ELECTRONIC RESEARCH ARCHIVE
Volume **29**, Number **5**, November **2021**
doi:10.3934/era.2021083
pp. **3551–3580**

CONTROLLABILITY OF NONLINEAR FRACTIONAL EVOLUTION SYSTEMS IN BANACH SPACES: A SURVEY

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(Communicated by Shouchuan Hu)

Abstract. This paper presents a survey for some recent research on the controllability of nonlinear fractional evolution systems (FESs) in Banach spaces. The prime focus is exact controllability and approximate controllability of several types of FESs, which include the basic systems with classical initial and nonlocal conditions, FESs with time delay or impulsive effect. In addition, controllability results via resolvent operator are reviewed in detail. At last, the conclusions of this work and the research prospect are presented, which provides a reference for further study.

1. **Introduction.** The theory of fractional calculus has a long-standing history, and it can be traced back to nearly four centuries ago, which firstly appeared in the correspondence between mathematicians Leibniz and L'Hospital about the definition of fractional derivative in the 17th century. After a long and tortuous development, fractional calculus has received considerable attention due mainly to its potential and wide applications in various kinds of scientific fields such as chemical physics, pure mathematics, signal processing, mechanics and engineering, viscoelasticity, biology, neural network model, fractal theory, etc. See [24, 56, 63, 75, 96, 102, 120, 121, 122] and references therein for further details. Actually, fractional calculus can describe mathematical models involving practical background with less parameters, and present a more vivid and accurate description over things than integral order ones [11, 25, 36, 40, 58, 92, 99, 114, 130]. Recent research trends and achievements in science and technology show that fractional differential equations including both ordinary and partial ones have more extensive applications than integral order differential equations [35, 50, 58, 65, 76, 77, 95, 103, 104, 125].

As we all know, control theory is an interdisciplinary branch of economics, engineering and mathematics that investigates and analyses some dynamical behaviors of various systems [78, 93, 116, 130, 131]. In addition, control theory of dynamical systems with impulse [8, 37, 46, 72] or time delay [38, 47, 109, 110] have been studied in the last decates. Some other excellent results for such systems are obtained on stability [14, 49, 41, 42, 43, 62, 70, 106, 105, 126, 127] and stabilization

2020 *Mathematics Subject Classification.* Primary: 93B05, 47D06, 34K30; Secondary: 35R11.
Key words and phrases. Controllability, fractional evolution systems, mild solutions, impulse, time delay, fixed point theorem.

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[9, 33, 34, 45, 53, 3, 112, 113], optimal control [1, 5, 12, 18, 30, 29], etc. As an important component of technology and mathematical control theory, controllability has already gained considerable attention. Among the methods to investigate the controllability of diverse nonlinear systems, fixed-point theory has been used widely and effectively, which was initiated by Tarnove in 1967 [94]. For the sake of using fixed-point theorem, the controllability problem of nonlinear systems is transformed to a fixed-point problem of corresponding nonlinear operator in a appropriate function space. Frequently used fixed-point theorems include Banach's fixed-point theorem [13], Schauder's fixed-point theorem [23, 32, 129], Darbo's fixed-point theorem [17, 26], Schaefer's fixed-point theorem [17], Krasnoselskii's fixed-point theorem [61, 101, 117], Sadovskii's fixed-point theorem [39, 68, 123], Mönch's fixed-point theorem [15, 48, 57, 124, 118], etc. It should be particularly noted that the controllability of fractional evolution systems (FESs) is an important issue for lots of practical problems since the fractional calculus can derive better results than integeral order one. The two most extensively studied subjects of controllability for FESs are exact controllability and approximate controllability.

On the other hand, time delay is a widespread phenomenon in the fields of science and engineering. So it can not be ignored in many practical situations [10, 52, 59, 66]. This is especially true for the evolution processes which are closely related to time. For example, the time delay is often inevitable in the process of pregnancy, maturation, and hatching at the different stages of population development. Sometimes, a minor delay may restrict the system seriously, or affect the structure and performance of the system to a great extent, and even lead to instability of the system. Therefore, various dynamic behaviors of FESs with time delay have been investigated during the last few decades, such as optimal controllability, stability and stabilization problems. With the continuous development of fractional calculus, more and more attention has been paid to the controllability of various kinds of FESs with time delay in recent years [13, 15, 22, 32, 124]. In addition, there is another inevitable factor in practice, that is, the phenomenon of impulse which widely exists in the real world. For example, it can be used to describe the sudden fluctuation of population caused by disease, famine, etc, in population model. The impulse phenomena also appear in the timing fishing or replenishment of population ecosystem, the external stimulation in nervous system, the closing of switch in circuit system and so on. The development of impulsive differential equations and related theories has been growing rapidly in recent decades [55, 107, 108, 119]. As we know, there are many processes in nature that are influenced by impulses as well as memory and heredity such as neural network and population dynamics model, etc. Obviously, they are most appropriately described by using fractional impulsive differential systems. Therefore, the investigation on controllability of FESs with impulsive effect has important theoretical significance and practical value. Nevertheless, analyzing controllability for FESs with impulse is more complicated than that for integral order ones. In the study of the fractional (impulsive) controllability, how to represent mild solutions of FESs is the first important step. Up to now, many divergences and academic controversies on the mild solutions to fractional impulsive systems still exist due to the fact that fractional derivatives have heredity, nonlocal behavior and memory property. Fortunately, some relevant literatures [87] have corrected these issues and given their correct expressions.

By the analysis and comparison of the relevant literatures, this work will present a comprehensive overview for the controllability on some classes of FESs, such as some basic systems with classical initial and nonlocal conditions, FESs with time delay or impulse. An outline of the rest of this work is arranged as follows. In Section 2, some notations, definitions and lemmas are listed. Section 3 discusses the controllability of some basic systems. Section 4 considers the controllability of some classes of systems with time delay. In Section 5, controllability of systems with impulsive effects is presented. Controllability results via resolvent operator theory is discussed in Section 6. At last, Section 7 provides a conclusion of the present paper and the research prospect in this area.

2. Preliminaries.

This section mainly lists some necessary notations, definitions and lemmas, which will be used throughout the present paper. The preliminaries here can be found in, for example, [39, 73, 74].

Denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers, $\mathbb{Z}^+$ the set of positive integer numbers and by $\mathbb{C}$ the set of complex numbers. Let $A : \mathcal{D} \subseteq X \to X$ be a closed linear operator defined on a Banach space X, where $\mathcal{D}$ stands for the domain of the operator A. Obviously, $\mathcal{D}$ is a Banach space equipped with the graph norm $\|x\|_{\mathcal{D}} = \|x\| + \|Ax\|$. Denote by $L^p(J, X)$ $(p \geq 1)$ the space of p-th power integrable functions from interval J to Banach space X, $L^1_{loc}(J)$ the space of locally integrable functions on J and by $C(J, X)$ the space of X-valued continuous functions on J. $\widehat{x}(\lambda)$ stands for the Laplace transform of some function x.

Definition 2.1. The Riemann-Liouville standard fractional integral with fractional order $q \in \mathbb{R}^+$ of a continuous function $x : (0, +\infty) \to \mathbb{R}$ is defined as follows:

$$I_t^q x(t) = D_t^{-q} x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s) ds,$$

provided that the right side integral is pointwise defined on $(0, +\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative with fractional order $q \in \mathbb{R}^+$ of a continuous function $x : (0, +\infty) \to \mathbb{R}$ is defined by

$${}^{RL}D_t^q x(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-q-1} x(s) ds = \frac{d^n}{dt^n} D_t^{-(n-q)},$$

where $n - 1 \leq q < n \in \mathbb{Z}^+$, provided that the right side integral is pointwise defined on $(0, +\infty)$.

As an example, for $\alpha > -1$, one can obtain ${}^{RL}D_t^q x^\alpha = \dfrac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-q)} x^{\alpha-q}$.

As early as the 19th century, Riemann-Liouville fractional derivative has been well developed in the works of Abel, Riemann and Liouville. Up to now, a complete theoretical system has been established. However, such theory had led to many difficulties in the application of some practical problems. In order to avoid these difficulties and to facilitate modeling the problems of engineering and physics, a new definition of fractional derivative was proposed by Caputo in 1967:

Definition 2.3. The Caputo fractional derivative with fractional order $q \in \mathbb{R}^+$ of a continuous function $x : (0, +\infty) \to \mathbb{R}$ is defined by

$${}^{C}D_t^q x(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} x^{(n)}(s) ds = D_t^{-(n-q)} \frac{d^n}{dt^n} x(t),$$

where $n - 1 < q < n \in \mathbb{Z}^+$, provided that the right side integral is pointwise defined on $(0, +\infty)$.

It is not difficult to see that the Caputo derivative of a constant is equal to zero. Especially, if $q \in (0,1)$, then

$$^{C}D_t^q x(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x'(s) ds,$$

which can be used to describe the state of the system at a given time depends on past events. Apart from that, the solutions of Caputo-type equations can approximate any given smooth function arbitrarily [6].

Definition 2.4. The Hilfer fractional derivative with fractional order $0 \leq \nu \leq 1$ and $0 < \mu < 1$ of a continuous function $x : (0, +\infty) \to \mathbb{R}$ is defined by

$$D_t^{\nu,\mu} x(t) = I_t^{\nu(1-\mu)} \frac{d}{dt} I_t^{(1-\nu)(1-\mu)} x(t),$$

for functions such that the expression on the right hand side exists.

Obviously, when $\nu = 0,\ 0 < \mu < 1$, the Hilfer fractional derivative corresponds to the classical Riemann-Liouville fractional derivative; when $\nu = 1,\ 0 < \mu < 1$, it corresponds to the classical Caputo fractional derivative.

Lemma 2.5. *Suppose that $x \in C(J) \cap L^1(J)$ with a fractional derivative of order $q \in \mathbb{R}^+$. Then the following equality holds:*

$$I_t^{q\,RL} D_t^{\,q} x(t) = x(t) + c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n},$$

where $c_i \in \mathbb{R},\ i = 1, 2, \cdots, n$ and $n - 1 < q \leq n \in \mathbb{Z}^+$.

Lemma 2.6. *Let $n - 1 < q \leq n \in \mathbb{Z}^+$. Then the following equality holds:*

$$I_t^{q\,C} D_t^{\,q} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R},\ i = 0, 1, 2, \cdots, n-1$.

Since the classical solution of a fractional evolution system satisfies a convolution equation on the halfline, it is natural to employ the theory of Laplace transform for its study. Therefore, mild solutions of the considered FESs can be obtained mainly by applying the Laplace transform technique. For this reason, we present the following properties:

Lemma 2.7. *Let $q \in \mathbb{R}^+$. m denotes the smallest integer greater than or equal to q. Then*

(i) $\widehat{g_q}(\lambda) = \dfrac{1}{\lambda^q}$, *where*

$$g_q(t) = \begin{cases} \dfrac{1}{\Gamma(q)} t^{q-1}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

(ii) $\widehat{I_t^q x}(\lambda) = \dfrac{1}{\lambda^q} \widehat{x}(\lambda)$.

(iii) $\widehat{^{RL}D_t^q x}(\lambda) = \lambda^q \widehat{x}(\lambda) - \sum_{k=0}^{m-1} (g_{m-q} * x)^{(k)}(0) \lambda^{m-1-k}$.

(iv) $\widehat{^{C}D_t^q x}(\lambda) = \lambda^q \widehat{x}(\lambda) - \sum_{k=0}^{m-1} x^{(k)}(0) \lambda^{q-1-k}$.

The definitions of mild solutions for various kinds of C_0-semigroup-based control systems are usually given by some probability density functions. To this end, the following preparations are necessary:

Definition 2.8. The Mainardi's Wright function Ψ_q $(0<q<1)$ is defined by

$$\Psi_q(\theta)=\sum_{n=0}^{\infty}\frac{(-\theta)^n}{n!\Gamma(-qn+1-q)}=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-\theta)^n}{(n-1)!}\Gamma(nq)\sin(n\pi q),\ \theta\in\mathbb{C}.$$

Remark 1. If $\theta\in\mathbb{R}^+$, then

$$\Psi_q(\theta)=\frac{1}{\pi q}\sum_{n=1}^{\infty}(-\theta)^{n-1}\frac{\Gamma(1+qn)}{n!}\sin(n\pi q),\ q\in(0,1).$$

Lemma 2.9. *Let $0<q<1$, then*

(i) $\Psi_q(t)\geq 0,\ t\in(0,\infty)$.
(ii) $\displaystyle\int_0^{\infty}\frac{q}{t^{q+1}}\Psi_q(t^{-q})e^{-\lambda t}dt=e^{-\lambda^q},\ Re(\lambda)\geq 0$.
(iii) $\displaystyle\int_0^{\infty}\Psi_q(t)t^r dt=\frac{\Gamma(1+r)}{\Gamma(1+qr)},\ r\in(-1,\infty)$.
(iv) $\displaystyle\int_0^{\infty}\frac{1}{q}t^{-1-\frac{1}{q}}\Psi_q(t^{-\frac{1}{q}})dt=1$.

3. **Controllability of some basic systems.** During the past four decades, controllability problems of various dynamical systems, including integeral order and fractional order derivatives, have been widely investigated in finite-dimensional and infinite-dimensional spaces [4, 27, 26, 57, 60, 69, 123]. Among such controllability problems, the two most fundamental types are exact controllability and approximate controllability. The exact controllability enables to steer the considered systems to arbitrary final state. If there exists a control such that the systems can be steered to zero point, we call it null controllability [86], which can be regarded as a special case of exact controllability in some way. Under the assumption that the invertible controllability operator can be constructed, the controllability problem can be transformed into a fixed point problem.

In this section, we consider the controllability of some basic nonlinear fractional control systems with local conditions and nonlocal conditions. First, consider the following systems:

$$^{C}D_t^q x(t)=Ax(t)+f(t,x(t))+Bu(t),\quad t\in J=[0,b],\tag{3.1}$$

with classical initial condition

$$x(0)=x_0,\tag{3.2}$$

where $^{C}D_t^q$ represents the Caputo derivative of order $0<q\leq 1$, A is unbounded and generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on Banach space X. B is a bounded linear operator and f is a nonlinear operator on X. If x is a classical solution of (3.1)-(3.2), then

$$x(t)\in\mathcal{D},\ \forall t\in[0,b],$$

which indicates that the systems will not be steered to arbitrary final state of X due to the fact $\mathcal{D}\neq X$ in most cases. Consequently, only some types of mild solutions are considered in the study of the control systems. With this prerequisite, we give the following notion of exact controllability:

Definition 3.1. The fractional control system (3.1)-(3.2) is said to be exactly controllable on J if, for every $x_0,\ x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the mild solution x of systems (3.1)-(3.2) satisfies $x(b) = x_1$.

It is worth mentioning that the introduction of mild solution in the investigation of fractional controllability problems is the first step. However, Hernández et al. [20] pointed out that the definition of mild solutions used in some papers was not appropriate for FESs as it is just a simple extension of mild solutions for integral order evolution equations. In 2002, El-Borai [16] firstly introduced some probability density functions and gave the fundamental solutions of fractional evolution equations, which is a significant contribution to the construction of mild solutions for FESs. Since then, mild solutions for most of fractional controllability problems are constructed in terms of such probability density functions. For example, with regard to (3.1)-(3.2), the mild solution can be given as follows:

Definition 3.2. For each $u \in L^2(J, U)$, a mild solution of the system (3.1)-(3.2) we mean the function $x \in C(J, X)$ which satisfies

$$x(t) = \mathcal{T}(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)f(s, x(s))ds + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)Bu(s)ds,$$

where $\mathcal{T}(\cdot)$ and $\mathcal{S}(\cdot)$ are given by

$$\mathcal{T}(t) = \int_0^\infty \xi_q(\theta)T(t^q\theta)d\theta, \quad \mathcal{S}(t) = q\int_0^\infty \theta\xi_q(\theta)T(t^q\theta)d\theta$$

and for $\theta \in (0, \infty)$,

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\Psi_q(\theta^{-\frac{1}{q}}) \geq 0.$$

Ψ_q is the Mainardi's Wright function defined by Definition 2.8, and $\xi_q(\theta)$ is a probability density function defined on $(0, \infty)$ satisfying

$$\int_0^\infty \xi_q(\theta)d\theta = 1, \quad \int_0^\infty \theta^\nu\xi_q(\theta)d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+q\nu)}, \ \nu \in [0, 1].$$

We are now in the position to describe the exact controllability of (3.1)-(3.2) based on the above definition of mild solution. Suppose that the following hypothesis holds:

(C) The linear operator $W : L^2(J, U) \to X$ defined by

$$Wu = \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)Bu(s)ds$$

has a bounded inverse operator W^{-1} which takes values in $L^2(J, U)/kerW$.

Then, for $\forall x \in C(J, X)$, we can construct the corresponding control function

$$u_x(t) = W^{-1}\left(x_1 - \mathcal{T}(b)x_0 - \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)f(s, x(s))ds\right)(t), \quad t \in J, \tag{3.3}$$

and condition (C) infers $x(b) = x_1$, which means that system (3.1)-(3.2) is exactly controllable on J.

Note that the formula of Definition 3.2 contains characteristic solution operators associated with semigroup $T(t)$, probability density functions and Mainardi's

Wright-type function. It is essentially different from the simple extension of mild solutions for integral order evolution systems. On the basis of Definition 3.2, J. Wang and Y. Zhou [101] established some results of exact controllability for system (3.1)-(3.2) by utilizing properties of noncompact characteristic solution operators and fixed point strategies.

Theorem 3.3. *([101]) Suppose that the following assumptions hold:*

(i) The operator A generates a strongly continuous semigroup $T(t)$ in X, and there exists a constant M_1 such that $\sup_{t\in J}\|T(t)\| \leq M_1$.

(ii) The linear operator $B : L^2(J,U) \to L^1(J,X)$ is bounded, $W : L^2(J,U) \to X$ defined by

$$Wu = \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)Bu(s)ds$$

has an inverse operator W^{-1} which takes values in $L^2(J,U)/kerW$ and there exist two positive constants $M_2, M_3 > 0$ such that $\| B \|_{L_b(U,X)} \leq M_2$ and

$$\|W^{-1}\|_{L_b(X,L^2(J,U)/kerW)} \leq M_3.$$

(iii) $f : J\times X \to X$ is continuous and there exists a constant $q_1 \in (0,q)$ and $L_f \in L^{\frac{1}{q_1}}(J,\mathbb{R}^+)$ such that

$$\|f(t,x_1) - f(t,x_2))\| \leq L_f(t)\|x_1 - x_2\|,\ t\in J,\ x_i \in X,\ i = 1,2.$$

(iv) $\mathcal{S}(t)$ is continuous in the sense of uniformly operator topology for $t > 0$.

(v) For all bounded subsets $\mathfrak{B}$, the set

$$\Pi_{h,\delta} = \left\{ q\int_0^{t-h}\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)f(s,x(s))d\theta ds : x\in\mathfrak{B}\right\}$$

is relatively compact in X for any $h \in (0,t)$ and $\delta > 0$.

Then system (3.1)-(3.2) is exactly controllable on J provided that

$$c\left[1 + \frac{M_3M_2M_1b^q}{\Gamma(1+q)}\right] < 1,$$

where $c = \dfrac{M_4M_1q}{\Gamma(1+q)},\ M_4 = \kappa\|L_f\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)},\ \kappa = \left[(\dfrac{1-q_1}{q-q_1})b^{\frac{q-q_1}{1-q_1}}\right]^{1-q_1}.$

Using the similar approach in [101], J. Du et al. [15], studied the exact controllability of some fractional delay control systems, and A. Kumar et al. [31], A. Debbouche et al. [13], investigated the fractional impulsive control problems or such problems with delay. Some fixed point theorems, such as Banach's fixed-point theorem, Dhage's fixed-point theorem and Krasnoselskii's fixed-point theorem, are used widely in establishing the controllability results for the aforementioned papers in which the Lipschitz conditions are all required. The premise of these results is that the semigroups generated by infinitesimal generators are noncompact. This can be illustrated by the following simple example:

Lemma 3.4. *([101]) Consider the following fractional differential systems*

$$\begin{cases} \dfrac{\partial^{\frac{2}{3}}}{\partial t^{\frac{2}{3}}}x(t,y) = x_y(t,y) + \omega\mu(t,y) + \dfrac{e^{-t}}{k+e^t}x(t,y), \quad t\in J = [0,1], \\ x(t,0) = x(t,1) = 0, \\ x(0,y) = 0, \quad 0 < y < 1, \end{cases}$$

where $\omega > 0, k \geq 1$, *and* $\mu : J \times (0,1) \to (0,1)$ *is continuous in* t. *Take* $X = C([0,1])$ *and let* A *be* $Ax = x'$, $x \in D(A)$, *where* $D(A) = \{x \in X : x' \in X, x(0) = x(1) = 0\}$.

Obviously, A is an infinitesimal generator of a noncompact semigroup $\{T(t) : t \geq 0\}$ which is defined by $T(t)x(s) = x(t+s)$ for $x \in X$. Define $f(t, x(t))(y) = \dfrac{e^{-t}}{k + e^t} x(t, y)$. Then f satisfies Lipschitz condition:

$$\|f(t, x_1 - f(t, x_2))\| \leq \frac{1}{1+k}\|x_1 - x_2\|,\ x_1, x_2 \in X,\ t \in J.$$

It is easy to see that all conditions of Theorem 3.3 are satisfied. Hence, the given system is exactly controllable on J.

Authors in [101] used the equicontinuity of semigroup $T(t)$ rather than its compactness in infinite-dimensional spaces, which can avoid a technical mistake happening while the compactness of $T(t)$ is employed. Controllability result in [97] was once established under the hypothesis that the semigroup associated with the linear part is compact. Note that Lipschitz continuity and some compact conditions were imposed on the nonlinear function f in [101] to guarantee the contraction and complete continuity of corresponding integral operators, and then Krasnoselskii's fixed-point theorem ensured the existence of mild solution for system (3.1)-(3.2). As a stronger condition than continuity, Lipschitz continuity is regarded only as an idealized one in many cases, which is difficult to be applied to practical problems. In addition, it seems difficult to verify the validity of the compact condition (v) in Theorem 3.3. Based on these observations, authors in [123] proposed a new concept of exact controllability, and derived some sufficient conditions ensuring the exact controllability of system (3.1)-(3.2) without Lipschitz continuity and compact conditions imposed on nonlinearity, but relying on the differentiability of resolvent operator.

Definition 3.5. ([123]) The fractional control system (3.1)-(3.2) is called exactly controllable on interval $J = [0, b]$, if for any $x_0,\ x_1 \in X$, there exists a control $u \in L^2(J, U)$ and a constant $\tau \in (0, b]$ such that a mild solution x of system (3.1)-(3.2) on $I = [0, \tau]$ satisfies $x(\tau) = x_1$.

Theorem 3.6. *([123]) Suppose that the following hypotheses are satisfied:*

(i) $f \in C(I \times X; \mathcal{D})$ *and takes bounded sets into bounded sets.*

(ii) The linear operator $B : L^2(J, U) \to L^1(J, \mathcal{D})$ *is bounded,* $W(t)$ *defined by*

$$W(t)u = B_u(t) + \int_0^t \dot{S}(t-s)B_u(s)ds,\ t \in J,$$

where $B_u(t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}Bu(s)ds$, *has an induced inverse operator* $W^{-1}(t)$ *which takes values in* $L^2(J, U)/kerW(t)$ *for every* $t \in J$ *and there exist two positive constants* $M_1, M_2 > 0$ *such that* $\| B \|_{L(U,\mathcal{D})} \leq M_1$ *and*

$$\sup_{t \in J} \|W^{-1}(t)\|_{L(X, L^2(J,U)/kerW(t))} \leq M_2.$$

(iii) There exists a positive constant L *such that* $\alpha(f(t, D)) \leq L\alpha(D)$, $t \in J$, *for any bounded subset* $D \subset X$.

Then the fractional control system (3.1)-(3.2) is exactly controllable on J *if* $x_0 \in \mathcal{D}$.

It is shown that in Definition 3.5, the time when objective system is steered to x_1 may be reached before the end value b. When τ equals b, Definition 3.5 is the same as Definition 3.1. Obviously, it is easy to see that hypotheses (i) and (iii) in Theorem 3.6 naturally hold if f is Lipschitz continuous or compact. From this point of view, the obtained results in [123] have wider applications. In addition, it should be stressed that whether a analytic resolvent operator is differentiable has not been demonstrated strictly at present, which is somewhat surprising to us because the given resolvent operator is of course continuously differentiable on $(0,\infty)$. For more details about resolvent operator theory, please refer to [74] and references therein. Hence, in order to suppress the hypotheses that f and x_0 are $\mathcal{D}$-valued in (i) of Theorem 3.6, the resolvent operator is supposed to be analytic instead of differentiability and thus the controllability result is also derived in [123]. To this end, it needs to replace only the hypotheses (i) and (ii) of Theorem 3.6 with the following assumptions:

(i) $f \in C(I \times X; X)$ and takes bounded sets into bounded sets.

(ii) The linear operator $B : L^2(J,U) \to L^1(J,X)$ is bounded, $W(t)$ defined by

$$W(t)u = S(t)B_u(t) + \int_0^t \dot{S}(t-s)(B_u(s) - B_u(t))ds, \ t \in J$$

has an induced inverse operator $W^{-1}(t)$ which takes values in $L^2(J,U)/kerW(t)$ for every $t \in J$ and there exist two positive constants $M_1, M_2 > 0$ such that

$$\| B \|_{L(U,X)} \leq M_1, \ \ \sup_{t\in J} \|W^{-1}(t)\|_{L(X,L^2(J,U)/kerW(t))} \leq M_2.$$

Next, we consider the system (3.1) with nonlocal conditions. In practice, nonlocal conditions are usually more accurate than classical initial conditions for physical estimation. For example, when observing the diffusion of a small amount of gas in the tube, there is more than one initial observation value, that is, the initial value can be determined by multiple observation points. The powerful tool to describe such phenomenon is the differential equations with nonlocal conditions. Therefore, more and more studies on the exact controllability of FESs with nonlocal conditions emerge in large numbers. Generally, nonlocal conditions for many practical problems can be described by the following formula:

$$x(0) + g(x) = x_0, \tag{3.4}$$

Definition 3.7. ([100]) The fractional control system (3.1) and (3.4) is said to be nonlocally exactly controllable on J iff, for every $x_0, \ x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that a mild solution x of system (3.1) and (3.4) satisfies $x(b) + g(x) = x_1$.

Different from (3.3), using hypothesis (C), for an arbitrary function $x \in C(J,X)$, define the corresponding control as follows

$$u_x(t) = W^{-1}\left(x_1 - g(x) - \mathcal{T}(b)(x_0 - g(x)) - \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)f(s,x(s))ds\right)(t).$$

It is not difficult to deduce that, by using this control, the mild solution of (3.1) and (3.4) satisfies $x(b) + g(x) = x_1$, which indicates the objective system is nonlocally exactly controllable on J. [100] obtained two nonlocally controllable results for system (3.1) and (3.4) by using noncompactness measure in $C(J,X)$. J. Liang and H. Yang [48] considered the nonlinear fractional integro-differential equations

$$^{C}D_t^q x(t) + Ax(t) = f(t, x(t), Gx(t)) + Bu(t), \quad t \in J = [0,b] \tag{3.5}$$

with nonlocal condition

$$x(0) = \sum_{k=1}^{m} c_k x(t_k), \ 0 < t_1 < t_2 < \cdots < t_m \leq b, \tag{3.6}$$

which covers condition (3.2) as a special case. Here $-A$ generates a C_0-semigroup on X. The control function $u \in L^2(J, U)$. U is a Banach space. f is imposed on some growth assumptions. Under the assumption of noncompactness of C_0-semigroup, exact controllability results of system (3.5) and (3.6) are derived. It is noted that the nonlocal condition in (3.6) is usually used to describe the diffusion phenomenon of small amount of gas in a transparent tube.

Thirdly, consider the following fractional evolution equations of mixed type

$${}^C D_t^q x(t) + Ax(t) = f(t, x(t), (Sx)(t), (Tx)(t)) + Bu(t), \quad t \in J = [0, b] \tag{3.7}$$

with nonlocal condition (3.4), where S and T are given integrals with scalar kernel $k(t, s)$ and $h(t, s)$, respectively. Compared with [123], [80] established some results on exact controllability for (3.4) and (3.7) by imposing some growth conditions on nonlinearity f and nonlocal term g rather than the Lipschitz continuity, which can be described as follows:

(i) For $r > 0$, there exist constants $q_1 \in [0, q)$ and functions $\varphi_r \in L^{\frac{1}{q_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t, x, Sx, Tx)\| \leq \varphi_r(t), \quad a.e.\ t \in J, \quad \forall x \in X \ satisfying\ \|x\| \leq r.$$

Moreover, suppose $\liminf_{r\to\infty} \dfrac{\|\varphi_r(s)\|_{L^{1/q_1}}}{r} < +\infty$.

(ii) The nonlocal term $g : C(J, E) \to X$ is compact and continuous. There exists a nondecreasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\| g(x) \| \leq \psi(r)$ for all $x \in \Omega_r = \{x \in C(J, X) : \|x\|_C \leq r\}$ $(r > 0)$ and $\liminf_{r\to\infty} \dfrac{\psi(r)}{r} < +\infty$.

It should be pointed out that, different from [123], [80] used the methods including a new estimation of the measure of noncompactness and a fixed point theorem with respect to a convex-power condensing operator. Obviously, condition (3.4) reduces to condition (3.6) when $g(x) = \sum_{k=1}^{m} c_k x(t_k)$ and $x_0 = \theta$. For more outstanding achievements about exact controllability of fractional differential systems under the assumptions that the nonlinearity satisfies Lipschitz continuity, compactness or some other growth conditions, we refer to [15, 31] and references therein.

Generally speaking, the controllability of the objective system is usually transformed into a fixed point problem for some nonlinear integral operator in an suitable function space. It is worth noting that the compactness conditions play a crucial role in the proof of their results. Sometimes, the compactness of the operator semigroup is required to obtain the controllability of the systems. Unfortunately, this is the case only in finite-dimensional spaces [98] since the inverse of control operator may not exist if the state space is infinite-dimensional. Similar technical errors due to the compactness of semigroup $T(t)$ have also been pointed out by Hernández et al. [19]. It can be observed that if the semigroup associated with the system is compact, then the control operator must also be compact. At the same time, it is also required that the control operator is reversible on a quotient space and the inverse operator is bounded, which means that the assumptions are too strong. From this point of view, approximate controllability is more attractive and popular as an extension of exact controllability, which avoids quite a lot limitations in some practical applications by contrast.

Compared with the exact controllability, approximate controllability can steer the considered system to arbitrary small neighborhood of final state. It is stressed here that exact controllability and approximate controllability coincide when the space is finite-dimensional. On the premise that the corresponding linear systems are approximate controllable, many scholars have studied the approximate controllability of various fractional semilinear evolution systems such as the time-delay systems [22, 32], impulsive systems [117], stochastic equations [86, 85], neutral equations [68] and so on. Generally speaking, the major techniques to investigate approximate controllability can be classified into three categories. The first one is to use iterative and approximate techniques, that is called sequential approach also [44, 90, 89]. The second one is the range conditions of the operator B [32, 69]. The last one, which is also the most popular one so far, is the method to use resolvent conditions based on resolvent operator or solution operator of the considered systems [82, 68, 83].

Definition 3.8. The fractional system (3.1)-(3.2) is said to be approximately controllable on J if $\overline{K_b(f)} = X$, where $K_b(f)$ is called the reachable set of system (3.1)-(3.2) at time b and is defined as

$$K_b(f) = \{x(b,u) \in X : u \in L^2(J,U),\ x(\cdot,u) \textit{ is the mild solution of } (3.1)-(3.2)\}.$$

We can also use the following equivalent expression:

Definition 3.9. The fractional system (3.1)-(3.2) is said to be approximately controllable on J if, for any desired final state $x_1 \in X$ and arbitrary $\varepsilon > 0$, there exists a control $u \in L^2(J,U)$ such that $x(t)$ satisfies $\|x(b) - x_1\| < \varepsilon$, where $x(t)$ is a corresponding solution of system (3.1)-(3.2).

With regard to the method to use resolvent conditions, it is particularly important to note that the approximate controllability results are based on the supposition that the corresponding linear dynamical systems are approximately controllable. For this reason, consider the following linear fractional systems corresponding to (3.1) and (3.2):

$$\begin{cases} {}^CD_t^q x(t) = Ax(t) + Bu(t), \quad t \in J = [0,b], \\ x(0) = x_0. \end{cases} \tag{3.8}$$

Note that approximate controllability of (3.8) is an extension of approximate controllability of linear first-order control systems. Then it is natural to introduce the control operator associated with (3.8) as

$$\Gamma_0^b = \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)BB^*\mathcal{S}^*(b-s)ds,$$

where B^* and $\mathcal{S}^*$ denote the adjoint of B and $\mathcal{S}$, respectively.

Lemma 3.10. *([84]) The linear fractional system (3.8) is approximately controllable on J iff $\alpha(\alpha I + \Gamma_0^b)^{-1} \to 0$ as $\alpha \to 0^+$ in the strong operator topology.*

For fractional systems (3.1) and (3.2), R. Sakthivel et al. in [84] established a new set of sufficient conditions guaranteeing the approximate controllability by using the semigroup theory and fixed point strategy, which is given as follows:

Theorem 3.11. *Assume that the following hypotheses hold:*

(i) $T(t)$ *is a compact operator for each* $t \in [0, b]$.

(ii) *For each* $t \in [0, b]$, *the function* $f(t, \cdot) : X \to X$ *is continuous and for each* $x \in C([0, b], X)$, *the function* $f(\cdot, x) : [0, b] \to X$ *is strongly measurable.*

(iii) *There exists a constant* $q_1 \in [0, q]$ *and* $m \in L^{\frac{1}{q_1}}([0, b], \mathbb{R}^+)$ *such that* $| f(t, x) | \leq m(t)$ *for all* $x \in X$ *and almost all* $t \in [0, b]$.

(iv) *The function* $f : J \times X \to X$ *is continuous and uniformly bounded, and there exists* $N > 0$ *such that* $\| f(t, x) \| \leq N$ *for all* $(t, x) \in J \times X$.

Then the semilinear fractional system (3.1)-(3.2) is approximately controllable on J.

Notice that Theorem 3.11 requires the compactness of operator $T(t)$, which ensures that operators $\mathcal{S}(t)$ and $\mathcal{T}(t)$ are also compact for every $t > 0$. This helps to guarantee the existence of the mild solution to systems (3.1) and (3.2) in the sense of Definition 3.2 via Schauder's fixed-point theorem. In addition, the selection of subsequences and the compactness of some relevant operators (see Theorem 3.3 in [84] for details) via resolvent conditions are also based on this compactness assumption. Further, by employing the similar technique adopted in Theorem 3.11, authors in [84] extended the approximate controllability results to nonlocal FESs (3.1) and (3.4):

Theorem 3.12. *Assume that all the assumptions of Theorem 3.11 hold and, in addition, the following hypothesis holds:*

(v) $g : C([0, b], X) \to X$ *is a given function satisfying that there exists a constant* $L > 0$ *such that* $\| g(x) - g(y) \| \leq L \| x - y \|$, *for any* $x, y \in C([0, b], X)$.

Then the semilinear fractional system (3.1) with nonlocal condition (3.4) is approximately controllable on J.

On the basis of the ideas presented in [84], S. Ji [23] further investigated the approximate controllability of the same nonlocal fractional systems by resolvent conditions and approximation method:

Theorem 3.13. *([23]) Assume that the following hypotheses are satisfied:*

(i) $T(t)$ *is compact for every* $t > 0$.

(ii) $f : [0, b] \times X \to X$ *is continuous and there exists a positive constant* L_1 *such that* $\| f(t, x) \| \leq L_1$ *for all* $(t, x) \in [0, b] \times X$.

(iii) $g : C([0, b]; X) \to X$ *is continuous and there exists a positive constant* L_2 *such that* $\| g(x) \| \leq L_1$ *for all* $x \in C([0, b]; X)$. *In addition, there exists a* $\delta = \delta(r) > 0$ *such that* $g(x) = g(y)$ *for any* $x, y \in W_r$ *with* $x(s) = y(s)$ *for* $s \in [\delta, b]$, *where* $W_r = \{x \in C(J; X) : x(t) \in B_r\}$, $B_r = \{x \in X : \|x\| \leq r\}$.

(iv) $B : U \to X$ *is a linear bounded operator.*

Then the fractional control system (3.1) with nonlocal condition (3.4) is approximately controllable on J.

Here, the nonlinear term f and nonlocal term g are supposed to be only continuous functions, which are different from that in Theorem 3.12. Furthermore, it is noted that, in infinite-dimensional spaces, C_0-semigroup $T(t)$ is compact only when $t > 0$, that is, $T(t)$ is not compact at $t = 0$. In order to overcome this difficulty, author in [23] established a compact set of approximate solutions on the basis of approximate semigroup $T(\frac{1}{n})$, and derived the approximate controllability of (3.1) and (3.4). From this point of view, the results in [23] improved that in [84]. Such similar techniques also be used in [64].

4. **Controllability of systems with time delay.** As we all know, the current states of some practical systems clearly depend on the past history, which is the so-called phenomenon of time delay. It occurs frequently and is inevitable in numerous practical systems of the real world. In fact, time delay is closely related to various evolution equations. Hence, the effect of time delay must be considered if we intend to describe and analyze evolution systems accurately. With the development of the applications for fractional calculus, the research on the controllability of FESs with time delay is more and more extensive [32, 57, 89].

According to that the time delay is finite or not, such systems in the existing literatures on fractional controllability can be classified into two types. One is FES with finite time delay, and the other is FES with infinite time delay. There are many excellent results on the controllability of FESs with finite time delay [13, 86, 88]. Compared with the spaces chosen to investigate infinite time delay systems, the Banach spaces selected to study finite time delay systems are regular and relatively simple.

Consider the following fractional finite time delay system:

$$ {}^{C}D_t^q x(t) = Ax(t) + A_1 x(t-h) + Bu(t) + f(t, x(t-h)), \quad t \in J = [0,b], \tag{4.1} $$

with initial condition

$$ x(t) = \phi(t), \quad t \in [-h, 0], \tag{4.2} $$

where $\frac{1}{2} < q \leq 1$, A is the infinitesimal generator of a C_0-semigroup $T(t)$ on Hilbert space V. A_1 is a bounded linear operator on V. U is a Hilbert space and $u(t) \in U$. B is a bounded linear operator and f is a nonlinear operator on V. $\phi \in C([-h,0], V)$.

Based on the ideas of [90, 128], authors in [44] investigated the approximate controllability of system (4.1)-(4.2) by using iterative and approximate techniques, i.e. sequential approach. Their main result is given as below:

Theorem 4.1. *([44]) Suppose that the following hypotheses hold:*

- *(i) The semigroup $T(t)$ generated by A is continuous and uniformly bounded on V, i.e., there is a constant $M > 0$ such that $\sup\limits_{t\in[0,+\infty)} \|T(t)\| \leq M$.*
- *(ii) The nonlinear function $f(t,x)$ is continuous in t for all $x \in V$ and continuous with respect to x for almost all $t \in [0,b]$, and there exists a positive constant L, such that*

$$ \|f(t,x) - f(t,y)\| \leq L\|x-y\|_V, \ \forall x, y \in V. $$

- *(iii) For any $\varepsilon > 0$ and $\varphi \in Z = L^2([0,b], V)$, there exists a $u \in Y = L^2([0,b], U)$, such that $\|\mathcal{G}\varphi - \mathcal{G}Bu\|_V < \varepsilon$, $\|Bu\|_Z < \gamma \|\varphi\|_Z$, where*

$$ \mathcal{G}h = \int_0^b (b-s)^{q-1} \mathcal{S}(b-s) h(s) ds, \ h \in Z, $$

and γ is a positive constant which is independent of $\varphi \in Z$ and satisfies

$$ \frac{M(\|A_1\| + L)\gamma}{\Gamma(\alpha)} \sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}} E_\alpha(M(\|A_1\| + L) b^\alpha) < 1. $$

Then, system (4.1)-(4.2) is approximately controllable.

Different from the method of resolvent conditions mentioned in Section 3, the sequential approach utilized in [44] has many advantages. Hypothesis (i) in Theorem 4.1 indicates that C_0-semigroup $T(t)$ is not necessarily compact, which represents a significant difference by contrast. In fact, if the semigroup $T(t)$ is compact or differentiable for $t > 0$, then $T(t)$ is thus continuous in the uniform operator topology for $t > 0$ which means hypothesis (i) in Theorem 4.1 holds naturally [71]. Hypothesis (ii) is a contraction condition that guarantees the existence and uniqueness of mild solution for system (4.1)-(4.2). It should be pointed out that the uniform boundedness of nonlinearity is not necessary in comparison with resolvent conditions method. The most important step is to construct iterative sequence of the control u_n by induction in accordance to hypothesis (iii), which derives a Cauchy sequences Bu_n, and thus leads to the approximate controllability of system (4.1)-(4.2). Considering the existence of time delay in systems, it is necessary that b is greater than h since $x(t)$ can not be controlled on $[b-h, 0]$ with $b < h$. If $A_1 = 0$ and $h = 0$, system (4.1)-(4.2) degenerates into system (3.1)-(3.2) whose approximate controllability was studied by the method of resolvent conditions in [84].

Another interesting work on this subject is to study the relationship of approximate controllability between FES with delay and FES without delay. A. Shukla et al. [88] studied the approximate controllability of the following two classes of FESs of order $q \in (1, 2]$:

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + Bu(t) + f(t, x(t-h)), \quad t \in J = [0, b], \\ x(t) = \phi(t), \quad t \in [-h, 0], \\ x'(0) = x_0, \end{cases} \tag{4.3}$$

and the corresponding semilinear system without delay

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in J = [0, b], \\ x(0) = \phi(0), \\ x'(0) = x_0, \end{cases} \tag{4.4}$$

where $1 < q \leq 2$, A is the infinitesimal generator of cosine operator families $\{C_q(t)\}_{t\geq 0}$. X and U are separable Hilbert spaces. $Z = L^2([0, b], X)$, $Z_h = L^2([-h, b], X)$, $0 \leq h \leq b$ and $Y = L^2([0, b], U)$. The control function $u : [0, b] \to U$. $x(t) \in Z$, $\phi \in L^2([-h, 0], X)$. $B : Y \to Z$ is a bounded linear operator. The nonlinearity $f : [0, b] \times X \to V$ satisfies Lipschitz continuity.

Different from [44], [88] took advantage of relationships of the approximate controllability between system (4.3) and its corresponding semilinear system (4.4). The approximate controllability of system (4.3) can be obtained from that of the approximate controllability of system (4.4) under some suitable assumptions. The two above-mentioned controllability results in [88] can be characterized by the following two theorems:

Theorem 4.2. *([88]) If the following assumptions hold:*

(i) $C_q(t)$ is a compact semigroup.

(ii) f is Lipschitz continuous with respect the second variable i.e

$$\|f(t, x(t)) - g(t, w(t))\|_X \leq L_g \|x(t) - w(t)\|_X.$$

(iii) $R(G) \subset R(B)$, where G is the Nemytskii operator of the function $f(t, x(t))$, and $R(G)$, $R(B)$ denote the range of the operator G and operator B, respectively.

(iv) The system (4.4) is approximately controllable.

Then, the system (4.3) is approximately controllable.

Theorem 4.3. *([88]) Suppose that the assumptions (i), (ii) and (iv) in Theorem 4.2 hold. In addition, assumption (iii) in Theorem 4.2 is replaced by*

(iii) $'$: $R(G) \subseteq \overline{R(B)}$.

Then, the system (4.3) is approximately controllable.

Besides using Lipschitz continuity of nonlinearity to obtain the existence and uniqueness of mild solutions, some other types of fixed-point theorems are also used to study the controllability of fractional finite time delay systems. For instance, authors in [27] established some necessary and sufficient conditions for the exact controllability of certain linear fractional finite delay systems by using the Laplace transformation techniques and Mittag-Leffler function, and also presented a sufficient condition for the exact controllability of nonlinear fractional finite delay system via Schauder's fixed-point theorem. [26] investigated a implicit fractional finite delay system with multiple delays in control. Under the hypothesis that control Gramian matrix W is positive definite and nonlinear term is Lipschitz continuous and bounded, some sufficient conditions ensuring the exact controllability are obtained by Darbo's fixed-point theorem [26].

As we know, the theory of noncompactness measures is also a powerful tool to study the controllability of fractional time delay systems in infinite-dimensional Banach spaces. Consider the following fractional nonlocal semilinear evolution systems with finite time delay:

$$\begin{cases} D^q x(t) = Ax(t) + f(t, x(t), x_t) + Bu(t), \quad a.e.\ t \in J = [0, b], \\ x(t) + M(t, x) = \phi(t), \quad t \in [-h, 0], \end{cases} \tag{4.5}$$

where $q \in (0, 1)$, $x : J \to X$, $u \in L^2(I, U)$, and $B : U \to X$ is a bounded linear operator. X and U are Banach spaces. $A : \mathcal{D} \subset X \to X$ is a closed linear unbounded operator on X with dense domain $\mathcal{D}$. $\phi \in C([-h, 0], X)$.

Inspired by [57, 123], authors in [124] established a new result of exact controllability for system (4.5) by utilizing the theory of noncompactness measures and resolvent operators. Due to the existence of time delay, sometimes we need to choose or construct some appropriate spaces. Therefore, different from [123], [124] selects the complete space $L([-h, 0], X)$ to conquer the difficulties caused by time delay during the use of noncompactness measures technique in different spaces (see Lemma 4 and 5 in [124] for detais). Compared with [57], [74] deduces the mild solutions of system (4.5) by a resolvent operator family. In order to avoid the compact hypothesis on resolvent operator, differentiability of resolvent operator is used to derive the mild solution. Similar to [123], the definition of exact controllability described the characteristic of "fast time" in [124] (Definition 3.2, [123]). However, the two papers did not provide numerical simulation for this phenomenon of "fast time". Obviously, when t is taken as 0, initial condition of (4.5) reduces to the form of condition (3.4).

Generally speaking, for the sake of investigating approximate controllability of nonlinear FESs with delay or without delay, some restrictive assumptions need to be imposed on the system components, such as continuity or Lipschitz continuity of nonlinearity, control interval, compactness of semigroup, and range conditions of operator B. For instance, S. Kumar and N. Sukavanam [32] derived the approximate controllability for system (4.1)-(4.2) with $A_1 = 0$ by using the approach of range conditions of operator B,

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + Bu(t) + f(t, x(t-h)), \quad t \in J = [0, b], \\ x(t) = \phi(t), \quad t \in [-h, 0]. \end{cases} \tag{4.6}$$

Denote the range of the operator B by $R(B)$, and define a linear operator $\mathcal{L}$ from $Z = L^2([0,b],V)$ to V by $\mathcal{L}_\xi = \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)\xi(s)ds$. Motivated by [69], authors in [32] supposed the hypothesis (HB) as follows:

$$(HB)\ for\ each\ \xi \in Z,\ there\ exists\ a\ function\ \eta \in \overline{R(B)}\ such\ that\ \mathcal{L}_\xi = \mathcal{L}_\eta.$$

It should be pointed out that hypothesis (HB) is essential to obtain the approximate controllability for system (4.6), since it implies that the corresponding linear system of (4.6) is approximately controllable. Also, from hypothesis (HB), they deduced that there is a geometrical relation in Z between the range of operator B and the null space of operator $\mathcal{L}$,

$$Z = N_0(\mathcal{L}) \oplus N_0^\perp(\mathcal{L}) = N_0(\mathcal{L}) \oplus \overline{R(B)},$$

where $N_0^\perp(\mathcal{L})$ denotes the orthogonal complement space of null space $N_0(\mathcal{L})$. Due to the existence of time delay, mild solution is derived in the same way as in [88]. Else, the uniform boundedness of nonlinearity required in resolvent conditions method is replaced by Lipschitz continuity here. The technique of range conditions of operator B is rarely used by far, we refer to [22, 91] for further details.

As we know, the theory of differential equations with finite time delay has been developed extensively. However, there are many complex dynamic systems in practice which can not be described and analyzed accurately by using finite time delay. In recent years, the theory of differential equations with infinite delay has received a great deal of concerns and gained rapid progress due to its applications in science and engineering. The choice of phase space is of vital importance. How to choose a suitable phase space is essential to solve such problems with infinite time delay. The most effective phase spaces used so far are $\mathcal{C}_h$-class space and $\mathcal{C}_g$-class space [2, 21]. This is also true for FESs with infinite time delay. More recently, many controllability results about FESs with infinite delay were established [28, 68]. P. Muthukumar et al. [68] presented some sufficient conditions to obtain the approximate controllability for nonlinear fractional nonlocal neutral stochastic differential systems of order $\alpha \in (1,2)$ with infinite time delay and Poisson jumps by using Lebesgue dominated convergence theorem and the compactness of some operator families.

In addition, [28] studied the following fractional systems with infinite delay:

$$D_{0+}^{\xi,\eta}[x(t) - g(t,x_t)] = Ax(t) + f(t,x_t) + Bu(t), \quad t \in (0,b], \tag{4.7}$$

$$I_{0+}^{(1-\xi)(1-\eta)}x(t)|_{t=0} = \phi(0) \in \mathcal{C}_h, \quad t \in (-\infty,0], \tag{4.8}$$

$$I_{0+}^{(1-\xi)(1-\eta)}x(t)|_{t=0} = \phi(0) + q(z_{t_1}, z_{t_2}, z_{t_3}, \cdots, z_{t_n}) \in \mathcal{C}_h, \quad t \in (-\infty,0], \tag{4.9}$$

where $D_{0+}^{\xi,\eta}$ stands for Hilfer fractional derivative of order η and type ξ. $\xi \in [0,1]$ and $\eta \in (\frac{1}{2},1)$. $\mathcal{C}_h$ is an appropriate phase space. A is the infinitesimal generator of analytic semigroup $\{T(t)\}$. $0 < t_1 < t_2 < t_3 < \cdots < t_n \le b$, $q : \mathcal{C}_h^n \to \mathcal{C}_h$. Under appropriate assumptions applied to semigroup T and the power of A together with some growth conditions imposed on nonlinear function, exact controllability of system (4.7)-(4.8) was established by utilizing Mönch's fixed-point theorem. As an extension of system (4.7)-(4.8), the nonlocal neutral differential systems (4.7) and (4.9) was considered also in [28]. It should be remarked that the Hilfer fractional derivative degenerates into classical Riemann-Liouville fractional derivative if $\xi = 0$, $\eta \in (0,1)$, and it degenerates into classical Caputo derivative if $\xi = 1$, $\eta \in (0,1)$.

As we know, the general method to study the approximate controllability on FESs with infinite time delay is resolvent condition technique [83]. Differently, A. Shukla et al. in [89] studied the approximate controllability for the following semilinear FES of order $\alpha \in (1,2]$ with infinite delay based on the sequential approach:

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + Bu(t) + f(t, x_t), \quad t \in [0,b], \\ x(t) = x_0 = \phi \in \mathfrak{B}, \quad t \in (-\infty, 0]. \\ x'(0) = \psi \in Y, \end{cases} \tag{4.10}$$

where A is the infinitesimal generator of strongly continuous cosine family $\{C_\alpha(t)\}_{t\geq 0}$ on the Banach space Y and A belongs to $C^\alpha(Y; M, \omega)$ [74]. State $x(t)$ takes values in Y. $B : U \to Y$ is a bounded linear operator, U is a Banach space. The history function $x_t : (-\infty, 0] \to Y$ which is defined as $x_t(\theta) = x(t+\theta)$ belongs to $\mathfrak{B}$ for $t \geq 0$, where $\mathfrak{B}$ is a $\mathcal{C}_h$-class abstract phase space described axiomatically, and maps $(-\infty, 0]$ into Y endowed with a seminorm $\|\cdot\|_{\mathfrak{B}}$ which satisfies some necessary axioms.

Theorem 4.4. *([89]) Assume the following conditions hold:*

(i) The nonlinear function $f(t,x) : [0,b] \times \mathfrak{B} \to Y$ satisfies the Caratheodory condition and there exists a positive constant L such that

$$\|f(t,x) - f(t,y)\|_Y \leq L\|x-y\|_{\mathfrak{B}}, \ \forall x, y \in \mathfrak{B}, \ t \in [0,b].$$

(ii) For any given $\varepsilon > 0$ and $p \in L^2([0,b], Y)$, there exists some $u \in U$ such that $\|\mathcal{L}p - \mathcal{L}Bu\|_Y < \varepsilon$, where $\mathcal{L}p = \int_0^b (I^{\alpha-1}C_\alpha)(b-s)p(s)ds$, for $p \in L^2([0,b], Y)$.

(iii) $\|Bu\|_{L^2([0,b],Y)} \leq \lambda \|p\|_{L^2([0,b],Y)}$, where λ is a positive constant independent of p.

(iv) The constant λ satisfies $\dfrac{Mb^q}{\Gamma(q)}\lambda L \ exp(M\dfrac{b^q L}{\Gamma(q)}) < 1$.

Then, the fractional infinite time delay system (4.10) is approximately controllable.

It should be pointed out that hypothesis (ii) in Theorem 4.4 plays an important role in ensuring the approximate controllability for correponding linear system. Authors in [89] defined a solution mapping φ from $L^2([0,b], Y)$ to $C([0,b], Y)$ as $(\varphi Bu)(t) = x_t(Bu)$ to deduce $\overline{K_b(0)} = Y$. This is crucial to derive the approximate controllability of system (4.10) provided that $\overline{K_b(0)} \subset \overline{K_b(f)}$. Hypothesis on nonlinearity and construction of the sequence of control are similar to that in [44].

5. **Controllability of systems with impulsive effects.** In real applications, a system (such as signal processing systems, computer networks, automatic control systems, and telecommunications) is often affected by abrupt changes and instantaneous disturbances at a certain moment. These systems are often described by impulsive differential systems which contain some continuous-time differential equations and some jumping operators. Since impulsive differential systems involved in piecewise continuous function spaces, some properties of continuous function spaces may not be applicable if they are extended to piecewise continuous spaces, such as Ascoli-Arzela theorem and the properties of noncompactness measure. For the sake of describing the evolution processes more reasonably and accurately, the influence of abrupt changes and instantaneous disturbances must be fully considered.

In many cases, impulse phenomenon and memory effect cross each other in evolution processes. For instance, impulsive system is suitable to describe the population model which is suddenly affected by disease and famine. However, the current number of population is closely related to the previous population base, gender ratio and age structure. These phenomena with memory effect can be described more precisely by fractional impulsive differential systems. Hence, it is believed that fractional differential equations with impulse and fractional impulsive control systems have great research significance and wide application background.

With regard to the study on controllability of fractional impulsive evolution equations, the most important step is to obtain the existence of mild solutions to the considered systems. The first systematic investigation for the mild solutions of fractional impulsive evolution equations was made by Mophou [67]. Although this type of mild solution has been quoted by many scholars, it is not suitable for their considered systems. The main reason is that these definitions based on that in [67] do not consider its memory and heredity. In recent years, some mathematicians have pointed out this error in the comments of [87].

Based on this, Z. Liu et al. [61] considered the following nonlinear fractional impulsive evolution systems:

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in J = [0, b] \setminus \{t_1, t_2, \cdots, t_k\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad i = 1, 2, \cdots, k, \\ x(0) = x_0, \end{cases} \tag{5.1}$$

where $q \in (0, 1]$, state $x(t) \in X$, control function $u \in L^2(J, U)$, and X, U are Banach spaces. A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on Banach space X. $f : J \times X \to X$ is a given X-value function. $I_i : X \to X$ is continuous, $0 = t_0 < t_1 < \cdots < t_i < \cdots < t_k < t_{k+1} = b$, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, $x(t_i^+)$ and $x(t_i^-)$ denote the right and the left limits of $x(t)$ at $t = t_i$ $(i = 1, 2, \cdots, k)$. They derived the mild solution of fractional impulsive system (5.1) satisfying

$$x(t) = \begin{cases} \mathcal{T}(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)(f(s, x(s)) + Bu(s))ds, \ t \in [0, t_1], \\ \mathcal{T}(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)(f(s, x(s)) + Bu(s))ds \\ \quad + \mathcal{T}(t-t_1)I_1(x(t_1^-)), \ t \in (t_1, t_2], \\ \quad \vdots \\ \mathcal{T}(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)(f(s, x(s)) + Bu(s))ds \\ \quad + \sum_{i=1}^k \mathcal{T}(t-t_i)I_i(x(t_i^-)), \ t \in (t_k, b]. \end{cases} \tag{5.2}$$

In addition to the hypotheses of Theorem 3.2 in [101], authors in [61] supposed that impulsive functions I_i $(i = 1, 2, \cdots, k)$ satisfy Lipschitz continuity, and defined a piecewise control function in view of (5.2). The eaxct controllability result of system (5.1) is established by using Krasnoselskii's fixed-point theorem.

Also by using the similar techniques as in [61] and with the Lipschitz continuity imposed on nonlocal terms, [79] and [115] further studied the exact controllability of the following fractional impulsive differential and integro-differential evolution equations with nonlocal conditions, respectively:

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in J = [0, b] \setminus \{t_1, t_2, \cdots, t_k\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad i = 1, 2, \cdots, k, \\ x(0) + g(x) = x_0, \end{cases} \tag{5.3}$$

and

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + Bu(t) + f(t, x(t), (Hx)(t)), \quad t \in J = [0, b] \setminus \{t_1, t_2, \cdots, t_k\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad i = 1, 2, \cdots, k, \\ x(0) + g(x) = x_0. \end{cases} \tag{5.4}$$

As for the case $g(x) \equiv 0$, similar results of exact controllability for system (5.4) was obtained in [81].

Recently, authors in [17] considered the following similar FESs with nonlocal initial condition and impulsive effects:

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in J = [0, b] \setminus \{t_1, t_2, \cdots, t_k\}, \\ \Delta x(t_i) = I_i(x(t_i)), \quad i = 1, 2, \cdots, k, \\ x(0) = g(x), \end{cases} \tag{5.5}$$

The approximate controllability was established by using Schaefer's fixed-point theorem, Darbo's fixed-point theorem and the theory of noncompactness measure. It should be further pointed out that [17] adopted the approximate technique to deal with system (5.5), which is different from the conventional method that certain fixed point theorem is applied directly to the corresponding integral operator. This method can overcome the difficulty caused by nonlocal conditions and reduce the assumptions on the impulsive terms effectively. In addition, unlike the hypotheses of [13, 79, 84, 115] in which the nonlinearity f and nonlocal function g were assumed to be Lipschitz continuous, the controllability result in [17] was obtained without the Lipschitz continuous or compact conditions on f and g.

At present, most of the fractional impulsive controllability results are involved in Caputo fractional derivative. However, there are still a few literatures on the controllability of FESs with Riemann-Liouville derivatives [54, 60]. In [54], Z. Liu et al. studied the approximate controllability of a class of fractional impulsive neutral evolution equations with Riemann-Liouville derivatives under the assumption

$$L_{ag}\|A^{-1}\| + \frac{L_{ag} M b^\alpha}{\Gamma(\alpha+1)} + \sum_{i=1}^{k} M d_i (t - t_i)^{\alpha - 1} + \frac{L_f M b^\alpha}{\Gamma(1+\alpha)} < 1. \tag{5.6}$$

Unfortunately, this assumption is unreasonable due to the possibility that operator A^{-1} may be unbounded and $(t - t_i)^{\alpha-1}$ may tend to infinity as $t \to t_i^+$.

In addition, Z. Liu et al. in [60] considered the following fractional impulsive evolution systems with Riemann-Liouville derivatives:

$$\begin{cases} {}^{RL} D_t^q x(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in J = [0, b] \setminus \{t_1, t_2, \cdots, t_k\}, \\ \triangle I_{t_i}^{1-q} x(t_i) = G_i(t_i^-, x(t_i^-)), \quad i = 1, 2, \cdots, k, \\ I_t^{1-q} x(t)|_{t=0} = x_0 \in X, \end{cases} \tag{5.7}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\ge 0}$ on Banach space X. $f, G_i : J \times X \to X$ are given continuous functions. $\triangle I_{t_i}^{1-q} x(t_i) = I_{t_i^+}^{1-q} x(t_i^+) - I_{t_i^-}^{1-q} x(t_i^-) = \Gamma(q)(\lim_{t \to t_i^+} (t-t_i)^{1-q} x(t) - \lim_{t \to t_i^-} (t-t_i)^{1-q} x(t))$. $I_{t_i^+}^{1-q} x(t_i^+)$ and $I_{t_i^-}^{1-q} x(t_i^-)$ denote the right and the left limits of $I_t^{1-q} x(t)$ at $t = t_i$ $(i = 1, 2, \cdots, k)$. The control u is in $L^p(J, U), p > \frac{1}{q}$ and U is a Banach space. B is a linear operator from $L^p(J, U)$ into $L^p(J, X)$.

In order to avoid the error (5.6), authors in [60] established some new sufficient conditions of approximate controllability for system (5.7). By introducing Banach space $PC_{1-q}(J,X) = \{x : (t-t_i)^{1-q}x(t) \in C((t_i,t_{i+1}],X)$ *and* $\lim\limits_{t\to t_i^+}(t-t_i)^{1-q}x(t)$ *exists*$, i = 0,1,2,\cdots,k\}$ with a weighted norm

$$\|x\|_{PC_{1-q}} = \max\{\sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{1-q}\|x(t)\|_X : \ i = 0,1,2,\cdots,k\},$$

they derived the existence and uniqueness of PC_{1-q}-mild solutions for systems (5.7) under the Lipschitz continuity exerted on the nonlinearity and impulsive items. Noticing that the semigroup $T(t)$ is differentiable on X, system (5.7) is proved to be approximately controllable by using iterative and approximate techniques. The controllability results are presented as below:

Theorem 5.1. *([60]) Suppose that the following hypotheses hold:*

(i) The function $f(\cdot,x) : J \to X$ *is measurable for all* $x \in X$ *and* $\|f(\cdot,0)\| \in L^p(J,\mathbb{R}^+)$. *Moreover, there exists a constant* $L > 0$ *such that*

$$\|f(t,x) - f(t,y)\| \le L(t-t_i)^{1-q}\|x-y\|_X$$

for a.e. $t \in (t_i,t_{i+1}], i = 0,1,2,\cdots,k,$ *and* $x,y \in X$.

(ii) There exist constants $0 < d_i < \Gamma(q)/(2M\sum\limits_{j=1}^{i}(t_j - t_{j-1})^{q-1}),\ i = 1,2,\cdots,k+1,$ $(M = \sup\limits_{t\in[0,+\infty)}\|T(t)\|)$ *such that* $\|G_i(t_i^-,x) - G_i(t_i^-,y)\| \le d_i\|x-y\|_X$, *for all* $x,y \in X$.

(iii) For any $\varepsilon > 0$ *and* $\varphi \in L^p(J,X)$, *there exists a control* $u \in L^p(J,U)$ *such that*

$$\|\mathcal{G}\varphi - \mathcal{G}Bu\|_X < \varepsilon,\ \|Bu\|_{L^p(J,X)} < N\|\varphi\|_{L^p(J,X)},$$

where

$$\mathcal{G}h = \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)h(s)ds,\ h(\cdot) \in L^p(J,X),$$

N *is a positive constant satisfying*

$$N(1-D^*)E_q(MLb)\frac{Mb^{1-\frac{1}{p}}}{\Gamma(q)}\left(\frac{p-1}{pq-1}\right)^{1-\frac{1}{p}} < 1,$$

and $D^* = \max\{[M\sum\limits_{j=1}^{i}d_j(t_j - t_{j-1})^{q-1})E_q(MLb)]/\Gamma(q) : i = 1,2,\cdots,k\}$.

Then, system (5.7) is approximately controllable on J, *if the* C_0*-semigroup* $T(t)$ *generated by* A *is differentiable on Banach space* X.

6. **Controllability results via resolvent operator.** Generally speaking, the approach of resolvent operator can be applied to the inhomogeneous equations to derive various variation of parameters formulas, and also it can lead to improved perturbation results and stronger properties of the variation of parameter formulas [74]. For equations with unbounded operators in infinite-dimensional space, the resolvent operator is more appropriate because it is a direct generalization of C_0-semigroups and cosine families. From the viewpoint of dependent parameters, it can be generally classified into two types: univariate resolvent operator [39, 68, 129] and bivariate resolvent operator [7, 13, 111].

Definition 6.1. ([7]) Let $A : D(A) \subseteq E \to E$ be a closed linear operator defined on a Banach space E and $\alpha, \beta > 0$. Let $\rho(A)$ be the resolvent set of A, we say that A is the generator of an (α, β)-resolvent operator family, if there exist $\omega \geq 0$ and a strongly continuous function $S_{\alpha,\beta}$: $[0, +\infty) \to \mathcal{L}(E)$ such that $S_{\alpha,\beta}(t)$ is exponentially bounded, $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$, and for all $u \in E$,

$$\lambda^{\alpha-\beta}(\lambda^\alpha I - A)^{-1}u = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}(t)udt, \ Re\lambda > \omega.$$

In this case, $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is called the (α, β)-resolvent operator family (also called (α, β)-resolvent operator function) generated by A.

Definition 6.2. ([74]) The bounded linear operator $\{S(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ on X is called a resolvent operator of the following integral equation:

$$u(t) = \int_0^t \sigma(t-s)Au(s)ds, \ t \geq 0,$$

where the scalar kernel $\sigma \in L^1_{loc}(\mathbb{R}^+)$ and $\sigma \not\equiv 0$, provided that:

(i) $S(t)$ is strongly continuous, $S(0) = I$;
(ii) $S(t)$ commutes with A, that is $S(t)\mathcal{D} \subset \mathcal{D}$, $AS(t)u = S(t)Au$ for each $u \in \mathcal{D}$ and each $t \geq 0$;
(iii) $S(t)u = u + \int_0^t \sigma(t-s)AS(s)uds$, for all $u \in \mathcal{D}$, $t \geq 0$.

For the special cases $\sigma(t) = 1$ and $\sigma(t) = t$, the resolvent operator $S(t)$ reduces to the C_0-semigroup e^{At} $(T(t))$ and the cosine operator family $Co(t)$, respectively. Denote $S(t)$ by $S_\alpha(t)$ with $\alpha \in (0, 2]$. It is not difficult to see that $S_{\alpha,1}(t) = S_\alpha(t)$, $S_1(t) = T(t)$ and $S_2(t) = Co(t)$.

Based on the subordination principle, if A is a infinitesimal generator of a C_0-semigroup $T(t)$ $(S_1(t))$ being exponentially bounded, then it must be a infinitesimal generator of a resolvent operator $S_\alpha(t)$ with $\alpha \in (0, 1)$. But the converse may not be true. Consequently, all the classic C_0-semigroup-based controllability results in the existing literature can be also obtained similarly via resolvent operator $S_\alpha(t)$ with $\alpha \in (0, 1)$. More specifically, the resolvent operator theory plays an important role on those FESs that can not generate a C_0-semigroup but admit a resolvent operator. For more details of resolvent operator theory, please refer to [74] and references therein.

Theoretically, for the evolution equations with order of fractional derivative $\alpha \in (0, 2]$, resolvent operator theory can be fully utilized and it can play an effective role in the investigation of controllability problems. But compared with classic C_0-semigroup-based fractional controllability consequences, the results obtained by resolvent operators is relatively few. Considering the case that the infinitesimal generator A defined on a dense domain is possiblely dependent on more than one variable, Debbouche and Baleanu [13] introduced a new concept called (α, u)-resolvent operator, and studied the exact controllability for a class of fractional nonlocal impulsive integro-differential control system by utilizing the fixed point theory and the properties of (α, u)-resolvent operator. It should be pointed out that all variables of the systems in [13] need to satisfy Lipschitz continuity condition, which is limited for practical problems. Under a new introduced concept of exact controllability and assumptions that f satisfies continuity instead of Lipschitz continuity, [123] obtained an interesting controllability result by using the theory of resolvent operator, fixed point strategy and Kuratowski's measure of noncompactness. Its result is an extension of the existing result of exact controllability to some degree.

Up to now, there are not many papers on controllability problems of FESs with order $\alpha \in (1,2)$. Especially, the resolvent operator theory plays an irreplaceable role in dealing with the evolution equations of order $\alpha \in (1,2)$. P. Muthukumar and K. Thiagu [68] presented some sufficient conditions to obtain the approximate controllability for nonlinear fractional nonlocal neutral impulsive stochastic differential systems of order $\alpha \in (1,2)$ with infinite delay and Poisson jumps by using Lebesgue dominated convergence theorem and the compactness of some operator families. Utilizing resolvent operator theory and fixed point techniques, R. Sakthivel et al. [83] derived some sufficient conditions to ensure the approximate controllability for a class of nonlinear fractional differential systems of order $\alpha \in (1,2)$ with nonlocal conditions. In addition, the same results are obtained for such systems with infinite time delay. Chang et al. [7] introduced (α,β)-resolvent operator $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ in the sense of Definition 6.1 to investigate the approximate controllability for two classes of FESs of Sobolev type with order $\alpha \in (1,2)$ via the norm continuity and compactness about $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ for suitable $\alpha, \beta > 0$.

Recently, K. Li et al. in [39] considered the fractional systems (3.1), (3.4) together with the initial condition

$$x'(0) = y_0, \tag{6.1}$$

An appropriate definition of mild solutions via Laplace transformation was introduced. Else, they derived some sufficient conditions to ensure the exact controllability for nonlocal problem (3.1), (3.4) and (6.1) by using Sadovskii's fixed-point theorem and vector-valued operator theory under the following hypotheses on nonlinearity together with some conditions imposed on operators B and W.

(i) $f : [0,b] \times X \to X$ satisfies the Carathéodory condition, i.e. $f(\cdot, x)$ is measurable for all $x \in X$, and $f(t,\cdot)$ is continuous for a.e. $t \in [0,b]$.
(ii) $f : [0,b] \times X \to X$ is compact.
(iii) There exists a function $l_f \in L^1([0,b]; R_+)$ such that

$$\| f(t,x) - f(t,z) \| \leq l_f(t) \| x - z \|, \quad x, z \in X.$$

Following [39], authors in [89] and [88] addressed the semilinear fractional control system of order $\alpha \in (1,2]$ with infinite delay and finite delay, respectively. They derived some sufficient conditions to ensure approximate controllability of the proposed systems. What's interesting is that the sequential method was used in [89] instead of the fixed point strategy.

For most practical problems, it is especially effective to solve the abstract differential equations of second order directly rather than transforming them into first-order systems. Travis and Webb [97] established the theory of strongly continuous sine and cosine operator families, in which they analyzed the advantages of such method. But for fractional controllability problems of order $\alpha \in (1,2]$, the major difficulty is how to give the appropriate definition of mild solutions for considered systems since the Mainardi's Wright-type function is only well defined for $\alpha \in (0,1)$. Fortunately, this problem was well solved by Y. Zhou and J. He [129]. They derived a new concept of mild solutions for the objective system (3.1), (3.2) and (6.1) by using the Laplace transformation and Mainardi's Wright-type function based on the theory of sine and cosine operator families:

Definition 6.3. ([129]) For each $u \in L^2(J;U)$, a mild solution of the system (3.1), (3.2) and (6.1), it means that the function $x \in C(J;X)$ satisfies the following integral equation

$$x(t) = C_q(t)x_0 + K_q(t)x_1 + \int_0^t (t-s)^{q-1} P_q(t-s) f(s, x(s)) ds$$

$$+ \int_0^t (t-s)^{q-1} P_q(t-s) B u(s) ds,$$

for each $t \in J$, where $q = \frac{\alpha}{2} \in (\frac{1}{2}, 1)$ and

$$C_q(t) = \int_0^\infty M_q(\theta) C(t^q\theta) d\theta,\ K_q(t) = \int_0^t C_q(s) ds,\ P_q(t) = \int_0^\infty q\theta M_q(\theta) S(t^q\theta) d\theta,$$

$M_q(t)$ denotes Mainardi's Wright-type function, $C(t)$ and $S(t)$ represents strongly continuous cosine operator family and sine operator family, respectively.

Their main results are as follows:

Theorem 6.4. *([129]) Assume the following conditions holds:*

(i) *$f(t,\cdot) : X \to X$ is continuous for a.e. $t \in J = [0,b]$, and $f(\cdot, x) : J \to X$ is strongly measurable for all $x \in X$.*

(ii) *There exists a function $k_f \in L^1(J, \mathbb{R}^+)$ such that*

$$| f(t,x) | \le k_f(t) \varphi(| x |),$$

for each $t \in J$ and for all $x \in X$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function satisfying

$$\liminf_{r \to \infty} \frac{\varphi(r)}{r} = 0.$$

(iii) *The linear operator $B : U \to X$ is bounded, $W : L^2(J,U) \to X$ defined by*

$$Wu = \int_0^b (b-s)^{q-1} P_q(b-s) B u(s) ds$$

has an inverse operator W^{-1} which takes values in $L^2(J,U)/kerW$ and there exist two positive constants $M_1, M_2 > 0$ such that $\| B \|_{L_b(U,X)} \le M_1$ and

$$\|W^{-1}\|_{L_b(X, L^2(J,U)/kerW)} \le M_2.$$

(iv) *For every $t \in J$ and for each $r > 0$, the set $\{P_q(t-s) f(s,x),\ s \in [0,t],\ x \in X,\ \|x\| \le r\}$ is relatively compact in X.*

Then, the evolution system (3.1), (3.2) and (6.1) is exactly controllable on J.

It is observed that the infinitesimal generator A in [129] generates a strongly continuous cosine operator family $C(t)$ (or denotes $S_2(t)$). As a result of subordination principle, it can also generate the resolvent operator $S_\alpha(t)$ associated with A similar to [39] for $\alpha \in (1,2)$, but the converse statement does not necessarily hold. Hence, [129] present a new approach to obtain the exact controllability of FESs with order $\alpha \in (1,2)$. Subsequently, based on the mild solution given in [129], authors in [82] further investigated the approximate controllability for fractional differential evolution equations of order $\alpha \in (1,2)$. Their main results are obtained by using the properties of solution operators deduced from cosine and sine operators families, Schauder's fixed-point theorem and Dhage's fixed-point theorem.

7. **Conclusions and future work.** The controllability is one of the fundamental problems for FESs. This work has provided a comprehensive survey on the elementary results and some recent progress of the controllability for FESs. Exact controllability and approximate controllability of some kinds of FESs are reviewed. Firstly, several basic FESs with classical initial and nonlocal conditions are considered. Some fundamental theory and sufficient conditions are presented to ensure the exact controllability and approximate controllability for such systems. Secondly, FESs with finite time delay and infinite time delay are discussed, respectively. We talked over three different methods for investigating approximate controllability of these time-delay systems. Thirdly, controllability results on some types of FESs with impulsive effects are discussed. At last, exact controllability and approximate controllability for FESs obtained by using resolvent operator theory have been carefully analyzed.

The interested directions of the theoretical study in the future on the controllability for FESs may be as follows:

(i) The systems studied on this topic will be more and more extensive and complicated. For instance, it is valuable to investigate controllability for hybrid FESs with delay, impulse or stochastic factors.
(ii) Besides fixed point theorem, more effective tools for studying controllability of FESs should be further developed.
(iii) Application area of controllability for FESs in reality need to be investigated in depth.

For our subsequent work, the following issues will continue to be focused on:

(1) It is noted that Lipschitz continuity and compact conditions are required in most of existing work on this area. How to relax such conditions imposed on nonlinear terms still need to be well investigated.
(2) Relationship of approximate controllability between FES with impulse and FES without impulse has not been fully described. Further research progress in this field is expected to be well achieved.
(3) When FES is affected by impulse and time delay simultaneously, the investigation of controllability on its nonlocal problem is still insufficient. In particular, to deal with such problems via resolvent operator should be further explored.
(4) Numerical simulation or other applications about the theoretical results, such as digital filters in Digital Signal Processing (DSP), should be paid more attention in the future work.

Acknowledgments. Research supported by the National Natural Science Foundation of China under grant 62073202, the Natural Science Foundation of Shandong Province under grant ZR2020MA007 and a project of Shandong Province Higher Educational Science and Technology Program of China under grant J18KA233.

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Received June 2021; revised August 2021; early access October 2021.

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