

BALANCE OF COMPLETE COHOMOLOGY IN EXTRIANGULATED CATEGORIES

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ABSTRACT. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category with a proper class ξ of \mathbb{E} -triangles. In this paper, we study the balance of complete cohomology in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, which is motivated by a result of Nucinkis that complete cohomology of modules is not balanced in the way the absolute cohomology Ext is balanced. As an application, we give some criteria for identifying a triangulated category to be Gorenstein and an Artin algebra to be F -Gorenstein.

1. Introduction. Exact categories and triangulated categories are two fundamental structures in different branches of mathematics. As expected, exact categories and triangulated categories are not independent of each other. In [19], Nakaoka and Palu introduced the notion of externally triangulated categories (extriangulated categories for short) as a simultaneous generalization of exact categories, triangulated categories and extension-closed subcategories of triangulated categories [19]. After that, the study of extriangulated categories has become an active topic, and up to now, many results on exact categories and triangulated categories can be unified in the same framework, e.g. see [15, 19]. Beligiannis developed in [5] a relative version of homological algebra in triangulated categories in analogy to relative homological algebra in abelian categories, in which the notion of a proper class of exact sequences is replaced by a proper class of triangles. Recently, the authors [15] studied a relative homological algebra in an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ which parallels the relative homological algebra given by A. Beligiannis [5] in a triangulated category. By specifying a class of \mathbb{E} -triangles, which is called a proper class

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ξ of \mathbb{E} -triangles, the authors introduced ξ -projective, ξ -injective, ξ - \mathcal{G} projective and ξ - \mathcal{G} injective dimensions, and discussed their properties.

It is well known that for any modules M and N over a ring R , a projective resolution of M and an injective coresolution of N lead to the same cohomology group $\text{Ext}_R^*(M, N)$. This implies that the absolute cohomology Ext is balanced, which is important and fundamental for classical homological algebra. Mislin [18] and Nucinkis [20] defined complete cohomology of modules. However, the complete cohomology of modules is not balanced in the way Ext is balanced by [20, Theorem 5.2]. Recently, the authors [14] developed a complete cohomology theory in an extriangulated category and demonstrated that this theory shared some basic properties of complete cohomology in the category of modules [6, 13, 18, 20, 24] and Tate cohomology in the triangulated category [1, 21, 22]. It seems natural to characterize when complete cohomology in extriangulated categories is balanced. The aim of this paper is to study this question.

We now outline the results of the paper. In Section 2, we summarize some preliminaries and basic facts about extriangulated categories which will be used throughout the paper.

From Section 3, we assume that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category with enough ξ -projectives and enough ξ -injectives satisfying the additional Condition 2.3. We first recall some definitions and basic properties of ξ -complete cohomology groups in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, and then we prove, under some conditions, that there are two long exact sequences of ξ -complete cohomology (see Theorems 3.9, 3.11 and 3.14).

In Section 4, we first show that for all objects M and N in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, if M has finite ξ - \mathcal{G} projective dimension and N has finite ξ - \mathcal{G} injective dimension, then ξ -complete cohomology groups $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, N)$ and $\widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, N)$ are isomorphic for any $i \in \mathbb{Z}$ (see Proposition 4.3), which improves [1, Theorem 4.11], [17, Theorem 2] and [21, Main Theorem]. As a result, we characterize when ξ -complete cohomology in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is balanced (see Theorem 4.4). As consequences, some criteria for a triangulated category to be Gorenstein and an Artin algebra to be F -Gorenstein are given (see Corollaries 4.6 and 4.8).

2. Preliminaries. We briefly recall some definitions and basic properties of extriangulated categories from [19]. We omit some details here, but the reader can find them in [19].

Let \mathcal{C} be an additive category equipped with an additive bifunctor

$$\mathbb{E} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Ab},$$

where Ab is the category of abelian groups. For any objects $A, C \in \mathcal{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an \mathbb{E} -extension. For an \mathbb{E} -extension $\delta \in \mathbb{E}(C, A)$, we briefly write

$$a_*\delta := \mathbb{E}(C, a)(\delta) \text{ and } c^*\delta := \mathbb{E}(c, A)(\delta).$$

Let \mathfrak{s} be a correspondence which associates an equivalence class

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$$

to any \mathbb{E} -extension $\delta \in \mathbb{E}(C, A)$. This \mathfrak{s} is called a *realization* of \mathbb{E} , if it makes the diagrams in [19, Definition 2.9] commutative. A triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is called an *extriangulated category* if it satisfies the following conditions.

1. $\mathbb{E} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Ab}$ is an additive bifunctor.
2. \mathfrak{s} is an additive realization of \mathbb{E} .

3. \mathbb{E} and \mathfrak{s} satisfy the compatibility conditions in [19, Definition 2.12].

Remark 2.1. Note that both exact categories and triangulated categories are ex-triangulated categories (see [19, Example 2.13]) and extension closed subcategories of ex-triangulated categories are again ex-triangulated (see [19, Remark 2.18]). However, there exist lots of ex-triangulated categories which are neither exact categories nor triangulated categories (see [19, Proposition 3.30] and [15, Remark 3.3]).

We will use the following terminology.

Definition 2.2. (see [19, Definitions 2.15 and 2.19]) Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an ex-triangulated category.

1. A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a *conflation* if it realizes some \mathbb{E} -extension $\delta \in \mathbb{E}(C, A)$. In this case, x is called an *inflation* and y is called a *deflation*.
2. If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ realizes $\delta \in \mathbb{E}(C, A)$, we write it in the following way:

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

and call it an \mathbb{E} -triangle. We usually do not write this “ δ ” if it is not used in the argument.

3. Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be any pair of \mathbb{E} -triangles. If a triplet (a, b, c) realizes $(a, c) : \delta \rightarrow \delta'$, then we write it as

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \overset{\delta}{\dashrightarrow} \\ a \downarrow & & b \downarrow & & c \downarrow \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \overset{\delta'}{\dashrightarrow} \end{array}$$

and call (a, b, c) a *morphism* of \mathbb{E} -triangles.

The following condition is analogous to the weak idempotent completeness in exact category (see [19, Condition 5.8]).

Condition 2.3. (Condition (WIC)) Consider the following conditions.

1. Let $f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C)$ be any composable pair of morphisms. If gf is an inflation, then so is f .
2. Let $f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C)$ be any composable pair of morphisms. If gf is a deflation, then so is g .

Example 2.4. (1) If \mathcal{C} is an exact category, then Condition (WIC) is equivalent to \mathcal{C} is weakly idempotent complete (see [7, Proposition 7.6]).

(2) If \mathcal{C} is a triangulated category, then Condition (WIC) is automatically satisfied.

Lemma 2.5. (see [19, Proposition 3.15]) Assume that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an ex-triangulated category. Let $A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \overset{\delta_1}{\dashrightarrow}$ and $A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \overset{\delta_2}{\dashrightarrow}$ be any

pair of \mathbb{E} -triangles. Then there is a commutative diagram in \mathcal{C}

$$\begin{array}{ccccc}
 & & A_2 & \xlongequal{\quad} & A_2 \\
 & & \downarrow m_2 & & \downarrow x_2 \\
 A_1 & \xrightarrow{m_1} & M & \xrightarrow{e_1} & B_2 \\
 \parallel & & \downarrow e_2 & & \downarrow y_2 \\
 A_1 & \xrightarrow{x_1} & B_1 & \xrightarrow{y_1} & C
 \end{array}$$

which satisfies $\mathfrak{s}(y_2^* \delta_1) = [A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2]$ and $\mathfrak{s}(y_1^* \delta_2) = [A_2 \xrightarrow{m_2} M \xrightarrow{e_2} B_1]$.

We recall the following definitions from [15, Section 3]. A class of \mathbb{E} -triangles ξ is closed under base change if for any \mathbb{E} -triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow^{\delta} \in \xi$$

and any morphism $c: C' \rightarrow C$, then any \mathbb{E} -triangle $A \xrightarrow{x'} B' \xrightarrow{y'} C' \dashrightarrow^{c^* \delta}$ belongs to ξ .

Dually, a class of \mathbb{E} -triangles ξ is closed under cobase change if for any \mathbb{E} -triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow^{\delta} \in \xi$$

and any morphism $a: A \rightarrow A'$, then any \mathbb{E} -triangle $A' \xrightarrow{x'} B' \xrightarrow{y'} C \dashrightarrow^{a_* \delta}$ belongs to ξ .

A class of \mathbb{E} -triangles ξ is called saturated if in the situation of Lemma 2.5, whenever

$A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \dashrightarrow^{\delta_2}$ and $A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2 \xrightarrow{y_2^* \delta_1} C$ belong to ξ , then the \mathbb{E} -triangle

$$A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \dashrightarrow^{\delta_1}$$

belongs to ξ .

An \mathbb{E} -triangle $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow^{\delta}$ is called split if $\delta = 0$. It is easy to see that it is split if and only if x is a section or y is a retraction. The class consisting of the split \mathbb{E} -triangles will be denoted by Δ_0 .

Definition 2.6. (see [15, Definition 3.1]) Let ξ be a class of \mathbb{E} -triangles which is closed under isomorphisms. Then ξ is called a proper class of \mathbb{E} -triangles if the following conditions hold:

1. ξ is closed under finite coproducts and $\Delta_0 \subseteq \xi$.
2. ξ is closed under base change and cobase change.
3. ξ is saturated.

Definition 2.7. (see [15, Definition 4.1]) An object $P \in \mathcal{C}$ is called ξ -projective if for any \mathbb{E} -triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow^{\delta}$$

in ξ , the induced sequence of abelian groups

$$0 \longrightarrow \mathcal{C}(P, A) \longrightarrow \mathcal{C}(P, B) \longrightarrow \mathcal{C}(P, C) \longrightarrow 0$$

is exact. Dually, we have the definition of ξ -injective objects.

We denote by $\mathcal{P}(\xi)$ (resp. $\mathcal{I}(\xi)$) the class of ξ -projective (resp. ξ -injective) objects of \mathcal{C} . It follows from the definition that the subcategories $\mathcal{P}(\xi)$ and $\mathcal{I}(\xi)$ are full, additive, closed under isomorphisms and direct summands.

An extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is said to have *enough ξ -projectives* (resp. *enough ξ -injectives*) provided that for each object A there exists an \mathbb{E} -triangle $K \twoheadrightarrow P \twoheadrightarrow A \dashrightarrow$ (resp. $A \twoheadrightarrow I \twoheadrightarrow K \dashrightarrow$) in ξ with $P \in \mathcal{P}(\xi)$ (resp. $I \in \mathcal{I}(\xi)$).

Let $K \twoheadrightarrow P \twoheadrightarrow A \dashrightarrow$ be an \mathbb{E} -triangle in ξ with $P \in \mathcal{P}(\xi)$, then we call K the *first ξ -syzygy* of A . An *n th ξ -syzygy* of A is defined as usual by induction. By Schanuel’s lemma ([15, Proposition 4.3]), any two ξ -syzygies of A are isomorphic modulo ξ -projectives.

The *ξ -projective dimension* $\xi\text{-pd}A$ of $A \in \mathcal{C}$ is defined inductively. If $A \in \mathcal{P}(\xi)$, then define $\xi\text{-pd}A = 0$. Next if $\xi\text{-pd}A > 0$, define $\xi\text{-pd}A \leq n$ if there exists an \mathbb{E} -triangle $K \rightarrow P \rightarrow A \dashrightarrow$ in ξ with $P \in \mathcal{P}(\xi)$ and $\xi\text{-pd}K \leq n - 1$. Finally we define $\xi\text{-pd}A = n$ if $\xi\text{-pd}A \leq n$ and $\xi\text{-pd}A \not\leq n - 1$. Of course we set $\xi\text{-pd}A = \infty$, if $\xi\text{-pd}A \neq n$ for all $n \geq 0$.

Dually we can define the *ξ -injective dimension* $\xi\text{-id}A$ of an object $A \in \mathcal{C}$.

Definition 2.8. (see [15, Definition 4.4]) A ξ -exact complex \mathbf{X} is a diagram

$$\cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \longrightarrow \cdots$$

in \mathcal{C} such that for each integer n , there exists an \mathbb{E} -triangle $K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xrightarrow{\delta_n} \dashrightarrow$ in ξ and $d_n = g_{n-1}f_n$.

Definition 2.9. (see [15, Definition 4.5]) Let \mathcal{W} be a class of objects in \mathcal{C} . An \mathbb{E} -triangle

$$A \twoheadrightarrow B \twoheadrightarrow C \dashrightarrow$$

in ξ is called $\mathcal{C}(-, \mathcal{W})$ -exact (resp. $\mathcal{C}(\mathcal{W}, -)$ -exact) if for any $W \in \mathcal{W}$, the induced sequence of abelian groups $0 \longrightarrow \mathcal{C}(C, W) \longrightarrow \mathcal{C}(B, W) \longrightarrow \mathcal{C}(A, W) \longrightarrow 0$ (resp. $0 \longrightarrow \mathcal{C}(W, A) \longrightarrow \mathcal{C}(W, B) \longrightarrow \mathcal{C}(W, C) \longrightarrow 0$) is exact in Ab .

Definition 2.10. (see [15, Definition 4.6]) Let \mathcal{W} be a class of objects in \mathcal{C} . A complex \mathbf{X} is called $\mathcal{C}(-, \mathcal{W})$ -exact (resp. $\mathcal{C}(\mathcal{W}, -)$ -exact) if it is a ξ -exact complex

$$\cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \longrightarrow \cdots$$

in \mathcal{C} such that there is a $\mathcal{C}(-, \mathcal{W})$ -exact (resp. $\mathcal{C}(\mathcal{W}, -)$ -exact) \mathbb{E} -triangle

$$K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xrightarrow{\delta_n} \dashrightarrow$$

in ξ for each integer n and $d_n = g_{n-1}f_n$.

A ξ -exact complex \mathbf{X} is called *complete $\mathcal{P}(\xi)$ -exact* (resp. *complete $\mathcal{I}(\xi)$ -exact*) if it is $\mathcal{C}(-, \mathcal{P}(\xi))$ -exact (resp. $\mathcal{C}(\mathcal{I}(\xi), -)$ -exact).

Definition 2.11. (see [15, Definition 4.7]) A *complete ξ -projective resolution* is a complete $\mathcal{P}(\xi)$ -exact complex

$$\mathbf{P} : \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \longrightarrow \cdots$$

in \mathcal{C} such that P_n is ξ -projective for each integer n . Dually, a *complete ξ -injective coresolution* is a complete $\mathcal{I}(\xi)$ -exact complex

$$\mathbf{I} : \cdots \longrightarrow I_1 \xrightarrow{d_1} I_0 \xrightarrow{d_0} I_{-1} \longrightarrow \cdots$$

in \mathcal{C} such that I_n is ξ -injective for each integer n .

Definition 2.12. (see [15, Definition 4.8]) Let \mathbf{P} be a complete ξ -projective resolution in \mathcal{C} . So for each integer n , there exists a $\mathcal{C}(-, \mathcal{P}(\xi))$ -exact \mathbb{E} -triangle

$K_{n+1} \xrightarrow{g_n} P_n \xrightarrow{f_n} K_n \xrightarrow{\delta_n} \rightarrow$ in ξ . The objects K_n are called ξ - \mathcal{G} projective for each integer n . Dually if \mathbf{I} is a complete ξ -injective coresolution in \mathcal{C} , there exists a $\mathcal{C}(\mathcal{I}(\xi), -)$ -exact \mathbb{E} -triangle $K_{n+1} \xrightarrow{g_n} I_n \xrightarrow{f_n} K_n \xrightarrow{\delta_n} \rightarrow$ in ξ for each integer n . The objects K_n are called ξ - \mathcal{G} injective for each integer n .

We denote by $\mathcal{GP}(\xi)$ (resp. $\mathcal{GI}(\xi)$) the class of ξ - \mathcal{G} projective (resp. ξ - \mathcal{G} injective) objects. It is obvious that $\mathcal{P}(\xi) \subseteq \mathcal{GP}(\xi)$ and $\mathcal{I}(\xi) \subseteq \mathcal{GI}(\xi)$.

Definition 2.13. (see [16, Definition 3.1]) Let M be an object in \mathcal{C} . A ξ -projective resolution of M is a ξ -exact complex $\mathbf{P} \rightarrow M$ such that $\mathbf{P}_n \in \mathcal{P}(\xi)$ for all $n \geq 0$. Dually, a ξ -injective coresolution of M is a ξ -exact complex $M \rightarrow \mathbf{I}$ such that $\mathbf{I}_n \in \mathcal{I}(\xi)$ for all $n \leq 0$.

We denote by $\text{Ch}(\mathcal{C})$ the category of complexes in \mathcal{C} ; the objects are complexes and morphisms are chain maps. We write the complexes homologically, so an object \mathbf{X} of $\text{Ch}(\mathcal{C})$ is of the form

$$\mathbf{X} := \cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}^{\mathbf{X}}} X_n \xrightarrow{d_n^{\mathbf{X}}} X_{n-1} \longrightarrow \cdots$$

The i th shift of \mathbf{X} is the complex $\mathbf{X}[i]$ with n th component \mathbf{X}_{n-i} and differential $d_n^{\mathbf{X}[i]} = (-1)^i d_{n-i}^{\mathbf{X}}$. Assume that \mathbf{X} and \mathbf{Y} are complexes in $\text{Ch}(\mathcal{C})$. A homomorphism $\varphi : \mathbf{X} \longrightarrow \mathbf{Y}$ of degree n is a family $(\varphi_i)_{i \in \mathbb{Z}}$ of morphisms $\varphi_i : X_i \longrightarrow Y_{i+n}$ for all $i \in \mathbb{Z}$. In this case, we set $|\varphi| = n$. All such homomorphisms form an abelian group, denoted by $\mathcal{C}(\mathbf{X}, \mathbf{Y})_n$, which is identified with $\prod_{i \in \mathbb{Z}} \mathcal{C}(X_i, Y_{i+n})$. We let $\mathcal{C}(\mathbf{X}, \mathbf{Y})$ be the complex of abelian groups with n th component $\mathcal{C}(\mathbf{X}, \mathbf{Y})_n$ and differential $d(\varphi_i) = d_{i+n}^{\mathbf{Y}} \varphi_i - (-1)^n \varphi_{i-1} d_i^{\mathbf{X}}$ for $\varphi = (\varphi_i) \in \mathcal{C}(\mathbf{X}, \mathbf{Y})_n$. We refer to [3, 8] for more details.

Remark 2.14. (see [14, Remark 3.3]) Let M and N be objects in \mathcal{C} .

1. Note that there are two ξ -projective resolutions $\mathbf{P}_M \longrightarrow M$ and $\mathbf{P}_N \longrightarrow N$ of M and N , respectively. A homomorphism $\beta \in \mathcal{C}(\mathbf{P}_M, \mathbf{P}_N)$ is *bounded above* if $\beta_i = 0$ for all $i \gg 0$. The subset $\overline{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_N)$, consisting of all bounded above homomorphisms, is a subcomplex with components

$$\overline{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_N)_n = \{(\varphi_i) \in \mathcal{C}(\mathbf{P}_M, \mathbf{P}_N)_n \mid \varphi_i = 0 \text{ for all } i \gg 0\}.$$

We set

$$\tilde{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_N) = \mathcal{C}(\mathbf{P}_M, \mathbf{P}_N) / \overline{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_N).$$

2. Note that there are two ξ -injective coresolutions $M \longrightarrow \mathbf{I}_M$ and $N \longrightarrow \mathbf{I}_N$ of M and N , respectively. A homomorphism $\beta \in \mathcal{C}(\mathbf{I}_M, \mathbf{I}_N)$ is *bounded below* if $\beta_i = 0$ for all $i \ll 0$. The subset $\underline{\mathcal{C}}(\mathbf{I}_M, \mathbf{I}_N)$, consisting of all bounded below homomorphisms, is a subcomplex with components

$$\underline{\mathcal{C}}(\mathbf{I}_M, \mathbf{I}_N)_n = \{(\varphi_i) \in \mathcal{C}(\mathbf{I}_M, \mathbf{I}_N)_n \mid \varphi_i = 0 \text{ for all } i \ll 0\}.$$

We set

$$\widetilde{\mathcal{C}}(\mathbf{I}_M, \mathbf{I}_N) = \mathcal{C}(\mathbf{I}_M, \mathbf{I}_N) / \underline{\mathcal{C}}(\mathbf{I}_M, \mathbf{I}_N).$$

Definition 2.15. (see [16, Definition 3.2]) Let M and N be objects in \mathcal{C} .

- (1) If we choose a ξ -projective resolution $\mathbf{P} \twoheadrightarrow M$ of M , then for any integer $n \geq 0$, the ξ -cohomology groups $\xi\text{xt}_{\mathcal{P}(\xi)}^n(M, N)$ are defined as

$$\xi\text{xt}_{\mathcal{P}(\xi)}^n(M, N) = H^n(\mathcal{C}(\mathbf{P}, N)).$$

- (2) If we choose a ξ -injective coresolution $N \twoheadrightarrow \mathbf{I}$ of N , then for any integer $n \geq 0$, the ξ -cohomology groups $\xi\text{xt}_{\mathcal{I}(\xi)}^n(M, N)$ are defined as

$$\xi\text{xt}_{\mathcal{I}(\xi)}^n(M, N) = H^n(\mathcal{C}(M, \mathbf{I})).$$

Remark 2.16. By [14, Lemma 3.2], one can see that $\xi\text{xt}_{\mathcal{P}(\xi)}^n(-, -)$ and $\xi\text{xt}_{\mathcal{I}(\xi)}^n(-, -)$ are cohomological functors for any integer $n \geq 0$, independent of the choice of ξ -projective resolutions and ξ -injective coresolutions, respectively. In fact, with the modifications of the usual proof, one obtains the isomorphism $\xi\text{xt}_{\mathcal{P}(\xi)}^n(M, N) \cong \xi\text{xt}_{\mathcal{I}(\xi)}^n(M, N)$, which is denoted by $\xi\text{xt}_{\xi}^n(M, N)$.

Throughout this paper, we always assume that $\mathcal{C} = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category and ξ is a proper class of \mathbb{E} -triangles in \mathcal{C} . We also assume that the extriangulated category \mathcal{C} has enough ξ -projectives and enough ξ -injectives satisfying the additional Condition 2.3.

3. ξ -complete cohomology and its long exact sequences. The goal of this section is to study long exact sequences of ξ -complete cohomology, which gives some preparations for the proof of the main result in the next section. To this end, we first recall some definitions and basic properties of ξ -complete cohomology in \mathcal{C} .

Definition 3.1. (see [14, Definition 3.4]) Let M and N be objects in \mathcal{C} , and let n be an integer.

1. Using ξ -projective resolutions, we define the n th ξ -complete cohomology group, denoted by $\widetilde{\xi\text{xt}}_{\mathcal{P}}^n(M, N)$, as

$$\widetilde{\xi\text{xt}}_{\mathcal{P}}^n(M, N) = H^n(\widetilde{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_N)),$$

where $\widetilde{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_N)$ is the complex defined in Remark 2.14(1).

2. Using ξ -injective coresolutions, we define the n th ξ -complete cohomology group, denoted by $\widetilde{\xi\text{xt}}_{\mathcal{I}}^n(M, N)$, as

$$\widetilde{\xi\text{xt}}_{\mathcal{I}}^n(M, N) = H^n(\widetilde{\mathcal{C}}(\mathbf{I}_M, \mathbf{I}_N)),$$

where $\widetilde{\mathcal{C}}(\mathbf{I}_M, \mathbf{I}_N)$ is the complex defined in Remark 2.14(2).

Definition 3.2. (see [14, Definition 4.3]) Let $M \in \mathcal{C}$ be an object. A ξ -complete resolution of M is a diagram

$$\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$$

of morphisms of complexes satisfying: (1) $\pi : \mathbf{P} \rightarrow M$ is a ξ -projective resolution of M ; (2) \mathbf{T} is a complete ξ -projective resolution; (3) $\nu : \mathbf{T} \rightarrow \mathbf{P}$ is a morphism such that $\nu_i = \text{id}_{T_i}$ for all $i \gg 0$.

The following lemma is a key result, which helps us to compute ξ -complete cohomology for objects having finite ξ - \mathcal{G} -projective dimension using ξ -complete resolutions.

Lemma 3.3. (see [14, Theorem 4.6]) *Let M and N be objects in \mathcal{C} . If M admits a ξ -complete resolution $\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$, then for any integer i , there exists an isomorphism*

$$\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, N) \cong H^i(\mathcal{C}(\mathbf{T}, N)).$$

Remark 3.4. Note that in the module categories and triangulated categories, the cohomology groups $H^n(\mathcal{C}(\mathbf{T}, N))$ are called Tate cohomology, see [1, 4] for more details. Motivated by this, for all objects M and N in \mathcal{C} , if M admits a ξ -complete resolution $\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$, then the cohomology group $H^n(\mathcal{C}(\mathbf{T}, N))$ can be defined as Tate cohomology group in \mathcal{C} for any integer n , which is also denoted by $\widetilde{\xi\text{xt}}_{\mathcal{P}}^n(M, N)$ by Lemma 3.3.

Now assume that M admits a ξ -complete resolution $\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$. For each $n \in \mathbb{Z}$, we have a comparison morphism

$$\widetilde{\varepsilon}_{\mathcal{P}}^n(M, N) : \xi\text{xt}_{\xi}^n(M, N) \rightarrow \widetilde{\xi\text{xt}}_{\mathcal{P}}^n(M, N)$$

given by

$$H^n\mathcal{C}(\nu, N) : H^n\mathcal{C}(\mathbf{P}, N) \rightarrow H^n\mathcal{C}(\mathbf{T}, N).$$

We denote by $\widetilde{\mathcal{GP}}(\xi)$ (resp. $\widetilde{\mathcal{GI}}(\xi)$) the full subcategory of \mathcal{C} whose objects have finite ξ - \mathcal{G} projective (resp. ξ - \mathcal{G} injective) dimension.

Lemma 3.5. (see [14, Lemma 4.5]) *Let $\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$ and $\mathbf{T}' \xrightarrow{\nu'} \mathbf{P}' \xrightarrow{\pi'} M'$ be ξ -complete resolutions of M and M' , respectively. For each morphism $\mu : M \rightarrow M'$, there exists a morphism $\bar{\mu}$, unique up to homotopy, making the right-hand square of the following diagram*

$$\begin{array}{ccccc} \mathbf{T} & \xrightarrow{\nu} & \mathbf{P} & \xrightarrow{\pi} & M \\ \downarrow \tilde{\mu} & & \downarrow \bar{\mu} & & \downarrow \mu \\ \mathbf{T}' & \xrightarrow{\nu'} & \mathbf{P}' & \xrightarrow{\pi'} & M' \end{array}$$

commutative, and for each choice of $\bar{\mu}$, there exists a morphism $\tilde{\mu}$, unique up to homotopy, making the left-hand square commute up to homotopy. In particular, if $\mu = \text{id}_M$, then $\tilde{\mu}$ and $\bar{\mu}$ are homotopy equivalences.

Remark 3.6. The assignment $(M, N) \mapsto \widetilde{\xi\text{xt}}_{\mathcal{P}}^n(M, N)$ defines a functor

$$\widetilde{\xi\text{xt}}_{\mathcal{P}}^n : \widetilde{\mathcal{GP}}(\xi)^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{Ab}$$

and the maps $\widetilde{\varepsilon}_{\mathcal{P}}^n(M, N)$ yields a morphism of functors $\widetilde{\varepsilon}_{\mathcal{P}}^n : \xi\text{xt}_{\xi}^n \rightarrow \widetilde{\xi\text{xt}}_{\mathcal{P}}^n$ such that both $\widetilde{\xi\text{xt}}_{\mathcal{P}}^n$ and $\widetilde{\varepsilon}_{\mathcal{P}}^n$ are independent of the choice of resolutions and liftings.

Lemma 3.7. *Let*

$$\mathbf{Q} : \cdots \longrightarrow Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} Q_{-1} \longrightarrow \cdots$$

be a complete ξ -projective resolution. If $M \in \widetilde{\mathcal{P}}(\xi)$ or $M \in \widetilde{\mathcal{I}}(\xi)$, then $\mathcal{C}(\mathbf{Q}, M)$ is exact.

Proof. It is easy to check by [15, Lemma 5.3] and [16, Lemma 4.5]. □

Proposition 3.8. (1) *Let $M \in \widetilde{\mathcal{GP}}(\xi)$. For any integer n the following are equivalent:*

- (i) $\xi\text{-}\mathcal{G}\text{pd}M \leq n$.
- (ii) The map $\tilde{\varepsilon}_{\mathcal{P}}^i(M, N) : \xi\text{xt}_{\xi}^i(M, N) \rightarrow \widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, N)$ is bijective for any $i \geq n + 1$ and any $N \in \mathcal{C}$.
- (2) If $\xi\text{-pd}M < \infty$, then $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, -) = 0$ for any $i \in \mathbb{Z}$.
- (3) If $\xi\text{-pd}M < \infty$, then $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(-, M) = 0$ for any $i \in \mathbb{Z}$.
- (4) If $\xi\text{-id}N < \infty$, then $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(-, N) = 0$ for any $i \in \mathbb{Z}$.

Proof. (1) (i) \Rightarrow (ii). Since $\xi\text{-}\mathcal{G}\text{pd}M \leq n$, one has a ξ -complete resolution

$$\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$$

of M such that ν_i is bijective for each $i \geq n$. It follows that $\tilde{\varepsilon}_{\mathcal{P}}^i(M, N) : \xi\text{xt}_{\xi}^i(M, N) \rightarrow \widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, N)$ is bijective for any $i \geq n + 1$ and any $N \in \mathcal{C}$, as desired.

(ii) \Rightarrow (i). Assume that $\tilde{\varepsilon}_{\mathcal{P}}^i(M, N)$ is bijective for any $i \geq n + 1$ and any $N \in \mathcal{C}$. In particular, for any $i \geq n + 1$ and any $P \in \mathcal{P}(\xi)$, one has $\xi\text{xt}_{\xi}^i(M, P) \cong \widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, P)$. But the latter is zero since each complete ξ -projective resolution is $\mathcal{C}(-, \mathcal{P}(\xi))$ -exact. Thus $\xi\text{xt}_{\xi}^i(M, P) = 0$ for any $i \geq n + 1$ and any $P \in \mathcal{P}(\xi)$. So $\xi\text{-}\mathcal{G}\text{pd}M \leq n$ by [16, Theorem 3.8].

(2) Assume that $\xi\text{-pd}M = n < \infty$. We can choose a ξ -projective resolution $\mathbf{P} \rightarrow M$ of length n . Then, in this case, $\mathbf{0} \xrightarrow{0} \mathbf{P} \xrightarrow{\pi} M$ is a ξ -complete resolution of M , and thus $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, -) = 0$ for any $i \in \mathbb{Z}$.

(3) and (4) follow from Lemma 3.7 directly. □

Now we show that there exists a long exact sequence of ξ -complete cohomology under some certain conditions.

Theorem 3.9. *Let $M \in \widetilde{\mathcal{GP}}(\xi)$ and consider an \mathbb{E} -triangle $\mathfrak{E} : A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A$ in ξ . Then there exists morphisms $\tilde{\partial}^n(M, \mathfrak{E})$, which are natural in M and \mathfrak{E} , such that the following sequence*

$$\begin{aligned} \cdots \longrightarrow \widetilde{\xi\text{xt}}_{\mathcal{P}}^n(M, A) \xrightarrow{\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, x)} \widetilde{\xi\text{xt}}_{\mathcal{P}}^n(M, B) \xrightarrow{\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, y)} \widetilde{\xi\text{xt}}_{\mathcal{P}}^n(M, C) \\ \xrightarrow{\tilde{\partial}^n(M, \mathfrak{E})} \widetilde{\xi\text{xt}}_{\mathcal{P}}^{n+1}(M, A) \longrightarrow \cdots \end{aligned}$$

is exact.

Moreover, the connecting map $\tilde{\partial}^n(M, \mathfrak{E})$ makes the following diagram

$$\begin{CD} \xi\text{xt}_{\xi}^n(M, C) @>\tilde{\partial}^n(M, \mathfrak{E})>> \xi\text{xt}_{\xi}^{n+1}(M, A) \\ @V\tilde{\varepsilon}_{\mathcal{P}}^n(M, C)VV @VV\tilde{\varepsilon}_{\mathcal{P}}^{n+1}(M, A)V \\ \widetilde{\xi\text{xt}}_{\mathcal{P}}^n(M, C) @>\tilde{\partial}^n(M, \mathfrak{E})>> \widetilde{\xi\text{xt}}_{\mathcal{P}}^{n+1}(M, A). \end{CD} \tag{1}$$

commutative for each $n \in \mathbb{Z}$.

Proof. Let $\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$ be a ξ -complete resolution of M . Then we have a commutative diagram of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{C}(\mathbf{P}, A) & \longrightarrow & \mathcal{C}(\mathbf{P}, B) & \longrightarrow & \mathcal{C}(\mathbf{P}, C) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{C}(\mathbf{T}, A) & \longrightarrow & \mathcal{C}(\mathbf{T}, B) & \longrightarrow & \mathcal{C}(\mathbf{T}, C) \longrightarrow 0. \end{array} \tag{2}$$

The rows are exact since all terms of \mathbf{P} and \mathbf{T} are ξ -projective. By the bottom row we get the desired long exact sequence, and by the commutativity of the diagram (2) and Lemma 3.3 we get the desired commutative diagram (1). The naturality in M follows from Lemma 3.5.

Assume that $\mathfrak{E}' : A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'} \gg$ is an \mathbb{E} -triangle and $\varphi : \mathfrak{E} \rightarrow \mathfrak{E}'$ is a morphism. Thus we have the following commutative diagram of \mathbb{E} -triangles

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \gg \\ a \downarrow & & b \downarrow & & c \downarrow \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \xrightarrow{\delta'} \gg. \end{array}$$

Since all terms of \mathbf{T} are ξ -projective, we have the following commutative diagram of exact complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{C}(\mathbf{T}, A) & \longrightarrow & \mathcal{C}(\mathbf{T}, B) & \longrightarrow & \mathcal{C}(\mathbf{T}, C) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{C}(\mathbf{T}, A') & \longrightarrow & \mathcal{C}(\mathbf{T}, B') & \longrightarrow & \mathcal{C}(\mathbf{T}, C') \longrightarrow 0. \end{array} \tag{3}$$

So the naturality in \mathfrak{E} holds by the commutativity of the diagram (3) and Lemma 3.3. □

Using standard arguments from relative homological algebra, one can prove the following version of the Horseshoe Lemma for ξ -complete resolutions. For convenience, we give the proof.

Lemma 3.10. (Horseshoe Lemma for ξ -complete resolutions) *Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg$ be an \mathbb{E} -triangle in ξ such that $\xi\text{-Gpd}A < \infty$ and $\xi\text{-Gpd}C < \infty$. Let $\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} A$ and $\mathbf{T}'' \xrightarrow{\nu''} \mathbf{P}'' \xrightarrow{\pi''} C$ be ξ -complete resolutions of A and C , respectively. Then there is a commutative diagram:*

$$\begin{array}{ccccc} \mathbf{T} & \longrightarrow & \mathbf{T}' & \longrightarrow & \mathbf{T}'' \\ \downarrow \nu & & \downarrow \nu' & & \downarrow \nu'' \\ \mathbf{P} & \longrightarrow & \mathbf{P}' & \longrightarrow & \mathbf{P}'' \\ \downarrow \pi & & \downarrow \pi' & & \downarrow \pi'' \\ A & \xrightarrow{x} & B & \xrightarrow{y} & C \end{array} \tag{4}$$

where the two upper rows are split \mathbb{E} -triangles in ξ and the columns are ξ -complete resolutions.

Proof. By [16, Lemma 3.3], we have a commutative diagram

$$\begin{array}{ccccc}
 \mathbf{P} & \longrightarrow & \mathbf{P}' & \longrightarrow & \mathbf{P}'' \\
 \downarrow \pi & & \downarrow \pi' & & \downarrow \pi'' \\
 A & \xrightarrow{x} & B & \xrightarrow{y} & C
 \end{array}$$

where $\mathbf{P}' \rightarrow B$ is a ξ -projective resolution of B . Setting $n = \max\{\xi\text{-Gpd}A, \xi\text{-Gpd}C\}$. Consider an \mathbb{E} -triangle $K_n \longrightarrow K'_n \longrightarrow K''_n \dashrightarrow$, where K_n (respectively, K'_n and K''_n) is the n th ξ -syzygy of A (respectively, B and C) obtained from the \mathbf{P} (respectively, \mathbf{P}' and \mathbf{P}''). Then K_n and K''_n are ξ - \mathcal{G} projective by [15, Proposition 5.2]. By a suitable adjustment (e.g. see [14, Proposition 4.4]), we can require that \mathbf{T} and \mathbf{T}'' are complete ξ -projective resolutions of K_n and K''_n , respectively. As a similar argument in proof of [15, Theorem 4.16], we can construct a complete ξ -projective resolution \mathbf{T}' of K'_n such that the desired commutative diagram (4) holds. \square

By Lemma 3.10, we can get the second long exact sequence of ξ -complete cohomology.

Theorem 3.11. *Let $\mathfrak{E} : A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow$ be an \mathbb{E} -triangle in ξ with $\xi\text{-Gpd}A < \infty$ and $\xi\text{-Gpd}C < \infty$. For each $N \in \mathcal{C}$, there exist morphisms $\tilde{\partial}^n(\mathfrak{E}, N)$, which are natural in N and \mathfrak{E} , such that the following sequence*

$$\begin{aligned}
 \cdots \longrightarrow \widetilde{\xi\text{xt}}_{\mathcal{P}}^n(C, N) &\xrightarrow{\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(y, N)} \widetilde{\xi\text{xt}}_{\mathcal{P}}^n(B, N) \xrightarrow{\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(x, N)} \widetilde{\xi\text{xt}}_{\mathcal{P}}^n(A, N) \\
 &\xrightarrow{\tilde{\partial}^n(\mathfrak{E}, N)} \widetilde{\xi\text{xt}}_{\mathcal{P}}^{n+1}(C, N) \longrightarrow \cdots
 \end{aligned}$$

is exact.

Moreover, the connecting map $\tilde{\partial}^n(\mathfrak{E}, N)$ makes the following diagram

$$\begin{array}{ccc}
 \xi\text{xt}_{\xi}^n(A, N) & \xrightarrow{\partial^n(\mathfrak{E}, N)} & \xi\text{xt}_{\xi}^{n+1}(C, N) \\
 \downarrow \tilde{\varepsilon}_{\mathcal{P}}^n(A, N) & & \downarrow \tilde{\varepsilon}_{\mathcal{P}}^{n+1}(C, N) \\
 \widetilde{\xi\text{xt}}_{\mathcal{P}}^n(A, N) & \xrightarrow{\tilde{\partial}^n(\mathfrak{E}, N)} & \widetilde{\xi\text{xt}}_{\mathcal{P}}^{n+1}(C, N)
 \end{array}$$

commutative for each $n \in \mathbb{Z}$.

Proof. Since A and C have finite ξ - \mathcal{G} projective dimension, we can construct the diagram (4) of ξ -complete resolutions. Moreover, since the two upper rows of (4) are split \mathbb{E} -triangles in ξ , by applying the functor $\mathcal{C}(-, N)$ we can get a commutative diagram of complexes

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{C}(\mathbf{P}'', N) & \longrightarrow & \mathcal{C}(\mathbf{P}', N) & \longrightarrow & \mathcal{C}(\mathbf{P}, N) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{C}(\mathbf{T}'', N) & \longrightarrow & \mathcal{C}(\mathbf{T}', N) & \longrightarrow & \mathcal{C}(\mathbf{T}, N) \longrightarrow 0
 \end{array}$$

with exact rows. By the bottom row we get the desired long exact sequence, and by the commutativity of the diagram and Lemma 3.3 we get the desired equality.

The naturality in N follows from Lemma 3.10. One can prove the naturality in \mathfrak{C} and its proof is similar to that of Theorem 3.9. \square

In the rest of this section, we will give the ξ -complete cohomology theory based on ξ - \mathcal{G} -injective objects. All arguments and proofs are similar to the above.

Definition 3.12. Let $N \in \mathcal{C}$ be an object. A ξ -complete coresolution of N is a diagram

$$N \xrightarrow{\iota} \mathbf{I} \xrightarrow{\mu} \mathbf{Q}$$

of morphisms of complexes satisfying: (1) $\iota : N \rightarrow \mathbf{I}$ is a ξ -injective coresolution of N ; (2) \mathbf{Q} is a complete ξ -injective coresolution; (3) $\mu : \mathbf{I} \rightarrow \mathbf{Q}$ is a morphism such that $\mu_i = \text{id}_{Q_i}$ for all $i \ll 0$.

An object N in \mathcal{C} admits a ξ -complete coresolution if and only if N has finite ξ - \mathcal{G} -injective dimension.

Now assume that N admits a ξ -complete coresolution $N \xrightarrow{\iota} \mathbf{I} \xrightarrow{\mu} \mathbf{Q}$. For each $n \in \mathbb{Z}$ and each $M \in \mathcal{C}$, we have

$$\widetilde{\xi\text{xt}}_{\mathcal{I}}^n(M, N) \cong H^n(\mathcal{C}(M, \mathbf{Q}))$$

These groups come equipped with comparison morphisms

$$\widetilde{\varepsilon}_{\mathcal{I}}^n(M, N) : \xi\text{xt}_{\xi}^n(M, N) \rightarrow \widetilde{\xi\text{xt}}_{\mathcal{I}}^n(M, N)$$

given by

$$H^n\mathcal{C}(M, \mu) : H^n\mathcal{C}(M, \mathbf{I}) \rightarrow H^n\mathcal{C}(M, \mathbf{Q}).$$

By a dual argument to the above, $\widetilde{\xi\text{xt}}_{\mathcal{I}}^n$ and $\widetilde{\varepsilon}_{\mathcal{I}}^n$ are independent of the choice of coresolutions and liftings.

Proposition 3.13. (1) Let $N \in \widetilde{\mathcal{GI}}(\xi)$. For any integer n the following are equivalent:

- (i) ξ - $\mathcal{G}\text{id}N \leq n$.
- (ii) The map $\widetilde{\varepsilon}_{\mathcal{I}}^i(M, N) : \xi\text{xt}_{\xi}^i(M, N) \rightarrow \widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, N)$ is bijective for any $i \geq n + 1$ and any $M \in \mathcal{C}$.
- (2) If $\xi\text{-id}N < \infty$, then $\widetilde{\xi\text{xt}}_{\mathcal{I}}^i(N, -) = 0$ for any $i \in \mathbb{Z}$.
- (3) If $\xi\text{-id}N < \infty$, then $\widetilde{\xi\text{xt}}_{\mathcal{I}}^i(-, N) = 0$ for any $i \in \mathbb{Z}$.
- (4) If $\xi\text{-pd}M < \infty$, then $\widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, -) = 0$ for any $i \in \mathbb{Z}$.

Theorem 3.14. (1) Let $N \in \widetilde{\mathcal{GI}}(\xi)$ and consider an \mathbb{E} -triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg \text{ in } \xi. \text{ Then there exists a long exact sequence}$$

$$\dots \xrightarrow{\widetilde{\xi\text{xt}}_{\mathcal{I}}^i(y, N) \sim^n} \widetilde{\xi\text{xt}}_{\mathcal{I}}^n(B, N) \xrightarrow{\widetilde{\xi\text{xt}}_{\mathcal{I}}^i(x, N) \sim^n} \widetilde{\xi\text{xt}}_{\mathcal{I}}^n(A, N) \longrightarrow \widetilde{\xi\text{xt}}_{\mathcal{I}}^{n+1}(C, N) \longrightarrow \dots$$

(2) Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg$ be an \mathbb{E} -triangle in ξ with $A, C \in \widetilde{\mathcal{GI}}(\xi)$. For each $M \in \mathcal{C}$, there exists a long exact sequence

$$\dots \xrightarrow{\widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, x) \sim^n} \widetilde{\xi\text{xt}}_{\mathcal{I}}^n(M, B) \xrightarrow{\widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, y) \sim^n} \widetilde{\xi\text{xt}}_{\mathcal{I}}^n(M, C) \longrightarrow \widetilde{\xi\text{xt}}_{\mathcal{I}}^{n+1}(M, A) \longrightarrow \dots$$

4. **The balance of ξ -complete cohomology.** Our goal in this section is to study the balance of ξ -complete cohomology. At first, we recall the following fact taking from [21].

Lemma 4.1. (see [21, Lemma 3.3]) *Given a commutative diagram*

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & M_{2,2} & \xrightarrow{d_{2,2}} & M_{1,2} & \xrightarrow{d_{1,2}} & M_{0,2} \\
 & & \downarrow e_{2,2} & & \downarrow e_{1,2} & & \downarrow e_{0,2} \\
 \cdots & \longrightarrow & M_{2,1} & \xrightarrow{d_{2,1}} & M_{1,1} & \xrightarrow{d_{1,1}} & M_{0,1} \\
 & & \downarrow e_{2,1} & & \downarrow e_{1,1} & & \downarrow e_{0,1} \\
 \cdots & \longrightarrow & M_{2,0} & \xrightarrow{d_{2,0}} & M_{1,0} & \xrightarrow{d_{1,0}} & M_{0,0}
 \end{array}$$

in Ab with all rows and columns exact. Then there are two complexes

$$\mathbf{C} : \cdots \rightarrow \text{Coker}d_{1,2} \rightarrow \text{Coker}d_{1,1} \rightarrow \text{Coker}d_{1,0} \rightarrow 0$$

and

$$\mathbf{D} : \cdots \rightarrow \text{Cokere}_{2,1} \rightarrow \text{Cokere}_{1,1} \rightarrow \text{Cokere}_{0,1} \rightarrow 0$$

with $H^n(\mathbf{C}) = H^n(\mathbf{D})$ for all n .

The following lemma is the crucial step in the study of the balance of ξ -complete cohomology, and the proof is modified from the construction of pinched complexes by Christensen and Jorgensen [9]. For the balance properties, see also [10, 11].

Lemma 4.2. *Assume that M is ξ - \mathcal{G} -projective with a complete ξ -projective resolution \mathbf{T} , and N is ξ - \mathcal{G} -injective with a complete ξ -injective coresolution \mathbf{Q} . Then there are isomorphisms*

$$H^i\mathcal{C}(\mathbf{T}, N) \cong H^i\mathcal{C}(M, \mathbf{Q})$$

for all $i \in \mathbb{Z}$.

Proof. Consider the following truncated complexes

$$\mathbf{T}_{\geq 0} : \cdots \rightarrow T_2 \rightarrow T_1 \rightarrow T_0 \rightarrow 0$$

and

$$\mathbf{Q}_{\leq 0} : 0 \rightarrow Q_0 \rightarrow Q_{-1} \rightarrow Q_{-2} \rightarrow \cdots$$

It is easy to see that $\mathbf{T} \rightarrow \mathbf{T}_{\geq 0} \rightarrow M$ is a ξ -complete resolution of M , and

$N \rightarrow \mathbf{Q}_{\leq 0} \rightarrow \mathbf{Q}$ is a ξ -complete coresolution of N . By Propositions 3.8 and 3.13,

$$H^i\mathcal{C}(\mathbf{T}, N) = \widetilde{\xi\text{xt}}^i_p(M, N) \cong \xi\text{xt}^i_\xi(M, N) \cong \widetilde{\xi\text{xt}}^i_{\mathbf{T}}(M, N) = H^i\mathcal{C}(M, \mathbf{Q})$$

for any $i \geq 1$.

Now consider the following ξ -exact complexes

$$\mathbf{T}_{\leq 0} : 0 \rightarrow M \rightarrow T_{-1} \rightarrow T_{-2} \rightarrow \cdots$$

and

$$\mathbf{Q}_{\geq 0} : \cdots \rightarrow Q_2 \rightarrow Q_1 \rightarrow N \rightarrow 0.$$

We have the following commutative diagram

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & \mathcal{C}(T_{-2}, Q_2) & \longrightarrow & \mathcal{C}(T_{-2}, Q_1) & \longrightarrow & \mathcal{C}(T_{-2}, N) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & \mathcal{C}(T_{-1}, Q_2) & \longrightarrow & \mathcal{C}(T_{-1}, Q_1) & \longrightarrow & \mathcal{C}(T_{-1}, N) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & \mathcal{C}(M, Q_2) & \longrightarrow & \mathcal{C}(M, Q_1) & \longrightarrow & \mathcal{C}(M, N) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
 \end{array}$$

Since $T_{-i} \in \mathcal{P}(\xi)$ and $Q_i \in \mathcal{I}(\xi)$ for any $i \geq 1$, all rows and columns are exact except the bottom row $\mathcal{C}(M, \mathbf{Q}_{\geq 0})$ and the far right column $\mathcal{C}(\mathbf{T}_{\leq 0}, N)$. By Lemma 4.1, the induced complexes

$$\mathcal{C}(M, \mathbf{Q}_{\geq 1}) : \cdots \rightarrow \mathcal{C}(M, Q_3) \rightarrow \mathcal{C}(M, Q_2) \rightarrow \mathcal{C}(M, Q_1) \rightarrow 0$$

and

$$\mathcal{C}(\mathbf{T}_{\leq -1}, N) : \cdots \rightarrow \mathcal{C}(T_{-3}, N) \rightarrow \mathcal{C}(T_{-2}, N) \rightarrow \mathcal{C}(T_{-1}, N) \rightarrow 0$$

have isomorphic cohomology groups, that is, for any $i \leq -2$,

$$H^i \mathcal{C}(\mathbf{T}, N) = H^i \mathcal{C}(\mathbf{T}_{\leq -1}, N) = H^i \mathcal{C}(M, \mathbf{Q}_{\geq 1}) = H^i \mathcal{C}(M, \mathbf{Q}).$$

For the complete ξ -injective coresolution $\mathbf{Q} : \cdots \rightarrow Q_2 \xrightarrow{d_2^{\mathbf{Q}}} Q_1 \xrightarrow{d_1^{\mathbf{Q}}} Q_0 \rightarrow \cdots$, there are \mathbb{E} -triangles $L_{i+1} \xrightarrow{s_i} Q_i \xrightarrow{t_i} L_i \xrightarrow{\delta_i}$ (here $L_1 = N$) in ξ such that $d_i^{\mathbf{Q}} = s_{i-1}t_i$ for all $i \in \mathbb{Z}$. Now consider the following commutative diagram

$$\begin{array}{ccccccc}
 \mathcal{C}(T_{-1}, Q_2) \oplus \mathcal{C}(T_{-2}, Q_1) & \xrightarrow{\rho_2} & \mathcal{C}(T_{-1}, Q_1) & \xrightarrow{\rho_1} & \mathcal{C}(T_0, Q_0) & \xrightarrow{\rho_0} & \mathcal{C}(T_1, Q_0) \oplus \mathcal{C}(T_0, Q_{-1}) \\
 \downarrow (0, \mathcal{C}(T_{-2}, t_1)) & & \downarrow \mathcal{C}(T_{-1}, t_1) & & \uparrow \mathcal{C}(T_0, s_0) & & \uparrow (\mathcal{C}(T_1, s_0)) \\
 \mathcal{C}(T_{-2}, N) & \xrightarrow{\mathcal{C}(d_{-1}^{\mathbf{T}}, N)} & \mathcal{C}(T_{-1}, N) & \xrightarrow{\mathcal{C}(d_0^{\mathbf{T}}, N)} & \mathcal{C}(T_0, N) & \xrightarrow{\mathcal{C}(d_1^{\mathbf{T}}, N)} & \mathcal{C}(T_1, N)
 \end{array} \tag{5}$$

where

$$\begin{aligned}
 \rho_2 &: \mathcal{C}(T_{-1}, Q_2) \oplus \mathcal{C}(T_{-2}, Q_1) \rightarrow \mathcal{C}(T_{-1}, Q_1) \\
 \alpha &:= (\alpha_{-1}, \alpha_{-2}) \mapsto d_2^{\mathbf{Q}}\alpha_{-1} + \alpha_{-2}d_{-1}^{\mathbf{T}} \\
 \rho_1 &: \mathcal{C}(T_{-1}, Q_1) \rightarrow \mathcal{C}(T_0, Q_0) \\
 \beta &\mapsto d_1^{\mathbf{Q}}\beta d_0^{\mathbf{T}} \\
 \rho_0 &: \mathcal{C}(T_0, Q_0) \rightarrow \mathcal{C}(T_1, Q_0) \oplus \mathcal{C}(T_0, Q_{-1}). \\
 \gamma &\mapsto (\gamma d_1^{\mathbf{T}}, d_0^{\mathbf{Q}}\gamma)
 \end{aligned}$$

Define

$$\begin{aligned}
 \Phi &: \text{Ker}\rho_1/\text{Im}\rho_2 \rightarrow \text{Ker}\mathcal{C}(d_0^{\mathbf{T}}, N)/\text{Im}\mathcal{C}(d_{-1}^{\mathbf{T}}, N). \\
 \alpha + \text{Im}\rho_2 &\mapsto t_1\alpha + \text{Im}\mathcal{C}(d_{-1}^{\mathbf{T}}, N)
 \end{aligned}$$

We first show that it is a well-defined map. Indeed, let $\alpha \in \text{Ker}\rho_1$. Then

$$0 = \rho_1(\alpha) = d_1^{\mathbf{Q}}\alpha d_0^{\mathbf{T}} = s_0 t_1 \alpha d_0^{\mathbf{T}} = \mathcal{C}(T_0, s_0)(t_1 \alpha d_0^{\mathbf{T}}).$$

Since $T_0 \in \mathcal{P}(\xi)$, $\mathcal{C}(T_0, s_0)$ is injective, and hence $\mathcal{C}(d_0^{\mathbf{T}}, N)(t_1 \alpha) = t_1 \alpha d_0^{\mathbf{T}} = 0$. Thus $t_1 \alpha \in \text{Ker}\mathcal{C}(d_0^{\mathbf{T}}, N)$. Moreover, if $\beta \in \text{Im}\rho_2$, then $t_1 \beta \in \text{Im}\mathcal{C}(d_{-1}^{\mathbf{T}}, N)$ by the commutativity of left square of (5). This shows that Φ is a well-defined map.

We next show that Φ is an isomorphism. Firstly, since $T_{-1} \in \mathcal{P}(\xi)$, there is an exact sequence

$$0 \longrightarrow \mathcal{C}(T_{-1}, L_2) \xrightarrow{\mathcal{C}(T_{-1}, s_1)} \mathcal{C}(T_{-1}, Q_1) \xrightarrow{\mathcal{C}(T_{-1}, t_1)} \mathcal{C}(T_{-1}, N) \longrightarrow 0.$$

In particular, the induced map $\mathcal{C}(T_{-1}, t_1)$ is surjective, and following this we easily get that Φ is surjective. Now assume that $\Phi(\alpha + \text{Im}\rho_2) = 0$, that is, $t_1 \alpha \in \text{Im}\mathcal{C}(d_{-1}^{\mathbf{T}}, N)$. Then there is $\beta \in \mathcal{C}(T_{-2}, N)$ with $t_1 \alpha = \mathcal{C}(d_{-1}^{\mathbf{T}}, N)(\beta)$. Similarly, since $T_{-2} \in \mathcal{P}(\xi)$, the induced map $\mathcal{C}(T_{-2}, t_1)$ is surjective, and hence there is $\gamma := (\gamma_{-1}, \gamma_{-2}) \in \mathcal{C}(T_{-1}, Q_2) \oplus \mathcal{C}(T_{-2}, Q_1)$ with $(0, \mathcal{C}(T_{-2}, t_1))(\gamma) = \beta$. By the commutativity of left square of (5), $\mathcal{C}(T_{-1}, t_1)(\alpha) = t_1 \alpha = \mathcal{C}(d_{-1}^{\mathbf{T}}, N)(\beta) = \mathcal{C}(d_{-1}^{\mathbf{T}}, N)((0, \mathcal{C}(T_{-2}, t_1))(\gamma)) = \mathcal{C}(T_{-1}, t_1)(\rho_2(\gamma))$, which shows that $\mathcal{C}(T_{-1}, t_1)(\alpha - \rho_2(\gamma)) = 0$, that is, $\alpha - \rho_2(\gamma) \in \text{Ker}\mathcal{C}(T_{-1}, t_1) = \text{Im}\mathcal{C}(T_{-1}, s_1)$. Thus there exists $\delta \in \mathcal{C}(T_{-1}, L_2)$ such that $\alpha - \rho_2(\gamma) = \mathcal{C}(T_{-1}, s_1)(\delta)$. By the surjectivity of $\mathcal{C}(T_{-1}, t_2)$, there is $\epsilon \in \mathcal{C}(T_{-1}, Q_2)$ with $\delta = \mathcal{C}(T_{-1}, t_2)(\epsilon)$. It follows that

$$\begin{aligned} \alpha &= \rho_2(\gamma) + s_1 \delta = d_2^{\mathbf{Q}}\gamma_{-1} + \gamma_{-2} d_{-1}^{\mathbf{T}} + s_1 t_1 \epsilon \\ &= d_2^{\mathbf{Q}}\gamma_{-1} + \gamma_{-2} d_{-1}^{\mathbf{T}} + d_2^{\mathbf{Q}}\epsilon = d_2^{\mathbf{Q}}(\gamma_{-1} + \epsilon) + \gamma_{-2} d_{-1}^{\mathbf{T}} \\ &= \rho_2(\gamma_{-1} + \epsilon, \gamma_{-2}) \end{aligned}$$

which means that $\alpha \in \text{Im}\rho_2$. This shows that Φ is injective, and hence Φ is an isomorphism.

Similarly, we can show that the following map

$$\begin{aligned} \Psi : \text{Ker}\mathcal{C}(d_1^{\mathbf{T}}, N)/\text{Im}\mathcal{C}(d_0^{\mathbf{T}}, N) &\rightarrow \text{Ker}\rho_0/\text{Im}\rho_1 \\ \alpha + \text{Im}\mathcal{C}(d_0^{\mathbf{T}}, N) &\mapsto s_0 \alpha + \text{Im}\rho_1 \end{aligned}$$

is an isomorphism.

On the other hand, we can prove that there is a commutative diagram as follows

$$\begin{array}{ccccccc} \mathcal{C}(T_{-1}, Q_2) \oplus \mathcal{C}(T_{-2}, Q_1) & \xrightarrow{\rho_2} & \mathcal{C}(T_{-1}, Q_1) & \xrightarrow{\rho_1} & \mathcal{C}(T_0, Q_0) & \xrightarrow{\rho_0} & \mathcal{C}(T_1, Q_0) \oplus \mathcal{C}(T_0, Q_{-1}) \\ \downarrow & & \downarrow & & \uparrow & & \uparrow \\ \mathcal{C}(M, Q_2) & \longrightarrow & \mathcal{C}(M, Q_1) & \longrightarrow & \mathcal{C}(M, Q_0) & \longrightarrow & \mathcal{C}(M, Q_{-1}) \end{array} \tag{6}$$

and the two rows have isomorphic cohomological groups. It follows that the bottom row of (5) and the bottom row of (6) have isomorphic cohomological groups, that is, $H^i\mathcal{C}(\mathbf{T}, N) \cong H^i\mathcal{C}(M, \mathbf{Q})$ for $i = 0, -1$. \square

More generally, we have the balance of ξ -complete cohomology as follows, which is the key result to prove the main result, and meanwhile generalizes [1, Theorem 4.11], [17, Theorem 2] and [21, Main Theorem].

Proposition 4.3. *Let $M \in \widetilde{\mathcal{GP}}(\xi)$ and $N \in \widetilde{\mathcal{GI}}(\xi)$. Then*

$$\widetilde{\xi}\text{xt}_{\mathcal{P}}^i(M, N) \cong \widetilde{\xi}\text{xt}_{\mathcal{I}}^i(M, N)$$

for any $i \in \mathbb{Z}$.

Proof. Assume that $\xi\text{-}\mathcal{G}\text{pd}M = m < \infty$ and $\xi\text{-}\mathcal{G}\text{id}N = n < \infty$. Let $\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$ be a ξ -complete resolution of M and $N \xrightarrow{\iota} \mathbf{I} \xrightarrow{\mu} \mathbf{Q}$ a ξ -complete coresolution of N .

For any $i \in \mathbb{Z}$, there are \mathbb{E} -triangles

$$K_{i+1} \longrightarrow T_i \longrightarrow K_i \dashrightarrow, \quad D_{i+1} \longrightarrow P_i \longrightarrow D_i \dashrightarrow$$

and

$$J_{i+1} \longrightarrow I_i \longrightarrow J_i \dashrightarrow, \quad L_{i+1} \longrightarrow Q_i \longrightarrow L_i \dashrightarrow$$

in ξ . Set $M = D_0$ and $N = J_1$. Then $K_m = D_m$ is $\xi\text{-}\mathcal{G}$ projective and $L_{-n+1} = J_{-n+1}$ is $\xi\text{-}\mathcal{G}$ injective.

Consider the truncations

$$\mathbf{T}_{\geq m} : \cdots \rightarrow T_{m+2} \rightarrow T_{m+1} \rightarrow T_m \rightarrow 0$$

and

$$\mathbf{Q}_{\leq -n} : 0 \rightarrow Q_{-n} \rightarrow Q_{-n-1} \rightarrow Q_{-n-2} \rightarrow \cdots$$

Then $\mathbf{T}[-m] \rightarrow \mathbf{T}_{\geq m}[-m] \rightarrow K_m$ is a ξ -complete resolution of K_m and $L_{-n+1} \rightarrow \mathbf{Q}_{\leq -n}[n] \rightarrow \mathbf{Q}[n]$ is a ξ -complete coresolution of L_{-n+1} . Thus we have

$$\begin{aligned} \widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, N) &= H^i\mathcal{C}(\mathbf{T}, N) = H^{i-m}\mathcal{C}(\mathbf{T}[-m], N) = \widetilde{\xi\text{xt}}_{\mathcal{P}}^{i-m}(K_m, N) \\ &\cong \widetilde{\xi\text{xt}}_{\mathcal{P}}^{i-m-n}(K_m, J_{-n+1}) \quad (\text{by Proposition 3.8(4) and Theorem 3.9}) \\ &= \widetilde{\xi\text{xt}}_{\mathcal{P}}^{i-m-n}(K_m, L_{-n+1}) = H^{i-m-n}\mathcal{C}(\mathbf{T}[-m], L_{-n+1}) \\ &\cong H^{i-m-n}\mathcal{C}(K_m, \mathbf{Q}[n]) \quad (\text{by Lemma 4.2}) \\ &= H^{i-m}\mathcal{C}(K_m, \mathbf{Q}) \\ &= \widetilde{\xi\text{xt}}_{\mathcal{I}}^i(K_m, N). \end{aligned}$$

Dually, using Proposition 3.13 and Theorem 3.14 we have

$$\widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, N) \cong \widetilde{\xi\text{xt}}_{\mathcal{I}}^{i-m}(D_m, N) = \widetilde{\xi\text{xt}}_{\mathcal{I}}^{i-m}(K_m, N).$$

Therefore, $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, N) \cong \widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, N)$ for any $i \in \mathbb{Z}$. □

Motivated by Gedrich and Gruenberg’s invariants of a ring [12], and Asadollahi and Salarian’s invariants to a triangulated category [1], the authors assigned in [16, Definition 4.2] two invariants to an extriangulated category \mathcal{C} :

$$\xi\text{-silp}\mathcal{C} = \sup\{\xi\text{-id}P \mid P \in \mathcal{P}(\xi)\},$$

$$\xi\text{-spli}\mathcal{C} = \sup\{\xi\text{-pd}I \mid I \in \mathcal{I}(\xi)\}.$$

It follows from [16, Proposition 4.3] that if both $\xi\text{-silp}\mathcal{C}$ and $\xi\text{-spli}\mathcal{C}$ are finite, then they are equal.

Recall from [16, Definition 4.1] that a full subcategory $\mathcal{X} \subseteq \mathcal{C}$ is called a *generating subcategory* of \mathcal{C} if for all $M \in \mathcal{C}$, $\mathcal{C}(\mathcal{X}, M) = 0$ implies that $M = 0$. Dually, a full subcategory $\mathcal{Y} \subseteq \mathcal{C}$ is called a *cogenerating subcategory* of \mathcal{C} if for all $N \in \mathcal{C}$, $\mathcal{C}(N, \mathcal{Y}) = 0$ implies that $N = 0$.

Next, we are in a position to prove the main result of this section.

Theorem 4.4. *Let \mathcal{C} be an extriangulated category, and let $\mathcal{P}(\xi)$ be a generating subcategory of \mathcal{C} and $\mathcal{I}(\xi)$ a cogenerating subcategory of \mathcal{C} . If $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, N) \cong \widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, N)$ for all objects M and N in \mathcal{C} , then $\xi\text{-spli}\mathcal{C} = \xi\text{-silp}\mathcal{C} < \infty$. The converses hold if \mathcal{C} satisfies the following condition:*

Condition (\star) : *If $N \in \mathcal{C}$ and $M \in \widetilde{\mathcal{P}}(\xi)$ such that $\xi\text{xt}_{\xi}^i(M, N) = 0$ for any $i \geq 1$, then $\mathcal{C}(M, N) \cong \xi\text{xt}_{\xi}^0(M, N)$. Dually, if $N \in \mathcal{C}$ and $M \in \widetilde{\mathcal{I}}(\xi)$ such that $\xi\text{xt}_{\xi}^i(N, M) = 0$ for any $i \geq 1$, then $\mathcal{C}(N, M) \cong \xi\text{xt}_{\xi}^0(N, M)$.*

Proof. Assume that $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, N) \cong \widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, N)$ for all objects M and N in \mathcal{C} . Let M be an object in $\mathcal{P}(\xi)$. Then $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, M) = 0$. Thus $\widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, M) = 0$, and hence $\xi\text{-id}M < \infty$ by [14, Theorem 3.10]. Similarly, one can show that $\xi\text{-pd}N < \infty$ for any object in $\mathcal{I}(\xi)$. So $\xi\text{-spli}\mathcal{C} = \xi\text{-silp}\mathcal{C} < \infty$ by [16, Proposition 4.3].

By hypothesis and [16, Theorem 4.7], we get that every object in \mathcal{C} has finite $\xi\text{-}\mathcal{G}$ projective dimension and finite $\xi\text{-}\mathcal{G}$ injective dimension. So $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, N) \cong \widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, N)$ for all objects M and N in \mathcal{C} by Proposition 4.3. □

Example 4.5. (see [16, Example 4.10]) (1) Assume that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an exact category and ξ is a class of exact sequences which is closed under isomorphisms. One can check that Condition (\star) in Theorem 4.4 is automatically satisfied.

(2) If \mathcal{C} is a triangulated category and the class ξ of triangles is closed under isomorphisms and suspension (see [5, Section 2.2] and [15, Remark 3.3(3)]), then Condition (\star) in Theorem 4.4 is also satisfied.

Recall from [1, Definition 4.6] that a triangulated category \mathcal{C} is called *Gorenstein* if there exists a positive integer n such that any object of \mathcal{C} has both $\xi\text{-}\mathcal{G}$ projective and $\xi\text{-}\mathcal{G}$ injective dimension less than or equal to n .

Corollary 4.6. *Let \mathcal{C} be a triangulated category, and let $\mathcal{P}(\xi)$ be a generating subcategory of \mathcal{C} and $\mathcal{I}(\xi)$ a cogenerating subcategory of \mathcal{C} . Then \mathcal{C} is Gorenstein if and only if $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, N) \cong \widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, N)$ for all objects M and N in \mathcal{C} .*

Proof. The result holds by Theorem 4.4, Example 4.5(2) and [16, Corollary 4.12]. □

Let R be a ring and $\text{Mod}R$ the category of left R -modules. Then it is clear that the class of projective left R -modules is a generating subcategory of $\text{Mod}R$ and the class of injective R -modules is a cogenerating subcategory of $\text{Mod}R$. If we assume that \mathcal{C} is the category of left R -modules and ξ is the class of all exact sequences in $\text{Mod}R$, then by Theorem 4.4 and Example 4.5(1), we have the following corollary, which has been proved by Nucinkis in [20].

Corollary 4.7. (see [20, Theorem 5.2]) *Let R be a ring. Then $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, N) \cong \widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, N)$ for all left R -modules M and N if and only if $\text{spli}R = \text{silp}R < \infty$.*

Let Λ be an Artin algebra over a commutative Artin ring k , and $\text{mod}\Lambda$ the category of finitely generated left Λ -modules. We denote by $(\text{mod}\Lambda)^{op}$ the opposite category of $\text{mod}\Lambda$. Suppose that F is an additive subbifunctor of the additive bifunctor $\text{Ext}_{\Lambda}^1(-, -) : (\text{mod}\Lambda)^{op} \times \text{mod}\Lambda \rightarrow \text{Ab}$. Recall from [2] that a short exact sequence $\eta : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $\text{mod}\Lambda$ is said to be F -exact if η is in $F(C, A)$. A Λ -module P (resp. I) in $\text{mod}\Lambda$ is said to be F -projective (resp.

F -injective) if for each F -exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, the sequence $0 \rightarrow \text{Hom}_\Lambda(P, A) \rightarrow \text{Hom}_\Lambda(P, B) \rightarrow \text{Hom}_\Lambda(P, C) \rightarrow 0$ (resp. $0 \rightarrow \text{Hom}_\Lambda(C, I) \rightarrow \text{Hom}_\Lambda(B, I) \rightarrow \text{Hom}_\Lambda(A, I) \rightarrow 0$) is exact. The full subcategory of $\text{mod}\Lambda$ consisting of all F -projective (resp. F -injective) modules is denoted by $\mathcal{P}(F)$ (resp. $\mathcal{I}(F)$). Assume that F has enough projectives and injectives. Then $(\text{mod}\Lambda, \varepsilon)$ is an exact category, where ε is the class of F -exact sequences.

On the other hand, it follows from [23, Theorem 3.4] that Λ is F -Gorenstein if and only if $\sup\{\text{pd}_F I \mid I \in \mathcal{I}(F)\} = \sup\{\text{id}_F P \mid P \in \mathcal{P}(F)\} < \infty$, where $\text{pd}_F I = \inf\{n \mid \text{Ext}_F^{n+1}(I, B) = 0 \text{ for any } B \in \text{mod}\Lambda\}$ and $\text{id}_F P = \inf\{n \mid \text{Ext}_F^{n+1}(A, P) = 0 \text{ for any } A \in \text{mod}\Lambda\}$. Thus we have the following corollary which characterizes when an Artin algebra is F -Gorenstein.

Corollary 4.8. *Let Λ be an Artin algebra and F an additive subbifunctor with enough projectives and injectives. If we assume that \mathcal{C} is the exact category $(\text{mod}\Lambda, \varepsilon)$ and $\xi = \varepsilon$ is the class of F -exact sequences in $\text{mod}\Lambda$, then Λ is F -Gorenstein if and only if $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, N) \cong \widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, N)$ for all finitely generated left Λ -modules M and N .*

Proof. By Theorem 4.4 and Example 4.5(1), one has that $\widetilde{\xi\text{xt}}_{\mathcal{P}}^i(M, N) \cong \widetilde{\xi\text{xt}}_{\mathcal{I}}^i(M, N)$ for all finitely generated left Λ -modules M and N if and only if $\sup\{\text{pd}_F I \mid I \in \mathcal{I}(F)\} = \sup\{\text{id}_F P \mid P \in \mathcal{P}(F)\} < \infty$. So the result holds by [23, Theorem 3.4]. \square

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