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ON MINIMAL 4-FOLDS OF GENERAL TYPE WITH $p_g \geq 2$

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Abstract. We show that, for any nonsingular projective 4-fold V of general type with geometric genus $p_g \geq 2$, the pluricanonical map φ_{33} is birational onto the image and the canonical volume $\mathrm{Vol}(V)$ has the lower bound $\frac{1}{480}$, which improves a previous theorem by Chen and Chen.

1. **Introduction.** Studying the behavior of pluricanonical maps of projective varieties has been one of the fundamental tasks in birational geometry. For varieties of general type, an interesting and critical problem is to find a positive integer m so that the pluricanonical map φ_m is birational onto the image. A momentous theorem given by Hacon-McKernan [13], Takayama [19] and Tsuji [20] says that for any integer $n > 0$, there is some constant r_n (we assume r_n to be the smallest one) such that the pluricanonical map φ_m is birational onto its image for all $m \geq r_n$ and for all minimal projective n-folds of general type. By using the birationality principle (see, for example, Theorem 2.2), an explicit upper bound of r_{n+1} is determined by that of r_n. Therefore, finding the explicit constant r_n for smaller n is the next problem. However, r_n is known only for $n \leq 3$, namely, $r_1 = 3$, $r_2 = 5$ by Bombieri [2] and $r_3 \leq 57$ by Chen-Chen [4, 5, 6] and Chen [11].

The first partial result concerning the explicit bound of r_4 was due to [6, Theorem 1.11] by Chen and Chen that φ_{35} is birational for all nonsingular projective 4-folds of general type with $p_g \geq 2$. It is mysterious whether the numerical bound "35" is optimal under the same assumption.

In this paper, we go on studying this question and prove the following theorem:

Theorem 1.1. *Let V be a nonsingular projective 4-fold of general type with $p_g(V) \geq 2$. Then*

(1) φ_m is birational for all $m \geq 33$;
(2) $\mathrm{Vol}(V) \geq \frac{1}{480}$.

Remark 1.2. As pointed out by Brown and Kasprzyk [3], the requirement on p_g in Theorem 1.1(2) is indispensable from the following list of canonical fourfolds, which are hypersurfaces in weighted projective spaces with at worst canonical singularities:

1. $X_{78} \subset \mathbb{P}(39, 14, 9, 8, 6, 1)$, $\mathrm{Vol}(X_{78}) = 1/3024$;
2. $X_{78} \subset \mathbb{P}(39, 13, 10, 8, 6, 1)$, $\mathrm{Vol}(X_{78}) = 1/3120$;
3. $X_{72} \subset \mathbb{P}(36, 11, 9, 8, 6, 1)$, $\mathrm{Vol}(X_{72}) = 1/2376$;
4. $X_{70} \subset \mathbb{P}(35, 14, 10, 6, 3, 1)$, $\mathrm{Vol}(X_{70}) = 1/1260$;
5. $X_{70} \subset \mathbb{P}(35, 14, 10, 5, 4, 1)$, $\mathrm{Vol}(X_{70}) = 1/1400$;

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6. $X_{68} \subset \mathbb{P}(34, 12, 8, 7, 5, 1)$, $\mathrm{Vol}(X_{68}) = 1/1680$.

Moreover, the following two hypersurfaces has $p_g \geq 2$ and φ_{17} is non-birational, so one may expect that 18 is the optimal lower bound of m such that φ_m is birational for a nonsingular projective 4-fold of general type with $p_g \geq 2$:

(1) $X_{36} \subset \mathbb{P}(18, 6, 5, 4, 1, 1)$;
(2) $X_{36} \subset \mathbb{P}(18, 7, 5, 3, 1, 1)$.

Throughout this paper, all varieties are defined over an algebraically closed field k of characteristic 0. We will frequently use the following symbols:

- ⋄ '$\sim$' denotes linear equivalence or $\mathbb{Q}$-linear equivalence;
- ⋄ '$\equiv$' denotes numerical equivalence;
- ⋄ '$|A| \succcurlyeq |B|$' or, equivalently, '$|B| \preccurlyeq |A|$' means $|A| \supseteq |B|+$ fixed effective divisors.

2. Preliminaries.

Let V be a nonsingular projective 4-fold of general type with geometric genus $p_g(V) := \dim_k H^0(V, \mathcal{O}_V(K_V)) \geq 2$, where K_V is a canonical divisor of V. By the minimal model program (see, for instance [1, 16, 17, 18]), we can find a minimal model Y of V with at worst $\mathbb{Q}$-factorial terminal singularities. Since the properties, which we study on V, are birationally invariant in the category of normal varieties with canonical singularities, we shall focus our study on Y instead.

2.1. Convention.

For an arbitrary linear system $|D|$ of positive dimension on a normal projective variety Z, we define *a generic irreducible element of* $|D|$ in the following way. We have $|D| = \mathrm{Mov}|D| + \mathrm{Fix}|D|$, where $\mathrm{Mov}|D|$ and $\mathrm{Fix}|D|$ denote the moving part and the fixed part of $|D|$ respectively. Consider the rational map $\varphi_{|D|} = \varphi_{\mathrm{Mov}|D|}$. We say that $|D|$ *is composed of a pencil* if $\dim \overline{\varphi_{|D|}(Z)} = 1$; otherwise, $|D|$ *is not composed of a pencil. A generic irreducible element of* $|D|$ is defined to be an irreducible component of a general member in $\mathrm{Mov}|D|$ if $|D|$ is composed of a pencil or, otherwise, a general member of $\mathrm{Mov}|D|$.

Keep the above settings. We say that $|D|$ can *distinguish different generic irreducible elements* X_1 *and* X_2 *of a linear system* $|M|$ *on* Z if neither X_1 nor X_2 is contained in $\mathrm{Bs}|D|$, and if $\overline{\varphi_{|D|}(X_1)} \nsubseteq \overline{\varphi_{|D|}(X_2)}, \overline{\varphi_{|D|}(X_2)} \nsubseteq \overline{\varphi_{|D|}(X_1)}$.

A nonsingular projective surface S of general type with $K_{S_0}^2 = u$ and $p_g(S_0) = v$ is referred to as *a* (u, v)*-surface*, where S_0 is the minimal model of S.

2.2. Setup for the map $\varphi_{1,Y}$.

Fix an effective divisor $K_1 \sim K_Y$. By Hironaka's theorem, we may take a series of blow-ups along nonsingular centers to obtain the model $\pi : Y' \to Y$ satisfying the following conditions:

(i) Y' is nonsingular and projective;

(ii) the moving part of $|K_{Y'}|$ is base point free so that

$$g_1 = \varphi_{1,Y} \circ \pi : Y' \to \overline{\varphi_{1,Y}(Y)} \subseteq \mathbb{P}^{p_g(Y)-1}$$

is a non-trivial morphism;

(iii) the union of $\pi^*(K_1)$ and all those exceptional divisors of π has simple normal crossing supports.

Take the Stein factorization of g_1. We get

$$Y' \xrightarrow{f_1} \Gamma \xrightarrow{s} \overline{\varphi_{1,Y}(Y)},$$

and hence the following diagram commutes:

$$\begin{array}{ccc} Y' & \xrightarrow{\quad f_1 \quad} & \Gamma \\ {\scriptstyle \pi}\downarrow & \searrow^{g_1} & \downarrow{\scriptstyle s} \\ Y & \overset{\varphi_{1,Y}}{\dashrightarrow} & \overline{\varphi_{1,Y}(Y)} \end{array}$$

We may write

$$K_{Y'} = \pi^*(K_Y) + E_\pi,$$

where E_π is a sum of distinct exceptional divisors with positive rational coefficients. Denote by $|M_1|$ the moving part of $|K_{Y'}|$. Since Y has at worst $\mathbb{Q}$-factorial terminal singularities, we may write

$$\pi^*(K_Y) \sim M_1 + E_1,$$

where E_1 is an effective $\mathbb{Q}$-divisor as well. One has $1 \leq \dim(\Gamma) \leq 4$.

If $\dim(\Gamma) = 1$, we have $M_1 \sim \sum_{i=1}^{b} F_i \equiv bF$, where F_i and F are smooth fibers of f_1 and $b = \deg f_{1*}\mathcal{O}_{Y'}(M_1) \geq p_g(Y) - 1 \geq 1$. More specifically, when $g(\Gamma) = 0$, we say that $|M_1|$ *is composed of a rational pencil* and when $g(\Gamma) > 0$, we say that $|M_1|$ *is composed of an irrational pencil.*

If $\dim(\Gamma) > 1$, by Bertini's theorem, we know that general members $T_i \in |M_1|$ are nonsingular and irreducible.

Denote by T' a generic irreducible element of $|M_1|$. Set

$$\theta_1 = \theta_{1,|M_1|} = \begin{cases} b, & \text{if } \dim(\Gamma) = 1; \\ 1, & \text{if } \dim(\Gamma) \geq 2. \end{cases}$$

So we naturally get

$$\pi^*(K_Y) \equiv \theta_1 T' + E_1.$$

2.3. **Notations.** Pick a generic irreducible element T' of $|M_1|$. Modulo further blow-ups on Y', which is still denoted as Y' for simplicity, we may have a birational morphism $\pi_T = \pi|_{T'} : T' \to T$ onto a minimal model T of T'. Let t_1 be the smallest positive integer such that $P_{t_1}(T) := \dim_k H^0(T, \mathcal{O}_T(t_1 K_T)) \geq 2$. Modulo a further blow-up of Y', we may assume that $\mathrm{Mov}|t_1 K_{T'}|$ is base point free.

Set $|N| = \mathrm{Mov}|t_1 K_{T'}|$ and let $\varphi_{t_1,T}$ be the t_1-canonical map: $T \dashrightarrow \mathbb{P}^{P_{t_1}(T)-1}$. Similar to the 4-fold case in Section 2.2, take the Stein factorization of the composition:

$$\varphi_{t_1,T} \circ \pi_T : T' \xrightarrow{j} \Gamma' \longrightarrow \overline{\varphi_{t_1,T}(T)}.$$

Denote by j the induced projective morphism with connected fibers from $\varphi_{t_1,T} \circ \pi_T$ by Stein factorization. Set

$$a_{t_1,T} = \begin{cases} c, & \text{if } \dim(\Gamma') = 1; \\ 1, & \text{if } \dim(\Gamma') \geq 2, \end{cases}$$

where $c = \deg j_*\mathcal{O}_{T'}(N) \geq P_{t_1}(T) - 1$. Let S be a generic irreducible element of $|N|$. Then we have

$$t_1 \pi_T^*(K_T) \equiv a_{t_1,T} S + E_N,$$

where E_N is an effective $\mathbb{Q}$-divisor. Denote by $\sigma : S \to S_0$ the contraction morphism of S onto its minimal model S_0.

Suppose that $|H|$ is a base point free linear system on S. Let C be a generic irreducible element of $|H|$. As $\pi_T^*(K_T)|_S$ is nef and big, by Kodaira's lemma, there is a rational number $\tilde{\beta} > 0$ such that $\pi_T^*(K_T)|_S \geq \tilde{\beta} C$.

Set

$$\begin{aligned} \beta = \beta(t_1, |N|, |H|) &= \sup\{\tilde{\beta} | \tilde{\beta} > 0 \ s.t. \ \pi_T^*(K_T)|_S \geq \tilde{\beta} C\} \\ \xi = \xi(t_1, |N|, |H|) &= (\pi_T^*(K_T) \cdot C)_{T'}. \end{aligned}$$

2.4. **Technical preparation.** We will use the following theorem which is a special form of Kawamata's extension theorem (see [15, Theorem A]).

Theorem 2.1. *(cf. [12, Theorem 2.2]) Let Z be a nonsingular projective variety on which D is a smooth divisor. Assume that $K_Z + D \sim A + B$ where A is an ample $\mathbb{Q}$-divisor and B is an effective $\mathbb{Q}$-divisor such that $D \nsubseteq \mathrm{Supp}(B)$. Then the natural homomorphism*

$$H^0(Z, m(K_Z + D)) \longrightarrow H^0(D, mK_D)$$

is surjective for any integer $m > 1$.

In particular, when Z is of general type and D, as a generic irreducible element, moves in a base point free linear system, the conditions of Theorem 2.1 are automatically satisfied. Keep the settings as in 2.2 and 2.3. Take $Z = Y'$ and $D = T'$.

If $|M_1|$ is composed of an irrational pencil, by [9, Lemma 2.5], we have

$$\pi^*(K_Y)|_{T'} = \pi_T^*(K_T). \tag{1}$$

If $|M_1|$ is not composed of an irrational pencil, then for a sufficiently large and divisible integer $n > 0$, we have

$$|n(\frac{1}{\theta_1} + 1)K_{Y'}| \succcurlyeq |n(K_{Y'} + T')|$$

and the homomorphism

$$H^0(Y', n(K_{Y'} + T')) \to H^0(T', nK_{T'})$$

is surjective. By [17, Theorem 3.3], $\mathrm{Mov}|nK_{T'}|$ is base point free, so one has

$$\mathrm{Mov}|nK_{T'}| = |n\pi_T^*(K_T)|.$$

It follows that

$$n(\frac{1}{\theta_1} + 1)\pi^*(K_Y)|_{T'} \geq M_{n(\frac{1}{\theta_1}+1)}|_{T'} \geq n\pi_T^*(K_T),$$

where the latter inequality holds by [7, Lemma 2.7]. Together with (1), we get the *canonical restriction inequality*:

$$\pi^*(K_Y)|_{T'} \geq \frac{\theta_1}{1 + \theta_1}\pi_T^*(K_T). \tag{2}$$

Similarly, one has

$$\pi_T^*(K_T)|_S \geq \frac{a_{t_1,T}}{t_1 + a_{t_1,T}}\sigma^*(K_{S_0}). \tag{3}$$

We will tacitly use the following type of birationality principle.

Theorem 2.2. *(cf. [5, 2.7]) Let Z be a nonsingular projective variety, A and B be two divisors on Z with $|A|$ being a base point free linear system. Take the Stein factorization of $\varphi_{|A|}$: $Z \xrightarrow{h} W \longrightarrow \mathbb{P}^{h^0(Z,A)-1}$ where h is a fibration onto a normal variety W. Then the rational map $\varphi_{|B+A|}$ is birational onto its image if one of the following conditions is satisfied:*

(i) $\dim \varphi_{|A|}(Z) \geq 2$, $|B| \neq \emptyset$ *and* $\varphi_{|B+A|}|_D$ *is birational for a general member* D *of* $|A|$.

(ii) $\dim \varphi_{|A|}(Z) = 1$, $\varphi_{|B+A|}$ *can distinguish different general fibers of* h *and* $\varphi_{|B+A|}|_F$ *is birational for a general fiber* F *of* h.

2.5. **Some useful lemmas.** The following results on surfaces will be used in our proof.

Lemma 2.3. *(see* [6, Lemma 2.5]*) Let* $\sigma : S \longrightarrow S_0$ *be the birational contraction onto the minimal model* S_0 *from a nonsingular projective surface* S *of general type. Assume that* S *is not a* $(1, 2)$*-surface and that* $\tilde{C}$ *is a curve on* S *passing through a very general point. Then* $(\sigma^*(K_{S_0}) \cdot \tilde{C}) \geq 2$.

Lemma 2.4. *(*[10, Lemma 2.5]*) Let* S *be a nonsingular projective surface. Let* L *be a nef and big* $\mathbb{Q}$*-divisor on* S *satisfying the following conditions:*

(1) $L^2 > 8$;

(2) $(L \cdot C_x) \geq 4$ *for all irreducible curves* C_x *passing through a very general point* $x \in S$.

Then $|K_S + \ulcorner L \urcorner|$ *gives a birational map.*

3. **Proof of the main theorem.** As an overall discussion, we keep the same settings as in 2.2 and 2.3.

3.1. **Separation properties of** $\varphi_{m,Y}$**.**

Lemma 3.1. *Let* Y *be a minimal 4-fold of general type with* $p_g(Y) \geq 2$. *Then* $|mK_{Y'}|$ *can distinguish different generic irreducible elements of* $|M_1|$ *for all* $m \geq 3$.

Proof. Suppose $m \geq 3$. As we have $mK_{Y'} \geq M_1$, we may just consider the case when $|M_1|$ is composed of a pencil. In particular, when $|M_1|$ is composed of a rational pencil, which is the case when $\Gamma \cong \mathbb{P}^1$, the global sections of $f_{1*}\mathcal{O}_{Y'}(M_1)$ can distinguish different points of Γ. So $|M_1|$, and consequently $|mK_{Y'}|$ can distinguish different general fibers of f_1. Hence we may just deal with the case when $|M_1|$ is composed of an irrational pencil. We have $M_1 \sim \sum_{i=1}^{b} T_i$, where T_i are smooth fibers of f_1 and $b \geq 2$. Pick two different generic irreducible elements T_1, T_2 of $|M_1|$. Then by Kawamata-Viehweg vanishing theorem ([14, 21]), one has

$$H^1(K_{Y'} + \ulcorner (m-2)\pi^*(K_Y) \urcorner + M_1 - T_1 - T_2) = 0,$$

and the surjective map

$$\begin{aligned} & H^0(Y', K_{Y'} + \ulcorner (m-2)\pi^*(K_Y) \urcorner + M_1) \\ \longrightarrow \; & H^0(T_1, (K_{Y'} + \ulcorner (m-2)\pi^*(K_Y) \urcorner + M_1)|_{T_1}) \qquad (4) \\ & \oplus H^0(T_2, (K_{Y'} + \ulcorner (m-2)\pi^*(K_Y) \urcorner + M_1)|_{T_2}). \qquad (5) \end{aligned}$$

Since $p_g(Y) \geq 2$, both K_{T_i} and $\pi^*(K_Y)$ are effective. So for general T_i, $\pi^*(K_Y)|_{T_i}$ is effective. As T_i is moving and $M_1|_{T_i} \sim 0$, both groups in (4) and (5) are non-zero. Therefore, $|mK_{Y'}|$ can distinguish different generic irreducible elements of $|M_1|$. □

Lemma 3.2. *Let* Y *be a minimal 4-fold of general type with* $p_g(Y) \geq 2$. *Pick a generic irreducible element* T' *of* $|M_1|$. *Then* $|mK_{Y'}||_{T'}$ *can distinguish different generic irreducible elements of* $|N|$ *for all*

$$m \geq 2t_1 + 4.$$

Proof. Suppose $m \geq 2t_1 + 4$. We have $K_{Y'} \geq \pi^*(K_Y) \geq T'$. Similar to the proof of Lemma 3.1, we consider the following two situations: (i) $|N|$ is not composed of a pencil or $|N|$ is composed of a rational pencil; (ii) $|N|$ is composed of an irrational pencil.

For (i), one has

$$|mK_{Y'}| \succcurlyeq |2(t_1+1)K_{Y'}| \succcurlyeq |(t_1+1)(K_{Y'}+T')|.$$

By Theorem 2.1, one has

$$|(t_1+1)(K_{Y'}+T')||_{T'} \succcurlyeq |(t_1+1)K_{T'}|.$$

As $(t_1+1)K_{T'} \geq N$, $|mK_{Y'}||_{T'}$ can distinguish different generic irreducible elements of $|N|$.

For (ii), it holds that

$$|mK_{Y'}| \succcurlyeq |2(t_1+2)K_{Y'}| \succcurlyeq |(t_1+2)(K_{Y'}+T')|.$$

Using Theorem 2.1 again, one gets

$$\begin{aligned} & |(t_1+2)(K_{Y'}+T')||_{T'} \\ \succcurlyeq\ & |(t_1+2)K_{T'}| \\ \succcurlyeq\ & |2K_{T'}+N| \\ \succcurlyeq\ & |K_{T'} + \ulcorner \pi_T^*(K_T) \urcorner + (N - S_1 - S_2) + S_1 + S_2|, \end{aligned}$$

where S_1 and S_2 are two different generic irreducible elements of $|N|$. The vanishing theorem implies the surjective map

$$\begin{aligned} & H^0(T', K_{T'} + \ulcorner \pi_T^*(K_T) \urcorner + N) \\ \to\ & H^0(S_1, (K_{T'} + \ulcorner \pi_T^*(K_T) \urcorner + N)|_{S_1}) \qquad (6) \\ & \oplus H^0(S_2, (K_{T'} + \ulcorner \pi_T^*(K_T) \urcorner + N)|_{S_2}), \qquad (7) \end{aligned}$$

where we note that $(N - S_i)|_{S_i}$ is linearly trivial for $i = 1, 2$. Since $p_g(T) > 0$, both groups in (6) and (7) are non-zero. So $|mK_{Y'}||_{T'}$ can distinguish different generic irreducible elements of $|N|$ for any $m \geq 2t_1 + 4$. □

Lemma 3.3. *Let Y be a minimal 4-fold of general type with $p_g(Y) \geq 2$. Pick a generic irreducible element T' of $|M_1|$ and a generic irreducible element S of $|N|$. Define*

$$|H| = \begin{cases} \mathrm{Mov}|K_S|, & \text{if } (K_{S_0}^2, p_g(S)) = (1,2) \text{ or } (2,3); \\ \mathrm{Mov}|2K_S|, & \text{otherwise.} \end{cases}$$

Then $|mK_{Y'}||_S$ can distinguish different generic irreducible elements of $|H|$ for all $m \geq 4(t_1+1)$.

Proof. Similar to the proof of Lemma 3.2, we have

$$|mK_{Y'}||_{T'} \succcurlyeq |4(t_1+1)K_{Y'}||_{T'} \succcurlyeq |2(t_1+1)K_{T'}|.$$

Since $t_1 K_{T'} \geq S$, we have

$$|2(t_1+1)K_{T'}||_S \succcurlyeq |2(K_{T'}+S)||_S \succcurlyeq |2K_S| \succcurlyeq |H|.$$

As $p_g(S) > 0$, $|H|$ is not composed of an irrational pencil, concluding the proof. □

3.2. **Two useful propositions.**

Proposition 3.4. *Let Y be a minimal 4-fold of general type with $p_g(Y) \geq 2$. Pick a generic irreducible element T' of $|M_1|$ and a generic irreducible element S of $|N|$. If S is not a $(1,2)$-surface, then $\varphi_{m,Y}$ is birational for all*

$$m > (2\sqrt{2}+1)(\frac{t_1}{a_{t_1,T}}+1)(1+\frac{1}{\theta_1}).$$

Proof. Suppose $m > (2\sqrt{2}+1)(\frac{t_1}{a_{t_1,T}}+1)(1+\frac{1}{\theta_1})$. Since

$$(m-1)\pi^*(K_Y) - T' - \frac{1}{\theta_1}E_1 \equiv (m-1-\frac{1}{\theta_1})\pi^*(K_Y)$$

is nef and big, and it has simple normal crossing support, Kawamata-Viehweg vanishing theorem implies

$$\begin{aligned} |mK_{Y'}||_{T'} &\succcurlyeq |K_{Y'} + \ulcorner (m-1)\pi^*(K_Y) - \frac{1}{\theta_1}E_1 \urcorner||_{T'} \\ &\succcurlyeq |K_{T'} + \ulcorner \big((m-1)\pi^*(K_Y) - T' - \frac{1}{\theta_1}E_1\big)|_{T'} \urcorner| \\ &= |K_{T'} + \ulcorner Q_{m,T'} \urcorner|, \end{aligned} \tag{8}$$

where $Q_{m,T'} = ((m-1)\pi^*(K_Y) - T' - \frac{1}{\theta_1}E_1)|_{T'} \equiv (m-1-\frac{1}{\theta_1})\pi^*(K_Y)|_{T'}$ is nef and big and has simple normal crossing support.

By the canonical restriction inequality (2), we may write

$$\pi^*(K_Y)|_{T'} \equiv \frac{\theta_1}{1+\theta_1}\pi_T^*(K_T) + E_{1,T'},$$

where $E_{1,T'}$ is certain effective $\mathbb{Q}$-divisor. As $t_1\pi_T^*(K_T) \equiv a_{t_1,T}S + E_N$, one may obtain that

$$\begin{aligned} &Q_{m,T'} - (m-1-\frac{1}{\theta_1})E_{1,T'} - S - \frac{1}{a_{t_1,T}}E_N \\ \equiv\ & ((m-1-\frac{1}{\theta_1})\cdot\frac{\theta_1}{1+\theta_1} - \frac{t_1}{a_{t_1,T}})\pi_T^*(K_T) \end{aligned}$$

is nef and big and has simple normal crossing support. So by Kawamata-Viehweg vanishing theorem, one has

$$\begin{aligned} |mK_{Y'}||_S &\succcurlyeq |K_{T'} + \ulcorner Q_{m,T'} - (m-1-\frac{1}{\theta_1})E_{1,T'} - \frac{1}{a_{t_1,T}}E_N \urcorner||_S \\ &\succcurlyeq |K_S + \ulcorner (Q_{m,T'} - (m-1-\frac{1}{\theta_1})E_{1,T'} - S - \frac{1}{a_{t_1,T}}E_N)|_S \urcorner| \\ &= |K_S + \ulcorner U_{m,S} \urcorner|, \end{aligned} \tag{9}$$

where

$$\begin{aligned} U_{m,S} &= (Q_{m,T'} - (m-1-\frac{1}{\theta_1})E_{1,T'} - S - \frac{1}{a_{t_1,T}}E_N)|_S \\ &\equiv ((m-1-\frac{1}{\theta_1})\cdot\frac{\theta_1}{1+\theta_1} - \frac{t_1}{a_{t_1,T}})\pi_T^*(K_T)|_S. \end{aligned}$$

By (3), we have

$$\pi_T^*(K_T)|_S \equiv \frac{a_{t_1,T}}{t_1+a_{t_1,T}}\sigma^*(K_{S_0}) + E_{t_1,S}$$

for some effective $\mathbb{Q}$-divisor $E_{t_1,S}$ on S, together with (9), one has

$$\begin{aligned} U_{m,S}^2 &= (((m-1-\frac{1}{\theta_1})\cdot\frac{\theta_1}{1+\theta_1}-\frac{t_1}{a_{t_1,T}})\pi_T^*(K_T)|_S)^2 \\ &\geq (((m-\frac{\theta_1+1}{\theta_1})\cdot\frac{\theta_1}{\theta_1+1}-\frac{t_1}{a_{t_1,T}})\cdot\frac{a_{t_1,T}}{t_1+a_{t_1,T}})^2\cdot K_{S_0}^2>8, \end{aligned}$$

where $U_{m,S}$ is nef and big. Hence the statement clearly follows from Lemma 2.3, Lemma 2.4, Lemma 3.1, Lemma 3.2 and Theorem 2.2. □

Proposition 3.5. *Let Y be a minimal 4-fold of general type with $p_g(Y)\geq 2$. Pick a generic irreducible element T' of $|M_1|$ and a generic irreducible element S of $|N|$. If S is neither a $(1,2)$-surface nor a $(2,3)$-surface, then $\varphi_{m,Y}$ is birational for all*

$$m\geq 6(t_1+1).$$

Proof. Suppose $m\geq 6(t_1+1)$. Following the same procedures as in the proof of Lemma 3.2 and Lemma 3.3, one has

$$|mK_{Y'}|\succcurlyeq|3(t_1+1)(K_{Y'}+T')|$$

and

$$|mK_{Y'}||_{T'}\succcurlyeq|3(t_1+1)K_{T'}|.$$

Furthermore, one has

$$\begin{aligned} |mK_{Y'}||_S &\succcurlyeq |3(t_1+1)K_{T'}||_S \\ &\succcurlyeq |3(K_{T'}+S)||_S \\ &= |3K_S|. \end{aligned}$$

By virtue of Bombieri's result in [2] that $|3K_S|$ gives a birational map unless S is a $(1,2)$-surface or a $(2,3)$-surface, together with Lemma 3.1, Lemma 3.2 and Theorem 2.2, the statement holds. □

3.3. **The case of** $\dim(\Gamma)\geq 2$**.** We follow Chen-Chen's approach in [6, Theorem 8.2] to deal with the case of $\dim(\Gamma)\geq 2$.

Theorem 3.6. *([6, Theorem 8.2]) Let Y be a minimal 4-fold of general type with $p_g(Y)\geq 2$. Assume that $\dim(\Gamma)\geq 2$, then $\varphi_{m,Y}$ is birational for all $m\geq 15$.*

Proof. By Theorem 2.2, we may just consider $\varphi_{m,Y'}|_{T'}$ for a general member $T'\in|M_1|$. As $\theta_1=1$, (2) gives

$$\pi^*(K_Y)|_{T'}\geq\frac{1}{2}\pi_T^*(K_T). \tag{10}$$

Pick a generic irreducible element S of $|M_1|_{T'}|$. It follows that

$$\pi^*(K_Y)|_{T'}\geq M_1|_{T'}\geq S.$$

Modulo $\mathbb{Q}$-linear equivalence, we have

$$K_{T'}\geq(\pi^*(K_Y)+T')|_{T'}\geq 2S. \tag{11}$$

Using Theorem 2.1, we get

$$\pi_T^*(K_T)|_S\geq\frac{2}{3}\sigma^*(K_{S_0}). \tag{12}$$

Thus, combining (10) and (12), one gets

$$\pi^*(K_Y)|_S\geq\frac{1}{3}\sigma^*(K_{S_0}).$$

By (8), we already have

$$|mK_{Y'}||_{T'} \succcurlyeq |K_{T'} + \ulcorner Q_{m,T'} \urcorner|,$$

where $Q_{m,T'} = ((m-1)\pi^*(K_Y) - T' - \frac{1}{\theta_1}E_1)|_{T'} \equiv (m-2)\pi^*(K_Y)|_{T'}$. As $\pi^*(K_Y)|_{T'} \equiv S + E_S$ for some effective $\mathbb{Q}$-divisor E_S on T' and

$$Q_{m,T'} - S - E_S \equiv (m-3)\pi^*(K_Y)|_{T'}$$

is nef and big, Kawamata-Viehweg vanishing theorem implies

$$\begin{aligned} |mK_{Y'}||_S &\succcurlyeq |K_{T'} + \ulcorner Q_{m,T'} - E_S \urcorner||_S \\ &\succcurlyeq |K_S + \ulcorner R'_{m,S} \urcorner|, \end{aligned}$$

where

$$\begin{aligned} R'_{m,S} &= (Q_{m,T'} - S - E_S)|_S \\ &\equiv (m-3)\pi^*(K_Y)|_S. \end{aligned}$$

Since $R'_{m,S} \equiv \frac{m-3}{3}\sigma^*(K_{S_0}) + E'_{m,S}$, where $E'_{m,S}$ is an effective $\mathbb{Q}$-divisor on S, by Lemma 2.4, $|K_S + \ulcorner R'_{m,S} \urcorner|$ gives a birational map whenever $m \geq 15$.

Since $\mathrm{Mov}|K_{T'}| \succcurlyeq |M_1|_{T'}|$, we may take $t_1 = 1$ and by the proof of Lemma 3.2 we know that $|mK_{Y'}||_{T'}$ distinguishes different generic irreducible elements of $|M_1|_{T'}|$ for $m \geq 6$. Therefore, $\varphi_{m,Y}$ is birational for all $m \geq 15$ in this case. □

3.4. **The case of** $\dim(\Gamma) = 1$**.**

Theorem 3.7. *Let Y be a minimal 4-fold of general type with $p_g(Y) \geq 2$. Assume that $\dim(\Gamma) = 1$, then $\varphi_{m,Y}$ is birational for all $m \geq 33$.*

Proof. We have $\theta_1 \geq 1$ and $p_g(T') > 0$. By Lemma 3.1, $|mK_{Y'}|$ distinguishes different generic irreducible elements of $|M_1|$ for all $m \geq 3$.

As an overall discussion, we study the linear system $|mK_{Y'}||_C$ for generic irreducible element C of $|H|$. Recall that, by (8) and (9), we already have

$$|mK_{Y'}||_S \succcurlyeq |K_S + \ulcorner U_{m,S} \urcorner|,$$

where $U_{m,S} \equiv \big((m-1-\frac{1}{\theta_1})\frac{\theta_1}{\theta_1+1} - \frac{t_1}{a_{t_1,T}}\big)\pi_T^*(K_T)|_S$ is a nef and big $\mathbb{Q}$-divisor on S. As we have $\pi_T^*(K_T)|_S \sim \beta C + E_H$ for some effective $\mathbb{Q}$-divisor E_H on S, applying Kawamata-Viehweg vanishing theorem, we may get

$$\begin{aligned} |mK_{Y'}||_C &\succcurlyeq |K_S + \ulcorner U_{m,S} - \frac{1}{\beta}E_H \urcorner||_C \\ &= |K_C + \ulcorner U_{m,S} - C - \frac{1}{\beta}E_H \urcorner|_C| \\ &= |K_C + \mathcal{D}_m|, \end{aligned} \tag{13}$$

where $\mathcal{D}_m = \ulcorner U_{m,S} - C - \frac{1}{\beta}E_H \urcorner|_C$ with

$$\deg \mathcal{D}_m \geq \big((m-1-\frac{1}{\theta_1})\frac{\theta_1}{\theta_1+1} - \frac{t_1}{a_{t_1,T}} - \frac{1}{\beta}\big)(\pi_T^*(K_T)|_S \cdot C).$$

Thus, whenever $m > \big(\frac{2}{\xi} + \frac{t_1}{a_{t_1,T}} + \frac{1}{\beta} + 1\big) \cdot \frac{\theta_1+1}{\theta_1}$, $|mK_{Y'}||_C$ gives a birational map.

Therefore, by Lemma 3.2, Lemma 3.3 and Theorem 2.2, $\varphi_{m,Y}$ is birational provided that

$$m \geq 4t_1 + 4 \text{ and } m > \frac{4}{\xi} + \frac{2t_1}{a_{t_1,T}} + \frac{2}{\beta} + 2.$$

Now we study this problem according to the value of $p_g(T)$.

Case 1. $p_g(T) \geq 2$

Clearly, we may take $t_1 = 1$ and so $a_{t_1,T} = 1$. From [8, Section 2, Section 3], we know that one of the following cases occurs:

(1) $\beta \geq 1$, $\xi \geq 1$; (correspondingly, $d \geq 2$ in [8, 3.1, 3.2])
(2) $\beta \geq \frac{1}{2}$, $\xi \geq 2$; (correspondingly, $d = 1$ and $b > 0$ in [8, 2.8, 2.10, 3.3])
(3) $\beta \geq \frac{1}{4}$, $\xi \geq 1$; (correspondingly, $d = 1$, $b = 0$ and $(1,1)$-surface case in [8, 2.13, 3.8])
(4) $\beta \geq \frac{1}{2}$, $\xi \geq \frac{2}{3}$; (correspondingly, $d = 1$, $b = 0$ and $(1,2)$-surface case in [8, 2.15, 3.7])
(5) $\beta \geq \frac{1}{2}$, $\xi \geq 1$; (correspondingly, $d = 1$, $b = 0$ and $(2,3)$-surface case in [8, 2.12, 3.6])
(6) $\beta \geq \frac{1}{4}$, $\xi \geq 2$. (correspondingly, $d = 1$, $b = 0$ and other surface case in [8, 2.11, 3.5])

So $\varphi_{m,Y}$ is birational for all $m \geq 17$.

Case 2. $p_g(T) = 1$

According to [6, Corollary 4.10], T must be of one of the following types: (i) $P_4(T) = 1, P_5(T) \geq 3$; (ii) $P_4(T) \geq 2$.

For Type (i), we have $t_1 = 5$ and set $|N| = \mathrm{Mov}|5K_{T'}|$. When $|N|$ is composed of a pencil, we have $a_{t_1,T} \geq 2$ and S is exactly the general fiber of the induced fibration from $\varphi_{t_1,T} \circ \pi_T$. If S is not a $(1,2)$-surface, by Proposition 3.4, $\varphi_{m,Y}$ is birational for all $m \geq 27$. If S is a $(1,2)$-surface, by [12, Proposition 4.1, Case 2], we have $\beta \geq \frac{2}{7}$ and $\xi \geq \frac{2}{7}$, and hence $\varphi_{m,Y}$ is birational for $m \geq 29$. When $|N|$ is not composed of a pencil, we have $a_{t_1,T} \geq 1$. Using the case by case argument of [12, Proposition 4.2, Proposition 4.3], to give an exact list, (β,ξ) must be one of the following: $(1/5,3/7)$, $(1/5,2/3)$, $(1/3,1/3)$, $(1/5,5/13)$, $(1/5,1)$, $(1/2,1/3)$, $(1/5,1/2)$, $(2/5,1/3)$, $(1/4,1/3)$. Hence $\varphi_{m,Y}$ is birational for all $m \geq 33$.

For Type (ii), we have $t_1 = 4$ and set $|N| = \mathrm{Mov}|4K_{T'}|$. When $|N|$ is composed of a pencil and the generic irreducible element S of $|N|$ is neither a $(1,2)$-surface nor a $(2,3)$-surface, by Proposition 3.5, $\varphi_{m,Y}$ is birational for all $m \geq 30$. When $P_4(T) = h^0(T, \mathcal{O}_T(4K_T)) = 2$ and $|N|$ is composed of a rational pencil of $(1,2)$-surfaces, the case by case argument of [12, Proposition 4.6, Proposition 4.7] tells that (β,ξ) must be one of the following: $(2/7,2/7)$, $(1/5,2/5)$, $(2/5,2/7)$, $(1/5,1/3)$, $(1/5,2/3)$, $(1/5,5/12)$, $(1/3,2/7)$, $(1/4,2/7)$. Hence $\varphi_{m,Y}$ is birational for all $m \geq 33$. Otherwise, the case by case argument of [12, Proposition 4.5] tells that (β,ξ) must be one of the following: $(1/4,2/5)$, $(1/5,2/5)$, $(1/3,1/3)$. Hence $\varphi_{m,Y}$ is birational for all $m \geq 31$.

In conclusion, $\varphi_{m,Y}$ is birational for all $m \geq 33$. □

3.5. **The canonical volume of 4-folds.**

Theorem 3.8. *Let Y be a minimal 4-fold of general type with $p_g(Y) \geq 2$. Then* $\mathrm{Vol}(Y) \geq \frac{1}{480}$.

Proof. We have $\mathrm{Vol}(Y) = K_Y^4 = (\pi^*(K_Y))^4$.

Recall that $\pi^*(K_Y) \equiv \theta_1 T' + E_1$. One has

$$\mathrm{Vol}(Y) \geq \theta_1(\pi^*(K_Y))^3 \cdot T' = \theta_1(\pi^*(K_Y)|_{T'})^3.$$

As we also have (2) and $t_1\pi_T^*(K_T) \equiv a_{t_1,T}S + E_N$, it follows that

$$\mathrm{Vol}(Y) \quad \geq \quad \theta_1 \cdot (\frac{\theta_1}{1+\theta_1})^3(\pi_T^*(K_T))^3$$

$$\begin{aligned} &\geq \theta_1 \cdot (\frac{\theta_1}{1+\theta_1})^3 \cdot \frac{a_{t_1,T}}{t_1}(S \cdot (\pi_T^*(K_T))^2) \\ &= \theta_1 \cdot (\frac{\theta_1}{1+\theta_1})^3 \cdot \frac{a_{t_1,T}}{t_1}(\pi_T^*(K_T)|_S)^2. \end{aligned}$$

By (3) and $\pi_T^*(K_T)|_S \geq \beta C$, we may get

$$\mathrm{Vol}(Y) \geq \theta_1 \cdot (\frac{\theta_1}{1+\theta_1})^3 \cdot \frac{a_{t_1,T}}{t_1} \cdot (\frac{a_{t_1,T}}{t_1+a_{t_1,T}})^2 K_{S_0}^2$$

and

$$\begin{aligned} \mathrm{Vol}(Y) &\geq \theta_1 \cdot (\frac{\theta_1}{1+\theta_1})^3 \cdot \frac{a_{t_1,T}}{t_1} \cdot \beta(\pi_T^*(K_T)|_S \cdot C) \\ &= \theta_1 \cdot (\frac{\theta_1}{1+\theta_1})^3 \cdot \frac{a_{t_1,T}}{t_1} \cdot \beta\xi. \end{aligned}$$

Now we estimate the canonical volume according to the same classification of T and S as in Subsection 3.3 and Subsection 3.4.

(I) The case of $\dim(\Gamma) \geq 2$

Remember that in this case, $\theta_1 = 1, t_1 = 1, a_{t_1,T} = 2$ (by (11)) and $\pi_T^*(K_T)|_S \geq \frac{2}{3}\sigma^*(K_{S_0})$ (by (12)). So we have $\mathrm{Vol}(Y) \geq \frac{1}{9}$.

(II) The case of $\dim(\Gamma) = 1$

Subcase (II-1). $p_g(T) \geq 2$.

As in Theorem 3.7, Case 1, $t_1 = 1, a_{t_1,T} = 1$, so we correspondingly have the estimation as follows:

(1) $\beta \geq 1$, $\xi \geq 1$, then $\mathrm{Vol}(Y) \geq \frac{1}{8}$;
(2) $\beta \geq \frac{1}{2}$, $\xi \geq 2$, then $\mathrm{Vol}(Y) \geq \frac{1}{8}$;
(3) $\beta \geq \frac{1}{4}$, $\xi \geq 1$, then $\mathrm{Vol}(Y) \geq \frac{1}{32}$;
(4) $\beta \geq \frac{1}{2}$, $\xi \geq \frac{2}{3}$, then $\mathrm{Vol}(Y) \geq \frac{1}{24}$;
(5) $\beta \geq \frac{1}{2}$, $\xi \geq 1$, then $\mathrm{Vol}(Y) \geq \frac{1}{16}$;
(6) $\beta \geq \frac{1}{4}$, $\xi \geq 2$, then $\mathrm{Vol}(Y) \geq \frac{1}{16}$.

Subcase (II-2). $p_g(T) = 1$.

We follow the same classification of T as in Theorem 3.7, Case 2.

Recall that for Type (i), we have $t_1 = 5$. When $|N|$ is composed of a pencil and the general fiber S of the induced fibration from $\varphi_{t_1,T} \circ \pi_T$ is not a $(1,2)$-surface, we have $a_{t_1,T} \geq 2$, $\beta \geq \frac{1}{7}$, $\xi \geq (\frac{2}{7}\sigma^*(K_{S_0}) \cdot C) \geq \frac{4}{7}$, and thus $\mathrm{Vol}(Y) \geq \frac{1}{245}$. When $|N|$ is composed of a pencil and the general fiber S is a $(1,2)$-surface, we have $a_{t_1,T} \geq 2$, $\beta \geq \frac{2}{7}$, $\xi \geq \frac{2}{7}$, and hence $\mathrm{Vol}(Y) \geq \frac{1}{245}$. When $|N|$ is not composed of a pencil, we have $a_{t_1,T} \geq 1$. The two cases corresponding to $(\beta, \xi) = (\frac{1}{5}, \frac{5}{13})$ in Theorem 3.7 Case 2 both have $K_{S_0}^2 \geq 4$, where S_0 is the minimal model of a generic irreducible element S of $|N|$. So $\mathrm{Vol}(Y) \geq \frac{1}{360}$. The corresponding lower bounds of $\mathrm{Vol}(Y)$ to those of (β, ξ) (except $(\beta, \xi) = (\frac{1}{5}, \frac{5}{13})$) are as follows: $\frac{3}{1400}, \frac{1}{300}, \frac{1}{360}, \frac{1}{200}, \frac{1}{240}, \frac{1}{400}, \frac{1}{300}, \frac{1}{480}$.

For Type (ii), we have $t_1 = 4$. When $|N|$ is not composed of a pencil, then $\beta \geq \frac{1}{4}, \xi \geq \frac{2}{5}$ and $\mathrm{Vol}(Y) \geq \frac{1}{320}$. When $|N|$ is composed of a pencil of $(2,3)$-surfaces, then $\beta \geq \frac{1}{5}, \xi \geq \frac{2}{5}$ and $\mathrm{Vol}(Y) \geq \frac{1}{400}$. When $|N|$ is composed of a pencil of surfaces with $K_{S_0}^2 \geq 2$, then $\mathrm{Vol}(Y) \geq \frac{1}{400}$. When $P_4(T) = h^0(T, \mathcal{O}_T(4K_T)) = 2$ and $|N|$ is composed of a rational pencil of $(1,2)$-surfaces, the corresponding lower bounds of $\mathrm{Vol}(Y)$ to those of (β, ξ) are as follows: $\frac{1}{392}, \frac{1}{400}, \frac{1}{280}, \frac{1}{480}, \frac{1}{240}, \frac{1}{384}, \frac{1}{336}, \frac{1}{448}$. Otherwise, $\beta \geq \frac{1}{3}, \xi \geq \frac{1}{3}$ and $\mathrm{Vol}(Y) \geq \frac{1}{288}$.

So we have shown $\mathrm{Vol}(Y) \geq \frac{1}{480}$. □

3.6. **Proof of Theorem 1.1.**

Proof. Theorem 3.6, Theorem 3.7 and Theorem 3.8 directly implies Theorem 1.1. □

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