q}-\alpha sq^*}M^{q^*}\right)\right).
\end{aligned}$$

Therefore we see that if $M=2C\|u_0\|_{\mathcal{D}(\mathbb{L}^{-1})}$ and

$$M\geq 2CT^{2s\alpha-s+\frac{s(q-1)}{q}}\left(T^{\frac{s}{q}}A|\Omega|^{\frac12}+T^{\frac{s}{q}-\alpha sq^*}M^{q^*}\right),$$

then $\mathbf{H}$ is invariant in $\mathbb{S}$.

Claim 2. $\mathbf{H}:\mathbb{S}\to\mathbb{S}$ is a contraction map. Let $u,v\in\mathbb{S}$, and using Lemma 2.1b), one has for every $t\in(0,T]$,

$$\begin{aligned}
&\|\mathbf{H}u(t)-\mathbf{H}v(t)\|_{L^2(\Omega)}\\
&\leq \frac{1}{s^\alpha}\int_0^t (t^s-\tau^s)^{\alpha-1}\left\|E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)f_q(u)-f_q(v)\right\|_{L^2(\Omega)}\mathrm{d}(\tau^s)\\
&\leq \frac{\bar{C}}{s^\alpha}\int_0^t (t^s-\tau^s)^{\alpha-1}\|f_q(u)-f_q(v)\|_{L^2(\Omega)}\mathrm{d}(\tau^s).
\end{aligned} \tag{41}$$

As a consequence of the mean value theorem, we have, for $0 \leq \theta \leq 1$,

$$\begin{aligned}\big|f_q(u) - f_q(v)\big| &= \big|f'(\theta u + (1-\theta)v)(u-v)\big| \\ &\leq q\Big[1 + (q-1)\log\big|\theta u + (1-\theta)v\big|\Big]\big|\theta u + (1-\theta)v\big|^{q-2}\big|u-v\big| \\ &\leq q\big|\theta u + (1-\theta)v\big|^{q-2}\big|u-v\big| \\ &\quad + q(q-1)\log\big|\theta u + (1-\theta)v\big|\big|\theta u + (1-\theta)v\big|^{q-2}\big|u-v\big|,\end{aligned}$$

where for $f_q(y) = |y|^{q-1}\log|y|^q$, then we have used $f_q'(y) = q\big[1+(q-1)\log|y|\big]|y|^{q-2}$. By recalling Lemma 4.6, we arrive at

$$\begin{aligned}\big|f_q(u) - f_q(v)\big| &\leq q\big|\theta u + (1-\theta)v\big|^{q-2}\big|u-v\big| \\ &\quad + q(q-1)\left(A + \big|\theta u + (1-\theta)v\big|^{q-2+\varepsilon}\right)\big|u-v\big| \\ &\leq q\big|u+v\big|^{q-2}\big|u-v\big| \\ &\quad + q(q-1)A\big|u-v\big| + q(q-1)\big|u+v\big|^{q-2+\varepsilon}\big|u-v\big|. \end{aligned} \tag{42}$$

We then use Hölder's inequality to get

$$\begin{aligned}\int_\Omega\Big[\big|u+v\big|^{q-2}\big|u-v\big|\Big]^2\mathrm{d}x &= \int_\Omega\big|u+v\big|^{2(q-2)}\big|u-v\big|^2\mathrm{d}x \\ &\leq \left(\int_\Omega\big|u+v\big|^{2(q-1)}\mathrm{d}x\right)^{\frac{q-2}{q-1}}\left(\int_\Omega\big|u-v\big|^{2(q-1)}\mathrm{d}x\right)^{\frac{1}{q-1}} \\ &\leq C\left[\big\|u\big\|^{2(q-1)}_{L^{2(q-1)}} + \big\|v\big\|^{2(q-1)}_{L^{2(q-1)}}\right]^{\frac{q-2}{q-1}}\big\|u-v\big\|^2_{L^{2(q-1)}(\Omega)}.\end{aligned}$$

Similarly, we estimate

$$\begin{aligned}&\int_\Omega\Big[\big|u+v\big|^{q-2+\varepsilon}\big|u-v\big|\Big]^2\mathrm{d}x \\ &= \int_\Omega\big|u+v\big|^{2(q-2+\varepsilon)}\big|u-v\big|^2\mathrm{d}x \\ &\leq \left(\int_\Omega\big|u+v\big|^{\frac{2(q-2+\varepsilon)(q-1)}{q-2}}\mathrm{d}x\right)^{\frac{q-2}{q-1}}\left(\int_\Omega\big|u-v\big|^{2(q-1)}\mathrm{d}x\right)^{\frac{1}{q-1}} \\ &\leq C\left[\big\|u\big\|^{2q^{**}}_{L^{2q^{**}}(\Omega)} + \big\|v\big\|^{2q^{**}}_{L^{2q^{**}}(\Omega)}\right]^{\frac{q-2}{q-1}}\big\|u-v\big\|^2_{L^{2(q-1)}(\Omega)},\end{aligned} \tag{43}$$

for putting $q^{**} = (q-1) + \frac{\varepsilon(q-1)}{q-2}$ with $q \geq 1$. Therefore, by combining (42) - (43), we obtain

$$\begin{aligned}\big\|f_q(u) - f_q(v)\big\|_{L^2(\Omega)} &\leq CA\big\|u-v\big\|^2_{L^{2(q-1)}(\Omega)} \\ &\quad + C\left[\big\|u\big\|^{q-1}_{L^{2(q-1)}(\Omega)} + \big\|v\big\|^{q-1}_{L^{2(q-1)}(\Omega)}\right]^{\frac{q-2}{q-1}}\big\|u-v\big\|_{L^{2(q-1)}(\Omega)} \\ &\quad + C\left[\big\|u\big\|^{q^{**}}_{L^{2q^{**}}(\Omega)} + \big\|v\big\|^{q^{**}}_{L^{2q^{**}}(\Omega)}\right]^{\frac{q-2}{q-1}}\big\|u-v\big\|_{L^{2(q-1)}(\Omega)}.\end{aligned}$$

Since, $q^{**} = (q-1) + \frac{\varepsilon(q-1)}{q-2}$, we can choose ε so large enough such that $\frac{1}{2} \leq q^{**} \leq 1$ and we also have $1 \leq 2(q-1) \leq 2$, for $\frac{3}{2} \leq q \leq 2$, from Lemma 4.7 for $\zeta = 0$, one obtains

$$\begin{cases} L^2(\Omega) \hookrightarrow L^{2(q-1)}(\Omega), & \text{since } 1 \le 2(q-1) \le 2, \\ L^2(\Omega) \hookrightarrow L^{2q^{**}}(\Omega), & \text{since } 1 \le 2q^{**} \le 2. \end{cases}$$

By choosing $M > 0$ such that $\max\left\{ \|u\|_{C([0,T];L^2(\Omega))} ; \|v\|_{C([0,T];L^2(\Omega))} \right\} \le M$, then we get

$$\big\| f_q(u) - f_q(v) \big\|_{L^2(\Omega)} \le C_{q,\varepsilon}(M) \big\| u - v \big\|_{L^2(\Omega)}, \tag{44}$$

whereupon $C_{q,\varepsilon}(M) := C\left(A + 2M^{q-2} + 2M^{q-2+\varepsilon}\right)$. Inserting the result of (44) into (41), and using Hölder inequality, we get that

$$\begin{aligned}
&\|\mathbf{H}u(t) - \mathbf{H}v(t)\|_{L^2(\Omega)} \\
&\le C_{q,\varepsilon}(M) \int_0^t (t^s - \tau^s)^{\alpha-1} \big\| u - v \big\|_{L^2(\Omega)} \mathrm{d}(\tau^s) \\
&\le C_{q,\varepsilon}(M) \left(\int_0^t (t^s - \tau^s)^{\frac{(\alpha-1)q}{q-1}} \mathrm{d}(\tau^s) \right)^{\frac{q-1}{q}} \left(\int_0^t \tau^{-\alpha s q} \mathrm{d}(\tau^s) \right)^{\frac{1}{q}} \big\| u - v \big\|_{X^{\alpha s}(0,T;L^2(\Omega))} \\
&\le C_{q,\varepsilon}(M) \left(\frac{q-1}{\alpha q - 1} \right)^{\frac{q-1}{q}} t^{\frac{\alpha s q - s}{q-1}} \left(\frac{1}{1 - \alpha s q} \right)^{\frac{1}{q}} t^{1-\alpha s q} \big\| u - v \big\|_{X^{\alpha s}(0,T;L^2(\Omega))},
\end{aligned}$$

for some constant q satisfies $\frac{1}{\alpha} < q < \frac{1}{\alpha s}$. Then we get $\frac{\alpha s q - s}{q-1} + 1 - \alpha s q + \alpha s > 0$ and the estimate holds for every $t \in (0, T]$

$$\|\mathbf{H}u - \mathbf{H}v\|_{X^{\alpha s}(0,T;L^2(\Omega))} \le C_{q,\varepsilon}(M) T^{\frac{\alpha s q - s}{q-1} + 1 - \alpha s q + \alpha s} \big\| u - v \big\|_{X^{\alpha s}(0,T;L^2(\Omega))}. \tag{45}$$

Choosing T small enough such that $C_{q,\varepsilon}(M) T^{\frac{\alpha s q - s}{q-1} + 1 - \alpha s q + \alpha s} < 1$, it follows that $\mathbf{H}$ is a contraction map on $\mathbb{S}$. So, we invoke the principle of contraction mapping to assert that the map $\mathbf{H}$ has a unique fixed point u in $\mathbb{S}$. □

Since we already know that the mild solution of ($\mathbb{P}$) does exist, the question is whether it will continue (continuation to a bigger interval of existence) and in what situation it is non-continuation by blowup. In answer to these questions is our purpose in the next results. Firsts, we consider the following definition.

Definition 4.9 (Continuation, see [8, 37])**.** Given a mild solution $u \in X^{\alpha s}((0,T]; L^2(\Omega))$ of ($\mathbb{P}$), we say that $u^\star$ is a *continuation* of u in $(0, T^\star]$ for $T^\star > T$ if it is satisfying

$$\begin{cases} u^\star \in X^{\alpha s}((0,T]; L^2(\Omega)) \text{ is a mild solution of } (\mathbb{P}) \text{ for all } t \in (0, T^\star], \\ u^\star(x,t) = u(x,t) \quad \text{whenever} \quad t \in [0, T^\star], x \in \Omega. \end{cases}$$

Theorem 4.10. *Suppose that the assumptions of the Theorem 4.8 are satisfied. Then, the solution (unique the weak solution) on the interval $(0,T]$ of Problem ($\mathbb{P}$) is extended to $(0, T^\star]$, for $T^\star > T$. So this extended function is also the weak solution (unique) of Problem ($\mathbb{P}$) on $(0, T^\star]$.*

Proof. Let $u : [0,T] \to L^2(\Omega)$ be a mild solution of Problem ($\mathbb{P}$) (T be the time from Theorem 4.8). Fix $M > 0$, and for $T^\star > T$, ($T^\star$ depending on M), we shall prove that $u^\star : [0, T^\star] \to L^2(\Omega)$ is a mild solution of Problem ($\mathbb{P}$). Assume the following estimates hold:

$$\mathscr{P}_1 := T^{-\alpha s} (T^\star)^{2\alpha s} \|u_0\|_{L^2(\Omega)} \le \frac{M}{3}, \tag{46}$$

$$\mathscr{P}_2 := C(T^\star)^{2s\alpha-s+\frac{s(q-1)}{q}}\left((T^\star)^{\frac{s}{q}}A|\Omega|^{\frac{1}{2}}+(T^\star)^{\frac{s}{q}-\alpha sq^*}M^{q^*}\right)\le\frac{M}{3}, \quad (47)$$

$$\mathscr{P}_3 := C_{q,\varepsilon}(M)(T^\star)^{\frac{\alpha sq-s}{q-1}+1-\alpha sq+\alpha s}\le\frac{M}{3}, \quad (48)$$

where $C_{q,\varepsilon}(M)$ defined in the proof of the Theorem 4.8.

For $T^\star\ge T>0$ and $M>0$, let us define

$$\mathbb{S}^\star := \left\{u^\star\in X^{\alpha s}((0,T^\star];L^2(\Omega)) : \begin{array}{|ll} u^\star(\cdot,t)=u(\cdot,t), & \forall t\in(0,T], \\ \|u^\star-u(\cdot,T)\|_{X^{\alpha s}([T,T^\star];L^2(\Omega))}\le M, & \forall t\in[T,T^\star]. \end{array}\right\}$$

Step 1. We show that $\mathbf{H}$ is defined as in (35) be the operator on $\mathbb{S}^\star$. Indeed, let $u^\star\in\mathbb{S}^\star$, we have the following cases:

• If $t\in(0,T]$, then by virtue of the Theorem (4.8), we have the Problem ($\mathbb{P}$) has a unique solution and we also have

$$u^\star(\cdot,t)=u(\cdot,t), \text{ and we have } \mathbf{H}(u^\star)=\mathbf{H}(u)=u.$$

Thus $\|\mathbf{H}u^\star-\mathbf{H}u\|_{X^{\alpha s}((0,T];L^2(\Omega))}$ is vanish in $\mathbb{S}^\star$ for all $t\in(0,T]$.

• If $t\in[T,T^\star]$, we have

$$\begin{aligned}
&\left\|\mathbf{H}u^\star(t)-u(\cdot,T)\right\|_{L^2(\Omega)}\\
&\le\left\|\left(E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)-E_{\alpha,1}\left(-\mathbb{L}\frac{T^{\alpha s}}{s^\alpha}\right)\right)u_0\right\|_{L^2(\Omega)}\\
&\quad+\frac{1}{s^\alpha}\left\|\int_0^t(t^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)f_q(u^\star)\mathrm{d}(\tau^s)\right.\\
&\quad\left.-\int_0^T(T^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(T^s-\tau^s)^\alpha}{s^\alpha}\right)f_q(u^\star)\mathrm{d}(\tau^s)\right\|_{L^2(\Omega)}\\
&\le\left\|\left(E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)-E_{\alpha,1}\left(-\mathbb{L}\frac{T^{\alpha s}}{s^\alpha}\right)\right)u_0\right\|_{L^2(\Omega)}\\
&\quad+\frac{1}{s^\alpha}\left\|\int_T^t(t^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)f_q(u^\star)\mathrm{d}(\tau^s)\right\|_{L^2(\Omega)}\\
&\quad+\frac{1}{s^\alpha}\left\|\int_0^T\left[(t^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)\right.\right.\\
&\quad\left.\left.-(T^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(T^s-\tau^s)^\alpha}{s^\alpha}\right)\right]f_q(u^\star)\mathrm{d}(\tau^s)\right\|_{L^2(\Omega)}\\
&=:\sum_{i=1}^3\|\mathcal{H}_i(\cdot,t)\|_{L^2(\Omega)}, \quad \text{(respectively)}.
\end{aligned}$$

Estimating the term $\|\mathcal{H}_1\|_{L^2(\Omega)}$, using Lemma 2.3, we have for all $t\in[T,T^\star]$

$$\begin{aligned}
&\|\mathcal{H}_1(\cdot,t)\|^2_{L^2(\Omega)}\\
&=\left\|\left(E_{\alpha,1}\left(-\mathbb{L}\frac{t^{\alpha s}}{s^\alpha}\right)-E_{\alpha,1}\left(-\mathbb{L}\frac{T^{\alpha s}}{s^\alpha}\right)\right)u_0\right\|^2_{L^2(\Omega)}\\
&=\sum_{p=1}^\infty\left[E_{\alpha,1}\left(-\frac{t^{\alpha s}}{s^\alpha}\lambda_p\right)-E_{\alpha,1}\left(-\frac{T^{\alpha s}}{s^\alpha}\lambda_p\right)\right]^2(u_0,\mathrm{e}_p)^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{p=1}^{\infty} \left[\int_0^{\infty} W_\alpha(z) \left| \exp\left(-z \frac{t^{\alpha s}}{s^\alpha} \lambda_p\right) - \exp\left(-z \frac{T^{\alpha s}}{s^\alpha} \lambda_p\right) \right| \mathrm{d}z \right]^2 (u_0, \mathrm{e}_p)^2 \\
&\le \sum_{p=1}^{\infty} \left[\int_0^{\infty} W_\alpha(z) \exp\left(-z \frac{T^{\alpha s}}{s^\alpha} \lambda_p\right) \left| \exp\left(-z \frac{(t^{\alpha s} - T^{\alpha s})}{s^\alpha} \lambda_p\right) - 1 \right| \mathrm{d}z \right]^2 (u_0, \mathrm{e}_p)^2 \\
&\le \sum_{p=1}^{\infty} \left[\frac{t^{\alpha s} - T^{\alpha s}}{s^\alpha} \lambda_p \int_0^{\infty} W_\alpha(z) \left(z \frac{T^{\alpha s}}{s^\alpha} \lambda_p \right)^{-1} z \mathrm{d}z \right]^2 (u_0, \mathrm{e}_p)^2 \\
&\le \left[\left(t^{\alpha s} - T^{\alpha s}\right) T^{-\alpha s} \int_0^{\infty} W_\alpha(z) \mathrm{d}z \right]^2 \sum_{p=1}^{\infty} (u_0, \mathrm{e}_p)^2 \le (t-T)^{2\alpha s} T^{-2\alpha s} \|u_0\|_{L^2(\Omega)}^2,
\end{aligned} \tag{49}
$$

where, we have use the inequalities

$$
a^c - b^c \le (a-b)^c, \quad \text{for } a > b > 0, c \in (0,1),
$$

and for $z > 0$

$$
1 - e^{-z} \le z, \quad \text{and } z e^{-z} \le 1.
$$

Hence, we get that

$$
t^{\alpha s} \|\mathcal{H}_1(\cdot, t)\|_{L^2(\Omega)} \le (t-T)^{\alpha s} T^{-\alpha s} (T^\star)^{\alpha s} \|u_0\|_{L^2(\Omega)} \le T^{-\alpha s} (T^\star)^{2\alpha s} \|u_0\|_{L^2(\Omega)}.
$$

From (46), this implies that the following estimate holds

$$
\|\mathcal{H}_1\|_{X^{\alpha s}((0,T^\star];L^2(\Omega))} \le T^{-\alpha s} (T^\star)^{2\alpha s} \|u_0\|_{L^2(\Omega)} = \mathscr{P}_1 \le \frac{M}{4}. \tag{50}
$$

Similar to (39), we have the following estimate for all $t \in [T, T^\star]$ (recall that $q^* = q - 1 + \varepsilon \in [\frac{1}{2}, 1]$)

$$
\begin{aligned}
\|\mathcal{H}_2(\cdot, t)\|_{L^2(\Omega)} &\le C (t-T)^{s\alpha - s + \frac{s(q-1)}{q}} \left((t-T)^{\frac{s}{q}} A |\Omega|^{\frac{1}{2}} + (t-T)^{\frac{s}{q} - \alpha s q^*} M^{q^*} \right) \\
&\le C t^{-\alpha s} (T^\star)^{2s\alpha - s + \frac{s(q-1)}{q}} \left((T^\star)^{\frac{s}{q}} A |\Omega|^{\frac{1}{2}} + (T^\star)^{\frac{s}{q} - \alpha s q^*} M^{q^*} \right),
\end{aligned}
$$

where, we have used the fact that $\|u\|_{X^{\alpha s}((T,T^\star];L^2(\Omega))} \le M$. Using (47), we infer that

$$
\begin{aligned}
\|\mathcal{H}_2\|_{X^{\alpha s}((0,T^\star];L^2(\Omega))} &\le C (T^\star)^{2s\alpha - s + \frac{s(q-1)}{q}} \left((T^\star)^{\frac{s}{q}} A |\Omega|^{\frac{1}{2}} + (T^\star)^{\frac{s}{q} - \alpha s q^*} M^{q^*} \right) \\
&= \mathscr{P}_2 \le \frac{M}{4}.
\end{aligned} \tag{51}
$$

We continue with the estimate of the third norm, using (5) and Lemma 2.1, for all $t \in [T, T^\star]$, one obtains

$$
\begin{aligned}
&\left| (t^s - \tau^s)^{\alpha - 1} E_{\alpha,\alpha} \left(-\frac{(t^s - \tau^s)^\alpha}{s^\alpha} \lambda_p \right) - (T^s - \tau^s)^{\alpha - 1} E_{\alpha,\alpha} \left(-\frac{(T^s - \tau^s)^\alpha}{s^\alpha} \lambda_p \right) \right| \\
&= \left| \int_{T^s - \tau^s}^{t^s - \tau^s} z^{\alpha - 2} E_{\alpha, \alpha - 1} \left(-\frac{z^\alpha}{s^\alpha} \lambda_p \right) \mathrm{d}z \right| \le \bar{C} \int_{T^s - \tau^s}^{t^s - \tau^s} z^{\alpha - 2} \mathrm{d}z \\
&\le C \left((T^s - \tau^s)^{\alpha - 1} - (t^s - \tau^s)^{\alpha - 1} \right) \le C \left(T^s - \tau^s \right)^{\alpha - 1}.
\end{aligned}
$$

Hence, we deduce that

$$\begin{aligned}\|\mathcal{H}_3(\cdot,t)\|_{L^2(\Omega)} &= \frac{1}{s^\alpha}\bigg\| \int_0^T \bigg[(t^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right) \\ &\qquad - (T^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(T^s-\tau^s)^\alpha}{s^\alpha}\right)\bigg] f_q(u^\star)\mathrm{d}(\tau^s)\bigg\|_{L^2(\Omega)} \\ &\le \frac{T}{s^\alpha}\int_0^T \left(\sum_{p=1}^\infty \bigg|(t^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\lambda_p\right)\right. \\ &\qquad \left. - (T^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{(T^s-\tau^s)^\alpha}{s^\alpha}\lambda_p\right)\bigg|^2 \left(f_q(u^\star),\mathrm{e}_p\right)^2\right)^{\frac12}\mathrm{d}(\tau^s) \\ &\le C\int_0^T\left(\sum_{p=1}^\infty (T^s-\tau^s)^{2\alpha-2}\left(f_q(u^\star),\mathrm{e}_p\right)^2\right)^{\frac12}\mathrm{d}(\tau^s) \\ &= C\int_0^T (T^s-\tau^s)^{\alpha-1}\|f_q(u^\star)\|_{L^2(\Omega)}\mathrm{d}(\tau^s).\end{aligned}$$

In the same way as in (39), using Hölder's inequality and using the embedding $L^2(\Omega)\hookrightarrow L^{2q^*}(\Omega)$, for $q^*\in[\frac12,1]$, we obtain for $t\in[T,T^\star]$

$$t^{\alpha s}\big\|\mathcal{H}_3(\cdot,t)\big\|_{L^2(\Omega)} \le C(T^\star)^{2s\alpha-s+\frac{s(q-1)}{q}}\left((T^\star)^{\frac{s}{q}}A|\Omega|^{\frac12}+(T^\star)^{\frac{s}{q}-\alpha s q^*}M^{q^*}\right).$$

From (47), we get

$$\begin{aligned}\|\mathcal{H}_3\|_{X^{\alpha s}((0,T^\star];L^2(\Omega))} &\le C(T^\star)^{2s\alpha-s+\frac{s(q-1)}{q}}\left((T^\star)^{\frac{s}{q}}A|\Omega|^{\frac12}+(T^\star)^{\frac{s}{q}-\alpha s q^*}M^{q^*}\right) \\ &= \mathscr{P}_2 < \frac{M}{3}.\end{aligned} \tag{52}$$

It follows from (50), (51), (52) that, for every $t\in[T,T^\star]$

$$\|\mathbf{H}u^\star - u(\cdot,T)\|_{X^{\alpha s}((0,T^\star];L^2(\Omega))} < \frac{M}{3}+\frac{M}{3}+\frac{M}{3} < M.$$

We have shown that $\mathbf{H}$ is a map $\mathbb{S}^\star$ into $\mathbb{S}^\star$.

Step 2. We show that $\mathbf{H}$ is a contraction on $\mathbb{S}^\star$. Let $u,v\in\mathbb{S}^\star$, for $T\le t\le T^\star$, one obtains

$$\mathbf{H}u(t)-\mathbf{H}v(t) = \frac{1}{s^\alpha}\int_T^t (t^s-\tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)(f_q(u)-f_q(v))\mathrm{d}(\tau^s),$$

where we note that $\mathbf{H}u(t)-\mathbf{H}v(t)=0$, for all $t\in(0,T]$. Then, for all $t\in[0,T^\star]$, similar to (45), we have

$$\begin{aligned}&\|\mathbf{H}u(t)-\mathbf{H}v(t)\|_{L^2(\Omega)} \\ &\le \int_T^t (t^s-\tau^s)^{\alpha-1}\left\|E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t^s-\tau^s)^\alpha}{s^\alpha}\right)(f_q(u)-f_q(v))\right\|_{L^2(\Omega)}\mathrm{d}(\tau^s) \\ &\le C_{q,\varepsilon}(M)(T^\star)^{\frac{\alpha s q-s}{q-1}+1-\alpha s q+\alpha s}\big\|u-v\big\|_{X^{\alpha s}(0,T;L^2(\Omega))}.\end{aligned}$$

Hence, from (48) we deduce that

$$\begin{aligned}\|\mathbf{H}u-\mathbf{H}v\|_{X^{\alpha s}((0,T^\star];L^2(\Omega))} &\le C_{q,\varepsilon}(M)(T^\star)^{\frac{\alpha s q-s}{q-1}+1-\alpha s q+\alpha s}\big\|u-v\big\|_{X^{\alpha s}((0,T];L^2(\Omega))} \\ &= \mathscr{P}_3\big\|u-v\big\|_{X^{\alpha s}((0,T];L^2(\Omega))}.\end{aligned}$$

Thus, for all $T^\star > 0$, there is no loss of generality, we may assume that $0 < M < 3$, we infer that

$$\|\mathbf{H}u - \mathbf{H}v\|_{X^{\alpha s}((0,T^\star];L^2(\Omega))} \leq \frac{M}{3}\|u - v\|_{X^{\alpha s}((0,T];L^2(\Omega))}.$$

This implies that $\mathbf{H}$ is a $\frac{M}{3}$-contraction. By the Banach contraction principle follows that $\mathbf{H}$ has a unique fixed point $u^\star$ of $\mathbf{H}$ in $\mathbb{S}^\star$, which is a continuation of u. This finishes the proof. □

The next results are on global existence or non-continuation by a blow-up and depend continuously on the initial data.

Definition 4.11 (*Maximal existence time*, see [8,37])**.** Let $u(x,t)$ be a weak solution of ($\mathbb{P}$). We define the maximal existence time $T_{\max}$ of $u(x,t)$ as follows:

(i) If $u(x,t)$ exists for all $0 \leq t < \infty$, then $T_{\max} = \infty$.
(ii) If there exists $T \in (0,\infty)$ such that $u(x,t)$ exists for $0 \leq t < T$, but does not exist at $t = T$, then $T_{\max} = T$.

Definition 4.12 (*Finite time blow-up*, see [8, 37])**.** Let $u(x,t)$ be a weak solution of ($\mathbb{P}$). We say $u(x,t)$ blows up in finite-time if the maximal existence time $T_{\max}$ is finite and

$$\lim_{t\to T_{\max}^-} \|u(\cdot,t)\|_{L^2(\Omega)} = \infty.$$

Theorem 4.13. *Assume the conditions of Theorem 4.8 holds. For $u_0 \in \mathcal{D}(\mathbb{L}^{-1})$, then there exists the maximal time $T_{\max} > 0$ such that $u \in C((0,T_{\max}];L^2(\Omega))$ be the mild solution of* ($\mathbb{P}$). *Then, either $T_{\max} = \infty$ or $\limsup_{t\to T_{\max}^-}\|u(\cdot,t)\|_{L^2(\Omega)} = \infty$, if $T_{\max} < \infty$.*

Remark 3. As an immediate consequence of Theorem 4.10, we guarantee the existence of a maximal time.

Proof. Let $u_0 \in \mathcal{D}(\mathbb{L}^{-1})$ and define

$$T_{\max} := \sup\{T > 0 : \text{there exits a solution on } (0,T]\}.$$

Assume that $T_{\max} < \infty$. Now suppose there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset [0,T_{\max})$ such that $t_n \to T_{\max}$ and $\|u(\cdot,t_n)\|_{L^2(\Omega)} \leq M$, for some $M > 0$. Given $\varepsilon > 0$ fix $N \in \mathbb{N}$ such that for all $n,j > N$, $0 < t_n < t_j < T_{\max}$. Now we solve our problem with initial data $u(x,t_n) =: u_n(x)$ and we extend our solution to the interval $[t_n,T_{\max}]$. Indded, we have

$$\begin{aligned}
&\|u(\cdot,t_j) - u(\cdot,t_n)\|_{L^2(\Omega)} \\
&\leq \left\|\left(E_{\alpha,1}\left(-\mathbb{L}\frac{t_j^{\alpha s}}{s^\alpha}\right) - E_{\alpha,1}\left(-\mathbb{L}\frac{t_n^{\alpha s}}{s^\alpha}\right)\right)u_n\right\|_{L^2(\Omega)} \\
&\quad + \frac{1}{s^\alpha}\int_{t_n}^{t_j}(t_j^s - \tau^s)^{\alpha-1}\left\|E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t_j^s-\tau^s)^\alpha}{s^\alpha}\right)f_q(u^\star)\right\|_{L^2(\Omega)}\mathrm{d}(\tau^s) \\
&\quad + \frac{1}{s^\alpha}\left\|\int_0^{t_n}\left[(t_n^s - \tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(t_n^s-\tau^s)^\alpha}{s^\alpha}\right)\right.\right. \\
&\qquad\qquad \left.\left. - (T_{\max}^s - \tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(T_{\max}^s-\tau^s)^\alpha}{s^\alpha}\right)\right]f_q(u^\star)\mathrm{d}(\tau^s)\right\|_{L^2(\Omega)} \\
&\quad + \frac{1}{s^\alpha}\left\|\int_0^{t_j}\left[(T_{\max}^s - \tau^s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mathbb{L}\frac{(T_{\max}^s-\tau^s)^\alpha}{s^\alpha}\right)\right.\right.
\end{aligned}$$

$$- (t_j^s - \tau^s)^{\alpha-1} E_{\alpha,\alpha}\Big(- \mathbb{L}\frac{(t_j^s-\tau^s)^\alpha}{s^\alpha}\Big)\Big] f_q(u^\star)\mathrm{d}(\tau^s)\Big\|_{L^2(\Omega)} =: \sum_{i=1}^4 \mathcal{Y}_i.$$

Similarly to (49), we have that

$$\mathcal{Y}_1 := \left\| \left(E_{\alpha,1}\left(-\mathbb{L}\frac{t_j^{\alpha s}}{s^\alpha}\right) - E_{\alpha,1}\left(-\mathbb{L}\frac{t_n^{\alpha s}}{s^\alpha}\right)\right) u_n \right\|_{L^2(\Omega)} \le |t_j - t_n|^{\alpha s} t_n{}^{-\alpha s} \|u_n\|_{L^2(\Omega)}.$$

In the same way as (51), we get

$$\begin{aligned}\mathcal{Y}_2 &= \frac{1}{s^\alpha}\int_{t_n}^{t_j} (t_j^s-\tau^s)^{\alpha-1} \left\| E_{\alpha,\alpha}\left(- \mathbb{L}\frac{(t_j^s-\tau^s)^\alpha}{s^\alpha}\right) f_q(u^\star)\right\|_{L^2(\Omega)} \mathrm{d}(\tau^s) \\ &\le C|t_j-t_n|^{s\alpha - s + \frac{s(q-1)}{q}} \left(|t_j-t_n|^{\frac{s}{q}} A|\Omega|^{\frac12} + |t_j - t_n|^{\frac{s}{q}-\alpha s q^*} M^{q^*}\right).\end{aligned}$$

Similar to (52), we have

$$\begin{aligned}\mathcal{Y}_3 + \mathcal{Y}_4 \le C &\left((T_{\max})^{\frac{s}{q}} A|\Omega|^{\frac12} + (T_{\max})^{\frac{s}{q}-\alpha s q^*} M^{q^*}\right) \\ &\times \left(|T_{\max} - t_n|^{s\alpha - s + \frac{s(q-1)}{q}} + |T_{\max} - t_j|^{s\alpha - s+\frac{s(q-1)}{q}}\right).\end{aligned}$$

Thus, since $\{t_n\}_{n\in\mathbb{N}^*}$ is convergent we can take $N := N(\varepsilon) \in \mathbb{N}^*$ with $j \ge n \ge N$ such that $|t_j - t_n|$ is as small as we want, and we have

- $|t_j - t_n|^{\alpha s} t_n{}^{-\alpha s} \|u_n\|_{L^2(\Omega)} < \dfrac{\varepsilon}{3}$,
- $C|t_j-t_n|^{s\alpha - s + \frac{s(q-1)}{q}} \left(|t_j-t_n|^{\frac{s}{q}} A|\Omega|^{\frac12} + |t_j - t_n|^{\frac{s}{q}-\alpha s q^*} M^{q^*}\right) < \dfrac{\varepsilon}{3}$,
- $C \left((T_{\max})^{\frac{s}{q}} A|\Omega|^{\frac12} + (T_{\max})^{\frac{s}{q}-\alpha s q^*} M^{q^*}\right)$
  $\times \left(|T_{\max} - t_n|^{s\alpha - s + \frac{s(q-1)}{q}} + |T_{\max} - t_j|^{s\alpha - s+\frac{s(q-1)}{q}}\right) < \dfrac{\varepsilon}{3}$.

Hence, given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\big\|u(\cdot, t_j) - u(\cdot,t_n)\big\|_{L^2(\Omega)} < \varepsilon, \quad \text{for } j, n \ge N.$$

It follows that $\{u(\cdot,t_n)\}_{n\in\mathbb{N}} \subset L^2(\Omega)$ is a Cauchy sequences and for $\{t_n\}_{n\in\mathbb{N}^*}$ is arbitrary we have proved the existence of the limit

$$\lim_{t\to T_{\max}^-} \|u(\cdot,t)\|_{L^2(\Omega)} < \infty.$$

We invoke Theorem 4.10 to deduce that the solution can extend to some larger interval (u can be continued beyond $T_{\max}$), we then contradict the definition of $T_{\max}$. Thus, either $T_{\max} = \infty$ or if $T_{\max} < \infty$ then $\lim_{t\to T_{\max}^-} \|u(\cdot,t)\|_{L^2(\Omega)} = \infty$. The proof is finished. □

5. **Conclusions.** We have achieved the blow-up result for the diffusion equation ($\mathbb{P}$) with (C-LC) of (H-BO) when the source function has a special logarithmic form. Besides, the results of Local well-posedness are also presented when the source function is linear and nonlinear (satisfying global Lipschitz condition). Although we can obtain the better regularity estimates, this will require much smoother properties of the source function f. We will try to achieve this in future works.

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