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SYNCHRONIZATION FOR A CLASS OF COMPLEX-VALUED MEMRISTOR-BASED COMPETITIVE NEURAL NETWORKS(CMCNNS) WITH DIFFERENT TIME SCALES

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Abstract. In this paper, the synchronization problem of complex-valued memristive competitive neural networks(CMCNNs) with different time scales is investigated. Based on differential inclusions and inequality techniques, some novel sufficient conditions are derived to ensure synchronization of the drive-response systems by designing a proper controller. Finally, a numerical example is provided to illustrate the usefulness and feasibility of our results.

1. **Introduction.** Since it was first put forward by Chua[3], memristor has attracted increasing attention in recent years[19, 16]. As the fourth fundamental circuit element except resistor, inductor, and capacitor in circuity, the prototype of practical memristor device was successfully developed by Hewlett-Packard Labs in 2008[17]. Moreover, memristor has many good properties, such as low power consumption, high density and good scalability. More importantly, the connection weight of the memristor is not fixed, it depends on the voltage applied in the corresponding state. Therefore, many researchers hope to use memristor as an artificial synapse to build a device similar to brain function. In view of its good charzcteristic, now memristor has been widely used to model memristor-based neural networks(MNNs)[24, 6, 15].

As a collective dynamical behavior, synchronization extensively exists in life, society and neural systems. Synchronization plays important role in the activity of the brain and nervous system[14]. So synchronization of memristor-based neural networks is significant[22, 1]. Liao et al.[22] discussed effects of initial conditions of memristor synapses on the synchronization of the coupled memristor neural circuits. Theoretical analysis and simulations show that the memristor synapse has played an important role in the synchronization of the coupled FitzHugh-Nagumo neural circuits. Ascoli et al.[1] found that the history of the memristor plays a critical role in the synchronous oscillations in the network and enhance synchronizaiton.

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In recent years, neurals networks have attracted a lot of researchers in different research areas[20, 21, 4, 27, 7, 8]. It is worth mentioning that MCNNs with different time scales, which are extensions of conventional neural networks. It is a kind of unsupervised learning neural networks, which refers to the whole interconnection between input and output of the single layer neural networks[25]. MCNNs contain two types of state variables, including the aspects of long-term memory (LTM)and short-term memory(STM)[10], corresponding to the fast changes of the neural network states and the slow changes of the synapses by external stimuli, respectively. And up to now, various dynamical behaviors for competitive neural networks have been investigated[6, 15, 12, 11, 13] and have been successfully applied to control theory, signal processing, pattern recognition and optimization design and so on[25, 11].

We noticed that the results mentioned above have been achieved within the real number domain. However, as we all know, sometimes, it is unreasonable to deal with some problems only in the real number domain, such as symmetry detection, XOR problems, electromagnetic wave imaging and so on[23, 18], which are more convenient and reasonable to deal with complex-value system. Thus, it is meaningful to study MCNNs as the generalization and extension for real-valued systems. In recent years, people have made a lot of achievements[9, 26, 2] in the field of complex number. Liu et al.[9] discussed global anti-synchronization of CMNNs with time delays by constructing an appropriate Lyapunov function. The proposed results of this paper are less conservative than existing literatures due to the characteristics of complex-valued memristive neural networks(CMNNs). Zhu et al.[26] investigated the synchronization of CMNNs with time delay by using the theory of the pinning control method, which control partial neurons instead of all neurons, and achieved new conclusions and progress. However, to the best of our knowledge, few scholars consider the synchronization problem of CMCNNs.

Based on the above analysis, this paper aims to investigate the synchronization problem of CMCNNs. By designing a proper controller, we achieve asymptotically stable of the error system such that achieve synchronization of the drive-response system. The contributions of this article can be summarized as follows.

(1)Different from the neural networks discussed earlier, the systems considered in this article are discussed based on complex-valued, which are an extension of the general real-valued networks.

(2) Different from asymptotic or exponential synchronization, it is shown that both the STM and LTM play a regulatory role in the systems so that the systems can show better performance.

(3) In this paper, the sufficiency of the synchronization of CMCNNs is derived by constructing a proper controller and use some inequality techniques.

The rest of the paper is organized as follows. In Section 2, some useful assumptions, definitions and lemmas needed in the paper are presented. In Section 3, a controller is designed to investigate the synchronization of CMCNNs by constructing a proper Lyapunov functional. In Section 4, a numerical example is given to illustrate the effectiveness of the obtained results. Finally, some conclusions are drawn.

2. **Preliminaries.** In this paper, the solutions of all the systems considered below are intended in Filippov's sense[5]. R^n and C^n denote the $n-$dimensional Euclidean space and complex space, respectively. $co[a, b]$ represents closure of the convex hull

of R^n generated by real numbers a and b. Now, the model of CMCNNs with different time scales to be introduced as follows:

$$\begin{cases} STM : \varepsilon \dot{z}_k(t) = -z_k(t) + \sum\limits_{l=1}^{n} a_{kl}(z_k(t)) f_l\big(z_l(t)\big) \\ \qquad\qquad + \sum\limits_{l=1}^{n} b_{kl}(z_k(t)) f_l\big(z_l(t-\tau(t))\big) + H_k m_k(t), l = 1, 2, \cdots, n, \\ LTM : \dot{m}_k(t) = -m_k(t) + f_k(z_k(t)),\ k = 1, 2, \cdots, n, \end{cases} \tag{1}$$

where $z(t) = (z_1(t), z_2(t), ..., z_n(t))^T \in C^n$. Let $z_k(t) = x_k(t) + iy_k(t)$, $x_k(t)$ and $y_k(t)$ are real part and imaginary part of $z_k(t)$, i is used to denote imaginary unit and satisfies $i = \sqrt{-1}$. $a_{kl}(z_k(t))$ and $b_{kl}(z_k(t))$ are complex-valued memristive connection weights; $f(z(.)) = (f_1(z_1(.)), f_2(z_2(.)), ..., f_n(z_n(.)))^T$ stands for the complex-valued activation function with $f(0) = 0$; $\tau(t)$ is used to express the variable time delay, which satisfies $0 \le \tau(t) \le \tau, \dot{\tau}(t) \le \gamma < 1$, where τ and γ are positive constants.

The initial conditions of system (1) are assumed to be

$$z(s) = \phi(s), -\tau \le s \le 0$$

where $\phi(s) = (\phi_1(s), \phi_2(s), ..., \phi_n(s))^T \in C([-\tau, 0], C^n)$. The memristive connection weights of (1) satisfy the following conditions:

$$a_{kl}(z_k(t)) = \begin{cases} \hat{a}_{kl},\ |z_k(t)| > T_k, \\ \check{a}_{kl},\ |z_k(t)| \le T_k, \end{cases} \quad b_{kl}(z_k(t)) = \begin{cases} \hat{b}_{kl},\ |z_k(t)| > T_k, \\ \check{b}_{kl},\ |z_k(t)| \le T_k, \end{cases}$$

for $t > 0$, where the switching jumps $T_k > 0, \hat{a}_{kl}, \check{a}_{kl}, \hat{b}_{kl}$ and $\check{b}_{kl}$ are the complex-valued connected weights.

Remark 1. From above analysis, we know that the connection weights $a_{kl}(z_k(t))$ and $b_{kl}(z_k(t))$ in system (1) are complex-valued and discontinuous due to the characteristics of state-dependent switched nonlinear dynamical system. Therefore, we will study the characteristics of solutions for differential equations with discontinuous right-hand sides by using the theory of Filippov in this paper.

In this paper, consider system (1) as drive system and corresponding response system can be described as follows;

$$\begin{cases} STM : \varepsilon \dot{\tilde{z}}_k(t) = -\tilde{z}_k(t) + \sum\limits_{l=1}^{n} a_{kl}(\tilde{z}_k(t)) f_l(\tilde{z}_l(t)) \\ \qquad\qquad + \sum\limits_{l=1}^{n} b_{kl}(\tilde{z}_k(t)) f_l\big(\tilde{z}_l(t-\tau(t))\big) + H_k \tilde{m}_k(t) \\ \qquad\qquad + u_k(t), l = 1, 2, \cdots, n, \\ LTM : \dot{\tilde{m}}_k(t) = -\tilde{m}_k(t) + f_k(\tilde{z}_k(t)),\ k = 1, 2, \cdots, n, \end{cases} \tag{2}$$

with initial condition $\tilde{z}(s) = \varphi(s) \in C([-\tau, 0], C^n)$. $u_k(t)$ is the control input to be designed.

Before proceeding further,$u_k(t), a_{kl}(z(t)), b_{kl}(z(t))$ and activation function f_k $(z_k(t))$ can be separated into real and imaginary parts as

$$u_k(t) = u_k^R(t) + iu_k^I(t);$$
$$a_{kl}(z(t)) = a_{kl}^R(z(t)) + ia_{kl}^I(z(t));$$
$$b_{kl}(z(t)) = b_{kl}^R(z(t) + ib_{kl}^I(z(t));$$
$$f_k(z_k(t)) = f_k^R(x_k(t), y_k(t)) + if_k^I(x_k(t), y_k(t)),$$

where $f_k^R(x_k(t),y_k(t)), f_k^I(x_k(t),y_k(t)) : R^2 \to R$, and they are odd functions. And the following assumption need to be introduced.

Assumption 1. *Suppose that $f_k^R(x_k(t),y_k(t))$ and $f_k^I(x_k(t),y_k(t))$ satisfy the following conditions.*

(1)The partial derivatives of $f_k^R(.,.)$ and $f_k^I(.,.)$ with respect to x,y : $\partial f_k^R/\partial x$, $\partial f_k^R/\partial y, \partial f_k^I/\partial x$ and $\partial f_k^I/\partial y$ are exist and continuous.

(2) The partial derivatives $\partial f_k^R/\partial x, \partial f_k^R/\partial y, \partial f_k^I/\partial x$ and $\partial f_k^I/\partial y$ are bounded, namely, there exist positive constants $\lambda_k^{RR}, \lambda_k^{RI}, \lambda_k^{IR}$, and λ_k^{II} such that

$$|\partial f_k^R/\partial x| \le \lambda_k^{RR}, |\partial f_k^R/\partial y| \le \lambda_k^{RI},$$

$$|\partial f_k^I/\partial x| \le \lambda_k^{IR}, |\partial f_k^I/\partial y| \le \lambda_k^{II}.$$

Then we have

$$|f_k^R(\tilde{x}_k(t),\tilde{y}_k(t)) - f_k^R(x_k(t),y_k(t))| \le \lambda_k^{RR}|\tilde{x}(t)-x(t)| + \lambda_k^{RI}|\tilde{y}(t)-y(t)|,$$

$$|f_k^I(\tilde{x}_k(t),\tilde{y}_k(t)) - f_k^I(x_k(t),y_k(t))| \le \lambda_k^{IR}|\tilde{x}(t)-x(t)| + \lambda_k^{II}|\tilde{y}(t)-y(t)|.$$

Under Assumption 1, separating system (1) into real and imaginary parts as follows

$$\begin{cases} STM : \varepsilon \dot{x}_k^R(t) = -x_k^R(t) + \sum\limits_{l=1}^n a_{kl}^R(x_k(t)) f_l^R(x_l(t),y_l(t)) \\ \quad - \sum\limits_{l=1}^n a_{kl}^I(x_k(t)) f_l^I(x_l(t),y_l(t)) + \sum\limits_{l=1}^n b_{kl}^R(x_k(t)) f_l^R(x_l(t-\tau(t)),y_l(t-\tau(t))) \\ \qquad\qquad - \sum\limits_{l=1}^n b_{kl}^I(x_k(t)) f_l^I(x_l(t-\tau(t)),y_l(t-\tau(t))) + H_k^R m_k^R(t) \\ \qquad\qquad -H_k^I m_k^I(t), l=1,2,\cdots,n, \\ LTM : \dot{m}_k^R(t) = -m_k^R(t) + f_k^R(x_k(t),y_k(t)), k=1,2,\cdots,n, \end{cases} \tag{3}$$

$$\begin{cases} STM : \varepsilon \dot{y}_k^I(t) = -y_k^I(t) + \sum\limits_{l=1}^n a_{kl}^R(y_k(t)) f_l^I(x_l(t),y_l(t)) \\ \quad + \sum\limits_{l=1}^n a_{kl}^I(y_k(t)) f_l^R(x_l(t),y_l(t)) + \sum\limits_{l=1}^n b_{kl}^R(y_k(t)) f_l^I(x_l(t-\tau(t)),y_l(t-\tau(t))) \\ \qquad\qquad + \sum\limits_{l=1}^n b_{kl}^I(y_k(t)) f_l^R(x_l(t-\tau(t)),y_l(t-\tau(t))) + H_k^R m_k^I(t) \\ \qquad\qquad +H_k^I m_k^R(t), l=1,2,\cdots,n, \\ LTM : \dot{m}_k^I(t) = -m_k^I(t) + f_k^I(x_k(t),y_k(t)), k=1,2,\cdots,n, \end{cases} \tag{4}$$

with initial conditions $x(s) = \phi^R(s) \in C([-\tau,0],R^n)$ and $y(s) = \phi^I(s) \in C([-\tau,0], R^n)$, respectively.

The memristive connection weights of (3) and (4) satisfy the following conditions:

$$a_{kl}^R(x_k(t)) = \begin{cases} \hat{a}_{kl}^R, & |x_k(t)| > T_k, \\ \check{a}_{kl}^R, & |x_k(t)| \le T_k, \end{cases} \quad a_{kl}^I(y_k(t)) = \begin{cases} \hat{a}_{kl}^I, & |y_k(t)| > T_k, \\ \check{a}_{kl}^I, & |y_k(t)| \le T_k, \end{cases}$$

$$b_{kl}^R(x_k(t)) = \begin{cases} \hat{b}_{kl}^R, & |x_k(t)| > T_k, \\ \check{b}_{kl}^R, & |x_k(t)| \le T_k, \end{cases} \quad b_{kl}^I(y_k(t)) = \begin{cases} \hat{b}_{kl}^I, & |y_k(t)| > T_k, \\ \check{b}_{kl}^I, & |y_k(t)| \le T_k, \end{cases}$$

where the switching jumps $T_k > 0$, connections weights $\hat{a}_{kl}^R, \check{a}_{kl}^R, \hat{a}_{kl}^I, \check{a}_{kl}^I, \hat{b}_{kl}^R, \check{b}_{kl}^R, \hat{b}_{kl}^I$ and $\check{b}_{kl}^I$ are constants, $k,l=1,...,n$.

Remark 2. Here, we transform a complex-valued system into two equivalent real-valued system. Similarly, the inequalities satisfied by activation function in Assumption 1 are equivalent to the Lipschitz continuity condition in the complex domain. The purpose of this process is to facilitate our discussion using the relevant theorems in the field of real numbers.

Because the memristor-based connection weights in (3) and (4) are discontinuous, then by differential inclusions feature for system with the discontinuous right-hand sides, (3) and (4) will be written as follows:

$$\begin{cases} STM : \varepsilon \dot{x}_k^R(t) \in -x_k^R(t) + \sum\limits_{l=1}^{n} co[a_{kl}^{-R}, a_{kl}^{+R}] f_l^R(x_l(t), y_l(t)) \\ \qquad - \sum\limits_{l=1}^{n} co[a_{kl}^{-I}, a_{kl}^{+I}] f_l^I(x_l(t), y_l(t)) \\ \qquad + \sum\limits_{l=1}^{n} co[b_{kl}^{-R}, b_{kl}^{+R}] f_l^R(x_l(t-\tau(t)), y_l(t-\tau(t))) \\ \qquad - \sum\limits_{l=1}^{n} co[b_{kl}^{-I}, b_{kl}^{+I}] f_l^I(x_l(t-\tau(t)), y_l(t-\tau(t))) + H_k^R m_k^R(t) \\ \qquad - H_k^I m_k^I(t)), l = 1, 2, \cdots, n, \\ LTM : \dot{m}_k^R(t) = -m_k^R(t) + f_k^R(x_k(t), y_k(t)), k = 1, 2, \cdots, n, \end{cases} \tag{5}$$

$$\begin{cases} STM : \varepsilon \dot{y}_k^I(t) \in -y_k^I(t) + \sum\limits_{l=1}^{n} co[a_{kl}^{-R}, a_{kl}^{+R}] f_l^I(x_l(t), y_l(t)) \\ \qquad + \sum\limits_{l=1}^{n} co[a_{kl}^{-I}, a_{kl}^{+I}] f_l^R(x_l(t), y_l(t)) \\ \qquad + \sum\limits_{l=1}^{n} co[b_{kl}^{-R}, b_{kl}^{+R}] f_l^I(x_l(t-\tau(t)), y_l(t-\tau(t))) \\ \qquad + \sum\limits_{l=1}^{n} co[b_{kl}^{-I}, b_{kl}^{+I}] f_l^R(x_l(t-\tau(t)), y_l(t-\tau(t))) + H_k^R m_k^I(t) \\ \qquad + H_k^I m_k^R(t), l = 1, 2, \cdots, n, \\ LTM : \dot{m}_k^I(t) = -m_k^I(t) + f_k^I(x_k(t), y_k(t)), k = 1, 2, \cdots, n, \end{cases} \tag{6}$$

where

$$a_{kl}^{+R} = \max\{\hat{a}_{kl}^R, \check{a}_{kl}^R\}, a_{kl}^{-R} = \min\{\hat{a}_{kl}^R, \check{a}_{kl}^R\}, b_{kl}^{+R} = \max\{\hat{b}_{kl}^R, \check{b}_{kl}^R\}, b_{kl}^{-R} = \min\{\hat{b}_{kl}^R, \check{b}_{kl}^R\},$$

$$a_{kl}^{+I} = \max\{\hat{a}_{kl}^R, \check{a}_{kl}^R\}, a_{kl}^{-I} = \min\{\hat{a}_{kl}^R, \check{a}_{kl}^R\}, b_{kl}^{+I} = \max\{\hat{b}_{kl}^R, \check{b}_{kl}^R\}, b_{kl}^{-I} = \min\{\hat{b}_{kl}^R, \check{b}_{kl}^R\}.$$

Or equivalently, for $k, l = 1, ..., n$, then there exists $\acute{a}_{kl}^R \in co[a_{kl}^{-R}, a_{kl}^{+R}], \acute{a}_{kl}^I \in co[a_{kl}^{-I}, a_{kl}^{+I}], \acute{b}_{kl}^R \in co[b_{kl}^{-R}, b_{kl}^{+R}], \acute{b}_{kl}^I \in co[b_{kl}^{-I}, b_{kl}^{+I}]$ such that

$$\begin{cases} STM : \varepsilon \dot{x}_k^R(t) = -x_k^R(t) + \sum\limits_{l=1}^{n} \acute{a}_{kl}^R f_l^R(x_l(t), y_l(t)) \\ \qquad - \sum\limits_{l=1}^{n} \acute{a}_{kl}^I f_l^I(x_l(t), y_l(t)) + \sum\limits_{l=1}^{n} \acute{b}_{kl}^R f_l^R(x_l(t-\tau(t)), y_l(t-\tau(t))) \\ \qquad - \sum\limits_{l=1}^{n} \acute{b}_{kl}^I f_l^I(x_l(t-\tau(t)), y_l(t-\tau(t))) + H_k^R m_k^R(t) \\ \qquad - H_k^I m_k^I(t), l = 1, 2, \cdots, n, \\ LTM : \dot{m}_k^R(t) = -m_k^R(t) + f_k^R(x_k(t), y_k(t)), k = 1, 2, \cdots, n, \end{cases} \tag{7}$$

$$\begin{cases} STM : \varepsilon \dot{y}_k^I(t) = -y_k^I(t) + \sum\limits_{l=1}^{n} \acute{a}_{kl}^R f_l^I(x_l(t), y_l(t)) \\ \qquad + \sum\limits_{l=1}^{n} \acute{a}_{kl}^I f_l^R(x_l(t), y_l(t)) + \sum\limits_{l=1}^{n} \acute{b}_{kl}^R f_l^I(x_l(t-\tau(t)), y_l(t-\tau(t))) \\ \qquad + \sum\limits_{l=1}^{n} \acute{b}_{kl}^I f_l^R(x_l(t-\tau(t)), y_l(t-\tau(t))) + H_k^R m_k^I(t) \\ \qquad + H_k^I m_k^R(t), l = 1, 2, \cdots, n, \\ LTM : \dot{m}_k^I(t) = -m_k^I(t) + f_k^I(x_k(t), y_k(t)), k = 1, 2, \cdots, n. \end{cases} \tag{8}$$

Similar to the system (1), separating system (2) into real and imaginary parts as follows

$$\begin{cases} STM : \varepsilon \dot{\tilde{x}}_k^R(t) = -\tilde{x}_k^R(t) + \sum\limits_{l=1}^{n} a_{kl}^R(\tilde{x}_k(t)) f_l^R(\tilde{x}_l(t), \tilde{y}_l(t)) \\ \quad - \sum\limits_{l=1}^{n} a_{kl}^I(\tilde{x}_k(t)) f_l^I(\tilde{x}_l(t), \tilde{y}_l(t)) + \sum\limits_{l=1}^{n} b_{kl}^R(\tilde{x}_k(t)) f_l^R(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) \\ \qquad - \sum\limits_{l=1}^{n} b_{kl}^I(\tilde{x}_k(t)) f_l^I(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) + H_k^R \tilde{m}_k^R(t) \\ \qquad - H_k^I \tilde{m}_k^I(t) + u_k^R(t), l = 1, 2, \cdots, n, \\ LTM : \dot{\tilde{m}}_k^R(t) = -\tilde{m}_k^R(t) + f_k^R(\tilde{x}_k(t), \tilde{y}_k(t)), k = 1, 2, \cdots, n, \end{cases} \tag{9}$$

$$\begin{cases} STM : \varepsilon \dot{\tilde{y}}_k^I(t) = -\tilde{y}_k^I(t) + \sum\limits_{l=1}^{n} a_{kl}^R(\tilde{y}_k(t)) f_l^I(\tilde{x}_l(t), \tilde{y}_l(t)) \\ \quad + \sum\limits_{l=1}^{n} a_{kl}^I(\tilde{y}_k(t)) f_l^R(\tilde{x}_l(t), \tilde{y}_l(t)) + \sum\limits_{l=1}^{n} b_{kl}^R(\tilde{y}_k(t)) f_l^I(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) \\ \qquad + \sum\limits_{l=1}^{n} b_{kl}^I(\tilde{y}_k(t)) f_l^R(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) + H_k^R \tilde{m}_k^I(t) \\ \qquad + H_k^I \tilde{m}_k^R(t) + u_k^I(t), l = 1, 2, \cdots, n, \\ LTM : \dot{\tilde{m}}_k^I(t) = -\tilde{m}_k^I(t) + f_k^I(\tilde{x}_k(t), \tilde{y}_k(t)), k = 1, 2, \cdots, n, \end{cases} \tag{10}$$

with initial conditions $\tilde{x}(s) = \tilde{\phi}^R(s) \in C([-\tau, 0], R^n)$ and $\tilde{y}(s) = \tilde{\phi}^I(s) \in C([-\tau, 0], R^n)$, respectively. By differential inclusions feature for system with the discontinuous right-hand sides, (9) and (10) will be written as follows:

$$\begin{cases} STM : \varepsilon \dot{\tilde{x}}_k^R(t) \in -\tilde{x}_k^R(t) + \sum\limits_{l=1}^{n} co[a_{kl}^{-R}, a_{kl}^{+R}] f_l^R(\tilde{x}_l(t), \tilde{y}_l(t)) \\ \qquad - \sum\limits_{l=1}^{n} co[a_{kl}^{-I}, a_{kl}^{+I}] f_l^I(\tilde{x}_l(t), \tilde{y}_l(t)) \\ \qquad + \sum\limits_{l=1}^{n} co[b_{kl}^{-R}, b_{kl}^{+R}] f_l^R(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) \\ \qquad - \sum\limits_{l=1}^{n} co[b_{kl}^{-I}, b_{kl}^{+I}] f_l^I(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) + H_k^R \tilde{m}_k^R(t) \\ \qquad - H_k^I \tilde{m}_k^I(t) + u_k^R(t), l = 1, 2, \cdots, n, \\ LTM : \dot{\tilde{m}}_k^R(t) = -\tilde{m}_k^R(t) + f_k^R(\tilde{x}_k(t), \tilde{y}_k(t)), k = 1, 2, \cdots, n, \end{cases} \tag{11}$$

$$\begin{cases} STM: \varepsilon \dot{\tilde{y}}_k^I(t) \in -\tilde{x}_k^I(t) + \sum_{l=1}^{n} co[a_{kl}^{-R}, a_{kl}^{+R}] f_l^I(\tilde{x}_l(t), \tilde{y}_l(t)) \\ \qquad\qquad + \sum_{l=1}^{n} co[a_{kl}^{-I}, a_{kl}^{+I}] f_l^R(\tilde{x}_l(t), \tilde{y}_l(t)) \\ \qquad\qquad + \sum_{l=1}^{n} co[b_{kl}^{-R}, b_{kl}^{+R}] f_l^I(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) \\ \qquad\qquad + \sum_{l=1}^{n} co[b_{kl}^{-I}, b_{kl}^{+I}] f_l^R(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) + H_k^R \tilde{m}_k^I(t) \\ \qquad\qquad + H_k^I \tilde{m}_k^R(t) + u_k^I(t), l = 1, 2, \cdots, n, \\ LTM: \dot{\tilde{m}}_k^I(t) = -\tilde{m}_k^I(t) + f_k^I(\tilde{x}_k(t), \tilde{y}_k(t)), k = 1, 2, \cdots, n. \end{cases} \tag{12}$$

Or equivalently, for $k, l = 1, ..., n$, then there exists $\grave{a}_{kl}^R \in co[a_{kl}^{-R}, a_{kl}^{+R}], \grave{a}_{kl}^I \in co[a_{kl}^{-I}, a_{kl}^{+I}], \grave{b}_{kl}^R \in co[b_{kl}^{-R}, b_{kl}^{+R}], \grave{b}_{kl}^I \in co[b_{kl}^{-I}, b_{kl}^{+I}]$ such that

$$\begin{cases} STM: \varepsilon \dot{\tilde{x}}_k^R(t) = -\tilde{x}_k^R(t) + \sum_{l=1}^{n} \grave{a}_{kl}^R f_l^R(\tilde{x}_l(t), \tilde{y}_l(t)) \\ \qquad\qquad - \sum_{l=1}^{n} \grave{a}_{kl}^I f_l^I(\tilde{x}_l(t), \tilde{y}_l(t)) + \sum_{l=1}^{n} \grave{b}_{kl}^R f_l^R(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) \\ \qquad\qquad - \sum_{l=1}^{n} \grave{b}_{kl}^I f_l^I(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) + H_k^R \tilde{m}_k^R(t) \\ \qquad\qquad - H_k^I \tilde{m}_k^I(t) + u_k^R(t), l = 1, 2, \cdots, n, \\ LTM: \dot{\tilde{m}}_k^R(t) = -\tilde{m}_k^R(t) + f_k^R(\tilde{x}_k(t), \tilde{y}_k(t)), k = 1, 2, \cdots, n, \end{cases} \tag{13}$$

$$\begin{cases} STM: \varepsilon \dot{\tilde{y}}_k^I(t) = -\tilde{x}_k^I(t) + \sum_{l=1}^{n} \grave{a}_{kl}^R f_l^I(\tilde{x}_l(t), \tilde{y}_l(t)) \\ \qquad\qquad + \sum_{l=1}^{n} \grave{a}_{kl}^I f_l^R(\tilde{x}_l(t), \tilde{y}_l(t)) + \sum_{l=1}^{n} \grave{b}_{kl}^R f_l^I(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) \\ \qquad\qquad + \sum_{l=1}^{n} \grave{b}_{kl}^I f_l^R(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) + H_k^R \tilde{m}_k^I(t) \\ \qquad\qquad + H_k^I \tilde{m}_k^R(t) + u_k^I(t), l = 1, 2, \cdots, n, \\ LTM: \dot{\tilde{m}}_k^I(t) = -\tilde{m}_k^I(t) + f_k^I(\tilde{x}_k(t), \tilde{y}_k(t)), k = 1, 2, \cdots, n. \end{cases} \tag{14}$$

Let $e_k^R(t) = \tilde{x}_k(t) - x_k(t), e_k^I(t) = \tilde{y}_k(t) - y_k(t), h_k^R(t) = \tilde{m}_k^R(t) - m_k^R(t), h_k^I(t) = \tilde{m}_k^I(t) - m_k^I(t) (k, l = 1, 2, ..., n)$ and make the following assumption:

Assumption 2.

$$\begin{aligned} &(1) co[a_{kl}^{-R}, a_{kl}^{+R}] f_l^R(\tilde{x}_l(t), \tilde{y}_l(t)) - co[a_{kl}^{-R}, a_{kl}^{+R}] f_l^R(x_l(t), y_l(t)) \\ &\quad \subseteq co[a_{kl}^{-R}, a_{kl}^{+R}](f_l^R(\tilde{x}_l(t), \tilde{y}_l(t)) - f_l^R(x_l(t), y_l(t))), \\ &(2) co[a_{kl}^{-I}, a_{kl}^{+I}] f_l^I(\tilde{x}_l(t), \tilde{y}_l(t)) - co[a_{kl}^{-I}, a_{kl}^{+I}] f_l^I(x_l(t), y_l(t)) \\ &\quad \subseteq co[a_{kl}^{-I}, a_{kl}^{+I}](f_l^I(\tilde{x}_l(t), \tilde{y}_l(t)) - f_l^I(x_l(t), y_l(t))), \\ &(3) co[b_{kl}^{-R}, b_{kl}^{+R}] f_l^R(\tilde{x}_l(t), \tilde{y}_l(t)) - co[b_{kl}^{-R}, b_{kl}^{+R}] f_l^R(x_l(t), y_l(t)) \\ &\quad \subseteq co[b_{kl}^{-R}, b_{kl}^{+R}](f_l^R(\tilde{x}_l(t), \tilde{y}_l(t)) - f_l^R(x_l(t), y_l(t))), \\ &(4) co[b_{kl}^{-I}, b_{kl}^{+I}] f_l^I(\tilde{x}_l(t), \tilde{y}_l(t)) - co[b_{kl}^{-I}, b_{kl}^{+I}] f_l^I(x_l(t), y_l(t)) \\ &\quad \subseteq co[b_{kl}^{-I}, b_{kl}^{+I}](f_l^I(\tilde{x}_l(t), \tilde{y}_l(t)) - f_l^I(x_l(t), y_l(t))). \end{aligned}$$

Then real and imaginary parts of the error system can be obtained by (5)-(6) and (11)-(12) as follows:

$$\begin{cases} STM: \varepsilon\dot{e}_k^R(t) \in -e_k^R(t) + \sum_{l=1}^{n} co[a_{kl}^{-R}, a_{kl}^{+R}]g_l^R(e_l^R(t), e_l^I(t)) \\ \qquad - \sum_{l=1}^{n} co[a_{kl}^{-I}, a_{kl}^{+I}]g_l^I(e_l^R(t), e_l^I(t)) \\ \qquad + \sum_{l=1}^{n} co[b_{kl}^{-R}, b_{kl}^{+R}]g_l^R(e_l^R(t-\tau(t)), e_l^I(t-\tau(t))) \\ \qquad - \sum_{l=1}^{n} co[b_{kl}^{-I}, b_{kl}^{+I}]g_l^I(e_l^R(t-\tau(t)), e_l^I(t-\tau(t))) + H_k^R h_k^R(t) \\ \qquad - H_k^I h_k^I(t) + u_k^R(t), l = 1, 2, \cdots, n, \\ LTM: \dot{h}_k^R(t) = -h_k^R(t) + g_k^R(e_k^R(t), e_k^I(t)), k = 1, 2, \cdots, n, \end{cases} \tag{15}$$

$$\begin{cases} STM: \varepsilon\dot{e}_k^I(t) \in -e_k^I(t) + \sum_{l=1}^{n} co[a_{kl}^{-R}, a_{kl}^{+R}]g_l^I(e_l^R(t), e_l^I(t)) \\ \qquad + \sum_{l=1}^{n} co[a_{kl}^{-I}, a_{kl}^{+I}]g_l^R(e_l^R(t), e_l^I(t)) \\ \qquad + \sum_{l=1}^{n} co[b_{kl}^{-R}, b_{kl}^{+R}]g_l^I(e_l^R(t-\tau(t)), e_l^I(t-\tau(t))) \\ \qquad + \sum_{l=1}^{n} co[b_{kl}^{-I}, b_{kl}^{+I}]g_l^R(e_l^R(t-\tau(t)), e_l^I(t-\tau(t))) + H_k^R h_k^I(t) \\ \qquad + H_k^I h_k^R(t) + u_k^I(t), l = 1, 2, \cdots, n, \\ LTM: \dot{h}_k^I(t) = -h_k^I(t) + g_k^I(e_k^R(t), e_k^I(t)), k = 1, 2, \cdots, n, \end{cases} \tag{16}$$

with initial conditions $\psi^R(s) = \varphi^R(s) - \phi^R(s)$ and $\psi^I(s) = \varphi^I(s) - \phi^I(s)$, *respectively*. Where

$$g_l^R(e_l^R(t), e_l^I(t)) = f_l^R(\tilde{x}_l(t), \tilde{y}_l(t)) - f_l^R(x_l(t), y_l(t));$$

$$g_l^I(e_l^R(t), e_l^I(t)) = f_l^I(\tilde{x}_l(t), \tilde{y}_l(t)) - f_l^I(x_l(t), y_l(t));$$

$$g_l^R(e_l^R(t-\tau(t)), e_l^I(t-\tau(t)))$$

$$= f_l^R(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) - f_l^R(x_l(t-\tau(t)), y_l(t-\tau(t)));$$

$$g_l^I(e_l^R(t-\tau(t)), e_l^I(t-\tau(t))) = f_l^I(\tilde{x}_l(t-\tau(t)), \tilde{y}_l(t-\tau(t))) - f_l^I(x_l(t-\tau(t)), y_l(t-\tau(t))).$$

Or equivalently, for $k, l = 1, ..., n$, then there exists $a_{kl}^R \in co[a_{kl}^{-R}, a_{kl}^{+R}], a_{kl}^I \in co[a_{kl}^{-I}, a_{kl}^{+I}], b_{kl}^R \in co[b_{kl}^{-R}, b_{kl}^{+R}], b_{kl}^I \in co[b_{kl}^{-I}, b_{kl}^{+I}]$ such that

$$\begin{cases} STM: \varepsilon\dot{e}_k^R(t) = -e_k^R(t) + \sum_{l=1}^{n} a_{kl}^R g_l^R(e_l^R(t), e_l^I(t)) \\ \qquad - \sum_{l=1}^{n} a_{kl}^I g_l^I(e_l^R(t), e_l^I(t)) + \sum_{l=1}^{n} b_{kl}^R g_l^R(e_l^R(t-\tau(t)), e_l^I(t-\tau(t))) \\ \qquad - \sum_{l=1}^{n} b_{kl}^I g_l^I(e_l^R(t-\tau(t)), e_l^I(t-\tau(t))) + H_k^R h_k^R(t) \\ \qquad - H_k^I h_k^I(t) + u_k^R(t), l = 1, 2, \cdots, n, \\ LTM: \dot{h}_k^R(t) = -h_k^R(t) + g_k^R(e_k^R(t), e_k^I(t)), k = 1, 2, \cdots, n, \end{cases} \tag{17}$$

$$\begin{cases} STM: \varepsilon\dot{e}_k^I(t) = -e_k^I(t) + \sum_{l=1}^{n} a_{kl}^R g_l^I(e_l^R(t), e_l^I(t)) \\ \qquad\qquad + \sum_{l=1}^{n} a_{kl}^I g_l^R(e_l^R(t), e_l^I(t)) + \sum_{l=1}^{n} b_{kl}^R g_l^I(e_l^R(t-\tau(t)), e_l^I(t-\tau(t))) \\ \qquad\qquad + \sum_{l=1}^{n} b_{kl}^I g_l^R(e_l^R(t-\tau(t)), e_l^I(t-\tau(t))) + H_k^R h_k^I(t) \\ \qquad\qquad + H_k^I h_k^R(t) + u_k^I(t), l = 1, 2, \cdots, n, \\ LTM: \dot{h}_k^I(t) = -h_k^I(t) + g_k^I(e_k^R(t), e_k^I(t)), l = 1, 2, \cdots, n. \end{cases} \tag{18}$$

Lemma 2.1. *For any vector $x, y \in R^n$ and a matrix $M > 0 \in R^{n\times n}$, one has the inequality that*

$$2x^T y \le x^T M x + y^T M^{-1} y. \tag{19}$$

In this paper, the control inputs in (17) and (18) are taken as follows:

$$u^R(t) = K^R e^R(t), u^I(t) = K^I e^I(t), \tag{20}$$

where

$$e^R(t) = (e_1^R(t), e_2^R(t), ..., e_n^R(t))^T, e^I(t) = (e_1^I(t), e_2^I(t), ..., e_n^I(t))^T,$$
$$K^R = diag(k_1^R, k_2^R, ..., k_n^R), K^I = diag(k_1^I, k_2^I, ..., k_n^I),$$

and the controller gains to be determined.

Definition 2.2. The system (17) and (18) are asymptotically stable for any given initial conditions they satisfy:

$$\lim_{t\to\infty} \|e(t)\|^2 = 0, \lim_{t\to\infty} \|h(t)\|^2 = 0,$$

where

$$e(t) = ((e^R(t))^T, e^I(t))^T)^T, h(t) = (h^R(t))^T, h^I(t))^T)^T,$$
$$h^R(t) = (h_1^R(t), h_2^R(t), ..., h_n^R(t))^T, h^I(t) = (h_1^I(t), h_2^I(t), ..., h_n^I(t))^T.$$

Then drive-response systems (1) and (2) are said to be synchronized.

3. **Main results.**

In the section, we present synchronization problems of CMCNNs.

Theorem 3.1. *Under Assumptions 1-2, the systems (1) and (2) can be asymptotically synchronized with control inputs (20), if there exist constants $r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_0, r_1^*, r_2^*, r_5^*, r_6^*, r_9^*, r_0^* > 0$, diagonal matrix $P, Q > 0$ such that $T^R, T^I > 0$, where*

$$\begin{aligned} T^R =& \frac{1}{\varepsilon}[2I - 2K^R - 2A^R\lambda^{RR} - A^R\lambda^{RI}(A^R)^T + 2A^I\lambda^{IR} + A^I\lambda^{II}(A^I)^T \\ &- r_1(B^R\lambda^{RR})(B^R\lambda^{RR})^T - r_1^*(B^R\lambda^{RI})(B^R\lambda^{RI})^T - r_2(B^I\lambda^{IR})(B^I\lambda^{IR})^T \\ &- r_2^*(B^I\lambda^{II})(B^I\lambda^{II})^T - r_3H^R(H^R)^T + r_4H^I(H^I)^T - \lambda^{IR}I - \lambda^{RR}I] \\ &- r_9\lambda^{RR}I - r_0\lambda^{IR}I - P, \end{aligned}$$

$$\begin{aligned} T^I =& \frac{1}{\varepsilon}[2I - 2K^I - \lambda^{RI}I + \lambda^{II}I - A^R\lambda^{IR}(A^R)^T - 2A^R\lambda^{II} \\ &- A^I\lambda^{RR}(A^I)^T - 2A^I\lambda^{RI} - r_5(B^R\lambda^{IR})(B^R\lambda^{IR})^T - r_5^*(B^R\lambda^{II})(B^R\lambda^{II})^T \\ &- r_6(B^I\lambda^{RR})(B^I\lambda^{RR})^T - r_6^*(B^I\lambda^{RI})(B^I\lambda^{RI})^T - r_7H^R(H^R)^T - r_8H^I(H^I)^T] \\ &- r_9^*\lambda^{RI}I - r_0^*\lambda^{II}I - Q. \end{aligned}$$

Proof. Consider the following Lyapunov function

$$\begin{aligned} V(t,e(t),h(t)) =&(e^R(t))^T e^R(t) + (e^I(t))^T e^I(t) + (h^R(t))^T h^R(t) + (h^I(t))^T h^I(t) \\ &+ \int_{t-\tau(t)}^{t} (e^R(s))^T P e^R(s) ds + \int_{t-\tau(t)}^{t} e^I(s))^T Q e^I(s) ds \end{aligned} \tag{21}$$

Then, calculating the time derivative of $V(t,e(t))$ along the trajectories of (17) and (18), that is

$$\begin{aligned} \dot{V}(t,e(t),h(t)) \le& 2(e^R(t))^T \dot{e}^R(t) + 2(e^I(t))^T \dot{e}^I(t) + 2(h^R(t))^T \dot{h}^R(t) + 2(h^I(t))^T \dot{h}^I(t) \\ &+ (e^R(t))^T P e^R(t) - (e^R(t-\tau(t)))^T P e^R(t-\tau(t))(1-\gamma) \\ &+ (e^I(t))^T Q e^I(t) - (e^I(t-\tau(t)))^T Q e^I(t-\tau(t))(1-\gamma) \end{aligned} \tag{22}$$

According to Assumption 1-2 and Lemma 2.1, we have

$$\begin{aligned} 2(e^R(t))^T \dot{e}^R(t) =& 2(e^R(t))^T \frac{1}{\varepsilon}[-e^R(t) + A^R g^R(e(t)) - A^I g^I(e(t)) \\ &+ B^R g^R(e(t-\tau(t))) - B^I g^I(e(t-\tau(t))) \\ &+ H^R h^R(t) - H^I h^I(t) + u^R(t)] \end{aligned} \tag{23}$$

$$\begin{aligned} 2(e^R(t))^T A^R g^R(e(t)) &\le 2(e^R(t))^T A^R(\lambda^{RR} e^R(t) + \lambda^{RI} e^I(t)) \\ &= 2(e^R(t))^T A^R \lambda^{RR} e^R(t) + 2(e^R(t))^T A^R \lambda^{RI} e^I(t) \\ &\le 2(e^R(t))^T A^R \lambda^{RR} e^R(t) + (e^R(t))^T A^R \lambda^{RI} (A^R)^T e^R(t) \\ &\quad + (e^I(t))^T \lambda^{RI} e^I(t) \end{aligned} \tag{24}$$

$$\begin{aligned} 2(e^R(t))^T A^I g^I(e(t)) &\le 2(e^R(t))^T A^I(\lambda^{IR} e^R(t) + \lambda^{II} e^I(t)) \\ &= 2(e^R(t))^T A^I \lambda^{IR} e^R(t) + 2(e^R(t))^T A^I \lambda^{II} e^I(t) \\ &\le 2(e^R(t))^T A^I \lambda^{IR} e^R(t) + (e^R(t))^T A^I \lambda^{II} (A^I)^T e^R(t) \\ &\quad + (e^I(t))^T \lambda^{II} e^I(t) \end{aligned} \tag{25}$$

$$\begin{aligned} 2(e^R(t))^T B^R g^R(e(t-\tau(t))) \le& (e^R(t))^T (B^R \lambda^{RR}) M_1 (B^R \lambda^{RR})^T e^R(t) \\ &+ (e^R(t-\tau(t)))^T M_1^{-1} e^R(t-\tau(t)) \\ &+ (e^R(t))^T (B^R \lambda^{RI}) M_1^* (B^R \lambda^{RI})^T e^R(t) \\ &+ (e^I(t-\tau(t)))^T M_1^{*-1} e^I(t-\tau(t)) \end{aligned} \tag{26}$$

Choose

$$M_1 = \begin{pmatrix} r_1 & 0 \\ 0 & r_1 \end{pmatrix}, M_1^* = \begin{pmatrix} r_1^* & 0 \\ 0 & r_1^* \end{pmatrix},$$

So (26) can be simplified as

$$\begin{aligned} 2(e^R(t))^T B^R g^R(e(t-\tau(t))) \le& r_1 (e^R(t))^T (B^R \lambda^{RR})(B^R \lambda^{RR})^T e^R(t) \\ &+ \frac{1}{r_1}(e^R(t-\tau(t)))^T e^R(t-\tau(t)) \\ &+ r_1^* (e^R(t))^T (B^R \lambda^{RI})(B^R \lambda^{RI})^T e^R(t) \\ &+ \frac{1}{r_1^*}(e^I(t-\tau(t)))^T e^I(t-\tau(t)) \end{aligned}$$

(Note: The following inequalities use the same method.)

$$\begin{aligned}2(e^R(t))^T B^I g^I(e(t-\tau(t))) \leq & r_2(e^R(t))^T (B^I\lambda^{IR})(B^I\lambda^{IR})^T e^R(t) \\ & + \frac{1}{r_2}(e^R(t-\tau(t)))^T e^R(t-\tau(t)) \\ & + r_2^*(e^R(t))^T (B^I\lambda^{II})(B^I\lambda^{II})^T e^R(t) \\ & + \frac{1}{r_2^*}(e^I(t-\tau(t)))^T e^I(t-\tau(t))\end{aligned} \tag{27}$$

$$2(e^R(t))^T H^R h^R(t) \leq r_3(e^R(t))^T H^R(H^R)^T e^R(t) + \frac{1}{r_3}(h^R(t))^T h^R(t) \tag{28}$$

$$2(e^R(t))^T H^I h^I(t) \leq r_4(e^R(t))^T H^I(H^I)^T e^R(t) + \frac{1}{r_4}(h^I(t))^T h^I(t) \tag{29}$$

Similarly,

$$\begin{aligned}2(e^I(t))^T \dot{e}^I(t) = & 2(e^I(t))^T \frac{1}{\varepsilon}[-e^I(t) + A^R g^I(e(t)) + A^I g^R(e(t)) \\ & + B^R g^I(e(t-\tau(t))) + B^I g^R(e(t-\tau(t))) \\ & + H^R h^I(t) + H^I h^R(t) + u^I(t)]\end{aligned} \tag{30}$$

$$\begin{aligned}2(e^I(t))^T A^R g^I(e(t)) \leq & \ 2(e^I(t))^T A^R(\lambda^{IR} e^R(t) + \lambda^{II} e^I(t)) \\ = & \ 2(e^I(t))^T A^R \lambda^{IR} e^R(t) + 2(e^I(t))^T A^R \lambda^{II} e^I(t) \\ \leq & \ (e^I(t))^T A^R \lambda^{IR}(A^R)^T e^I(t) + (e^R(t))^T \lambda^{IR} e^R(t) \\ & + 2(e^I(t))^T A^R \lambda^{II} e^I(t)\end{aligned} \tag{31}$$

$$\begin{aligned}2(e^I(t))^T A^I g^R(e(t)) \leq & \ 2(e^I(t))^T A^I(\lambda^{RR} e^R(t) + \lambda^{RI} e^I(t)) \\ = & \ 2(e^I(t))^T A^I \lambda^{RR} e^R(t) + 2(e^I(t))^T A^I \lambda^{RI} e^I(t) \\ \leq & \ (e^I(t))^T A^I \lambda^{RR}(A^I)^T e^I(t) + (e^R(t))^T \lambda^{RR} e^R(t) \\ & + 2(e^I(t))^T A^I \lambda^{RI} e^I(t)\end{aligned} \tag{32}$$

$$\begin{aligned}2(e^I(t))^T B^R g^I(e(t-\tau(t))) \leq & r_5(e^I(t))^T (B^R\lambda^{IR})(B^R\lambda^{IR})^T e^I(t) \\ & + \frac{1}{r_5}(e^R(t-\tau(t)))^T e^R(t-\tau(t)) \\ & + r_5^*(e^I(t))^T (B^R\lambda^{II})(B^R\lambda^{II})^T e^I(t) \\ & + \frac{1}{r_5^*}(e^I(t-\tau(t)))^T e^I(t-\tau(t))\end{aligned} \tag{33}$$

$$\begin{aligned}2(e^I(t))^T B^I g^R(e(t-\tau(t))) \leq & r_6(e^I(t))^T (B^I\lambda^{RR})(B^I\lambda^{RR})^T e^I(t) \\ & + \frac{1}{r_6}(e^R(t-\tau(t)))^T e^R(t-\tau(t)) \\ & + r_6^*(e^I(t))^T (B^I\lambda^{RI})(B^I\lambda^{RI})^T e^I(t) \\ & + \frac{1}{r_6^*}(e^I(t-\tau(t)))^T e^I(t-\tau(t))\end{aligned} \tag{34}$$

$$2(e^I(t))^T H^R h^I(t) \leq r_7(e^I(t))^T H^R(H^R)^T e^I(t) + \frac{1}{r_7}(h^I(t))^T h^I(t) \tag{35}$$

$$2(e^I(t))^T H^I h^R(t) \leq r_8(e^I(t))^T H^I(H^I)^T e^I(t) + \frac{1}{r_8}(h^R(t))^T h^R(t) \tag{36}$$

$$2(h^R(t))^T\dot{h}^R(t) = 2(h^R(t))^T(-h^R(t) + g^R(e(t))) \tag{37}$$

$$\begin{aligned}2(h^R(t))^T g^R(e(t)) \leq & 2(h^R(t))^T(\lambda^{RR}e^R(t) + \lambda^{RI}e^I(t)) \\ \leq & \frac{1}{r_9}(h^R(t))^T h^R(t) + r_9\lambda^{RR}(e^R(t))^T e^R(t) \\ & + \frac{1}{r_9^*}(h^R(t))^T h^R(t) + r_9^*\lambda^{RI}(e^I(t))^T e^I(t)\end{aligned} \tag{38}$$

$$2(h^I(t))^T\dot{h}^I(t) = 2(h^I(t))^T(-h^I(t) + g^I(e(t))) \tag{39}$$

$$\begin{aligned}2(h^I(t))^T g^I(e(t)) \leq & 2(h^I(t))^T(\lambda^{IR}e^R(t) + \lambda^{II}e^I(t)) \\ \leq & \frac{1}{r_0}(h^I(t))^T h^I(t) + r_0\lambda^{IR}(e^R(t))^T e^R(t) \\ & + \frac{1}{r_0^*}(h^I(t))^T h^I(t) + r_0^*\lambda^{II}(e^I(t))^T e^I(t)\end{aligned} \tag{40}$$

Substituting(23)-(40) to (22) yield

$$\begin{aligned}\dot{V}(t) \leq & -(e^R(t))^T\{\frac{1}{\varepsilon}[2I - 2K^R - 2A^R\lambda^{RR} - A^R\lambda^{RI}(A^R)^T \\ & + 2A^I\lambda^{IR} + A^I\lambda^{II}(A^I)^T \\ & - r_1(B^R\lambda^{RR})(B^R\lambda^{RR})^T - r_1^*(B^R\lambda^{RI})(B^R\lambda^{RI})^T - r_2(B^I\lambda^{IR})(B^I\lambda^{IR})^T \\ & - r_2^*(B^I\lambda^{II})(B^I\lambda^{II})^T - r_3H^R(H^R)^T + r_4H^I(H^I)^T - \lambda^{IR}I - \lambda^{RR}I] \\ & - r_9\lambda^{RR}I - r_0\lambda^{IR}I - P\}e^R(t) \\ & - (e^I(t))^T\{\frac{1}{\varepsilon}[2I - 2K^I - \lambda^{RI}I + \lambda^{II}I - A^R\lambda^{IR}(A^R)^T - 2A^R\lambda^{II} \\ & - A^I\lambda^{RR}(A^I)^T - 2A^I\lambda^{RI} - r_5(B^R\lambda^{IR})(B^R\lambda^{IR})^T - r_5^*(B^R\lambda^{II})(B^R\lambda^{II})^T \\ & -r_6(B^I\lambda^{RR})(B^I\lambda^{RR})^T - r_6^*(B^I\lambda^{RI})(B^I\lambda^{RI})^T - r_7H^R(H^R)^T - r_8H^I(H^I)^T] \\ & - r_9^*\lambda^{RI}I - r_0^*\lambda^{II}I - Q\}e^I(t) \\ & + (h^R(t))^T[-2I + \frac{I}{\varepsilon r_3} + \frac{I}{\varepsilon r_8} + \frac{I}{r_9} + \frac{I}{r_9^*}]h^R(t) \\ & + (h^I(t))^T[-2I + \frac{I}{\varepsilon r_4} + \frac{I}{\varepsilon r_7} + \frac{I}{r_0} + \frac{I}{r_0^*}]h^I(t) \\ & + (e^R(t-\tau(t)))^T[-\frac{I}{\varepsilon r_1} - \frac{I}{\varepsilon r_2} + \frac{I}{\varepsilon r_5} + \frac{I}{\varepsilon r_6} - (1-\gamma)P]e^R(t-\tau(t)) \\ & + (e^I(t-\tau(t)))^T[-\frac{I}{\varepsilon r_1^*} - \frac{I}{\varepsilon r_2^*} + \frac{I}{\varepsilon r_5^*} + \frac{I}{\varepsilon r_6^*} - (1-\gamma)Q]e^I(t-\tau(t))\end{aligned} \tag{41}$$

where I is the identity matrix of appropriate dimension.

It is easy to know that there are real numbers $r_3, r_4, r_7, r_8, r_9, r_0, r_9^*$, and r_0^* such that

$$\begin{aligned}\frac{1}{\varepsilon r_3} + \frac{1}{r_8} + \frac{1}{r_9} + \frac{1}{r_9^*} - 2 < 0, \\ \frac{1}{\varepsilon r_4} + \frac{1}{r_7} + \frac{1}{r_0} + \frac{1}{r_0^*} - 2 < 0.\end{aligned} \tag{42}$$

Letting

$$
\begin{aligned}
-\frac{I}{\varepsilon r_1}-\frac{I}{\varepsilon r_2}+\frac{I}{\varepsilon r_5}+\frac{I}{\varepsilon r_6}&=(1-\gamma)P,\\
-\frac{I}{\varepsilon r_1^*}-\frac{I}{\varepsilon r_2^*}+\frac{I}{\varepsilon r_5^*}+\frac{I}{\varepsilon r_6^*}&=(1-\gamma)Q,\\
\lambda^R=\min\{\lambda_{min}(T^R),2-\frac{1}{\varepsilon r_3}+\frac{1}{r_8}&+\frac{1}{r_9}+\frac{1}{r_9^*}\},\\
\lambda^I=\min\{\lambda_{min}(T^I),2-\frac{1}{\varepsilon r_4}+\frac{1}{r_7}&+\frac{1}{r_0}+\frac{1}{r_0^*}\}.
\end{aligned}
\tag{43}
$$

From (41)-(43), it can be seen that

$$
\dot V(t)\le-\lambda^R(\|e^R(t)\|^2+\|h^R(t)\|^2)-\lambda^I(\|e^I(t)\|^2+\|h^I(t)\|^2). \tag{44}
$$

Moreover, in (44), the equality holds if and only if $\|e^R(t)\|^2+\|h^R(t)\|^2=0, \|e^I(t)\|^2+\|h^I(t)\|^2=0$, i.e., $\|e^R(t)\|^2=0, \|e^I(t)\|^2=0, \|h^R(t)\|^2=0$, and $\|h^I(t)\|^2=0$. It can be concluded from Lyapunov stability theory that

$$
\lim_{t\to\infty}\|e(t)\|^2=0, \lim_{t\to\infty}\|h(t)\|^2=0.
$$

According to Definition 2.2 ,the trivial solution of system (17) and (18) are asymptotically stable. We can conclude that the neural networks (1) and (2) can be synchronized with control inputs (20). The proof is complete. □

Remark 3. From Theorem 3.1, we can see that the existence of the variable delay affects the value of the matrix P and Q and then affects the value of the matrix T^R and T^I. In the following, we will give two corollaries to explain how T^R and T^I will change when $\tau(t)=\tau$ (τ is a constant) and $\tau(t)=0$.

Corollary 1. *Under Assumptions 1-2, and the controllers (20) when $\tau(t)=\tau>0$, system (1) and (2) can be asymptotically synchronized, if there exist constants $r_1,r_2,r_3,r_4,r_5,r_6,r_7,r_8,r_9,r_0,r_1^*,r_2^*,r_5^*,r_6^*,\ r_9^*,r_0^*>0$, diagonal matrix $P,Q>0$ such that $T^R,T^I>0$, where*

$$
\begin{aligned}
T^R=&\frac{1}{\varepsilon}[2I-2K^R-2A^R\lambda^{RR}-A^R\lambda^{RI}(A^R)^T+2A^I\lambda^{IR}+A^I\lambda^{II}(A^I)^T\\
&-r_1(B^R\lambda^{RR})(B^R\lambda^{RR})^T-r_1^*(B^R\lambda^{RI})(B^R\lambda^{RI})^T-r_2(B^I\lambda^{IR})(B^I\lambda^{IR})^T\\
&-r_2^*(B^I\lambda^{II})(B^I\lambda^{II})^T-r_3H^R(H^R)^T+r_4H^I(H^I)^T-\lambda^{IR}I-\lambda^{RR}I]\\
&-r_9\lambda^{RR}I-r_0\lambda^{IR}I-P,
\end{aligned}
$$

$$
\begin{aligned}
T^I=&\frac{1}{\varepsilon}[2I-2K^I-\lambda^{RI}I+\lambda^{II}I-A^R\lambda^{IR}(A^R)^T-2A^R\lambda^{II}\\
&-A^I\lambda^{RR}(A^I)^T-2A^I\lambda^{RI}-r_5(B^R\lambda^{IR})(B^R\lambda^{IR})^T-r_5^*(B^R\lambda^{II})(B^R\lambda^{II})^T\\
&-r_6(B^I\lambda^{RR})(B^I\lambda^{RR})^T-r_6^*(B^I\lambda^{RI})(B^I\lambda^{RI})^T-r_7H^R(H^R)^T-r_8H^I(H^I)^T]\\
&-r_9^*\lambda^{RI}I-r_0^*\lambda^{II}I-Q.
\end{aligned}
$$

Proof. The proof process is similar to Theorem 3.1 by taking

$$
\begin{aligned}
-\frac{I}{\varepsilon r_1}-\frac{I}{\varepsilon r_2}+\frac{I}{\varepsilon r_5}+\frac{I}{\varepsilon r_6}&=P,\\
-\frac{I}{\varepsilon r_1^*}-\frac{I}{\varepsilon r_2^*}+\frac{I}{\varepsilon r_5^*}+\frac{I}{\varepsilon r_6^*}&=Q,
\end{aligned}
$$

and is omitted here. □

Corollary 2. *Under Assumptions 1-2, and the controllers (20) when* $\tau(t) = 0$, *system (1) and (2) can be asymptotically synchronized, if there exist constants* $r_3, r_4, r_7, r_8, r_9, r_0, r_9^*, r_0^* > 0$, *such that* $T^R, T^I > 0$, *where*

$$\begin{aligned} T^R =& \frac{1}{\varepsilon}[2I - 2K^R - 2(A^R + B^R)\lambda^{RR} - (A^R + B^R)\lambda^{RI} + 2(A^I + B^I)\lambda^{IR} \\ &+ (A^I + B^I)\lambda^{II} - r_3 H^R (H^R)^T + r_4 H^I (H^I)^T - (A^R + B^R)\lambda^{IR} \\ &- (A^I + B^I)\lambda^{RR}] - r_9\lambda^{RR} I - r_0 \lambda^{IR} I, \end{aligned}$$

$$\begin{aligned} T^I =& \frac{1}{\varepsilon}[2I - 2K^I - (A^R + B^R)\lambda^{RI} + (A^I + B^I)\lambda^{II} - (A^R + B^R)\lambda^{IR} \\ &- 2(A^R + B^R)\lambda^{II} - (A^I + B^I)\lambda^{RR} - 2(A^I + B^I)\lambda^{RI} - r_7 H^R (H^R)^T \\ &- r_8 H^I (H^I)^T] - r_9^* \lambda^{RI} I - r_0^* \lambda^{II} I. \end{aligned}$$

Proof. The proof process is similar to Theorem 3.1 and is omitted here. □

4. A numerical example.

In the section, a numerical example is given to demonstrate the validity of the above results . Consider the following memristor-based complex-valued competitive neural networks:

$$\begin{cases} STM : \varepsilon \dot{z}_k(t) = -z_k(t) + \sum\limits_{l=1}^{2} a_{kl}(z_k(t)) f_l\big(z_l(t)\big) \\ \qquad\qquad + \sum\limits_{l=1}^{2} b_{kl}(z_k(t)) f_l\big(z_l(t - \tau(t))\big) + H_k m_k(t), l = 1, 2 \\ LTM : \dot{m}_k(t) = -m_k(t) + f_k(z_k(t)),\ k = 1, 2, \end{cases} \tag{45}$$

where

$$a_{11}^R(x_1(t)) = \begin{cases} -1.5, |x_1| < 1, \\ -0.4, |x_1| \geq 1, \end{cases} \quad a_{12}^R(x_1(t)) = \begin{cases} -0.2, |x_1| < 1, \\ -1.0, |x_1| \geq 1, \end{cases}$$

$$a_{21}^R(x_2(t)) = \begin{cases} -0.7, |x_2| < 1, \\ -1.4, |x_2| \geq 1, \end{cases} \quad a_{22}^R(x_2(t)) = \begin{cases} -2.0, |x_2| < 1, \\ -0.8, |x_2| \geq 1, \end{cases}$$

$$a_{11}^I(y_1(t)) = \begin{cases} -1.5, |y_1| < 1, \\ -0.06, |y_1| \geq 1, \end{cases} \quad a_{12}^I(y_1(t)) = \begin{cases} -1.5, |y_1| < 1, \\ -0.28, |y_1| \geq 1, \end{cases}$$

$$a_{21}^I(y_2(t)) = \begin{cases} -0.6, |y_2| < 1, \\ -0.25, |y_2| \geq 1, \end{cases} \quad a_{22}^I(y_2(t)) = \begin{cases} -1.0, |y_2| < 1, \\ -1.6, |y_2| \geq 1, \end{cases}$$

$$b_{11}^R(x_1(t)) = \begin{cases} 0.2, |x_1| < 1, \\ 1.0, |x_1| \geq 1, \end{cases} \quad b_{12}^R(x_1(t)) = \begin{cases} -3.0, |x_1| < 1, \\ -0.5, |x_1| \geq 1, \end{cases}$$

$$b_{21}^R(x_2(t)) = \begin{cases} -1.6, |x_2| < 1, \\ -2.0, |x_2| \geq 1, \end{cases} \quad b_{22}^R(x_2(t)) = \begin{cases} -1.0, |x_2| < 1, \\ -1.25, |x_2| \geq 1, \end{cases}$$

$$b_{11}^I(y_1(t)) = \begin{cases} 1.2, |y_1| < 1, \\ 0.6, |y_1| \geq 1, \end{cases} \quad b_{12}^I(y_1(t)) = \begin{cases} 1.25, |y_1| < 1, \\ 0.13, |y_1| \geq 1, \end{cases}$$

$$b_{21}^I(y_2(t)) = \begin{cases} 1.4, |y_2| < 1, \\ 0.03, |y_2| \geq 1, \end{cases} \quad b_{22}^I(y_2(t)) = \begin{cases} -1.5, |y_2| < 1, \\ -0.7, |y_2| \geq 1, \end{cases}$$

with initial values $z_1(t) = -0.8 - 0.5i, z_2(t) = 0.46 + 0.5i, m_1 = 0.82 - 0.14i, m_2 = -0.4 + 0.82i$. Consider system (45) as drive system and corresponding response system can be described as follows;

$$\begin{cases} STM : \varepsilon\dot{\tilde{z}}_k(t) = -\tilde{z}_k(t) + \sum_{l=1}^{2} a_{kl}(\tilde{z}_k(t)) f_l\big(\tilde{z}_l(t)\big) \\ \qquad\qquad + \sum_{l=1}^{2} b_{kl}(\tilde{z}_k(t)) f_l\big(\tilde{z}_l(t-\tau(t))\big) + H_k \tilde{m}_k(t), l = 1, 2 \\ LTM : \dot{\tilde{m}}_k(t) = -\tilde{m}_k(t) + f_k(\tilde{z}_k(t)),\ \ k = 1, 2, \end{cases} \tag{46}$$

with initial values $\tilde{z}_1(t) = 0.15-0.63i, \tilde{z}_2(t) = -0.36+0.35i, \tilde{m}_1 = -0.63+0.2i, \tilde{m}_2 = 0.46 - 0.9i$. Now we denote $e^R_k(t) = \tilde{x}_k - x_k(t), e^I_k(t) = \tilde{y}_k - y_k(t), \dot{h}^R_k(t) = \tilde{m}^R_k(t) - m^R_k(t)$ and $\dot{h}^I_k(t) = \dot{\tilde{m}}^I_k(t) - m^I_k(t) (k = 1, 2)$,

Let $\varepsilon = 0.9, \tau(t) = [(e^t)/(1 + e^t)]$ such that $\dot{\tau}(t) \leq \gamma = 0.25$, the activation function is consider as $f(z(.)) = \tanh(x(.)) + i \tanh(y(.))$. And choose $\lambda^{RR}_1 = \lambda^{IR}_1 = \lambda^{RI}_1 = \lambda^{II}_1 = \lambda^{RR}_2 = \lambda^{IR}_2 = \lambda^{RI}_2 = \lambda^{II}_2 = 1$. Meanwhile, let $H^R_1 = 0.5, H^R_2 = 1.0, H^I_1 = 1.5, H^I_2 = 1.8, K^R = diag(-7, -7), K^I = diag(-11, -11)$.

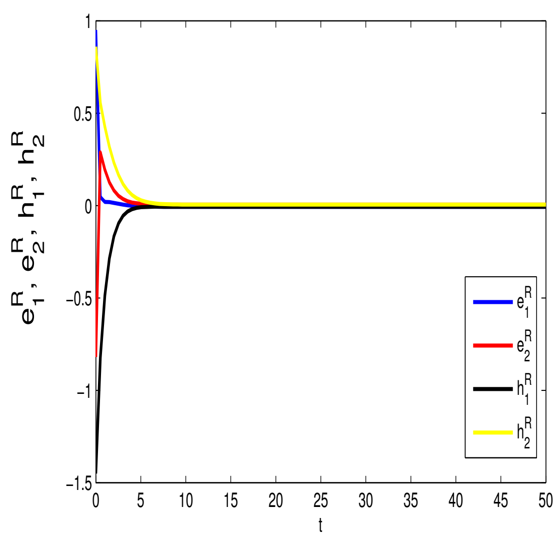

FIGURE 1. Synchronization errors of $e^R_1, e^R_2, h^R_1, h^R_2$ of system (45) and (46) with the controllers (20).

When the controllers is the form of (20), then by simple computation we get

$$A^R = \begin{pmatrix} -1.5 & -0.2 \\ -0.7 & -2 \end{pmatrix}, A^I = \begin{pmatrix} -1.5 & -1.5 \\ -0.6 & -1 \end{pmatrix},$$

$$B^R = \begin{pmatrix} 0.2 & 3 \\ -1.6 & -1 \end{pmatrix}, B^I = \begin{pmatrix} -1.2 & 1.25 \\ 1.4 & -1.5 \end{pmatrix}.$$

And choose

$$r_1 = r_2 = \frac{1}{2}, r_3 = 2, r_4 = 3, r_5 = r_6 = \frac{1}{3}, r_7 = r_8 = 2, r_9 = 3, r_0 = 2,$$

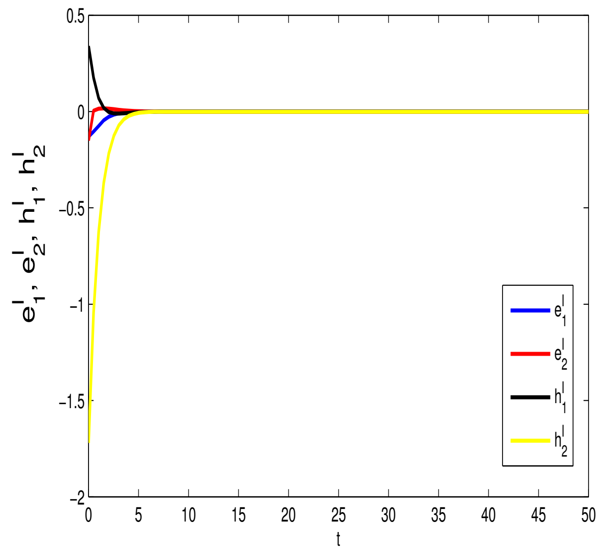

FIGURE 2. Synchronization errors of $e_1^I, e_2^I, h_1^I, h_2^I$ of system (45) and (46) with the controllers (20).

$$r_1^* = r_2^* = \frac{1}{3}, r_5^* = r_6^* = \frac{1}{4}, r_9^* = 3, r_0^* = 2.$$

so from Theorem 3.1, we have

$$T^R = \begin{pmatrix} 5.85 & 4.53 \\ 7.64 & 7.72 \end{pmatrix}, T^I = \begin{pmatrix} 4.47 & 3.96 \\ 3.07 & 4.41 \end{pmatrix}.$$

Then, all the conditions of Theorem 3.1 are satisfied. Under the controllers (20), the synchronization errors of real parts and imaginary parts are depicted by Figure 1. Therefore, according to Theorem 3.1, the systems (45) and (46) are synchronized.

5. **Conclusions.** This paper is concerned with synchronization of CMCNNs with different time scales. Firstly, we improved the model: (1) improved the ordinary neural network model to CMCNNs with different time scales; (2) extended the a common real-valued system to a complex-valued system. Then, we achieved the synchronization problem of the drive and response systems by designing a proper controller. In theory, the control design is operable and can be easily realized. Moreover, our results are more general and extend the previously known results. Finally, the effectiveness of our results has been demonstrated by Section 4. In further research, the main results of this paper can be extended to no time-delay for the feedback controller. We will also explore more dynamical behaviors of CMCNNs, for example, finite-time synchronization, fixed-time synchronization and anti-synchronization.

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