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GLOBAL STABILITY OF TRAVELING WAVE FRONTS IN A TWO-DIMENSIONAL LATTICE DYNAMICAL SYSTEM WITH GLOBAL INTERACTION

Cui-Ping Cheng*

School of Science, Shanghai Institute of Technology
Shanghai 201418, China

Ruo-Fan An

School of Mathematics, University of Mining and Technology
Xuzhou, Jiangsu 221008, China

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Abstract. This paper is concerned with the traveling wave fronts for a lattice dynamical system with global interaction, which arises in a single species in a 2D patchy environment with infinite number of patches connected locally by diffusion and global interaction by delay. We prove that all non-critical traveling wave fronts are globally exponentially stable in time, and the critical traveling wave fronts are globally algebraically stable by the weighted energy method combined with the comparison principle and the discrete Fourier transform.

1. **Introduction.** Lattice differential equations (LDEs) are systems of ordinary differential equations with a discrete spatial structure, which can naturally arise in various fields, such as image processing, neural networks, patten recognition and chemical reaction theory. These can be seen in [9, 14, 29, 37] and the references therein. Recently, there is a particular interest on studying the species population living in a patchy environment consisting of all integer nodes, see [7, 8, 34, 35].

Inspired by Bates [1], Chow [9], Weng et al. [34] and many other excellent survey papers, the authors in [7] considered a single-species population with two-age classes distributed over a patchy environment consisting of all integer nodes of a 2D lattice and derived the following system:

$$\begin{aligned}\frac{dw_{k,j}(t)}{dt} = \quad & D_m[w_{k+1,j}(t) + w_{k-1,j}(t) + w_{k,j+1}(t) + w_{k,j-1}(t) - 4w_{k,j}(t)] \\ & -d_m w_{k,j}(t) + \tfrac{\mu}{(2\pi)^2} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q) b(w_{k-l,j-q}(t-r)), \quad (1.1)\end{aligned}$$

where $w_{k,j}(t)$ denote the densities of matured population in the (k,j)-th patch and time $t \geq 0$, D_m and d_m represent the diffusion coefficient and the death rate of the matured population, respectively, $r > 0$ is the maturation time of species.

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* Corresponding author: Cui-Ping Cheng.

$\mu = e^{-\int_0^r d(z)dz}$ and $\alpha = \int_0^r D(z)dz$ represent the impact of the death rate of immature and the effect of the dispersal rate of immature on the mature population, respectively, where $d(z)$ and $D(z)$ are the death rate and diffusion rate of the immature population at age $z \in (0, r)$, respectively. While

$$\beta_\alpha(l) = 2e^{-2\alpha} \int_0^\pi \cos(l\omega_1) e^{2\alpha \cos \omega_1} d\omega_1,$$
$$\gamma_\alpha(l) = 2e^{-2\alpha} \int_0^\pi \cos(l\omega_2) e^{2\alpha \cos \omega_2} d\omega_2,$$

for any $l \in \mathbb{Z}$ and satisfy:

(i): $\beta_\alpha(l) = \beta_\alpha(|\ l\ |)$, $\gamma_\alpha(l) = \gamma_\alpha(|\ l\ |)$ for $l \in \mathbb{Z}$, that is $\beta_\alpha(l), \gamma_\alpha(l)$ is isotropic function for any $\alpha \geq 0$;

(ii): $\frac{1}{2\pi} \sum\limits_{l=-\infty}^{\infty} \beta_\alpha(l) = 1$, $\frac{1}{2\pi} \sum\limits_{l=-\infty}^{\infty} \gamma_\alpha(l) = 1$;

(iii): $\beta_\alpha(l) \geq 0$, $\gamma_\alpha(l) \geq 0$ if $\alpha = 0$ and $l \in \mathbb{Z}$; $\beta_\alpha(l) > 0$, $\gamma_\alpha(l) > 0$, if $\alpha > 0$ and $l \in \mathbb{Z}$.

(iv): $\frac{1}{2\pi} \sum\limits_{l=-\infty}^{\infty} \beta_\alpha(l) e^{\lambda l \cos\theta} = e^{2\alpha(cosh(\lambda \cos\theta)-1)}$, $\frac{1}{2\pi} \sum\limits_{l=-\infty}^{\infty} \beta_\alpha(l) e^{\lambda l \sin\theta} = e^{2\alpha(cosh(\lambda \sin\theta)-1)}$

Assume the birth function b satisfies the following assumptions:

(H1): $b \in C^2(\mathbb{R}^+)$ and $b(w) \leq b'(0)w$ for $w \in \mathbb{R}^+$;

(H2): $\mu b(w) = d_m w$ has only two real roots 0 and w^+, and b is non-decreasing on $[0, w^+]$;

(H3): $0 < b'(w^+) < \frac{d_m}{\mu} < b'(0)$;

(H4): For $w \in (0, w^+)$, $\mu b(w) > d_m w$, $b'(w) \geq 0$ and $b''(w) \leq 0$.

The authors [7] studied the existence of the asymptotic speed of propagation, the existence of monotone traveling waves and the minimal wave and its relation with the asymptotic speed of propagation. Recently, Xu [35] further showed that for any given admissible speed, all the wave profiles propagating toward a fixed direction of (1.1) have the same asymptotic behavior when they approach the limiting states, which plays a very important role in the stability of traveling waves. Meanwhile, in the past decades, there are various surveys focusing on the existence, uniqueness, asymptotic behavior and stability of traveling waves for LDEs and its continuum RDEs([1–4, 6–9, 11, 13, 16, 17, 19–22, 37–39]).

In this paper, we are concerned with the stability of traveling waves of (1.1) under the assumptions $(H1)-(H4)$. In view of the symmetry, we only consider the case $\theta \in [0, \frac{\pi}{2}]$. For fixed $\theta \in [0, \frac{\pi}{2}]$ such that $\tan\theta \in \mathbb{Q}$ and the Cauchy problem (1.1)with initial data

$$w_{k,j}(t)|_{t=s} = w^0_{k,j}(s), \text{for } s \in [-r, 0], k, j \in \mathbb{Z}, \tag{1.2}$$

where

$$\begin{aligned} &w^0_{k,j}(s) \to 0, \text{for all } s \in [-r, 0], \text{as } k\cos\theta + j\sin\theta \to -\infty; \\ &w^0_{k,j}(s) \to w^+, \text{for all } s \in [-r, 0], \text{as } k\cos\theta + j\sin\theta \to \infty, \end{aligned} \tag{1.3}$$

we prove that the global solution $\{w_{k,j}(t)\}_{k,j\in\mathbb{Z}}$ of (1.1) and (1.2) converges exponentially to a traveling wavefront $\phi(k\cos\theta + j\sin\theta + ct)$ for $c > c_*$ (which is the minimal wave speed); while for $c = c_*$ the global solution converges to the traveling solution $\phi(k\cos\theta + j\sin\theta + c_*t)$ algebraically in time, when the initial perturbation around the wave.

As we know, many techniques are developed to investigate the stability of traveling waves such as the spectral analysis method, the weighted energy method ([5, 8, 12, 14, 15, 18, 23–28, 30, 37, 38]) and the comparison principle combining the squeezing technique ([4,6,19–22,31]). Recently, Mei et al. [25] and Huang et al. [15] obtained the *global stability* of traveling wave fronts with noncritical speed and critical speed of nonlocal reaction-diffusion equations via the weighted energy method together with the comparison principle and Green function or Fourier transform. Zhang [38] applied this method to nonlocal LDEs in one dimension. However, there seems to be not much progress on the stability of traveling waves of system (1.1).

Particularly in [8], we only consider the case that the immature population is non-mobile, that is $D(a) = 0$ for $0 < a < r$. In this case $\alpha = 0$, (1.1) reduces to

$$\begin{aligned}\frac{dw_{k,j}(t)}{dt} = \quad & D_m[w_{k+1,j}(t) + w_{k-1,j}(t) + w_{k,j+1}(t) + w_{k,j-1}(t) - 4w_{k,j}(t)] \\ & -d_m w_{k,j}(t) + \mu b(w_{k,j}(t-r)). \end{aligned} \tag{1.4}$$

In monostable case, using weighted energy method, we derived that the Cauchy problem of (1.4) converges to a traveling wavefront when the initial perturbation around the wave is suitably small in a weighted norm. Due to the limitation of the key inequality, the result only holds for $c > \tilde{c} > c_*$.

The outline of this paper is as follows: in Section 2, we introduce some preliminaries, recall the result on the existence of traveling wave fronts of (1.1) and present our main result. Section 3 is devoted to the global stability of traveling wave fronts by using the weighted energy method combined with the semi-discrete Fourier transform.

2. **Preliminary and main result.**

Notations. Let $T > 0$ be a number and X be a Banach space. We denote by $C([0, T]; X)$ the space of the X valued continuous function on $[0, T]$, and by $L^1([0, T]; X)$ the space of the X valued L^1 functions on $[0, T]$. $C > 0$ denotes a generic constant, while $C_k (k = 1, 2, \cdots)$ represents a specific constant. l^∞ is the Banach space:

$$l^\infty = \left\{ c = \{c_{k,j}\}_{k,j\in\mathbb{Z}}, c_{k,j} \in \mathbb{R}; \|c\|_{l^\infty} := \sup_{k,j\in\mathbb{Z}} |c_{k,j}| < \infty \right\},$$

let l^1 denote the Banach space:

$$l^1 = \left\{ c = \{c_{k,j}\}_{i\in\mathbb{Z}}, c_{k,j} \in \mathbb{R}; \|c\|_{l^1} := \sum_{k,j\in\mathbb{Z}} |c_{k,j}| < \infty \right\},$$

and denote by l^2 the Hilbert space

$$l^2 = \left\{ c = \{c_{k,j}\}_{k,j\in\mathbb{Z}}, c_{k,j} \in \mathbb{R}; \|c\|_{l^2} := \left(\sum_{k,j\in\mathbb{Z}} |c_{k,j}|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

Further, $l^p_w (p \geq 1)$ denotes the weighted l^p-space for a weight $0 < w(x) \in C(\mathbb{R})$ with the norm

$$\|c\|_{l^p_w} := \left(\sum_{k,j\in\mathbb{Z}} w(k\cos\theta + j\sin\theta)|c_{k,j}|^p \right)^{\frac{1}{p}}.$$

For any $v = \{v_{k,j}\}_{k,j\in\mathbb{Z}} \in l^2$, its semi-discrete Fourier transform ([10,32]) is defined as

$$F[v](\omega) = \hat{v}(\omega) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} e^{-\mathbf{i}(k\omega_1+j\omega_2)} v_{k,j} \quad \omega = (\omega_1, \omega_2) \in [-\pi, \pi]^2,$$

and the inverse Fourier transform is given by

$$F^{-1}[\hat{v}] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{\mathbf{i}(k\omega_1+j\omega_2)}\hat{v}(\omega)d\omega_1 d\omega_2 \quad k, j \in \mathbb{Z},$$

where $\mathbf{i}$ is the imaginary unit.

A traveling wave of system (1.1) is $w_{k,j}(t) = \phi(x)$, $x = k\cos\theta + j\sin\theta + ct$ satisfying the following equations:

$$\begin{aligned} c\frac{d\phi(x)}{dx} &= D_m(\phi(x+\cos\theta) + \phi(x-\cos\theta) + \phi(x+\sin\theta) + \phi(x-sin\theta) - 4\phi(x)) \\ &\quad -d_m\phi(x) + \tfrac{\mu}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)b(\phi(x - l\cos\theta - q\sin\theta - cr)), \end{aligned} \tag{2.1}$$

where $c > 0$ is called the wave speed, θ is the direction of propagation and ϕ is the wave profile , subject to the boundary condition

$$\lim_{x\to-\infty}\phi(x) = 0, \lim_{x\to\infty}\phi(x) = w^+. \tag{2.2}$$

Denoting the characteristic equation at the trivial equilibrium $w^0 = 0$ by $\Delta(\lambda, c; \theta)$, we obtain

$$\begin{aligned} \Delta(\lambda, c; \theta) &= D_m(e^{\lambda\cos\theta} + e^{-\lambda\cos\theta} + e^{\lambda\sin\theta} + e^{-\lambda\sin\theta} - 4) - c\lambda - d_m \\ &\quad + \frac{\mu b'(0)}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)e^{-(\lambda l\cos\theta+\lambda q\sin\theta+\lambda cr)}. \end{aligned}$$

It is easy to see that $\Delta(\lambda, c; \theta)$ is well-defined and satisfies the following properties.

Lemma 2.1. *[7, Lemma 4.2] There exist a pair of c_* and λ_* such that*

- **(i):** $\Delta(\lambda_*, c_*; \theta) = 0$, $\frac{\partial}{\partial\lambda}\Delta(\lambda_*, c_*; \theta) = 0$;
- **(ii):** *for $0 < c < c_*$, and any $\lambda > 0$, $\Delta(\lambda, c; \theta) > 0$;*
- **(iii):** *for any $c > c_*$, the equation $\Delta(\lambda_*, c_*; \theta) = 0$ has two positive real solutions $0 < \lambda_1(c) < \lambda_2(c)$, such that $\Delta(\cdot, c; \theta) < 0$ in $(\lambda_1(c), \lambda_2(c))$, $\Delta(\cdot, c; \theta) > 0$ in $\mathbb{R}\backslash[\lambda_1(c), \lambda_2(c)]$.*

The existence of traveling wave fronts for (2.1) with the boundary condition (2.2) can be easily verified by using the monotone iteration technique combined with the sub-sup solutions, see [7].

Lemma 2.2. *[7, Theorem 5.4] Assume* (H1) − (H4) *hold. Then there exists $c_* > 0$ such that for every $c \geq c_*$, (2.1) has a monotone traveling wave $U : \mathbb{R} \to \mathbb{R}$ satisfying the boundary condition $\lim_{x\to-\infty}\phi(x) = 0$, $\lim_{x\to+\infty}\phi(x) = w^+$; For any $c \in (0, c_*)$, (2.1) has no nontrivial traveling wave solution satisfying $\phi(x) \in [0, w^+]$ for all $x \in \mathbb{R}$.*

Recently, Xu [35] derived the asymptotic behavior of traveling waves of (1.1), which is the key premise.

Lemma 2.3. *[35, Theorem 2.1] Assume* (H1) − (H4) *hold. For any $c \geq c_*$, let (c, ϕ) be a solution of (2.1) with boundary conditions (2.2), then*

$$\lim_{x\to-\infty}\phi(x) = (m - x)^l e^{\lambda_1(c)x}$$

for some $m \in \mathbb{R}$, where $l = 0$ when $c > c^*$ and $l = 1$ when $c = c^*$.

Now, define a weight function $\nu(x)$ as

$$\nu(x) = e^{-\lambda_*(x-x_*)}, \tag{2.3}$$

where λ_* is given in Lemma 2.1 and $x_* > 0$ is chosen to be sufficiently large such that Eq.(3.19) hold.

We now state our main stability result in this paper.

Theorem 2.4. *Assume that* (H1) – (H4) *hold. For a given traveling wave front* $\phi(x)$ *of* (1.1) *with speed* $c \geq c_*$, *if the initial data satisfies* $0 \leq w^0_{k,j}(s) \leq w^+$ *and condition* (1.3) *and the initial perturbation* $w^0_{k,j}(s) - \phi(k\cos\theta + j\sin\theta + cs) \in C^0\left([-r,0], l^1_\nu(\mathbb{Z}^2)\right)$ *and*

$$\frac{d}{ds}\left(w^0_{k,j}(s) - \phi(k\cos\theta + j\sin\theta + cs)\right) \in L^1\left([-r,0], l^1_\nu(\mathbb{Z}^2)\right),$$

then the unique solution $w_{k,j}(t)$ *of the corresponding Cauchy problem of* (1.1) *with the initial value* $w_{k,j}(s) = w^0_{k,j}(s)$ *exists globally and satisfies*

$$0 \leq w_{k,j}(t) \leq w^+, k, j \in \mathbb{Z}, t \geq 0,$$

and

$$w_{k,j}(t) - \phi(k\cos\theta + j\sin\theta + ct) \in C^0\left([0,+\infty), l^1_\nu\right).$$

Furthermore, for $0 < \kappa \leq 2$, *when* $c > c_*$, *the solution* $w_{k,j}(t)$ *converges to the traveling wave fronts* $\phi(k\cos\theta + j\sin\theta + ct)$ *exponentially,*

$$\sup_{k,j\in\mathbb{Z}} |w_{k,j}(t) - \phi(k\cos\theta + j\sin\theta + ct)| \leq C_1(1+t)^{-\frac{2}{\kappa}} e^{-\mu_0 t}, t \geq 0,$$

for some constant μ_0; *when* $c = c_*$, *the solution* $w_{k,j}(t)$ *converges to the traveling wavefronts* $\phi(k\cos\theta + j\sin\theta + c_*t)$ *algebraically,*

$$\sup_{k,j\in\mathbb{Z}} |w_{k,j}(t) - \phi(k\cos\theta + j\sin\theta + c_*t)| \leq C_3(1+t)^{-\frac{2}{\kappa}}, t \geq 0.$$

3. **Stability.** We first consider the following initial value problem:

$$\begin{cases} \dfrac{dw_{k,j}(t)}{dt} =& D_m[w_{k+1,j}(t) + w_{k-1,j}(t) + w_{k,j+1}(t) + w_{k,j-1}(t) - 4w_{k,j}(t)] \\ & - d_m w_{k,j}(t) + \dfrac{\mu}{(2\pi)^2}\displaystyle\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)b(w_{k-l,j-q}(t-r)), \\ w_{k,j}(s) =& w^0_{k,j}(s) \in C^0\left([-r,0],[0,w^+]\right), \end{cases} \tag{3.1}$$

where $k, j \in \mathbb{Z}$, $t > 0$ and $s \in [-r, 0]$. Since our analysis in this paper relies on the comparison principle, we now state the definition of super-sub solutions of (1.1) as follows:

Definition 3.1. A sequence of continuous differentiable functions $\{w_{k,j}(t)\}_{k,j\in\mathbb{Z}}$, $t \in [-r, l), l > 0$ is called supersolution (subsolution) of (1.1) on $[0, l]$, if

$$\frac{dw_{k,j}(t)}{dt} \geq (\leq) \quad D_m[w_{k+1,j}(t) + w_{k-1,j}(t) + w_{k,j+1}(t) + w_{k,j-1}(t) - 4w_{k,j}(t)]$$
$$-d_m w_{k,j}(t) + \frac{\mu}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)b(w_{k-l,j-q}(t-r)). \tag{3.2}$$

Theorem 3.2. *For any given function*

$$W^0 = \{w^0_{k,j}\}_{k,j\in Z}, \quad w^0_{k,j} \in C^0\left([-r,0],[0,w^+]\right), k,j\in\mathbb{Z}$$

and $w^0_{k,j}(s) - \phi(k\cos\theta + j\sin\theta + cs) \in C^0\left([-r,0], l^1_\nu(\mathbb{Z}^2)\right)$, *then* (1.1) *has a unique solution* $W(t) = \{w_{k,j}(t)\}_{k,j\in Z}$ *with* $w_{k,j} \in C^0([-r,\infty),[0,w^+])$,*and* $w_{k,j}(t) - \phi(k\cos\theta + j\sin\theta + ct) \in C^0\left([0,+\infty), l^1_\nu(\mathbb{Z})\right)$. *Furthermore, for any pair of sup-subsolution* $w^+_{k,j}(t)$ *and subsolution* $w^-_{k,j}(t)$ *of* (3.1) *on* $[0,\infty)$ *with* $0 \le w^-_{k,j}(t), w^+_{k,j}(t) \le w^+$, $t\in[-r,\infty)$ *and* $w^-_{k,j}(s) \le w^+_{k,j}(s)$, *for* $s\in[-r,0]$, *there hold* $0 \le w^-_{k,j}(t) \le w^+_{k,j}(t) \le w^+$ *for* $t\in[0,\infty)$.

Note that (3.1) is equivalent to

$$\begin{cases} w_{k,j}(t) = e^{-\delta t}w_{k,j}(0) + \displaystyle\int_0^t e^{-\delta s}\left(G_w(s) + H_w(s)\right)ds, \\ w_{k,j}(t) = w^0_{k,j}(s) \in C^0\left([-r,0],[0,1]\right), \end{cases}$$

where $\delta = 4D_m + d_m$,

$$G_w(s) = D_m\left[w_{k+1,j}(s) + w_{k-1,j}(s) + w_{k,j+1}(s) + w_{k,j-1}(s)\right],$$

and

$$H_w(s) = \frac{\mu}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)b(w_{k-l,j-q}(s-r)).$$

Theorem 3.2 can be verified using an argument in [7, Theorem 3.1] and [33, Lemma 3.2] and we omit it here.

Let

$$w^+_{k,j}(s) = \max\left\{w^0_{k,j}(s), \phi(k\cos\theta + j\sin\theta + cs)\right\},$$
$$w^-_{k,j}(s) = \min\left\{w^0_{k,j}(s), \phi(k\cos\theta + j\sin\theta + cs)\right\},$$

for $s\in[-r,0]$ and $k,j\in\mathbb{Z}$. It follows that

$$0 \le w^-_{k,j}(s) \le w^0_{k,j}(s) \le w^+_{k,j}(s) \le w^+, s\in[-r,0],$$

and

$$0 \le w^-_{k,j}(s) \le \phi(k\cos\theta + j\sin\theta + cs) \le w^+_{k,j}(s) \le w^+, s\in[-r,0]. \tag{3.3}$$

Denote $w^-_{k,j}(t)\left(w^+_{k,j}(t)\right)$ as the corresponding solutions of Eq.(3.1) with respect to the above mentioned initial data $w^-_{k,j}(s)\left(w^+_{k,j}(s)\right)$, i.e.

$$\begin{cases} \dfrac{dw^\pm_{k,j}(t)}{dt} = D_m[w^\pm_{k+1,j}(t) + w^\pm_{k-1,j}(t) + w^\pm_{k,j+1}(t) + w^\pm_{k,j-1}(t) - 4w^\pm_{k,j}(t)] \\ \qquad\qquad - d_m w^\pm_{k,j}(t) + \dfrac{\mu}{(2\pi)^2}\displaystyle\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)b(w^\pm_{k-l,j-q}(t-r)), t\ge 0, \\ w^\pm_{k,j}(s) = w^\pm_{k,j}(s), s\in[-r,0]. \end{cases}$$

By the comparison principle, we have

$$\begin{aligned} &0 \le w^-_{k,j}(t) \le w_{k,j}(t) \le w^+_{k,j}(t) \le w^+, t\ge 0, \\ &0 \le w^-_{k,j}(t) \le \phi(k\cos\theta + j\sin\theta + ct) \le w^+_{k,j}(t) \le w^+, t\ge 0. \end{aligned} \tag{3.4}$$

Thus

$$w^-_{k,j}(t) - \phi(x) \le w_{k,j}(t) - \phi(x) \le w^+_{k,j}(t) - \phi(x), \tag{3.5}$$

where $x = k\cos\theta + j\sin\theta + ct$. Now,we can prove the stability of traveling wavefronts in three steps.

Step 1. The convergence of $w^+_{k,j}(t)$ to $\phi(k\cos\theta + j\sin\theta + ct)$.

$$\begin{cases} z_{k,j}(t) = w^+_{k,j}(t) - \phi(k\cos\theta + j\sin\theta + ct), t \ge 0 \\ z^0_{k,j}(s) = w^+_{k,j}(s) - \phi(k\cos\theta + j\sin\theta + cs), s \in [-r, 0], \end{cases} \tag{3.6}$$

According to (3.3) and (3.4), we have $z^0_{k,j}(s) \ge 0$, and $z_{k,j}(t) \ge 0$. By simple calculation, $z_{k,j}(t)$ satisfies

$$\begin{cases} \dfrac{dz_{k,j}(t)}{dt} - D_m[z_{k+1,j}(t) + z_{k-1,j}(t) + z_{k,j+1}(t) + z_{k,j-1}(t) - 4z_{k,j}(t)] + d_m z_{k,j}(t) \\ \quad - \dfrac{b'(0)\mu}{(2\pi)^2} \displaystyle\sum_{l,q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q) z_{k-l,j-q}(t-r) \\ = \dfrac{\mu}{(2\pi)^2} \displaystyle\sum_{l,q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q) \left[b(z_{k-l,j-q}(t-r) + \phi) - b(\phi) - b'(\phi) z_{k-l,j-q}(t-r)\right] \\ \quad + \dfrac{\mu}{(2\pi)^2} \displaystyle\sum_{l,q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q)[b'(\phi) - b'(0)] z_{k-l,j-q}(t-r) \\ = \dfrac{\mu}{(2\pi)^2} \displaystyle\sum_{l,q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q) m_{k,j}(t-r) + \dfrac{\mu}{(2\pi)^2} \displaystyle\sum_{l,q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q) n_{k,j}(t-r), \\ z_{k,j}(s) = z^0_{k,j}(s), \end{cases} \tag{3.7}$$

where

$$m_{k,j}(t-r) = b(z+\phi) - b(\phi) - b'(\phi) z_{k,j}(t-r),$$
$$n_{k,j}(t-r) = [b'(\phi) - b'(0)] z_{k,j}(t-r),$$

with $\phi = \phi(x - l\cos\theta - q\sin\theta - cr)$. The properity $b''(\cdot) \le 0$ on $[0, w^+]$ leads to

$$b'(\phi) - b'(0) \le 0$$

and

$$m_{k,j}(t-r) = b(z+\phi) - b(\phi) - b'(\phi) z_{k,j}(t-r) = \frac{b''(\widetilde{\phi})}{2!} z^2 \le 0,$$

where $\widetilde{\phi}$ is some function between ϕ and $\phi + z$. Then we get the following inequality,

$$\frac{dz_{k,j}(t)}{dt} - D_m[z_{k+1,j}(t) + z_{k-1,j}(t) + z_{k,j+1}(t) + z_{k,j-1}(t) - 4z_{k,j}(t)] + d_m z_{k,j}(t)$$
$$- \frac{b'(0)\mu}{(2\pi)^2} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q) z_{k-l,j-q}(t-r) \le 0.$$

Let $\bar{z}_{k,j}(t)$ be the solution of the following equation:

$$\begin{cases} \dfrac{dz_{k,j}(t)}{dt} - D_m[z_{k+1,j}(t) + z_{k-1,j}(t) + z_{k,j+1}(t) + z_{k,j-1}(t) - 4z_{k,j}(t)] + d_m z_{k,j}(t) \\ - \dfrac{b'(0)\mu}{(2\pi)^2} \displaystyle\sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q) z_{k-l,j-q}(t-r) = 0, \\ \bar{z}_{k,j}(s) = z^0_{k,j}(s), s \in [-r, 0]. \end{cases} \tag{3.8}$$

Then according to the comparison principle, we have

$$0 \le z_{k,j}(t) \le \bar{z}_{k,j}(t), k, j \in \mathbb{Z}, t > 0. \tag{3.9}$$

Let $z^*_{k,j}(t) := \nu(x)\bar{z}_{k,j}(t)$, then $z^*_{k,j}(t)$ satisfies

$$\begin{cases} \dfrac{dz^*_{k,j}(t)}{dt} - D_m \left[e^{\lambda_* \cos\theta} z^*_{k+1,j}(t) + e^{-\lambda_* \cos\theta} z_{k-1,j}(t) \right. \\ \left. +e^{\lambda_* \sin\theta} z^*_{k,j+1}(t) + e^{-\lambda_* \sin\theta} z^*_{k,j-1}(t) \right] + (c\lambda_* + 4D_m + d_m) z^*_{k,j}(t) \\ = \dfrac{\mu b'(0)}{(2\pi)^2} \displaystyle\sum_{l,q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q) e^{-\lambda_*(k\cos\theta + l\sin\theta + cr)} z^*_{k-l,j-q}(t-r), \\ z^*_{k,j}(s) = e^{-\lambda_*(k\cos\theta + j\sin\theta + cs - x_*)} z^0_{k,j}(s), s \in [-r, 0]. \end{cases} \tag{3.10}$$

By Fourier transformation, one has

$$\begin{aligned} \mathbf{F}\left[e^{\lambda_* \cos\theta} z^*_{k+1,j}(t) \right] &= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} e^{-\mathbf{i}(k\omega_1 + j\omega_2)} e^{\lambda_* \cos\theta} z^*_{k+1,j}(t) \\ &= e^{\lambda_* \cos\theta} e^{\mathbf{i}\omega_1} \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} e^{-\mathbf{i}((k+1)\omega_1 + j\omega_2)} z^*_{k+1,j}(t) \\ &= e^{\lambda_* \cos\theta} e^{\mathbf{i}\omega_1} \widehat{z}^*(t, \omega), \end{aligned}$$

$$\begin{aligned} &\mathbf{F}\left[\frac{b'(0)\mu}{(2\pi)^2} \sum_{l,q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q) e^{-\lambda_*(l\cos\theta + q\sin\theta + cr)} z^*_{k-l,j-q}(t-r) \right] \\ = &\frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} e^{-\mathbf{i}(k\omega_1 + j\omega_2)} \left[\frac{b'(0)\mu}{(2\pi)^2} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q) e^{-\lambda_*(l\cos\theta + q\sin\theta + cr)} z^*_{k-l,j-q}(t-r) \right] \\ = &b'(0) \left\{ \frac{\mu}{(2\pi)^2} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q) e^{-\lambda_*(l\cos\theta + q\sin\theta + cr)} e^{-\mathbf{i}(l\omega_1 + q\omega_2)} \right\} \widehat{z}^*(t-r, \omega), \end{aligned}$$

where $\omega = (\omega_1, \omega_2)$. Taking the semi-discrete Fourier transform to (3.10), we obtain

$$\begin{cases} \dfrac{d\widehat{z}^*(t,\omega)}{dt} + A(\omega)\widehat{z}^*(t,\omega) = B(\omega)\widehat{z}^*(t-r,\omega), \\ \widehat{z}^*(s,\omega) = \widehat{z}^*_0(s,\omega), s \in [-r, 0], \end{cases} \tag{3.11}$$

where

$$A(\omega) = c\lambda_* + 4D_m + d_m - D_m \left[e^{\lambda_* \cos\theta} e^{\mathbf{i}\omega_1} + e^{-\lambda_* \cos\theta} e^{-\mathbf{i}\omega_1} + e^{\lambda_* \sin\theta} e^{\mathbf{i}\omega_2} + e^{-\lambda_* \sin\theta} e^{-\mathbf{i}\omega_2} \right],$$

$$B(\omega) = b'(0) \left\{ \frac{\mu}{(2\pi)^2} \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q) e^{-\lambda_*(l\cos\theta + q\sin\theta + cr)} e^{-\mathbf{i}(l\omega_1 + q\omega_2)} \right\},$$

$$\widehat{z}^*_0(s,\omega) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} e^{-\mathbf{i}(k\omega_1 + j\omega_2)} e^{-\lambda_*(k\cos\theta + j\sin\theta + cs - x_*)} z^0_{k,j}(s).$$

In order to obtain the decay estimates of $z^*_{k,j}(t)$, we need the following lemma.

Lemma 3.3. *Let $x(t)$ be the solution to the following scalar differential equations with delay*

$$\begin{cases} \dfrac{dx(t)}{dt} + k_1 x(t) = k_2 x(t-r), t > 0, r > 0, \\ x(s) = x_0(s), s \in [-r, 0]. \end{cases}$$

Then

$$x(t) = e^{-k_1(t+r)} e_r^{k_3 t} x_0(-r) + \int_{-r}^{0} e^{-k_1(t-s)} e_r^{k_3(t-r-s)} \left[z_0'(s) + k_1 z_0(s)\right] ds,$$

where $k_3 = k_2 e^{k_1 r}$ *and* $e_r^{k_3 t}$ *is the so called delayed exponential function in the form*

$$e_r^{k_3 t} = \begin{cases} 0, & -\infty < t < -r, \\ 1, & -r \le t < 0, \\ 1 + k_3 \dfrac{t}{1!}, & 0 \le t < r, \\ 1 + k_3 \dfrac{t}{1!} + k_3^2 \dfrac{(t-r)^2}{2!}, & r \le t < 2r, \\ \vdots & \vdots \\ 1 + k_3 \dfrac{t}{1!} + k_3^2 \dfrac{(t-r)^2}{2!} + \cdots + k_3^m \left[\dfrac{t-(m-1)r}{m!}\right]^m, & (m-1)r \le t < mr, \\ \vdots & \vdots \end{cases}$$

and $e_r^{k_3 t}$ *is a solution to the following linear homogeneous equation with pure delay*

$$\begin{cases} \dfrac{d}{dt} x(t) = k_3 x(t-r), & t \ge 0, \\ x(s) \equiv 1, & s \in [-r, 0]. \end{cases}$$

Furthermore, it is pointed in [36] that when $k_1 \ge k_2 \ge 0$, there exists a constant $\epsilon_1 = \epsilon_1(r)$ with $0 < \epsilon_1 < 1$ for $r > 0$, and $\epsilon_1 = 1$ for $r = 0$, and $\epsilon_1 = \epsilon_1(r) \to 0^+$ as $r \to +\infty$, such that

$$e^{-k_1 t} e_r^{k_3 t} \le C e^{-\epsilon_1 (k_1 - k_2) t}, t > 0. \tag{3.12}$$

In view of Lemma 3.3, the solution of (3.11) can be given as follows:

$$\begin{aligned} \widehat{z}^*(t,\omega) &= e^{-A(\omega)(t+r)} e_r^{\mathfrak{B}(\omega)t} \widehat{z}_0^*(-r,\omega) \\ &\quad + \int_{-r}^{0} e^{-A(\omega)(t-s)} e_r^{\mathfrak{B}(\omega)(t-r-s)} \left[\partial_s \widehat{z}_0^*(s,\omega) + A(\omega) \widehat{z}_0^*(s,\omega)\right] ds, \end{aligned} \tag{3.13}$$

where $\mathfrak{B}(\omega) = \mathbf{B} e^{\mathbf{A}(\omega) r}$. Applying the inverse Fourier transform to (3.13), we obtain

$$\begin{aligned} z_{k,j}^*(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{\mathbf{i}(k\omega_1 + j\omega_2)} e^{-A(\omega)(t+r)} e_r^{\mathfrak{B}(\omega)t} \widehat{z}_0^*(-r,\omega) d\omega_1 d\omega_2 \\ &+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-r}^{0} e^{\mathbf{i}(k\omega_1 + j\omega_2)} e^{-A(\omega)(t-s)} e_r^{\mathfrak{B}(\omega)(t-r-s)} \left[\partial_s \widehat{z}_0^*(s,\omega) + A(\omega) \widehat{z}_0^*(s,\omega)\right] ds d\omega_1 d\omega_2, \end{aligned} \tag{3.14}$$

Let

$$P_{k,j}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{\mathbf{i}(k\omega_1 + j\omega_2)} e^{-A(\omega)(t+r)} e_r^{\mathfrak{B}(\omega)t} \widehat{z}_0^*(-r,\omega) d\omega_1 d\omega_2,$$

and

$$\begin{aligned} &Q_{k,j}(t) \\ =& \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-r}^{0} e^{\mathbf{i}(k\omega_1 + j\omega_2)} e^{-A(\omega)(t-s)} e_r^{\mathfrak{B}(\omega)(t-r-s)} \left[\partial_s \widehat{z}_0^*(s,\omega) + A(\omega) \widehat{z}_0^*(s,\omega)\right] ds d\omega_1 d\omega_2. \end{aligned}$$

we first give an estimate of $P_{k,j}(t)$ in $l^\infty(\mathbb{Z}^2)$,

$$\|P_{k,j}(t)\| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left\|e^{-\mathbf{A}(\omega)(t+r)}\right\|\left\|e_r^{\mathfrak{B}(\omega)t}\right\||\widehat{z}_0^*(-r,\omega)|\,d\omega_1 d\omega_2. \tag{3.15}$$

For the estimation of $\|e^{-\mathbf{A}(\xi)t}\|$,one has

$$\begin{aligned}&\left\|e^{-A(\omega)t}\right\|\\ =&e^{-c_1 t}\left|\exp\left\{D_m t\left(e^{\lambda_*\cos\theta}e^{\mathbf{i}\omega_1}+e^{-\lambda_*\cos\theta}e^{-\mathbf{i}\omega_1}+e^{\lambda_*\sin\theta}e^{\mathbf{i}\omega_2}+e^{-\lambda_*\sin\theta}e^{-\mathbf{i}\omega_2}\right)\right\}\right|\\ =&e^{-c_1 t}\exp\left\{D_m t\left[\cos(\omega_1)\left(e^{\lambda_*\cos\theta}+e^{-\lambda_*\cos\theta}\right)+\cos(\omega_2)\left(e^{\lambda_*\sin\theta}+e^{-\lambda_*\sin\theta}\right)\right]\right\}\\ =&e^{-k_0 t}\exp\left\{-D_m t\left[\left(e^{\lambda_*\cos\theta}+e^{-\lambda_*\cos\theta}\right)(1-\cos\omega_1)+\left(e^{\lambda_*\sin\theta}+e^{-\lambda_*\sin\theta}\right)(1-\cos\omega_2)\right]\right\}\\ =&e^{-k_1 t},\end{aligned}$$

where

$$c_1 = c\lambda_* + 4D_m + d_m,$$
$$k_0 = c_1 - D_m\left[e^{\lambda_*\cos\theta}+e^{-\lambda_*\cos\theta}+e^{\lambda_*\sin\theta}+e^{-\lambda_*\sin\theta}\right],$$

and

$$k_1 = k_0 + D_m\left[\left(e^{\lambda_*\cos\theta}+e^{-\lambda_*\cos\theta}\right)(1-\cos\omega_1)+\left(e^{\lambda_*\sin\theta}+e^{-\lambda_*\sin\theta}\right)(1-\cos\omega_2)\right]$$

Meanwhile, due to

$$B(\omega) = b'(0)\left\{\frac{\mu}{(2\pi)^2}\sum_{l,q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)e^{-\lambda_*(l\cos\theta+q\sin\theta+cr)}e^{-\mathbf{i}(l\omega_1+q\omega_2)}\right\}$$

$$|B(\omega)| \le \frac{\mu b'(0)}{(2\pi)^2}\sum_{l,q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)e^{-\lambda_*(l\cos\theta+q\sin\theta+cr)} := k_2$$

$$\|\mathfrak{B}(\omega)\| \le |Be^{A(\omega)r}| \le k_2 e^{k_1 r} := k_3,$$

we have

$$|e_r^{\mathfrak{B}(\omega)t}| \le e_r^{k_3 t}.$$

Then

$$|P_{k,j}(t)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}e^{-k_1(t+r)}e_r^{k_3 t}|\widehat{z}_0^*(-r,\omega)|d\omega_1 d\omega_2. \tag{3.16}$$

Since when $c \ge c_*$,

$$\begin{aligned}k_1 \;=\;& k_0 + D_m\left[\left(e^{\lambda_*\cos\theta}+e^{-\lambda_*\cos\theta}\right)(1-\cos\omega_1)+\left(e^{\lambda_*\sin\theta}+e^{-\lambda_*\sin\theta}\right)(1-\cos\omega_2)\right]\\ =\;& c\lambda_* + d_m - D_m\left[e^{\lambda_*\cos\theta}+e^{-\lambda_*\cos\theta}+e^{\lambda_*\sin\theta}+e^{-\lambda_*\sin\theta}-4\right]\\ &+D_m\left[\left(e^{\lambda_*\cos\theta}+e^{-\lambda_*\cos\theta}\right)(1-\cos\omega_1)+\left(e^{\lambda_*\sin\theta}+e^{-\lambda_*\sin\theta}\right)(1-\cos\omega_2)\right]\\ \ge\;& \frac{\mu b'(0)}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)e^{-\lambda_*(l\cos\theta+q\sin\theta+cr)} = k_2,\end{aligned}$$

(3.12) and (3.16) imply that

$$|P_{k,j}(t)| \le \frac{C}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}e^{-\epsilon_1(k_1-k_2)t}|\widehat{z}_0^*(-r,\omega)|d\omega_1 d\omega_2, \tag{3.17}$$

for a constant $0 < \epsilon_1 < 1$.

Since $\frac{e^x+e^{-x}}{2} \ge 1$ for all $x \in \mathbb{R}$, we obtain

$$\exp\left\{-D_m t\left[\left(e^{\lambda_*\cos\theta}+e^{-\lambda_*\cos\theta}\right)(1-\cos\omega_1)+\left(e^{\lambda_*\sin\theta}+e^{-\lambda_*\sin\theta}\right)(1-\cos\omega_2)\right]\right\}$$

$$
\begin{aligned}
\leq\quad & \exp\{-2D_m t(1-\cos\omega_1)\}\exp\{-2D_m t(1-\cos\omega_2)\} \\
=\quad & \exp\{-2D_m t[1-(\cos\omega_1+\mathbf{i}\sin\omega_1)]\}\exp\{-2D_m t[1-(\cos\omega_2+\mathbf{i}\sin\omega_2)]\} \\
=\quad & \exp\left\{D_m t\left[e^{\mathbf{i}\omega_1}+e^{-\mathbf{i}\omega_1}+e^{\mathbf{i}\omega_2}+e^{-\mathbf{i}\omega_2}-4\right]\right\}.
\end{aligned}
$$

According to Taylor series expansion

$$
e^{\mathbf{i}\xi} = 1+\mathbf{i}\xi-\frac{\xi^2}{2!}-\frac{\mathbf{i}\xi^3}{3}+\frac{\xi^4}{4!}-\cdots,
$$

$$
e^{\mathbf{i}\xi}+e^{-\mathbf{i}\xi} = 2\left(1-\frac{\xi^2}{2!}+\frac{\xi^4}{4!}-\cdots\right).
$$

It follows that

$$
\frac{e^{-\mathbf{i}\xi}+e^{\mathbf{i}\xi}}{2}\leq 1-M|\xi|^\kappa+|\xi|^\kappa h(\xi),
$$

where $0<\kappa\leq 2$, h is bounded and $h(\xi)\to 0$ as $\xi\to 0$. Hence, there exist $0<M_1<M$ and $a_0>0$ such that

$$
\frac{e^{-\mathbf{i}\xi}+e^{\mathbf{i}\xi}}{2}\leq 1-M_1|\xi|^\kappa,\ \text{for } |\xi|\leq a_0,
$$

and $0<\delta<1$ such that

$$
\frac{e^{-\mathbf{i}\xi}+e^{\mathbf{i}\xi}}{2}=\cos\xi\leq 1-\delta,\ \text{for } |\xi|>a_0.
$$

Let $E_{a_0}=\{\xi\in[-\pi,\pi],|\xi|>a_0\}$. Then one has

$$
\begin{aligned}
&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\exp\left\{-\epsilon_1 D_m\left[4-\left(e^{-\mathbf{i}\omega_1}+e^{\mathbf{i}\omega_1}+e^{-\mathbf{i}\omega_2}+e^{\mathbf{i}\omega_2}\right)\right]t\right\}|\widehat{z}_0^*(-r,\omega)|d\omega_1 d\omega_2 \\
\leq&\|\widehat{z}_0^*(-r,\omega)\|_{L^\infty([-\pi,\pi]^2)}\times \\
&\int_{-\pi}^{\pi}\exp\left\{-\epsilon_1 D_m\left[2-\left(e^{-\mathbf{i}\omega_2}+e^{\mathbf{i}\omega_2}\right)\right]\right\}d\omega_2\int_{-\pi}^{\pi}\exp\left\{-\epsilon_1 D_m\left[2-\left(e^{-\mathbf{i}\omega_1}+e^{\mathbf{i}\omega_1}\right)\right]\right\}d\omega_1 \\
=&\|\widehat{z}_0^*(-r,\omega)\|_{L^\infty([-\pi,\pi]^2)}\left(\int_{-\pi}^{\pi}\exp\left\{-\epsilon_1 D_m\left[2-\left(e^{-\mathbf{i}\xi}+e^{\mathbf{i}\xi}\right)\right]\right\}d\xi\right)^2 \\
\leq&\|\widehat{z}_0^*(-r,\omega)\|_{L^\infty([-\pi,\pi]^2)}\left(\int_{|\xi|<a_0}e^{-2\epsilon_1 D_m M_1|\xi|^\kappa t}d\xi+\int_{\xi\in E_{a_0}}e^{-2\epsilon_1 D_m\delta t}d\xi\right)^2 \\
\leq&\|\widehat{z}_0^*(-r,\omega)\|_{L^\infty([-\pi,\pi]^2)}\left(\int_{|\xi|<a_0}e^{-2\epsilon_1 D_m M_1|\xi|^\kappa t}d\xi+e^{-2\epsilon_1 D_m\delta t}m(E_{a_0})\right)^2,
\end{aligned}
\tag{3.18}
$$

where $m(E_{a_0})$ is the measure of E_{a_0}.

By changing variables $\sigma=\xi t^{\frac{1}{\kappa}}$, one has

$$
\begin{aligned}
&\int_{|\xi|<a_0}e^{-\epsilon_1 D_m M_1|\xi|^\kappa t}d\xi+e^{-\epsilon_1 D_m\delta t}m(E_{a_0}) \\
\leq\quad & t^{-\frac{1}{\kappa}}\int_{|\sigma|\leq a_0 t^{\frac{1}{\kappa}}}e^{-\epsilon_1 D_m M_1|\sigma|^\kappa}d\sigma+e^{-\epsilon_1 D_m\delta t}m(E_{a_0}) \\
\leq\quad & Ct^{-\frac{1}{\kappa}},
\end{aligned}
$$

for some constant $C>0$.

Then

$$
\begin{aligned}
&\|P_{k,j}(t)\| \\
\leq\quad & \frac{C}{2\pi}e^{-\epsilon_1(k_0-k_2)t}\times \\
&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\exp\left\{-\epsilon_1 D_m\left[4-\left(e^{-\mathbf{i}\omega_1}+e^{\mathbf{i}\omega_1}+e^{-\mathbf{i}\omega_2}+e^{\mathbf{i}\omega_2}\right)\right]t\right\}|\widehat{z}_0^*(-r,\omega)|d\omega_1 d\omega_2
\end{aligned}
$$

$$\leq C\|\widehat{z}_0^*(-r,\omega)\|_{L^\infty([-\pi,\pi]^2)}t^{-\frac{2}{\kappa}}e^{-\epsilon_1(k_0-k_2)t}$$
$$\leq C\|z_{k,j}^*(-r)\|_{l^1(\mathbb{Z}^2)}t^{-\frac{2}{\kappa}}e^{-\epsilon_1(k_0-k_2)t}.$$

Since $P_{k,j}(t)$ has no singularity for t near zero, the term $t^{\frac{2}{\kappa}}$ can be replaced by $(1+t)^{\frac{2}{\kappa}}$, we get

$$\|P_{k,j}(t)\| \leq C\|z_{k,j}^*(-r)\|_{l^1(\mathbb{Z})}(1+t)^{-\frac{2}{\kappa}}e^{-\epsilon_1(k_0-k_2)t}.$$

The following inequality can be obtained similarly

$$\begin{aligned} &\|Q_{k,j}(t)\| \\ \leq\ & \frac{1}{2\pi}\int_{-r}^{0}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|e^{-A(\omega)(t-s)}\right|\left|e_r^{B(\omega)(t-r-s)}\right||\partial_s\widehat{z}_0^*(s,\omega)+A(\omega)\widehat{z}_0^*(s,\omega)|\,d\omega_1 d\omega_2 ds \\ \leq\ & C\int_{-r}^{0}\left(\|z_{k,j}^*(s)\|_{l^1(\mathbb{Z})}+\left\|\frac{d}{ds}z_{k,j}^*(s)\right\|_{l^1(\mathbb{Z})}\right)ds\,(1+t)^{-\frac{2}{\kappa}}\,e^{-\epsilon_1(k_0-k_2)t}. \end{aligned}$$

Consequently, we get the following decay estimate

$$\|z_{k,j}^*(t)\|_{l^\infty(\mathbb{Z})} \leq C(1+t)^{-\frac{2}{\kappa}}e^{-\epsilon_1(k_0-k_2)t}.$$

When $c > c_*$, one has $k_0 > k_2$. It then follows that

$$\|z_{k,j}^*(t)\|_{l^\infty(\mathbb{Z})} \leq C(1+t)^{-\frac{2}{\kappa}}e^{-\epsilon_1\mu_1 t},$$

where $\mu_1 = k_0 - k_2 > 0$.

When $c = c_*$, we have

$$\|z_{k,j}^*(t)\|_{l^\infty(\mathbb{Z})} \leq C(1+t)^{-\frac{2}{\kappa}},$$

due to $k_0 = k_2$.

According to (3.9),

$$z_{k,j}(t) \leq \bar{z}_{k,j}(t) = e^{\lambda_*(k\cos\theta+j\sin\theta+ct-x_*)}z_{k,j}^*(t),$$

let $H = \{k,j \in \mathbb{Z}, k\cos\theta + j\sin\theta < x_* - ct\}$. If $k,j \in H$, then $e^{\lambda_*(k+ct-\xi_*)} \leq 1$. Thus, the following decay for $z_{k,j}(t)$ is clear.

Lemma 3.4. *For any $r > 0$, it holds that*

(i): *when $c > c_*$, then*

$$\|z_{k,j}(t)\|_{l^\infty(H)} \leq C(1+t)^{-\frac{2}{\kappa}}e^{-\epsilon_1\mu_1 t}$$

for some $\mu_1 > 0$;

(ii): *when $c = c_*$, then*

$$\|z_{k,j}(t)\|_{l^\infty(H)} \leq C(1+t)^{-\frac{2}{\kappa}}.$$

Next we will prove $z_{k,j}(t)$ decay exponentially for $(k,j) \in \mathbb{Z}^2\backslash H$.

Lemma 3.5. *For $r > 0$, it holds that*

(i): *when $c > c_*$, then*

$$\|z_{k,j}(t)\|_{l^\infty(\mathbb{Z}^2\backslash H)} \leq C(1+t)^{-\frac{2}{\kappa}}e^{-\mu_2 t},$$

where μ_2 is a constant and satisfies

$$0 < \mu_2 < \min\left\{d_m - \mu b'(w^+), \epsilon_1\mu_1\right\};$$

(ii): *when $c = c_*$, then*

$$\|z_{k,j}(t)\|_{l^\infty(\mathbb{Z}^2\backslash H)} \leq C(1+t)^{-\frac{2}{\kappa}}.$$

Proof. Let $z_{k,j}(t) = w^+_{k,j}(t) - \phi(k\cos\theta + j\sin\theta + ct)$, we have

$$\begin{aligned}
&\frac{dz_{k,j}(t)}{dt} - D_m[z_{k+1,j}(t) + z_{k-1,j}(t) + z_{k,j+1}(t) + z_{k,j-1}(t) - 4z_{k,j}(t)] + d_m z_{k,j}(t)\\
= \; & \frac{\mu}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)\left(b(\phi+z) - b(\phi)\right).
\end{aligned}$$

Since $b''(\cdot) \le 0$, we have

$$b(\phi+z) - b(\phi) = b'(\phi)z + b''(\widetilde{\phi})z^2 \le b'(\phi)z,$$

where $\widetilde{\phi}$ is a function between ϕ and $\phi + z$. Consequently,

$$\begin{cases}
\dfrac{dz_{k,j}(t)}{dt} - D_m[z_{k+1,j}(t) + z_{k-1,j}(t) + z_{k,j+1}(t) + z_{k,j-1}(t) - 4z_{k,j}(t)] + d_m z_{k,j}(t)\\
\le \dfrac{\mu}{(2\pi)^2}\displaystyle\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)b'(\phi)z_{k-l,j-q}(t-r),\\
z_{k,j}(t)|_{k\cos\theta + j\sin\theta < x_* - ct} \le C_2(1+t)^{-\frac{2}{\kappa}}e^{-\epsilon_1\mu_1 t},\\
z_{k,j}(t)|_{t=s} = z^0_{k,j}(s), s \in [-r, 0], k, j \in \mathbb{Z},
\end{cases}$$

for some positive constant C_2. Let

$$\widetilde{z}(t) = C_3(1+r+t)^{-\frac{2}{\kappa}}e^{-\mu_2 t},$$

for $C_3 \ge C_2 \ge \max_{s\in[-r,0],k,j\in\mathbb{Z}} |z^0_{k,j}(s)|$, and μ_2 will be chosen later. Choose a large number x_* such that for $x > x_* >> 1$,

$$\begin{aligned}
&d_m - \frac{\mu}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)b'(\phi(x - l\cos\theta - q\sin\theta - cr))\\
\ge \; & \epsilon_0\left[d_m - \mu b'(w^+)\right],
\end{aligned} \tag{3.19}$$

where $0 < \epsilon_0 < 1$. We then obtain

$$\begin{aligned}
&\frac{d\widetilde{z}(t)}{dt} + d_m\widetilde{z}(t) - \frac{\mu}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)b'(\phi)\widetilde{z}(t-r)\\
= \; & C_3(1+r+t)^{-\frac{2}{\kappa}}e^{-\mu_2 t}\left\{d_m - \mu_2 - \frac{2}{\kappa}\frac{1}{1+r+t} - \frac{\mu}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)b'(\phi)\right.\\
&\left. - \left[e^{\mu_2 r}\left(\frac{1+t}{1+r+t}\right)^{-\frac{2}{\kappa}} - 1\right]\frac{\mu}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)b'(\phi)\right\}\\
\ge \; & C_3(1+r+t)^{-\frac{2}{\kappa}}e^{-\mu_2 t}\left\{\epsilon_0[d_m - \mu b'(w^+)] - \mu_2 - \frac{2}{\kappa}\frac{1}{(1+r+t)}\right.\\
&\left. - \left[e^{\mu_2 r}\left(\frac{1+t}{1+r+t}\right)^{-\frac{2}{\kappa}} - 1\right]\frac{\mu}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\beta_\alpha(l)\gamma_\alpha(q)b'(\phi)\right\}\\
\ge \; & 0,
\end{aligned}$$

by choosing $t \ge l_0 r$ for sufficiently large integer $l_0 >> 1$ and

$$0 < \mu_2 < \min\{d_m - \mu b'(w^+), \epsilon_1\mu_1\} \text{ for } c > c_*,$$
$$\mu_2 = 0, \text{ for } c = c_*.$$

Hence, we have

$$\begin{cases} \dfrac{d\widetilde{z}(t)}{dt} + d_m \widetilde{z}(t) \geq \dfrac{\mu}{(2\pi)^2} \displaystyle\sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta_\alpha(l)\gamma_\alpha(q) b'(\phi)\widetilde{z}(t-r), t > l_0 r, x > x_*, \\ \widetilde{z}(t)|_{x=x_*} = C_3(1+r+t)^{-\frac{2}{\kappa}} e^{-\mu_2 t} \geq C_2(1+t)^{-\frac{2}{\kappa}} e^{-\epsilon_1 \mu_1 t}, \\ \widetilde{z}(t) = C_3(1+r+t)^{-\frac{2}{\kappa}} e^{-\mu_2 t} > z^0_{k,j}(t), t \in [-r, l_0 r], k, j \in \mathbb{Z}. \end{cases}$$

Therefore, by the comparison principle, we can get

$$z_{k,j}(t) \leq \widetilde{z}(t).$$

The proof is complete. □

It then follows from Lemmas 3.4 and 3.5 that

Lemma 3.6. *For any $r > 0$, it holds that*

(i): *when $c > c_*$, then*

$$\|w^+_{k,j}(t) - \phi(x)\|_{l^\infty(\mathbb{Z}^2)} \leq C(1+t)^{-\frac{2}{\kappa}} e^{-\mu_2 t},$$

where μ_2 is a constant and satisfies

$$0 < \mu_2 < \min\left\{d_m - \mu b'(w^+), \epsilon_1 \mu_1\right\};$$

(ii): *when $c = c_*$, then*

$$\|w^+_{k,j}(t) - \phi(x)\|_{l^\infty(\mathbb{Z}^2)} \leq C(1+t)^{-\frac{2}{\kappa}}.$$

Step 2. The convergence of $w^-_{k,j}(t)$ to $\phi(x)$.

By a similar argument as in Step 1, the proof can be done.

Lemma 3.7. *For any $r > 0$, it holds that*

(i): *when $c > c_*$, then*

$$\|\phi(x) - w^-_{k,j}(t)\|_{l^\infty(\mathbb{Z}^2)} \leq C(1+t)^{-\frac{2}{\kappa}} e^{-\mu_2 t},$$

where μ_2 is a constant and satisfies

$$0 < \mu_2 < \min\left\{d_m - \mu b'(w^+), \epsilon_1 \mu_1\right\};$$

(ii): *when $c = c_*$, then*

$$\|\phi(x) - w^-_{k,j}(t)\|_{l^\infty(\mathbb{Z}^2)} \leq C(1+t)^{-\frac{2}{\kappa}}.$$

Step 3: The convergence of $w_{k,j}(t)$ to $\phi(x)$.

Lemma 3.8. *For any $r > 0$, it holds that*

(i): *when $c > c_*$, then*

$$\|w_{k,j}(t) - \phi(x)\|_{l^\infty(\mathbb{Z}^2)} \leq C(1+t)^{-\frac{2}{\kappa}} e^{-\epsilon_1 \mu t},$$

where $\mu > 0$ is a constant;

(ii): *when $c = c_*$, then*

$$\|w_{k,j}(t) - \phi(x)\|_{l^\infty(\mathbb{Z}^2)} \leq C(1+t)^{-\frac{2}{\kappa}}.$$

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E-mail address: chengcp0611@163.com
E-mail address: an18020536372@163.com