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IDENTITIES FOR LINEAR RECURSIVE SEQUENCES OF ORDER 2

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Dedicated to Professor Peter Bundschuh on the occasion of his 80th birthday

Abstract. We present here a general rule of construction of identities for recursive sequences by using sequence transformation techniques developed in [16]. Numerous identities are constructed, and many well known identities can be proved readily by using this unified rule. Various Catalan-like and Cassini-like identities are given for recursive number sequences and recursive polynomial sequences. Sets of identities for Diophantine quadruple are shown.

1. **Introduction.** Albert Girard published a class of identities in Amsterdam in 1629 and Edward Waring published similar material in Cambridge in 1762-1782, which are referred as Girard-Waring identities later. These identities may be derived from the earlier work of Sir Isaac Newton. Surveys and some applications of these identities can be found in Comtet [5] (P. 198), Gould [10], Shapiro and one of the authors [13], and the authors [15]. We now give a different approach to derive Girard-Waring identities by using the Binet formula of recursive sequences and divided differences. Meanwhile, this approach offers some formulas and identities that may have more wider applications. Finally, an application of the Girard-Waring identities to the sum of powers of consecutive integers is studied.

This paper starts from a review of the application of recursive sequences in the construction of a combinatorial identity called generalized Girard-Waring identity from the Binet formula and the generating function of a recursive sequence. By using the generalized Girard-Waring identity, the Binet type Girard-Waring identity is derived. Then the generalized Girard-Waring identity is used to develop several

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transformation formulas for recursive sequences of numbers and polynomials. All those results are shown in Nie, Chen, and the authors [16].

A recursive sequence constructed from a recursive relation starting from a few initial quantities models some real world problems or mathematical problems. As a natural mathematical model, recursive sequences are widely used in Combinatorics and Graph Theory, Number Theory, Fractal, Cryptography, etc. Many recursive number and polynomial sequences can be defined, characterized, evaluated, and/or classified by linear recurrence relations of various orders. Throughout this paper, a number sequence $\{a_n\}$ is called a linear recursive sequence of order 2 if it satisfies the following linear recurrence relation of order 2:

$$a_n = pa_{n-1} + qa_{n-2}, \quad n \geq 2, \tag{1}$$

for constants $p, q \in \mathbb{R}$ and $q \neq 0$ and initial conditions a_0 and a_1. Let α and β be two roots of the quadratic equation $x^2 - px - q = 0$, of which the left-hand side is called the characteristic polynomial of the recurrence relation. From He and Shiue [14] (see also in [11, 12, 17] for some applications), the general term of the sequence a_n can be presented by the following Binet formula.

$$a_n = \begin{cases} \left(\frac{a_1-\beta a_0}{\alpha-\beta}\right)\alpha^n - \left(\frac{a_1-\alpha a_0}{\alpha-\beta}\right)\beta^n, & if\ \alpha \neq \beta; \\ na_1\alpha^{n-1} - (n-1)a_0\alpha^n, & if\ \alpha = \beta. \end{cases} \tag{2}$$

In [16], the following expression of the general term of the recursive sequence defined by (1) is given.

Theorem 1.1. *([16]) Let (a_n) be the sequence defined by the recursive relation (1), and let α and β be two distinct roots of the characteristic polynomial of (1). Then we have the following generalized Girard-Waring identity:*

$$a_n = a_1p^{n-1} + \sum_{j=1}^{[n/2]} \frac{1}{j}\binom{n-j-1}{j-1}p^{n-2j-1}q^j\left(jpa_0 + (n-2j)a_1\right). \tag{3}$$

Particularly, if $a_0 = 0$ and $a_1 = 1$, (3) implies the Binet type Girard-Waring identity

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \sum_{j=0}^{[n/2]}\binom{n-j-1}{j}p^{n-2j-1}q^j \tag{4}$$

$$= \sum_{j=0}^{[n/2]}(-1)^j\binom{n-j-1}{j}(\alpha+\beta)^{n-2j-1}(\alpha\beta)^j, \tag{5}$$

where $p = \alpha + \beta$ and $q = -\alpha\beta$.

As a source of Binet Girard-Waring identity, the generalized Girard-Waring identity (3) and its extension to recursive polynomial sequences have many applications including a simple way in transferring recursive sequences of numbers and polynomials. Recall that the Chebyshev polynomials of the first kind defined by the recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \tag{6}$$

for all $n \geq 2$ with initial conditions $T_0(x) = 1$ and $T_1(x) = x$, from Theorem 1.1 we have

$$\begin{aligned} & T_n(x) \\ = \quad & 2^{n-1}x^n + n\sum_{j=1}^{[n/2]} \frac{1}{j}\binom{n-j-1}{j-1}(-1)^j 2^{n-2j-1}x^{n-2j}. \end{aligned} \tag{7}$$

Similarly, for Lucas numbers defined by

$$L_n = L_{n-1} + L_{n-2}$$

for all $n \geq 2$ and $L_0 = 2$ and $L_1 = 1$, we have

$$L_n = 1 + n\sum_{j=1}^{[n/2]} \frac{1}{j}\binom{n-j-1}{j-1}.$$

From the expressions of $T_n(x)$ and L_n, we may see that

$$T_n\left(-\frac{i}{2}\right) = \frac{i^{3n}}{2}L_n = \frac{(-i)^n}{2}L_n,$$

or equivalently,

$$L_n = 2i^n T\left(-\frac{i}{2}\right),$$

where $i = \sqrt{-1}$.

In general, [16] presents the following result for transferring a certain class of recursive sequences to the Chebyshev polynomial sequence of the first kind at certain points.

Theorem 1.2. *([16]) Let $\{a_n\}_{n\geq 0}$ be a sequence defined by (1) with $pa_0 = 2a_1$, $a_0 \neq 0$, and let $\{T_n(x)\}_{n\geq 0}$ be the Chebyshev polynomial sequence of the first kind defined by (6). Then*

$$a_n = \frac{2a_1 p^{n-1}}{(2x_0)^n}T_n(x_0), \tag{8}$$

where

$$x_0 = \pm\frac{ip}{2\sqrt{q}}.$$

Namely, a_n shown in (8) can be expressed as

$$a_n = (\mp i)^n a_0 q^{n/2} T_n\left(\pm\frac{ip}{2\sqrt{q}}\right). \tag{9}$$

Theorem 1.2 can be extended to recursive polynomial case as follows.

Corollary 1. *([16]) Let $\{a_n(x)\}_{n\geq 0}$ be a recursive polynomial sequence defined by*

$$a_n(x) = p(x)a_{n-1}(x) + qa_{n-2}(x),$$

where $p(x) \in \mathbb{R}[x]$ and $q \in \mathbb{R}$, with initial conditions $a_0(x)$ and $a_1(x)$ satisfying $p(x)a_0(x) = 2a_1(x)$. Then

$$a_n(x) = (\mp i)^n a_0(x) q^{n/2} T_n\left(\pm\frac{ip(x)}{2\sqrt{q}}\right),$$

where $T_n(x)$ is the nth Chebyshev polynomial of the first kind.

The proofs of Theorem 1.2 and Corollary 2 can be found in [16]. In addition, [16] considers the recursive polynomial sequences defined by (1) with initial conditions $a_0(x) = 0$ and $a_1(x) \neq 0$, where $p = p(x) \in \mathbb{R}[x]$ and $q \in \mathbb{R}$. For instance,

$$\hat{U}_{n+1}(x) = 2x\hat{U}_n(x) - \hat{U}_{n-1}(x) \tag{10}$$

for all $n \geq 1$, where initial conditions are $\hat{U}_0(x) = 0$ and $\hat{U}_1(x) = 1$. It is obvious that $\hat{U}_{n+1}(x) = U_n(x)$, the Chebyshev polynomials of the second kind. By using (3), we have

$$\hat{U}_n(x) = (2x)^{n-1} \sum_{j=0}^{[n/2]} \binom{n-j-1}{j} \left(-\frac{1}{4x^2}\right)^j. \tag{11}$$

If a_n is a sequence defined by (3), with the initial conditions $a_0 = 0$ and $a_1 \neq 0$, direct substitution shows that

$$\begin{aligned} a_n &= a_1 p^{n-1} + \sum_{j=1}^{[n/2]} \frac{1}{j} \binom{n-j-1}{j-1} p^{n-2j-1} q^j (n-2j) a_1 \\ &= a_1 p^{n-1} + \sum_{j=1}^{[n/2]} \binom{n-j-1}{j} p^{n-2j-1} q^j a_1 \\ &= a_1 p^{n-1} \sum_{j=0}^{[n/2]} \binom{n-j-1}{j} p^{-2j} q^j. \end{aligned} \tag{12}$$

Comparing with the rightmost sides of (11) and (12), the following result is obtained.

Theorem 1.3. *([16]) Let $\{a_n\}_{n\geq 0}$ be the sequence defined by (1) with $a_0 = 0$ and $a_1 \neq 0$, and let $\{U_n(x)\}_{n\geq 0}$ be the Chebyshev polynomial sequence of the second kind defined by (10), where $U_n(x) = \hat{U}_{n+1}(x)$. Then*

$$a_n = (\mp i)^{n-1} a_1 q^{(n-1)/2} U_{n-1}(x_0), \tag{13}$$

where

$$x_0 = \pm \frac{ip}{2\sqrt{q}}.$$

Namely,

$$a_n = (\mp i)^{n-1} a_1 q^{(n-1)/2} U_{n-1}\left(\pm \frac{ip}{2\sqrt{q}}\right).$$

Theorem 1.3 is extended in [16] to recursive polynomial case as Chebyshev polynomials of the second kind.

Corollary 2. *Let $\{a_n(x)\}_{n\geq 0}$ be the recursive polynomial sequence defined by*

$$a_n(x) = p(x) a_{n-1}(x) + q a_{n-2}(x),$$

where $p(x) \in \mathbb{R}[x]$ and $q \in \mathbb{R}$, with initial conditions $a_0(x) = 0$ and $a_1(x) \neq 0$. Then

$$a_n(x) = (\mp i)^{n-1} a_1(x) q^{(n-1)/2} U_{n-1}\left(\pm \frac{ip}{2\sqrt{q}}\right).$$

Numerous examples of the transformation of recursive number and polynomial sequences are shown in [16]. In the next section, we will see how those transformation help us to verify some well-known identities readily and to establish new identities.

2. **Identities constructed from the recursive sequence transformation.** The Catalan identity for Fibonacci numbers F_n, namely that $F_n^2 - F_{n+k}F_{n-k} = (-1)^{n-k}F_k^2$, and its special case of $k=1$, named the Cassini identity for Fibonacci numbers F_n, namely that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, are two facts about the Fibonacci numbers that one might call common mathematical knowledge (cf. [22, 30, 32]). We will in the following aim at presenting the Catalan-like and the Cassini-like identities in a more general context and in the process obtain similar results for related sequences by using the sequence transformation technique shown before.

From Theorem 1.3 we may get the Catalan-like identities of the recursive sequences defined by (1) as follows.

Theorem 2.1. *Let $\{a_n\}_{n\geq 0}$ be the sequence defined by (1) with $a_0 = 0$ and $a_1 \neq 0$. Then, we have Catalan-like identity*

$$a_n^2 - a_{n+k}a_{n-k} = (-q)^{n-k}a_k^2. \tag{14}$$

Particularly, for $k=1$, we have the Cassini-like identity for $(a_n)_{n\geq 0}$,

$$a_n^2 - a_{n+1}a_{n-1} = (-q)^{n-1}a_1^2. \tag{15}$$

Proof. Let $\{U_n(x)\}_{n\geq 0}$ be the Chebyshev polynomial sequence of the second kind defined by (10). Then, from (13) we have

$$U_n(x_0) = a_1^{-1}q^{-n/2}(\pm i)^n a_{n+1}, \tag{16}$$

where $x_0 = \pm ip/(2\sqrt{q})$. It is known (cf. Udrea [30]) that $U_n(x)$ satisfied the identity

$$U_n^2(x) - U_{n+k}(x)U_{n-k}(x) = U_{k-1}^2(x). \tag{17}$$

Substituting (16) into (17) yields

$$\begin{aligned} & a_1^{-2}q^{-n}(\pm i)^{2n}a_{n+1}^2 - a_1^{-1}q^{-(n+k)/2}(\pm i)^{n+k}a_{n+k+1}a_1^{-1}q^{-(n-k)/2}(\pm i)^{n-k}a_{n-k+1} \\ =& a_1^{-2}q^{-(k-1)}(\pm i)^{2(k-1)}a_k^2, \end{aligned} \tag{18}$$

which can be simplified to

$$a_{n+1}^2 - a_{n+k+1}a_{n-k+1} = q^{n-k+1}(\pm i)^{-2(n-k+1)}a_k^2 = (-q)^{n-k+1}a_k^2.$$

Thus, we obtain (14). When $k=1$, (14) implies (15). □

Similarly, we have an analogue for the recursive polynomial sequence.

Corollary 3. *Let $\{a_n(x)\}_{n\geq 0}$ be a recursive polynomial sequence defined by*

$$a_n(x) = p(x)a_{n-1}(x) + qa_{n-2}(x),$$

where $p(x) \in \mathbb{R}[x]$ and $q \in \mathbb{R}$, with initial conditions $a_0(x) = 0$ and $a_1(x) \neq 0$. Then, we have Catalan-like identity

$$a_n^2(x) - a_{n+k}(x)a_{n-k}(x) = (-q)^{n-k}a_k^2(x). \tag{19}$$

Particularly, for $k=1$, we have the Cassini-like identity

$$a_n^2(x) - a_{n+1}(x)a_{n-1}(x) = (-q)^{n-1}a_1^2(x). \tag{20}$$

Example 2.2. For the Fibonacci number sequence defined by $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$, $F_0 = 0$, and $F_1 = 1$, we have $q = 1$ and the Catalan-like identity

$$F_n^2 - F_{n+k}F_{n-k} = (-1)^{n-k}F_k^2$$

and the Cassini-like identity

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}.$$

As for the Fibonacci polynomial sequence defined by $F_n(x) = xF_{n-1}(x)+F_{n-2}(x)$ $(n \geq 2)$, $F_0(x) = 0$, and $F_1(x) = 1$, we have $q = 1$ and the Catalan-like identity

$$F_n^2(x) - F_{n+k}(x)F_{n-k}(x) = (-1)^{n-k}F_k^2(x)$$

and the Cassini-like identity

$$F_n^2(x) - F_{n+1}(x)F_{n-1}(x) = (-1)^{n-1}.$$

Example 2.3. For the Pell polynomial sequence defined by $P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$ $(n \geq 2)$, $P_0(x) = 0$, and $P_1(x) = 1$, and the Pell number sequence $(P_n = P_n(1))_{n\geq 0}$, we have the Catalan-like identities

$$P_n^2(x) - P_{n+k}(x)P_{n-k}(x) = (-1)^{n-k}P_k^2(x),$$
$$P_n^2 - P_{n+k}P_{n-k} = (-1)^{n-k}P_k^2,$$

and the Cassini-like identities

$$P_n^2(x) - P_{n+1}(x)P_{n-1}(x) = (-1)^{n-1},$$
$$P_n^2 - P_{n+1}P_{n-1} = (-1)^{n-1}.$$

Horadam and Mahon [22] prove the above identities by using a different approach.

Example 2.4. We call an integer $n \geq 2$ a balancing number (cf., for example, Behera and Panda [2]) if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{21}$$

for some $r \in \mathbb{N}$. Here r is called the balancer corresponding to the balancing number n. For example, $6, 35$, and 204 are balancing numbers with balancers $2, 14$, and 84, respectively.

It follows from (21) that, if n is a balancing number with balancer r, then

$$n^2 = \frac{(n+r)(n+r+1)}{2}$$

and thus

$$r = \frac{-(2n+1) + \sqrt{8n^2+1}}{2}.$$

Denote $(B_n)_{n\geq 0}$ the balancing number sequence and assume $B_0 = 0$. In [2] the recursive relation of the balancing number sequence is given as $B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 2$ with the initials $B_0 = 0$ and $B_1 = 1$. Hence, we have the Catalan-like numbers and the Cassini-like numbers for the balancing number sequence as

$$B_n^2 - B_{n+k}B_{n-k} = B_k^2$$
$$B_n^2 - B_{n+1}B_{n-1} = B_1^2 = 1.$$

The above two identities are given by Catarino, Campos, and Vasco [4] using a different approach.

The Chebyshev polynomials of the first kind $T_n(x)$ defined by (6) satisfies the Cassini-like identity (cf. [32])

$$T_n^2(x) - T_{n+1}(x)T_{n-1}(x) = 1 - x^2. \tag{22}$$

Then, by using Theorem 1.2 we may find the analogy Cassini-like identities for some other recursive sequences. More precisely, we have the following result.

Theorem 2.5. *Let $\{a_n\}_{n\geq 0}$ be a sequence defined by (1) with $pa_0 = 2a_1$, $a_0 \neq 0$. Then we have the Cassini-like identity for $(a_n)_{n\geq 0}$,*

$$a_n^2 - a_{n+1}a_{n-1} = (-q)^n \left(a_0^2 + \frac{a_1^2}{q}\right). \tag{23}$$

Similarly, let $\{a_n(x)\}_{n\geq 0}$ be a recursive polynomial sequence defined by

$$a_n(x) = p(x)a_{n-1}(x) + qa_{n-2}(x),$$

where $p(x) \in \mathbb{R}[x]$ and $q \in \mathbb{R}$, with initial conditions $a_0(x)$ and $a_1(x)$ satisfying $p(x)a_0(x) = 2a_1(x)$. Then

$$a_n^2(x) - a_{n+1}(x)a_{n-1}(x) = (-q)^n \left(a_0^2(x) + \frac{a_1^2(x)}{q}\right). \tag{24}$$

Proof. The transformation formula (8) shown in Theorem 1.2 with $x_0 = \pm ip/(2\sqrt{q})$ and $p = 2a_1/a_0$ gives

$$\begin{aligned} T_n(x_0) &= 2^{-1}a_1^{-1}p^{-(n-1)}(2x_0)^n a_n \\ &= a_0^{-1}p^{-n}\left(\pm\frac{ip}{\sqrt{q}}\right)^n a_n = a_0^{-1}q^{-n/2}(\pm i)^n a_n. \end{aligned}$$

Substituting the above expression of $T_n(x_0)$ into (22) and noting $p = 2a_1/a_0$ yields

$$\begin{aligned} & a_0^{-2}q^{-n}(\pm i)^{2n}a_n^2 - a_0^{-1}q^{-(n+1)/2}(\pm i)^{n+1}a_{n+1}a_0^{-1}q^{-(n-1)/2}(\pm i)^{n-1}a_{n-1} \\ =& 1 - \left(\pm\frac{ip}{2\sqrt{q}}\right)^2 = 1 + \frac{a_1^2}{a_0^2 q}. \end{aligned}$$

After simplifying the leftmost side of the last equation, we obtain the identity

$$a_0^{-2}(-q)^{-n}\left(a_n^2 - a_{n+1}a_{n-1}\right) = 1 + \frac{a_1^2}{a_0^2 q},$$

which generalizes (23). Identity (24) can be proved similarly. □

Remark 2.6. From the following examples, we will see many recursive sequences satisfy the condition $p = 2a_1/a_0$. It worth investigating the reason behind the fact.

Example 2.7. Consider the Lucas polynomial sequence defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x) \tag{25}$$

with initials $L_0(x) = 2$ and $L_1(x) = x$ satisfying $p(x) = x = 2L_1(x)/L_0(x)$ and Lucas numbers $L_n = L_n(1)$. From (23) and (24) we have the Cassini-like identities

$$\begin{aligned} & L_n^2(x) - L_{n+1}(x)L_{n-1}(x) = (-1)^n(4 + x^2), \\ & L_n^2 - L_{n+1}L_{n-1} = 5(-1)^n. \end{aligned}$$

The Fermat polynomial sequence is defined by

$$f_n(x) = xf_{n-1}(x) - 2f_{n-2}(x)$$

with initials $f_0(x) = 2$ and $f_1(x) = x$ satisfying $p = x = 2f_1(x)/f_0(x)$, and the Fermat number sequence is defined by $(f_n = f_n(1))_{n\geq 0}$. Then, from (23) and (24) we get the Cassini-like identities

$$\begin{aligned} & f_n^2(x) - f_{n+1}(x)f_{n-1}(x) = 2^n\left(4 - \frac{x^2}{2}\right) = 2^{n-1}(8 - x^2), \\ & f_n^2 - f_{n+1}f_{n-1} = 7 \cdot 2^{n-1}. \end{aligned}$$

For the Dickson polynomial sequence of the first kind defined by

$$D_n(x,a) = xD_{n-1}(x,a) - aD_{n-2}(x,a)$$

with initials $D_0(x,a) = 2$ and $D_1(x,a) = x$, it satisfies $p = x = 2D_1(x,a)/D_0(x,a)$. Then, from (23) we get the Cassini-like identity

$$D_n^2(x,a) - D_{n+1}(x,a)D_{n-1}(x,a) = a^n\left(4 - \frac{x^2}{a}\right) = a^{n-1}(4a - x^2), \tag{26}$$

which seems to be a new identity to the best of our knowledge.

For the Pell-Lucas polynomials A122075 [27] $Q_n(x)$ defined by (see Horadam and Mahon [22])

$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x) \tag{27}$$

for all $n \geq 2$ with initial conditions $Q_0(x) = 2$ and $Q_1(x) = 2x$, we have the Cassini-like identity

$$Q_n^2(x) - Q_{n+1}(x)Q_{n-1}(x) = (-1)^n(4 + 4x^2).$$

For the Viate polynomials of the second kind defined by (see Horadan [21])

$$v_n(x) = xv_{n-1}(x) - v_{n-2}(x) \tag{28}$$

for all $n \geq 2$ with the initial conditions $v_0(x) = 2$ and $v_1(x) = x$, we have the Cassini-like identity

$$v_n^2(x) - v_{n+1}(x)v_{n-1}(x) = 4 - x^2.$$

3. **An efficient and unified approach to construct product expansions and product expressions of recursive sequences.** The transformation technique presented in the previous section provides an efficient and unified way to construct product expansions and product expressions for two classes recursive sequences from the corresponding product expansions and product expressions for the sequence of Chebyshev polynomials of the first kind and the second kind. The first class is a recursive number sequence $\{a_n\}_{n\geq 0}$ defined by (1) with the initial conditions satisfying $pa_0 = 2a_1$ and $a_0 \neq 0$ (the conditions of Theorem 1.2) or satisfying $a_0 = 0$ and $a_1 \neq 0$ (the conditions of Theorem 1.3). The second class is a recursive polynomial sequence $\{a_n(x)\}_{n\geq 0}$ with the initial conditions satisfying $p(x)a_0(x) = 2a_1(x)$ and $a_0(x) \neq 0$ (the conditions of Corollary 1) or satisfying $a_0(x) = 0$ and $a_1(x) \neq 0$ (the conditions of Corollary 2).

We first discuss the transformation of product expansions, then the transformation of product expressions. We know that the Chebyshev polynomials of the first kind satisfies the following product expansion

$$2T_m(x)T_n(x) = T_{m+n}(x) + T_{m-n}(x), \quad m \geq n. \tag{29}$$

This follows quickly from the fact that $T_n(\cos\theta) = \cos n\theta$, $\theta = \arccos x$, and the addition theorem $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$ with $\alpha = m\arccos x$ and $\beta = n\arccos x$. Consequently, we have the following result.

Theorem 3.1. *Let $\{a_n(x)\}_{n\geq 0}$ be a recursive polynomial sequence defined by*

$$a_n(x) = p(x)a_{n-1}(x) + qa_{n-2}(x),$$

where $p(x) \in \mathbb{R}[x]$ and $q \in \mathbb{R}$, with initial conditions $a_0(x)$ and $a_1(x)$ satisfying $p(x)a_0(x) = 2a_1(x)$. Then,

$$2a_m(x)a_n(x) = a_0(x)\left(a_{m+n}(x) + (-q)^n a_{m-n}(x)\right). \tag{30}$$

Proof. We have the following analogy of (9) for the Chebyshev polynomial sequence of the first kind

$$a_n(x) = (\mp i)^n a_0(x) q^{n/2} T_n\left(\pm\frac{ip(x)}{2\sqrt{q}}\right),$$

from which

$$T_n\left(\pm\frac{ip(x)}{2\sqrt{q}}\right) = (\mp i)^{-n} a_0(x)^{-1} q^{-n/2} a_n(x).$$

Substituting the above expression for T_n into (29) yields (30). □

Example 3.2. For the Dickson polynomials shown in Example 2.7, from Theorem 3.1 and noting $a_0(x) = 2$ and $q = -a$, we have the identity

$$D_m(x,a)D_n(x,a) = D_{m+n}(x,a) + a^n D_{m-n}(x,a). \tag{31}$$

This identity seems new to our knowledge, although its special cases of $m = n$ and $m = n + 1$, i.e.,

$$D_n^2(x,a) - D_{2n}(x,a) = a^n D_0(x,a) = 2a^n,$$
$$D_n(x,a)D_{n+1}(x,a) - D_{2n+1}(x,a) = a^n D_1(x,a) = a^n x,$$

respectively, have been shown in Lidl, Mullen, and Turnwald, [23, p.11].

For the Pell-Lucas polynomials $Q_n(x)$ defined by (27), from Theorem 3.1 and noting $Q_0(x) = 2$ and $q = 1$, we have

$$Q_m(x)Q_n(x) = Q_{m+n}(x) + (-1)^n Q_{m-n}(x).$$

The special cases of $m = n$ and $m = n + 1$ are

$$Q_n^2(x) - Q_{2n}(x) = (-1)^n D_0(x) = 2(-1)^n,$$
$$Q_n(x)Q_{n+1}(x) - Q_{2n+1}(x) = (-1)^n Q_1(x) = 2(-1)^n x,$$

respectively.

For the Viate polynomials of the second kind defined by (28), from Theorem 3.1 and noting $v_0(x) = 2$ and $q = -1$, we have the identity

$$v_m(x)v_n(x) = v_{m+n}(x) + v_{m-n}(x).$$

The special cases of $m = n$ and $m = n + 1$ are

$$v_n^2(x) - v_{2n}(x) = D_0(x) = 2,$$
$$v_n(x)v_{n+1}(x) - v_{2n+1}(x) = Q_1(x) = x,$$

respectively.

For the Lucas polynomial sequence $\{L_n(x)\}$ defined by (25), we have the identity

$$L_m(x)L_n(x) = L_{m+n}(x) + (-1)^n L_{m-n}(x).$$

The special cases of $m = n$ and $m = n + 1$ are

$$L_n^2(x) - L_{2n}(x) = (-1)^n L_0(x) = 2(-1)^n,$$
$$L_n(x)L_{n+1}(x) - L_{2n+1}(x) = (-1)^n L_1(x) = (-1)^n x,$$

respectively.

Other examples for the Lucas number sequence are:

$$L_m L_n = L_{m+n} + (-1)^n L_{m-n},$$
$$L_n^2 - L_{2n} = (-1)^n L_0 = 2(-1)^n,$$
$$L_n L_{n+1} - L_{2n+1} = (-1)^n L_1 = (-1)^n.$$

For Chebyshev polynomials of the second kind, it is known that their products can be written as

$$U_m(x)U_n(x) = \sum_{k=0}^{n} U_{m-n+2k}(x) \tag{32}$$

for $m \geq n$. By this with $n = 2$, there is a recurrence formula for Chebyshev polynomials of the second kind,

$$\begin{aligned} &U_{m+2}(x) = U_2(x)U_m(x) - U_m(x) - U_{m-2}(x) \\ =&U_m(x)(U_2(x) - 1) - U_{m-2}(x). \end{aligned} \tag{33}$$

We now extend the above formula to the product formulas for a class of recursive sequences.

Theorem 3.3. *Let $\{a_n(x)\}_{n\geq 0}$ be a recursive polynomial sequence defined by*

$$a_n(x) = p(x)a_{n-1}(x) + qa_{n-2}(x),$$

where $p(x) \in \mathbb{R}[x]$ and $q \in \mathbb{R}$, with initial conditions $a_0(x) = 0$ and $a_1(x) \neq 0$. Then,

$$a_{m+1}(x)a_{n+1}(x) = a_1(x)\sum_{k=0}^{n}(-q)^{n-k}a_{m-n+2k+1}(x), \tag{34}$$

which is independent of $p(x)$. Particularly, for $n = 2$, we have the recursive formula

$$a_{m+3}(x) = a_{m+1}(x)\left(\frac{a_3(x)}{a_1(x)} + q\right) - q^2 a_{m-1}(x). \tag{35}$$

Proof. From Corollary 2, we have

$$a_n(x) = (\mp i)^{n-1}a_1(x)q^{(n-1)/2}U_{n-1}(x_0),$$

where $x_0 = \pm ip(x)/(2\sqrt{q})$. Hence,

$$U_n(x_0) = \frac{a_{n+1}(x)}{(\mp i)^n a_1(x)q^{n/2}} = (\pm i)^n a_1(x)^{-1}q^{-n/2}a_{n+1}(x).$$

Substituting the above expression of $U_n(x)$ into (32) yields

$$\begin{aligned} &(\pm i)^m a_1(x)^{-1}q^{-m/2}a_{m+1}(x)(\pm i)^n a_1(x)^{-1}q^{-n/2}a_{n+1}(x) \\ =&\sum_{k=0}^{n}(\pm i)^{m-n+2k}a_1(x)^{-1}q^{-(m-n+2k)/2}a_{m-n+2k+1}(x). \end{aligned}$$

The last equation can be simplified to

$$a_{m+1}(x)a_{n+1}(x) = a_1(x)\sum_{k=0}^{n}(\pm i)^{2(k-n)}q^{n-k}a_{m-n+2k+1}(x),$$

which implies (34). Formulas (35) follows after substituting $n = 2$. □

Remark 3.4. Clearly, Theorem 3.3 can be extended to the recursive number sequence $\{a_n\}_{n\geq 0}$ defined by

$$a_n = pa_{n-1} + qa_{n-2},$$

where $p, q \in \mathbb{R}$, with initial conditions $a_0 = 0$ and $a_1 \neq 0$.

Example 3.5. For the Fibonacci polynomial sequence $\{F_n(x)\}$ defined in Example 2.2 with $p(x) = x$, $q = 1$, $F_0(x) = 0$ and $F_1(x) = 1$, from (34) and (35) in Theorem 3.3 we have product expansion and recursive formula for Fibonacci polynomials.

$$F_{m+1}(x)F_{n+1}(x) = \sum_{k=0}^{n}(-1)^{n-k}F_{m-n+2k+1}(x),$$
$$F_{m+3}(x) = F_{m+1}(x)(F_3(x)+1) - F_{m-1}(x)$$
$$=(x^2+2)F_{m+1}(x) - F_{m-1}(x).$$

For Fibonacci numbers $F_n = F_n(1)$, we have

$$F_{m+1}F_{n+1} = \sum_{k=0}^{n}(-1)^{n-k}F_{m-n+2k+1},$$
$$F_{m+3} = F_{m+1}(F_3+1) - F_{m-1} = 3F_{m+1} - F_{m-1}.$$

For the Pell polynomial sequence defined in Example 2.3 with $q = 1$ and $P_1(x) = 1$, we have their product expansion and recursive formula as follows.

$$P_{m+1}(x)P_{n+1}(x) = \sum_{k=0}^{n}(-1)^{n-k}P_{m-n+2k+1}(x),$$
$$P_{m+3}(x) = P_{m+1}(x)(P_3(x)+1) - P_{m-1}(x)$$
$$=(4x^2+2)P_{m+1}(x) - P_{m-1}(x).$$

For the Pell number sequence $(P_n = P_n(1))_{n\geq 0}$, we have their product expansion and recursive formula as

$$P_{m+1}P_{n+1} = \sum_{k=0}^{n}(-1)^{n-k}P_{m-n+2k+1},$$
$$P_{m+3} = P_{m+1}(P_3+1) - P_{m-1} = 6P_{m+1} - P_{m-1}.$$

For the balancing numbers defined in Example 2.4 by $B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 2$ with the initials $B_0 = 0$ and $B_1 = 1$, noting $q = -1$ we have the product expansion and recursive formula for $(B_n)_{n\geq 0}$ as

$$B_{m+1}B_{n+1} = \sum_{k=0}^{n} B_{m-n+2k+1},$$
$$B_{m+3} = B_{m+1}(B_3 - 1) - B_{m-1} = 34B_{m+1} - B_{m-1}.$$

In Cahill, D'Errico, and Spence [3], the following product expressions for the Fibonacci number sequence and the Lucas number sequence were constructed:

$$F_n = \Pi_{k=1}^{n-1}\left(1 - 2i\cos\frac{\pi k}{n}\right), \tag{36}$$

$$L_n = \Pi_{k=1}^{n}\left(1 - 2i\cos\frac{(2k-1)\pi}{2n}\right) \tag{37}$$

for $n \geq 2$ and $n \geq 1$, respectively. In an earlier paper, Hoggatt and Bicknell [19] showed that the roots of Fibonacci polynomials $F_n(x)$ and Lucas polynomials $L_n(x)$ of degree n are $x = 2i\cos(k\pi)/n$ $(k = 1, 2, \ldots, n-1)$ and $x = 2i\cos(2k-1)\pi/(2n)$

$(k = 1, 2, \ldots, n)$, respectively, which imply the identities

$$F_n(x) = \Pi_{k=1}^{n-1}\left(x - 2i\cos\frac{\pi k}{n}\right), \tag{38}$$

$$L_n(x) = \Pi_{k=1}^{n}\left(x - 2i\cos\frac{(2k-1)\pi}{2n}\right). \tag{39}$$

We will show that identities (36)-(39) can be easily established by using our sequence transformation technique. Actually, they are special cases of the general results of the following two theorems.

In the handbook edited by Zwillinger [33], the product expression of Chebyshev polynomials of the first kind and the second kind are given as

$$T_n(x) = 2^{n-1}\Pi_{k=1}^{n}\left(x - \cos\frac{(2k-1)\pi}{2n}\right) \quad \text{and} \tag{40}$$

$$U_n(x) = 2^{n}\Pi_{k=1}^{n}\left(x - \cos\frac{k\pi}{n+1}\right). \tag{41}$$

Theorem 3.6. *Let $\{a_n\}_{n\geq 0}$ be a recursive number sequence defined by (1) with initial conditions a_0 and a_1 satisfying $pa_0 = 2a_1$. Then, we have*

$$a_n = \frac{1}{2}a_0\Pi_{k=1}^{n}\left(p \pm 2i\sqrt{q}\cos\frac{(2k-1)\pi}{2n}\right).$$

Similarly, let $\{a_n(x)\}_{n\geq 0}$ be a recursive polynomial sequence defined by

$$a_n(x) = p(x)a_{n-1}(x) + qa_{n-2}(x),$$

where $p(x) \in \mathbb{R}[x]$ ($p \in \mathbb{R}$) and $q \in \mathbb{R}$, with initial conditions $a_0(x)$ and $a_1(x)$ satisfying $p(x)a_0(x) = 2a_1(x)$. Then, we have

$$a_n(x) = \frac{1}{2}a_0(x)\Pi_{k=1}^{n}\left(p(x) \pm 2i\sqrt{q}\cos\frac{(2k-1)\pi}{2n}\right). \tag{42}$$

Proof. We now prove the formula (42) for the polynomial sequence, the corresponding result for the number sequence can be proved similarly. We have the following analogy of (9) for the Chebyshev polynomial sequence of the first kind

$$a_n(x) = (\mp i)^n a_0(x)q^{n/2}T_n\left(\pm\frac{ip(x)}{2\sqrt{q}}\right),$$

from this equation and (40) with the replacement $x \to \pm\frac{ip(x)}{2\sqrt{q}}$ we have

$$\begin{aligned}a_n(x) &= (\mp i)^n a_0(x)q^{n/2}2^{n-1}\Pi_{k=1}^{n}\left(\pm\frac{ip(x)}{2\sqrt{q}} - \cos\frac{(2k-1)\pi}{2n}\right)\\
&=(\mp i)^n a_0(x)q^{n/2}2^{n-1}\left(\pm\frac{i}{2\sqrt{q}}\right)^n\Pi_{k=1}^{n}\left(p(x) \pm 2i\sqrt{q}\cos\frac{(2k-1)\pi}{2n}\right)\\
&=\frac{1}{2}(-i^2)^n a_0(x)\Pi_{k=1}^{n}\left(p(x) \pm 2i\sqrt{q}\cos\frac{(2k-1)\pi}{2n}\right)\\
&=\frac{1}{2}a_0(x)\Pi_{k=1}^{n}\left(p(x) \pm 2i\sqrt{q}\cos\frac{(2k-1)\pi}{2n}\right).\end{aligned}$$

The equivalence of two product expression can be shown by setting $k \to n-k+1$. Indeed, we have

$$\begin{aligned}&\Pi_{k=1}^{n}\left(p(x)+2i\sqrt{q}\cos\frac{(2(n-k+1)-1)\pi}{2n}\right)\\=&\Pi_{k=1}^{n}\left(p(x)+2i\sqrt{q}\cos\left(\pi-\frac{(2k-1)\pi}{2n}\right)\right)\\=&\Pi_{k=1}^{n}\left(p(x)-2i\sqrt{q}\cos\frac{(2k-1)\pi}{2n}\right).\end{aligned}$$

□

Example 3.7. For the Lucas polynomial sequence $\{L_n(x)\}$ defined by (25) with $p(x)=x$, $q=1$, $a_0(x)=2$ and $a_1(x)=x$ satisfying $p(x)a_0(x)=2x=2a_1(x)$, from Theorem 3.6 we immediately have (39). For Lucas numbers $L_n=L_n(1)$, we also readily obtain (37).

For the Dickson polynomials shown in Example 2.7, from Theorem 3.6 and noting $a_0(x)=2$, $a_1(x)=D_1(x,a)=x$, and $q=-a$, we have the identity

$$\begin{aligned}&D_n(x,a)=\Pi_{k=1}^{n}\left(x\pm 2i\sqrt{-a}\cos\frac{(2k-1)\pi}{2n}\right)\\=&\Pi_{k=1}^{n}\left(x\mp 2\sqrt{a}\cos\frac{(2k-1)\pi}{2n}\right).\end{aligned}\tag{43}$$

Particularly, for $a=1$

$$\begin{aligned}&D_n(x,1)=\Pi_{k=1}^{n}\left(x+2\cos\frac{(2k-1)\pi}{2n}\right)\\=&\Pi_{k=1}^{n}\left(x-2\cos\frac{(2k-1)\pi}{2n}\right).\end{aligned}$$

Furthermore, $D_n(x,a)$ has roots $2\sqrt{a}\cos\frac{(2k-1)\pi}{2n}$ for $k=1,2,\ldots,n$.

For the Pell-Lucas polynomials $Q_n(x)$ defined by (27) with $p(x)=2x$, $q=1$, $Q_0(x)=2$ and $Q_1(x)=2x$ satisfying $p(x)Q_0(x)=4x=2Q_1(x)$, from Theorem 3.6 we have the product expression of $Q_n(x)$ as

$$\begin{aligned}Q_n(x)=&\Pi_{k=1}^{n}\left(2x\pm 2i\cos\frac{(2k-1)\pi}{2n}\right)\\=&2^n\Pi_{k=1}^{n}\left(x\pm i\cos\frac{(2k-1)\pi}{2n}\right),\end{aligned}$$

and $Q_n(x)$ has roots $i\cos\frac{(2k-1)\pi}{2n}$ for $k=1,2,\ldots,n$.

For the Viate polynomials of the second kind defined by (28) with $p(x)=x$, $q=-1$, $v_0(x)=2$ and $v_1(x)=x$ satisfying $p(x)v_0(x)=2x=2v_1(x)$, from Theorem 3.6 we have the product expression of $v_n(x)$ as

$$\begin{aligned}v_n(x)=&\Pi_{k=1}^{n}\left(p(x)\pm 2i\sqrt{-1}\cos\frac{(2k-1)\pi}{2n}\right)\\=&\Pi_{k=1}^{n}\left(x\mp 2\cos\frac{(2k-1)\pi}{2n}\right),\end{aligned}$$

and $v_n(x)$ has roots $2\cos\frac{(2k-1)\pi}{2n}$ for $k=1,2,\ldots,n$.

Theorem 3.8. *Let $\{a_n(x)\}_{n\geq 0}$ $(a_n))_{n\geq 0})$ be a recursive polynomial (number) sequence defined by*

$$a_n(x) = p(x)a_{n-1}(x) + qa_{n-2}(x) \quad (a_n = pa_{n-1} + qa_{n-2}),$$

where $p(x) \in \mathbb{R}[x]$ $(p \in \mathbb{R})$ and $q \in \mathbb{R}$, with initial conditions $a_0(x) = 0$ $(a_0 = 0)$ and $a_1(x) \neq 0$ $(a_1 \neq 0)$. Then, we have

$$a_n(x) = a_1(x)\Pi_{k=1}^{n-1}\left(p(x) \pm 2i\sqrt{q}\cos\frac{k\pi}{n}\right) \tag{44}$$

$$\left(a_n = a_1\Pi_{k=1}^{n-1}\left(p \pm 2i\sqrt{q}\cos\frac{k\pi}{n}\right)\right), \tag{45}$$

where the equivalence of the two expressions in (44) ((43)) can be seen from the transformation $k \to n-k+1$.

Proof. Let $\{U_n(x)\}_{n\geq 0}$ be the Chebyshev polynomial sequence of the second kind defined by (10). From Corollary 2 and (41) we have

$$\begin{aligned}
&a_n(x) = (\mp i)^{n-1}a_1(x)q^{(n-1)/2}U_{n-1}\left(\pm\frac{ip(x)}{2\sqrt{q}}\right)\\
=&(\mp i)^{n-1}a_1(x)q^{(n-1)/2}2^{n-1}\Pi_{k=1}^{n-1}\left(\pm\frac{ip(x)}{2\sqrt{q}} - \cos\frac{k\pi}{n}\right)\\
=&(\mp i)^{n-1}a_1(x)q^{(n-1)/2}2^{n-1}\left(\pm\frac{i}{2\sqrt{q}}\right)^{n-1}\Pi_{k=1}^{n-1}\left(p(x) \pm 2i\sqrt{q}\cos\frac{k\pi}{n}\right)\\
=&a_1(x)\Pi_{k=1}^{n-1}\left(p(x) \pm 2i\sqrt{q}\cos\frac{k\pi}{n}\right),
\end{aligned}$$

where the equivalence of the last two products can be shown by setting $k \to n-k$ into one product as

$$\begin{aligned}
&\Pi_{k=1}^{n-1}\left(p(x) + 2i\sqrt{q}\cos\frac{(n-k)\pi}{n}\right)\\
=&\Pi_{k=1}^{n-1}\left(p(x) + 2i\sqrt{q}\cos\left(\pi - \frac{k\pi}{n}\right)\right)\\
=&\Pi_{k=1}^{n-1}\left(p(x) - 2i\sqrt{q}\cos\frac{k\pi}{n}\right).
\end{aligned}$$

Similarly, for recursive number sequence $(a_n)_{n\geq 0}$ we have

$$a_n = a_1\Pi_{k=1}^{n-1}\left(p \pm 2i\sqrt{q}\cos\frac{k\pi}{n}\right).$$

□

Example 3.9. For the Fibonacci polynomial sequence $\{F_n(x)\}$ defined in Example 2.2 with $p(x) = x$, $q = 1$, $F_0(x) = 0$ and $F_1(x) = 1$, from Theorem 3.8 we immediately have (38). For Fibonacci numbers $F_n = F_n(1)$, we also readily obtain (36).

For the Pell polynomial sequence defined in Example 2.3 by $P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$ $(n \geq 2)$, $P_0(x) = 0$, and $P_1(x) = 1$, and the Pell number sequence ($P_n =$

$P_n(1))_{n\geq 0}$, we have their product expressions as

$$P_n(x) = \Pi_{k=1}^{n-1}\left(2x \pm 2i\cos\frac{k\pi}{n}\right) = 2^{n-1}\Pi_{k=1}^{n-1}\left(x \pm i\cos\frac{k\pi}{n}\right)$$
$$P_n = 2^{n-1}\Pi_{k=1}^{n-1}\left(1 \pm i\cos\frac{k\pi}{n}\right).$$

Furthermore, $P_n(x)$ has roots $i\cos\frac{k\pi}{n}$ for $k = 1, 2, \ldots, n$.

For the balancing polynomials defined by the recurrence $B_n(x) = 6xB_{n-1}(x) - B_{n-2}(x)$ $(n \geq 2)$, $B_0(x) = 0$, and $B_1(x) = 1$ (cf. Frontczak [9]), we have have their product expressions as

$$B_n(x) = \sum_{k=1}^{n-1}\left(6x \pm 2i\sqrt{-1}\cos\frac{k\pi}{n}\right) = \Pi_{k=1}^{n-1}\left(6x \mp 2\cos\frac{k\pi}{n}\right),$$

which was also proved by Ray [28] using a different approach.

For the balancing numbers defined by $B_n = B_n(1)$ or by $B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 2$ with the initials $B_0 = 0$ and $B_1 = 1$ (cf. Example 2.4), we have their product expression

$$B_n = \Pi_{k=1}^{n-1}\left(6 \pm 2i\sqrt{-1}\cos\frac{k\pi}{n}\right) = \Pi_{k=1}^{n-1}\left(6 \mp 2\cos\frac{k\pi}{n}\right).$$

This formula was proved in Ray [29] using a more complicated approach.

4. **Identities raised from a Diophantus problem.** Diophantus raised the following problem (Heath [18, pp.179–181] and [7, p. 513],): "To find four numbers such that the product of any two increased by unity is a square", for which he obtained the solution 1/16, 33/16, 68/16, 105/16.

Fermat [8] (cf. p. 251) found the solution $1, 3, 8, 120$. In 1968, J.H. van Lint raised the problem whether the number 120 is the unique (positive) integer n for which the set $\{1, 3, 8, n\}$ constitutes a solution for the above Diophantus' problem; in the same year, Baker and Davenport [1] studied this question and concluded that, in fact, 120 is the unique integer satisfying the problem raised by J.H. van Lint. In 1977, Hoggatt and Bergum [19] observed that $1, 3, 8$ are, respectively, the terms F_2, F_4, F_6 of the Fibonacci sequence and formulated the problem of finding a positive integer n such that $F_{2t^n}+1, F_{2t+2^n}+1, F_{2t+4^n}+1$ be perfect squares. Hoggatt and Bergum also obtained the number $n = 4F_{2t+1}F_{2t+2}F_{2t+3}$, which, for $t = 1$, gives exactly $n = 120$.

The result shown in [19] was generalized in Morgado [25] by showing that the product of any two distinct elements of the set

$$\{F_n, F_{n+2r}, F_{n+4r}, 4F_{n+r}F_{n+2r}F_{n+3r}\},$$

increased by $\pm F_a^2F_b^2$ with suitable integers a and b is a perfect square. In other words, this set is a Fibonacci quadruple. This result was fourthly generalized in Horadam [20] by presenting that the product of any two distinct elements of the set

$$\{w_n, w_{n+2r}, w_{n+4r}, 4w_{n+r}w_{n+2r}w_{n+3r}\}, \tag{46}$$

where $w_n = w_n(a, b; p, q) = pw_{n-1} - qw_{n-2}$, $w_0 = a$, and $w_1 = b$, increased by a suitable integer, is a perfect square. In other words, this set is a Diophantine quadruple, which is a generalization of Fibonacci quadruple because $F_n = w_n(0, 1; 1, -1)$. Related work can be found in Melham and Shannon [24] and Cooper [6].

Udrea [31] generalizes a result obtained in [25] for the Fibonacci sequence and $(U_n = U_n(x))_{n\geq 0}$, the sequence of Chebyshev polynomials of the second kind, more precisely, the product of any two distinct elements of the set

$$\{U_n, U_{n+2r}, U_{n+4r}, 4U_{n+r}U_{n+2r}U_{n+3r}\}, \quad n, r \in \mathbb{N}_0, \tag{47}$$

increased by $U_a^2 U_b^2$, for suitable nonnegative integer a and b is a perfect square.

Morgado [26] proved an analogue result for $(T_n = T_n(x))_{n\geq 0}$, the sequence of Chebyshev polynomials of the first kind. More precisely, the product of any two distinct elements of the set

$$\{T_n, T_{n+2r}, T_{n+4r}, 4T_{n+r}T_{n+2r}T_{n+3r}\}, \quad n, r \in \mathbb{N}_0, \tag{48}$$

increased by $((T_h - T_k)/2)^t$, where T_h and T_k, with $k > h \geq 0$, are suitable terms of the sequence $(T_n)_{n\geq 0}$, is a perfect square.

We may use the sequence transformation technique to find the increased integers of the Diophantine quadruple (46) when $n = 2m$, $m \geq 0$, for some special set $\{a, b, p, q\}$.

Theorem 4.1. *Let $\{a_n\}_{n\geq 0}$ be a sequence defined by $a_n = pa_{n-1} + a_{n-2}$, $n \geq 2$, with $a_0 = 0$ and $a_1 = 1$. Then, we have a Diophantine quadruple*

$$\{a_{2m}, a_{2m+2r}, a_{2m+4r}, 4a_{2m+r}a_{2m+2r}a_{2m+3r}\} \tag{49}$$

for $m, r \in \mathbb{N}$. More precisely, the product of any two distinct elements of the set increased by $(a_\alpha a_\beta)^2$ for suitable $\alpha, \beta \in \mathbb{N}$ is a perfect square.

Proof. From (2.1) of [30], the sequence of Chebyshev polynomials of the second kind satisfy

$$U_m U_{m+r+s} + U_{r-1}U_{s-1} = U_{m+r}U_{m+s}$$

for any $x \in \mathbb{C}$, and $m, r, s \in \mathbb{N}$, and noting (16),

$$U_n(x_0) = a_1^{-1} q^{-n/2} (\pm i)^n a_{n+1} = (\pm i)^n a_{n+1},$$

where $x_0 = \pm ip/(2\sqrt{q})$, we immediately have

$$\begin{aligned}&(\pm i)^m a_{m+1} (\pm i)^{m+r+s} a_{m+r+s+1} + (\pm i)^{r-1} a_r (\pm i)^{s-1} a_s \\ =&(\pm i)^{m+r} a_{m+r+1} (\pm i)^{m+s} a_{m+s+1}.\end{aligned}$$

In the last equation, canceling $(\pm i)^{2m+r+s}$ and then replacing $m+1$ by $2m$ yields

$$a_{2m}a_{2m+r+s} + a_r a_s = a_{2m+r}a_{2m+s}. \tag{50}$$

By setting $s = r$ in (50), one obtain

$$a_{2m}a_{2m+2r} + a_r^2 = a_{2m+r}^2 \tag{51}$$

for $m, r \in \mathbb{N}$. Let us replace $2m$ by $2m + 2r$ in (51). Then

$$a_{2m+2r}a_{2m+4r} + a_r^2 = a_{2m+3r}^2 \tag{52}$$

for $m, r \in \mathbb{N}$. Identities (51) and (52) prove the theorem partially with $\alpha = r$ and $\beta = 1$.

Let us replace r by $2r$ in (51). Then,

$$a_{2m}a_{2m+4r} + a_{2r}^2 = a_{2m+2r}^2 \tag{53}$$

for $m, r \in \mathbb{N}$, which proves the theorem partially with $\alpha = 2r$ and $\beta = 0$.

From identity (50), it follows

$$(a_r a_s)^2 = (a_{2m+r}a_{2m+s} - a_{2m}a_{2m+r+s})^2$$

and so

$$4a_{2m}a_{2m+r}a_{2m+s}a_{2m+r+s} + a_r^2 a_s^2 = (a_{2m+r}a_{2m+s} + a_{2m}a_{2m+r+s})^2 \tag{54}$$

for $m, r, s \in \mathbb{N}$. By setting $s = 2r$ into (54), one obtains

$$4a_{2m}a_{2m+r}a_{2m+2r}a_{2m+3r} + a_r^2 a_{2r}^2 = (a_{2m+r}a_{2m+2r} + a_{2m}a_{2m+3r})^2 \tag{55}$$

for $m, r \in \mathbb{N}$. Let us replace $2m$ by $2m + r$ in (55). Then it becomes

$$4a_{2m+r}a_{2m+2r}a_{2m+3r}a_{2m+4r} + a_r^2 a_{2r}^2 = (a_{2m+2r}a_{2m+3r} + a_{2m+r}a_{2m+4r})^2 \tag{56}$$

for $m, r \in \mathbb{N}$. Identities (55) and (56) prove the theorem partially with $\alpha = r$ and $\beta = 2r$.

Finally, in (54), by using the replacements $s \to r$ and $2m \to 2m + r$, we have

$$4a_{2m+r}a_{2m+2r}^2 a_{2m+3r} + a_r^4 = (a_{2m+2r}^2 + a_{2m+r}a_{2m+3r})^2 \tag{57}$$

for $m, r \in \mathbb{N}$, which proves the theorem of the product of a_{2m+2r} and $4a_{2m+r}a_{2m+2r}a_{2m+3r}$ increased by $(a_\alpha a_\beta)^2$ with $\alpha = \beta = r$. Thus, the proof of the theorem is completed by the identities (51)-(53) and (55)-(57). □

From the Diophantine quadruple (48) of the sequence of Chebyshev polynomials of the first kind we have the following Diophantine quadruple of recursive sequences.

Theorem 4.2. *Let $\{a_n\}_{n\geq 0}$ be a sequence defined by $a_n = 2pa_{n-1} + a_{n-2}$, $n \geq 2$, with $a_0 = 1$ and $a_1 = p$. Then, we have a Diophantine quadruple*

$$\{a_{2m}, a_{2m+4r}, a_{2m+8r}, 4a_{2m+2r}a_{2m+4r}a_{2m+6r}\} \tag{58}$$

for $m, r \in \mathbb{N}$. More precisely, the product of any two distinct elements of the set increased by $((a_\alpha - a_\beta)/2)^t$ for suitable integers $\beta > \alpha \geq 0$ is a perfect square.

Proof. The transformation formula (8) shown in Theorem 1.2 with $x_0 = \pm ip/(2\sqrt{q}) = \pm ip/2$ and $2pa_0 = 2p = 2a_1$ gives

$$T_n(x_0) = a_0^{-1} q^{-n/2} (\pm i)^n a_n = (\pm i)^n a_n. \tag{59}$$

Replacing n and r in (2.3)-(2.8) in [26], we have

$$T_{2n}T_{2n+4r} + \frac{1}{2}(T_0 - T_{4r}) = T_{2n+2r}^2, \tag{60}$$

$$T_{2n}T_{2n+8r} + \frac{1}{2}(T_0 - T_{8r}) = T_{2n+4r}^2, \tag{61}$$

$$T_{2n+4r}T_{2n+8r} + \frac{1}{2}(T_0 - T_{4r}) = T_{2n+6r}^2, \tag{62}$$

$$4T_{2n}T_{2n+2r}T_{2n+4r}T_{2n+6r} + \left[\frac{1}{2}(T_{2r} - T_{6r})\right]^2 = (T_{2n}T_{2n+6r} + T_{2n+2r}T_{2n+4r})^2, \tag{63}$$

$$4T_{2n+2r}T_{2n+4r}T_{2n+6r}T_{2n+8r} + \left[\frac{1}{2}(T_{2r} - T_{6r})\right]^2 = (T_{2n+2r}T_{2n+8r} + T_{2n+4r}T_{2n+6r})^2, \tag{64}$$

$$4T_{2n+2r}T_{2n+4r}^2 T_{2n+6r} + \left[\frac{1}{2}(T_0 - T_{4r})\right]^2 = (T_{2n+2r}T_{2n+6r} + T_{2n+4r}^2)^2. \tag{65}$$

By substituting $x = x_0 = \pm ip/2$ and then (59) into above identities (60)-(65), we obtain

$$a_{2n}a_{2n+4r} + \frac{1}{2}(a_0 - a_{4r}) = a_{2n+2r}^2, \tag{66}$$

$$a_{2n}a_{2n+8r} + \frac{1}{2}(a_0 - a_{8r}) = a_{2n+4r}^2, \tag{67}$$

$$a_{2n+4r}a_{2n+8r} + \frac{1}{2}(a_0 - a_{4r}) = a_{2n+6r}^2, \tag{68}$$

$$4a_{2n}a_{2n+2r}a_{2n+4r}a_{2n+6r} + \left[\frac{1}{2}(a_{2r} - a_{6r})\right]^2 = (a_{2n}a_{2n+6r} + a_{2n+2r}a_{2n+4r})^2, \tag{69}$$

$$4a_{2n+2r}a_{2n+4r}a_{2n+6r}a_{2n+8r} + \left[\frac{1}{2}(a_{2r} - a_{6r})\right]^2 = (a_{2n+2r}a_{2n+8r} + a_{2n+4r}a_{2n+6r})^2, \tag{70}$$

$$4a_{2n+2r}a_{2n+4r}^2a_{2n+6r} + \left[\frac{1}{2}(a_0 - a_{4r})\right]^2 = (a_{2n+2r}a_{2n+6r} + a_{2n+4r}^2)^2. \tag{71}$$

The proof is completed by (66)-(71). □

For $p = 3$, the corresponding sequence $(a_n)_{n\geq 0} = (1, 3, 19, 117, 721, \ldots)$ is the OEIS sequence A005667 (cf. [27]), which generates a Diophantine quadruple shown in Theorem 4.2.

For many other p, we may obtain some new sequences with a Diophantine quadruple. Here, the new sequences mean that they are not included by the OEIS [27].

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