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GROUND STATE AND NODAL SOLUTIONS FOR FRACTIONAL KIRCHHOFF EQUATION WITH PURE CRITICAL GROWTH NONLINEARITY

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Abstract. In this paper, we consider the existence of least energy nodal solution and ground state solution, energy doubling property for the following fractional critical problem

$$\begin{cases} -(a+b\|u\|_K^2)\mathcal{L}_K u + V(x)u = |u|^{2_\alpha^*-2}u + kf(x,u), & x\in\Omega,\\ u=0, & x\in\mathbb{R}^3\backslash\Omega, \end{cases}$$

where k is a positive parameter, $\mathcal{L}_K$ stands for a nonlocal fractional operator which is defined with the kernel function K. By using the nodal Nehari manifold method, we obtain a least energy nodal solution u and a ground state solution v to this problem when $k \gg 1$, where the nonlinear function $f:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function.

1. **Introduction and main results.** Our goal of this paper is to consider the existence of nodal solution and ground state solution for the following fractional Kirchhoff equation:

$$\begin{cases} -(a+b\|u\|_K^2)\mathcal{L}_K u + V(x)u = |u|^{2_\alpha^*-2}u + kf(x,u), & x\in\Omega,\\ u=0, & x\in\mathbb{R}^3\backslash\Omega, \end{cases} \tag{1}$$

where a, b, k are positive real numbers. Let $\alpha\in(\frac{3}{4},1)$ such that $2_\alpha^* = \frac{6}{3-2\alpha}\in(4,6)$ and $V(x)\in C(\mathbb{R}^3,\mathbb{R}^+)$, $\Omega\subset\mathbb{R}^3$ is a bounded domain with a smooth boundary $\partial\Omega$. $\|u\|_K^2 := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|u(x)-u(y)|^2K(x-y)dxdy$. The non-local integrodifferential operator $\mathcal{L}_K$ is defined as follows:

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^3}(u(x+y)+u(x-y)-2u(x))K(y)dy,\ x\in\mathbb{R}^3,$$

the kernel $K:\mathbb{R}^3\to(0,\infty)$ is a function with the properties that
(i) $mK\in L^1(\mathbb{R}^3)$, where $m(x)=\min\{|x|^2,1\}$;

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(ii) there exists $\lambda > 0$ such that $K(x) \geq \lambda |x|^{-(3+2\alpha)}$ for any $x \in \mathbb{R}^3 \backslash \{0\}$;
(iii) $K(x) = K(-x)$ for any $x \in \mathbb{R}^3 \backslash \{0\}$.
We note that when $K(x) = \frac{1}{|x|^{3+2\alpha}}$, the integrodifferential operator $\mathcal{L}_K$ is the fractional Laplacian operator $(-\Delta)^\alpha$:

$$(-\Delta)^\alpha u(x) = -\frac{C(\alpha)}{2} \int_{\mathbb{R}^3} \frac{(u(x+y) + u(x-y) - 2u(x))}{|y|^{3+2\alpha}} dy$$

and in this case

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u(x) - u(y)|^2 K(x-y) dx dy = \frac{2}{C(\alpha)} \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u(x)|^2 dx,$$

where $C(\alpha) = \left(\int_{\mathbb{R}^3} \frac{1-\cos \xi_1}{|\xi|^{3+2\alpha}} d\xi \right)^{-1}$ is the normalized constant.

When $a = 1$, $b = 0$ and $K(x) = \frac{1}{|x|^{3+2\alpha}}$, the fractional Kirchhoff equation is reduced to the undermentioned fractional nonlocal problem

$$\begin{cases} (-\Delta)^\alpha u + V(x) u = |u|^{2^*_\alpha - 2} u + k f(x,u), & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^3 \backslash \Omega. \end{cases} \tag{2}$$

Equation (2) is derived from the fractional Schrödinger equation and the nonlinearity $f(x,u)$ represents the particles interacting with each other. On the other hand, recently a great attention has been given to the so called fractional Kirchhoff equation (see [3, 10] etc.,):

$$(a + b \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2\alpha}} dx dy)(-\Delta)^\alpha u = f(x,u), \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain or $\Omega = \mathbb{R}^N$, $a > 0, b > 0$ and u satisfies some boundary conditions. Problem (3) is called to the stationary state of the fractional Kirchhoff equation

$$u_{tt} + (a + b \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2\alpha}} dx dy)(-\Delta)^\alpha u = f(x,u). \tag{4}$$

As a special significant case, the nonlocal aspect of the tension arises from nonlocal measurements of the fractional length of the string. For more mathematical and physical background on Schrödinger-Kirchhoff type problems, we refer the readers to [6] and the references therein.

In the remarkable work of Caffarelli and Silvestre [2], the authors express this nonlocal operator $(-\Delta)^\alpha$ as a Dirichlet-Neumann map for a certain elliptic boundary value problem with local differential operators defined on the upper half space. This technique is a valid tool to deal with the equations involving fractional operator in the respects of regularity and variational methods. However, we do not know whether the Caffarelli-Silvestre extension method can be applied to the general integro-differential operator $\mathcal{L}_K$. So the nonlocal feature of the integro-differential operator brings some difficulties to applications of variational method to problem (1). Some of these additional difficulties were overcome in [12, 13]. More studies on this integro-differential operator equation, for examples, positive, negative, sign-changing or ground state solutions, we refer the papers [4, 5, 9, 16] and their references. In [1], the authors considered the obstacle problems for $\mathcal{L}_K$ about the higher regularity of free boundaries. The obstacle problems for nonlinear integro-differential operators and applications are also available in papers [7, 8]. The appearance of nonlocal term not only makes it playing an important role in many physical applications, but also brings some difficulties and challenges in mathematical analysis. This fact makes the study of fractional Kirchhoff equation or similar problems particularly interesting. A lot of interesting results on the existence of nonlocal problems were obtained recently, specially on the existence of positive solutions, multiple solutions, ground states and semiclassical states, for examples, we refer [3, 15, 19] and the cited references.

In past few years, some researchers began to search for nodal solutions of Schrödinger type equation with critical growth nonlinearity and have got some interesting results. For example, Zhang [20] considered the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + u + k(x)\phi u = a(x)|u|^{p-2}u + u^5, & x \in \mathbb{R}^3, \\ -\Delta\phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{5}$$

where $p \in (4, 6)$, $k(x)$ and $a(x)$ are nonnegative functions. By using variational method, a ground state solution and a nodal solution for problem (5) were obtained.

Wang [17] studies the following Kirchhoff-type equation:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 dx)\Delta u = |u|^4 u + \lambda f(x, u), & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{6}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $\lambda, a, b > 0$ are fixed parameters, and $f(x, \cdot)$ is continuously differentiable for a.e. $x \in \Omega$. By combining constrained variational method with the degree theory, the existence of ground state and nodal solutions for above problem were obtained.

However, as for fractional Kirchhoff types equation, to the best of our knowledge, few results involved the existence and asymptotic behavior of ground state and nodal solutions in case of critical growth. If $k(x) \equiv 1$, the method used in [20] seems not valid for problem (1), because their result depends on the case $k \in L^p(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ for some $p \in (1, 2^*_\alpha)$. We employ minimization arguments on suitable nodal Nehari manifold $\mathcal{M}_k$ to construct a nodal solution by using various constrained method and qualitative deformation lemma given in [14].

It's worth noting that, the Brouwer degree method used in [18] strictly depends on the nonlinearity $f \in C^1(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, so we have to find new tricks to solve our modeling where we only allow $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. On the other hand, in our modeling, both of the nonlocal fractional operator $\mathcal{L}_K$ and nonlocal terms $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |u(x) - u(y)|^2 K(x - y)dxdy$ appear, we need to overcome the difficulties caused by the nonlocal terms under a suitable variational framework. In brief, our discussion is based on the standard nodal Nehari manifold method used in [14].

Throughout this paper, we let

$$E = \{u \in X : \int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x < \infty, u = 0 \text{ a.e. in } \mathbb{R}^3 \backslash \Omega\},$$

where the space X introduced by Servadei and Valdinoci ([12, 13]) denotes the linear space of Lebesgue measurable functions $u : \mathbb{R}^3 \to \mathbb{R}$ such that its restriction to Ω $u|_\Omega \in L^2(\Omega)$ and

$$((x, y) \to (u(x) - u(y))\sqrt{K(x - y)} \in L^2((\mathbb{R}^3 \times \mathbb{R}^3) \setminus (\Omega^c \times \Omega^c), dxdy)$$

with the following norm

$$||u||_X^2 = ||u||_{L^2}^2 + \int_Q |u(x)-u(y)|^2 K(x-y)dxdy,$$

where $Q = (\mathbb{R}^3 \times \mathbb{R}^3) \setminus (\Omega^c \times \Omega^c)$. Then, E is a Hilbert space with inner product

$$\langle u,v\rangle = \frac{a}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(u(x)-u(y))(v(x)-v(y))K(x-y)dxdy + \int_\Omega V(x)uvdx, \forall\, u,\ v \in E$$

and the norm $\|\cdot\|$ defined by

$$\|u\|^2 = \frac{a}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|u(x)-u(y)|^2 K(x-y)dxdy + \int_\Omega V(x)u^2dx.$$

$X_0 = \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^3\backslash\Omega\}$ is a Hilbert space with inner product $(u,v)_K = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(u(x)-u(y))(v(x)-v(y))K(x-y)dxdy$ and the Gagliardo norm $\|u\|_K^2 := (u,u)_K = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|u(x)-u(y)|^2K(x-y)dxdy$. In X_0 the norms $\|\cdot\|_K$ and $\|\cdot\|_X$ are equivalent. Thus due to $V(x) \geq 0$ in Ω, the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$ defined above still make sense for the Hilbert space E. Moreover, $\|u\|_K^2 \leq a\|u\|^2$. For more details , we refer to [12].
The following result for the space X_0 will be used repeatedly.

Lemma 1.1. *([12]) Let $s \in (0,1)$, $N > 2s$, Ω be an open bounded set of $\mathbb{R}^N$, $X_0 \hookrightarrow\hookrightarrow L^p(\Omega)$ for $2 < p < 2_\alpha^*$, and $X_0 \hookrightarrow L^{2_\alpha^*}(\Omega)$ is continuous. Then, $E \subset X_0$ has the same embedding properties.*

A weak solution $u \in E$ of (1) is defined to satisfy the following equation

$\frac{1}{2}(a+b\|u\|_K^2)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(u(x)-u(y))(v(x)-v(y))K(x-y)dxdy$
$+\int_\Omega V(x)u(x)v(x)dx - \int_\Omega |u(x)|^{2_\alpha^*-2}u(x)v(x)dx - k\int_\Omega f(x,u(x))v(x)dx = 0,$

for any $v \in E$, *i.e.*

$$\langle u,v\rangle + b\|u\|_K^2(u,v)_K - k\int_\Omega f(x,u)vdx - \int_\Omega |u|^{2_\alpha^*-2}uvdx = 0,\ \forall v \in E.$$

As for the function f, we assume $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfing the following hypotheses:

(f_1): $f(x,t)\cdot t > 0$ for $t \neq 0, \forall x \in \Omega$ and $\lim_{t\to 0}\frac{f(x,t)}{t} = 0$ for $x \in \Omega$ uniformly;
(f_2): There exists $q \in (4, 2_\alpha^*)$ such that $\lim_{t\to\infty}\frac{f(x,t)}{t^{q-1}} = 0$ for all $t \in \mathbb{R}\setminus\{0\}$ and for $x \in \Omega$ uniformly;
(f_3): $\frac{f(x,t)}{|t|^3}$ is an nondecreasing function with respect to t in $(-\infty,0)$ and $(0,+\infty)$ for a.e. $x \in \Omega$.

Remark 1. We note that under the conditions (f_1)-(f_3), it is easy to see the function $f(x,t) = t^3$ is an example satisfying all conditions (f_1)-(f_3).

The main results can be stated as follows.

Theorem 1.2. *Suppose that (f_1)-(f_3) are satisfied. Then, there exists $k^\star > 0$ such that for all $k \geq k^\star$, the problem (1) has a ground state nodal solution u_k.*

Remark 2. The ground state nodal solution u_k with $u_k^\pm \neq 0$ is a solution of (1) satisfying

$$J_k(u_k) = c_k := \inf_{u\in\mathcal{M}_k} J_k(u),$$

where $\mathcal{M}_k$ is defined by (7) in next section. For $u \in E$, $u^\pm$ is defined by

$$u^+ = \max\{u(x),0\}, \quad u^- = \min\{u(x),0\}.$$

We recall that the nodal set of a continuous function $u : \mathbb{R}^3 \to \mathbb{R}$ is the set $u^{-1}(0)$. Every connected component of $\mathbb{R}^3 \setminus u^{-1}(0)$ is called a nodal domain. For this equation talking about the nodal domain of u_k in somewhat is difficult for us, we leave it as future topics.

Theorem 1.3. *Suppose that* (f_1)-(f_3) *are satisfied. Then, there exists* $k^{\star\star} > 0$ *such that for all* $k \geq k^{\star\star}$*, the* $c^* = \inf_{u\in\mathcal{N}_k} J_k(u) > 0$ *is achieved by* v_k*, which is a ground state solution of* (1) *and*

$$J_k(u_k) > 2c^*,$$

where $\mathcal{N}_k = \{u \in E\backslash \ \{0\} | \langle J_k'(u), u\rangle = 0\}$*,* u_k *is the ground state nodal solution obtained in Theorem 1.1.*

Comparing with the literature, the above two results can be regarded as a supplementary of those in [3, 4, 14, 17].

The remainder of this paper is organized as follows. In Section 2, we give some useful preliminaries. In Section 3, we study the existence of ground state and nodal solutions of (1) and we prove Theorems 1.1-1.2.

2. Some technical lemmas.

We define the energy functional associated with equation (1) as follows:

$$J_k(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|_K^4 - k\int_\Omega F(x,u)dx - \frac{1}{2_\alpha^*}\int_\Omega |u|^{2_\alpha^*}dx, \ \forall u \in E.$$

According to our assumptions on $f(x,t)$, $J_k(u)$ belongs to $C^1(E,\mathbb{R})$ (see [10, 12]), then by direct computations, we have

$$\langle J_k'(u), v\rangle = \langle u, v\rangle + b\|u\|_K^2(u,v)_K - k\int_\Omega f(x,u)vdx - \int_\Omega |u|^{2_\alpha^*-2}uvdx, \ \forall u,v \in E.$$

Note that, since (1) involves pure critical nonlinearity $|u|^{2_\alpha^*-2}u$, it will prevent us using the standard arguments as in [14]. Hence, we need some tricks to overcome the lack of compactness of $E \hookrightarrow L^{2_\alpha^*}(\mathbb{R}^3)$.

For fixed $u \in E$ with $u^\pm \neq 0$, the function $\varphi_u : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is well defined by $\varphi_u(s,t) = J_k(su^+ + tu^-)$. We set

$$H(s,t) = \langle J_k'(su^+ + tu^-), su^+\rangle, \ \ G(s,t) = \langle J_k'(su^+ + tu^-), tu^-\rangle.$$

The argument generally used is to modify the method developed in [11]. The nodal Nehari manifold is defined by

$$\mathcal{M}_k = \{u \in E, u^\pm \neq 0 \text{ and } \langle J_k'(u), u^+\rangle = \langle J_k'(u), u^-\rangle = 0\}, \tag{7}$$

which is a subset of the Nehari manifold $\mathcal{N}_k$ and contains all nodal solutions of (1). Then, one needs to show the minimizer $u \in \mathcal{M}_k$ is a critical point of J_k. Because the problem (1) contains a nonlocal term $\|u\|_K^2$, the corresponding functional J_k does not have the decomposition

$$J_k(u) = J_k(u^+) + J_k(u^-),$$

it brings difficulties to construct a nodal solution.

The following result describes the shape of the nodal Nehari manifold $\mathcal{M}_k$.

Lemma 2.1. *Assume that* (f_1)-(f_3) *are satisfied, if* $u \in E$ *with* $u^\pm \neq 0$*, then there is a unique pair* $(s_u, t_u) \in (0,\infty) \times (0,\infty)$ *such that* $s_u u^+ + t_u u^- \in \mathcal{M}_k$*, which is also the unique maximum point of* φ_u *on* $[0,\infty) \times [0,\infty)$*. Furthermore, if* $\langle J_k'(u), u^\pm\rangle \leq 0$*, then* $0 < s_u, \ t_u \leq 1$*.*

Proof. From (f_1) and (f_2), for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ satisfying

$$|f(x,t)| \le \varepsilon|t| + C_\varepsilon |t|^{q-1}, \quad \forall t \in \mathbb{R}. \tag{8}$$

By above equality and Sobolev's embedding theorems, we have

$$\begin{aligned} H(s,t) :=& s^2\|u^+\|^2 + b\|su^+ + tu^-\|_K^2 (su^+ + tu^-, su^+)_K - \int_\Omega |su^+|^{2_\alpha^*} dx \\ &- k\int_\Omega f(x, su^+) su^+ dx \\ &- \frac{ast}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} [u^+(x)u^-(y) + u^-(x)u^+(y)]K(x-y)dxdy \\ \ge& s^2\|u^+\|^2 - C_1 s^{2_\alpha^*}\|u^+\|^{2_\alpha^*} - k\varepsilon C_2 s^2\|u^+\|^2 - kC_\varepsilon C_3 s^q \|u^+\|^q. \end{aligned} \tag{9}$$

By choosing $\varepsilon > 0$ small enough, we can deduce $H(s,t) > 0$ for $0 < s \ll 1$ and all $t \ge 0$. Similarly,

$$\begin{aligned} G(s,t) :=& t^2\|u^-\|^2 + b\|su^+ + tu^-\|_K^2 (su^+ + tu^-, tu^-)_K - \int_\Omega |tu^-|^{2_\alpha^*} dx \\ &- k\int_\Omega f(x, tu^-) tu^- dx \\ &- \frac{ast}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} [u^+(x)u^-(y) + u^-(x)u^+(y)]K(x-y)dxdy > 0, \end{aligned} \tag{10}$$

for $0 < t \ll 1$ and all $s \ge 0$. We denote $D(u) = \langle u^+, u^-\rangle = a(u^+, u^-)_K = -\frac{a}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}[u^+(x)u^-(y) + u^-(x)u^+(y)]K(x-y)dxdy \ge 0$. Hence, by choosing $\delta_1 > 0$ small, we have

$$H(\delta_1, t) > 0, \quad G(s, \delta_1) > 0. \tag{11}$$

For any $\delta_2 > \delta_1$, by condition (f_1), it is easy to see

$$\begin{aligned} H(\delta_2, t) \le& {\delta_2}^2\|u^+\|^2 + b\|\delta_2 u^+ + tu^-\|_K^2(\delta_2 u^+ + tu^-, \delta_2 u^+)_K \\ &- {\delta_2}^{2_\alpha^*}\int_{\mathbb{R}^3} |u^+|^{2_\alpha^*} dx + \delta_2 t D(u). \end{aligned}$$

Similarly, we have

$$\begin{aligned} G(s, \delta_2) \le& {\delta_2}^2\|u^-\|^2 + b\|su^+ + \delta_2 u^-\|_K^2(su^+ + \delta_2 u^-, \delta_2 u^-)_K \\ &- {\delta_2}^{2_\alpha^*}\int_{\mathbb{R}^3} |u^-|^{2_\alpha^*} dx + s\delta_2 D(u). \end{aligned}$$

By choosing $\delta_2 \gg 1$, we deduce

$$H(\delta_2, t) < 0, \quad G(s, \delta_2) < 0 \tag{12}$$

for all $s, t \in [\delta_1, \delta_2]$.

Following (11) and (12), we can use Miranda's Theorem (see Lemma 2.4 in [17]) to get a positive pair $(s_u, t_u) \in (0,\infty) \times (0,\infty)$ such that $s_u u^+ + t_u u^- \in \mathcal{M}_k$. Similar to the standard argument in [17], we can prove the pair (s_u, t_u) is unique maximum point of φ_u on $[0,+\infty) \times [0,+\infty)$.

Lastly, we will prove that $0 < s_u, t_u \le 1$ when $\langle J_k'(u), u^\pm\rangle \le 0$. Following from $s_u \ge t_u > 0$, by a direct computation, we have

$$\begin{aligned}
&s_u^2\|u^+\|^2 + s_u^2 D(u) + s_u^4 b(\|u^+\|_K^2 + \|u^-\|_K^2 + 2D(u))(\|u^+\|_K^2 + D(u))\\
\geq &s_u^2\|u^+\|^2 + s_u t_u D(u) + b(s_u^2\|u^+\|_K^2 + t_u^2\|u^-\|_K^2\\
&+ 2s_u t_u D(u))(s_u^2\|u^+\|_K^2 + s_u t_u D(u))\\
= &s_u^{2_\alpha^*}\int_{\mathbb{R}^3}|u^+|^{2_\alpha^*}dx + k\int_{\mathbb{R}^3} f(x, s_u u^+)s_u u^+ dx.
\end{aligned} \tag{13}$$

On the other hand, $\langle J_k'(u), u^+\rangle \leq 0$ implies that

$$\begin{aligned}
&\|u^+\|^2 + D(u) + b(\|u^+\|_K^2 + \|u^-\|_K^2 + 2D(u))(\|u^+\|_K^2 + D(u))\\
\leq &\int_{\mathbb{R}^3}|u^+|^{2_\alpha^*}dx + k\int_{\mathbb{R}^3} f(x, u^+)u^+ dx.
\end{aligned} \tag{14}$$

From (13) - (14), we can see that

$$\begin{aligned}
(\frac{1}{s_u^2} - 1)(\|u^+\|^2 + D(u)) \geq &(s_u^{2_\alpha^*-4} - 1)\int_{\mathbb{R}^3}|u^+|^{2_\alpha^*}dx\\
&+ k\int_{\mathbb{R}^3}\left[\frac{f(x, s_u u^+)}{(s_u u^+)^3} - \frac{f(x, u^+)}{(u^+)^3}\right](u^+)^4 dx.
\end{aligned}$$

So we have $s_u \leq 1$, which implies that $0 < s_u, t_u \leq 1$. □

Lemma 2.2. *There exists $k^\star > 0$ such that for all $k \geq k^\star$, the infimum $c_k = \inf_{u\in\mathcal{M}_k} J_k(u)$ is achieved.*

Proof. For any $u \in \mathcal{M}_k$, in view of the definitions of $\mathcal{M}_k$, it follows that

$$\|u^\pm\|^2 + b\|u\|_K^2(u, u^\pm)_K = k\int_{\mathbb{R}^3} f(x, u^\pm)u^\pm dx + \int_{\mathbb{R}^3}|u^\pm|^{2_\alpha^*}dx.$$

Hence, in view of (8), we have

$$\|u^\pm\|^2 \leq k\varepsilon C_1\|u^\pm\|^2 + kC_2\|u^\pm\|^q + C_3\|u^\pm\|^{2_\alpha^*}.$$

By choosing $\varepsilon > 0$ small enough, we can deduce

$$\|u^\pm\| \geq \rho \tag{15}$$

for some $\rho > 0$. From assumption (f_3), we have

$$f(x, t)t - 4F(x, t) \geq 0, \tag{16}$$

and $f(x, t)t - 4F(x, t)$ is nondecreasing in $(0, +\infty)$ and non-increasing in $(-\infty, 0)$ with respect to the variable t. Hence, combining with $\langle J_k'(u), u\rangle = 0$, we have

$$J_k(u) = \frac{1}{4}\|u\|^2 + (\frac{1}{4} - \frac{1}{2_\alpha^*})\int_{\mathbb{R}^3}|u|^{2_\alpha^*}dx + \frac{k}{4}\int_{\mathbb{R}^3}[f(x, u)u - 4F(x, u)]dx \geq \frac{1}{4}\|u\|^2.$$

So $c_k = \inf_{u\in\mathcal{M}_k} J_k(u) \geq 0$ is well defined.

Let $u \in E$ with $u^\pm \neq 0$ be fixed. According to Lemma 2.1, for each $k > 0$, there exists $s_k, t_k > 0$ such that $s_k u^+ + t_k u^- \in \mathcal{M}_k$. Hence, by (f_1) and the Sobolev's embedding theorem, we have

$$\begin{aligned}
&s_k^{2_\alpha^*}\int_{\mathbb{R}^3}|u^+|^{2_\alpha^*}dx + t_k^{2_\alpha^*}\int_{\mathbb{R}^3}|u^-|^{2_\alpha^*}dx \leq \|s_k u^+ + t_k u^-\|^2 + b\|s_k u^+ + t_k u^-\|_K^4\\
\leq &2s_k^2\|u^+\|^2 + 2t_k^2\|u^-\|^2 + 4bs_k^4\|u^+\|_K^4 + 4bt_k^4\|u^-\|_K^4,
\end{aligned}$$

which implies (s_k, t_k) is bounded and furthermore by taking any sequence $(s_{k_n}, t_{k_n}) \to (s_0, t_0)$ as $k_n \to \infty$, we claim that $s_0 = t_0 = 0$. In fact, if $s_0 > 0$ or $t_0 > 0$. Thanks to $s_{k_n}u^+ + t_{k_n}u^- \in \mathcal{M}_{k_n}$, we get

$$\begin{aligned}
&\|s_{k_n}u^+ + t_{k_n}u^-\|^2 + b\|s_{k_n}u^+ + t_{k_n}u^-\|_K^4 \\
=& \int_{\mathbb{R}^3} |s_{k_n}u^+ + t_{k_n}u^-|^{2^*_\alpha} dx + k_n \int_{\mathbb{R}^3} f(s_{k_n}u^+ + t_{k_n}u^-)(s_{k_n}u^+ + t_{k_n}u^-)dx.
\end{aligned} \tag{17}$$

Because $k_n \to \infty$ and $\{s_{k_n}u^+ + t_{k_n}u^-\}$ is bounded in E , we have a contradiction with the equality (17). On the other hand, we have

$$0 \le c_k \le J_k(s_k u^+ + t_k u^-) \le s_k^2\|u^+\|^2 + t_k^2\|u^-\|^2 + 2bs_k^4\|u^+\|_K^4 + 2bt_k^4\|u^-\|_K^4,$$

so $\lim_{k\to\infty} c_k = 0$. By the definition of c_k, we can find a sequence $\{u_n\} \subset \mathcal{M}_k$ satisfing $\lim_{n\to\infty} J_k(u_n) = c_k$, which converges to $u_k = u^+ + u^- \in E$. By standard arguments, we have

$$\begin{aligned}
&u_n^\pm \to u^\pm \text{ in } L^p(\mathbb{R}^3) \ \ \forall p \in (2, 2^*_\alpha), \\
&u_n^\pm(x) \to u^\pm(x) \text{ a.e. } x \in \mathbb{R}^3.
\end{aligned}$$

Denote $\beta := \frac{s}{3}(S)^{\frac{3}{2s}}$, where

$$S := \inf_{u\in E\setminus\{0\}} \frac{\|u\|^2}{(\int_{\mathbb{R}^3} |u|^{2^*_\alpha} dx)^{\frac{2}{2^*_\alpha}}}.$$

Sobolev embedding theorem insures that $\beta > 0$. So that there exists $k^\star > 0$ such that $c_k < \beta$ for all $k \ge k^\star$. Fix $k \ge k^\star$, in view of Lemma 2.1, we have $J_k(su_n^+ + tu_n^-) \le J_k(u_n)$.

By $u_n^\pm \rightharpoonup u^\pm$ in E, we have

$$\|u_n^\pm\|^2 - \|u_n^\pm - u^\pm\|^2 = 2\langle u_n^\pm, u^\pm\rangle - \|u^\pm\|^2.$$

By taking $n \to \infty$ in both sides of above equality, there holds

$$\lim_{n\to\infty} \|u_n^\pm\|^2 = \lim_{n\to\infty} \|u_n^\pm - u^\pm\|^2 + \|u^\pm\|^2.$$

On the other hand, by (8) we have

$$\int_{\mathbb{R}^3} F(x, su_n^\pm)dx \to \int_{\mathbb{R}^3} F(x, su^\pm)dx.$$

Then,

$$\begin{aligned}
&\liminf_{n\to\infty} J_k(su_n^+ + tu_n^-) \\
&\qquad \ge \frac{s^2}{2} \lim_{n\to\infty} (\|u_n^+ - u^+\|^2 + \|u^+\|^2) + \frac{t^2}{2} \lim_{n\to\infty} (\|u_n^- - u^-\|^2 + \|u^-\|^2) \\
&\qquad\quad + \frac{bs^4}{4}[\lim_{n\to\infty} \|u_n^+ - u^+\|_K^2 + \|u^+\|_K^2]^2 + \frac{bt^4}{4}[\lim_{n\to\infty} \|u_n^- - u^-\|_K^2 + \|u^-\|_K^2]^2 \\
&\qquad\quad + st \liminf_{n\to\infty} D(u_n) + \frac{bs^2t^2}{4a^2} \liminf_{n\to\infty} D^2(u_n) + \frac{bs^2t^2}{2} \liminf_{n\to\infty} (\|u_n^+\|_K^2 \|u_n^-\|_K^2) \\
&\qquad\quad + \frac{bs^3t}{2a} \liminf_{n\to\infty} (D(u_n)\|u_n^+\|_K^2) + \frac{bst^3}{2a} \liminf_{n\to\infty} (D(u_n)\|u_n^-\|_K^2) \\
&\qquad\quad - \frac{s^{2^*_\alpha}}{2^*_\alpha} \lim_{n\to\infty} (|u_n^+ - u^+|_{2^*_\alpha}^{2^*_\alpha} + |u^+|_{2^*_\alpha}^{2^*_\alpha}) - \frac{t^{2^*_\alpha}}{2^*_\alpha} \lim_{n\to\infty} (|u_n^- - u^-|_{2^*_\alpha}^{2^*_\alpha} + |u^-|_{2^*_\alpha}^{2^*_\alpha}) \\
&\qquad\quad - k \int_{\mathbb{R}^3} F(x, su^+)dx - k \int_{\mathbb{R}^3} F(x, tu^-)dx,
\end{aligned}$$

where $|*|_2$ and $|*|_{2^*_\alpha}$ are the norm in $L^2(\mathbb{R}^3)$ and $L^{2^*_\alpha}(\mathbb{R}^3)$ repeatedly. So, Fatou's Lemma follows that

$$\begin{aligned}
&\liminf_{n\to\infty} J_k(su_n^+ + tu_n^-)\\
\geq& J_k(su^+ + tu^-) + \frac{s^2}{2}\lim_{n\to\infty}\|u_n^+ - u^+\|^2 + \frac{t^2}{2}\lim_{n\to\infty}\|u_n^- - u^-\|^2\\
&- \frac{s^{2^*_\alpha}}{2^*_\alpha}\lim_{n\to\infty}|u_n^+ - u^+|_{2^*_\alpha}^{2^*_\alpha} - \frac{t^{2^*_\alpha}}{2^*_\alpha}\lim_{n\to\infty}|u_n^- - u^-|_{2^*_\alpha}^{2^*_\alpha}\\
&+ \frac{bs^4}{2}\lim_{n\to\infty}\|u_n^+ - u^+\|_K^2\|u^+\|_K^2 + \frac{bs^4}{4}(\lim_{n\to\infty}\|u_n^+ - u^+\|_K^2)^2\\
&+ \frac{bt^4}{2}\lim_{n\to\infty}\|u_n^- - u^-\|_K^2\|u^-\|_K^2 + \frac{bt^4}{4}(\lim_{n\to\infty}\|u_n^- - u^-\|_K^2)^2\\
=& J_k(su^+ + tu^-) + \frac{s^2}{2}A_1 - \frac{s^{2^*_\alpha}}{2^*_\alpha}B_1 + \frac{t^2}{2}A_2 - \frac{t^{2^*_\alpha}}{2^*_\alpha}B_2\\
&+ \frac{bs^4}{2}A_3\|u^+\|_K^2 + \frac{bs^4}{4}A_3^2 + \frac{bt^4}{2}A_4\|u^-\|_K^2 + \frac{bt^4}{4}A_4^2,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \lim_{n\to\infty}\|u_n^+ - u^+\|^2, \quad A_2 = \lim_{n\to\infty}\|u_n^- - u^-\|^2,\\
B_1 &= \lim_{n\to\infty}|u_n^+ - u^+|_{2^*_\alpha}^{2^*_\alpha}, \quad B_2 = \lim_{n\to\infty}|u_n^- - u^-|_{2^*_\alpha}^{2^*_\alpha},\\
A_3 &= \lim_{n\to\infty}\|u_n^+ - u^+\|_K^2, \quad A_4 = \lim_{n\to\infty}\|u_n^- - u^-\|_K^2.
\end{aligned}$$

From the inequality above, we deduce that

$$\begin{aligned}
&J_k(su^+ + tu^-) + \frac{s^2}{2}A_1 - \frac{s^{2^*_\alpha}}{2^*_\alpha}B_1 + \frac{t^2}{2}A_2 - \frac{t^{2^*_\alpha}}{2^*_\alpha}B_2\\
&+ \frac{bs^4}{2}A_3\|u^+\|_K^2 + \frac{bs^4}{4}A_3^2 + \frac{bt^4}{2}A_4\|u^-\|_K^2 + \frac{bt^4}{4}A_4^2 \leq c_k.
\end{aligned} \tag{18}$$

We next prove $u^\pm \neq 0$. Because the claim $u^- \neq 0$ is similar, so we only prove $u^+ \neq 0$. Indirectly, we suppose that $u^+ = 0$ and so $A_1 \geq \rho$ from (15). By letting $t = 0$ in (18), the case $B_1 = 0$ is done. So we only study the case $B_1 > 0$ at length. In this case, by the definition of S, we deduce

$$\beta = \frac{\alpha}{3}S^{\frac{3}{2\alpha}} \leq \frac{\alpha}{3}\left(\frac{A_1}{(B_1)^{\frac{2}{2^*_\alpha}}}\right)^{\frac{3}{2\alpha}}.$$

It happens that,

$$\begin{aligned}
\frac{\alpha}{3}\left(\frac{A_1}{(B_1)^{\frac{2}{2^*_\alpha}}}\right)^{\frac{3}{2\alpha}} &= \max_{s\geq 0}\left\{\frac{s^2}{2}A_1 - \frac{s^{2^*_\alpha}}{2^*_\alpha}B_1\right\}\\
&\leq \max_{s\geq 0}\left\{\frac{s^2}{2}A_1 - \frac{s^{2^*_\alpha}}{2^*_\alpha}B_1 + \frac{bs^4}{2}A_3\|u^+\|_K^2 + \frac{bs^4}{4}A_3^2\right\}.
\end{aligned}$$

The inequality (18) and $c_k < \beta$ follows that

$$\beta \leq \max_{s\geq 0}\{\frac{s^2}{2}A_1 - \frac{s^{2^*_\alpha}}{2^*_\alpha}B_1 + \frac{bs^4}{2}A_3\|u^+\|_K^2 + \frac{bs^4}{4}A_3^2\} \leq c_k < \beta,$$

which is a contradiction. Thus $u^+ \neq 0$ are claimed.

Then, we consider the key point to the proof of Theorem 1.1, that is $B_1 = B_2 = 0$ and then c_k is achieved by $u_k = u^+ + u^- \in \mathcal{M}_k$.

Similarly, we only prove $B_1 = 0$. Indirectly, we suppose that $B_1 > 0$. We have two cases.

Case 1: $B_2 > 0$. Let $\bar{s}$ and $\bar{t}$ be the numbers such that

$$\begin{aligned}&\frac{\bar{s}^2}{2}A_1 - \frac{\bar{s}^{2^*_\alpha}}{2^*_\alpha}B_1 + \frac{b\bar{s}^4}{2}A_3\|u^+\|_K^2 + \frac{b\bar{s}^4}{4}A_3^2\\ =&\max_{s\geq 0}\{\frac{s^2}{2}A_1 - \frac{s^{2^*_\alpha}}{2^*_\alpha}B_1 + \frac{bs^4}{2}A_3\|u^+\|_K^2 + \frac{bs^4}{4}A_3^2\},\end{aligned}$$

$$\begin{aligned}&\frac{\bar{t}^2}{2}A_2 - \frac{\bar{t}^{2^*_\alpha}}{2^*_\alpha}B_2 + \frac{b\bar{t}^4}{2}A_4\|u^-\|_K^2 + \frac{b\bar{t}^4}{4}A_4^2\\ =&\max_{t\geq 0}\{\frac{t^2}{2}A_2 - \frac{t^{2^*_\alpha}}{2^*_\alpha}B_2 + \frac{bt^4}{2}A_4\|u^-\|_K^2 + \frac{bt^4}{4}A_4^2\}.\end{aligned}$$

Since φ_u is continuous, we have $(s_u, t_u) \in [0, \bar{s}] \times [0, \bar{t}]$ satisfying

$$\varphi_u(s_u, t_u) = \max_{(s,t)\in[0,\bar{s}]\times[0,\bar{t}]} \varphi_u(s, t).$$

If $t > 0$ small enough, then, $\varphi_u(s, 0) < J_k(su^+) + J_k(tu^-) \leq \varphi_u(s, t)$ for all $s \in [0, \bar{s}]$. Thus there is $t_0 \in [0, \bar{t}]$ such that $\varphi_u(s, 0) \leq \varphi_u(s, t_0)$ for all $s \in [0, \bar{s}]$. Thus, $(s_u, t_u) \notin [0, \bar{s}] \times \{0\}$. Similarly, $(s_u, t_u) \notin \{0\} \times [0, \bar{t}]$.
By direct computation, we get

$$\frac{s^2}{2}A_1 - \frac{s^{2^*_\alpha}}{2^*_\alpha}B_1 + \frac{bs^4}{2}A_3\|u^+\|_K^2 + \frac{bs^4}{4}A_3^2 > 0, \tag{19}$$

$$\frac{t^2}{2}A_2 - \frac{t^{2^*_\alpha}}{2^*_\alpha}B_2 + \frac{bt^4}{2}A_4\|u^-\|_K^2 + \frac{bt^4}{4}A_4^2 > 0, \tag{20}$$

for all $(s, t) \in (0, \bar{s}] \times (0, \bar{t}]$. Hence, there holds

$$\begin{aligned}\beta \leq&\frac{\bar{s}^2}{2}A_1 - \frac{\bar{s}^{2^*_\alpha}}{2^*_\alpha}B_1 + \frac{b\bar{s}^4}{2}A_3\|u^+\|_K^2 + \frac{b\bar{s}^4}{4}A_3^2\\ &+ \frac{t^2}{2}A_2 - \frac{t^{2^*_\alpha}}{2^*_\alpha}B_2 + \frac{bt^4}{2}A_4\|u^-\|_K^2 + \frac{bt^4}{4}A_4^2,\end{aligned}$$

$$\begin{aligned}\beta \leq&\frac{\bar{t}^2}{2}A_2 - \frac{\bar{t}^{2^*_\alpha}}{2^*_\alpha}B_2 + \frac{bs^4}{2}A_3\|u^+\|_K^2 + \frac{bs^4}{4}A_3^2\\ &+ \frac{s^2}{2}A_1 - \frac{s^{2^*_\alpha}}{2^*_\alpha}B_1 + \frac{b\bar{t}^4}{2}A_4\|u^-\|_K^2 + \frac{b\bar{t}^4}{4}A_4^2.\end{aligned}$$

In view of (18), it follows that $\varphi_u(s, \bar{t}) \leq 0\ \ \forall s \in [0, \bar{s}]$ and $\varphi_u(\bar{s}, t) \leq 0\ \ \forall t \in [0, \bar{t}]$. That is, $(s_u, t_u) \notin \{\bar{s}\} \times [0, \bar{t}]$ and $(s_u, t_u) \notin [0, \bar{s}] \times \{\bar{t}\}$. Hence,we can deduce that $(s_u, t_u) \in (0, \bar{s}) \times (0, \bar{t})$. By (18), (19) and (20), we deduce

$$\begin{aligned}c_k \geq&J_k(s_u u^+ + t_u u^-) + \frac{s_u^2}{2}A_1 - \frac{s_u^{2^*_\alpha}}{2^*_\alpha}B_1 + \frac{t_u^2}{2}A_2 - \frac{t_u^{2^*_\alpha}}{2^*_\alpha}B_2\\ &+ \frac{bs_u^4}{2}A_3\|u^+\|_K^2 + \frac{bs_u^4}{4}A_3^2 + \frac{bt_u^4}{2}A_4\|u^-\|_K^2 + \frac{bt_u^4}{4}A_4^2\\ >&J_k(s_u u^+ + t_u u^-) \geq c_k.\end{aligned}$$

It is impossible. The proof of Case 1 is completed.

Case 2: $B_2 = 0$. From the definition of J_k, it is easy to show that there exists $t_0 \in [0, \infty)$ such that $\varphi_u(s,t) \le 0$, for all $(s,t) \in [0,\bar{s}] \times [t_0, \infty)$. Thus, there is $(s_u, t_u) \in [0,\bar{s}] \times [0,\infty)$ satisfying

$$\varphi_u(s_u, t_u) = \max_{(s,t)\in[0,\bar{s}]\times[0,\infty)} \varphi_u(s,t).$$

We need to prove that $(s_u, t_u) \in (0,\bar{s}) \times (0,\infty)$. Similarly, it is noticed that $\varphi_u(s,0) < \varphi_u(s,t)$ for $s \in [0,\bar{s}]$ and $0 < t \ll 1$, that is $(s_u,t_u) \notin [0,\bar{s}] \times \{0\}$. Also, for s small enough, we get $\varphi_u(0,t) < \varphi_u(s,t)$ for $t \in [0,\infty)$, that is $(s_u,t_u) \notin \{0\} \times [0,\infty)$. We note that

$$\begin{aligned}\beta \le& \frac{\bar{s}^2}{2}A_1 - \frac{\bar{s}^{2^*_\alpha}}{2^*_\alpha}B_1 + \frac{b\bar{s}^4}{2}A_3\|u^+\|_K^2 + \frac{b\bar{s}^4}{4}A_3^2 \\ &+ \frac{t^2}{2}A_2 - \frac{t^{2^*_\alpha}}{2^*_\alpha}B_2 + \frac{bt^4}{2}A_4\|u^-\|_K^2 + \frac{bt^4}{4}A_4^2.\end{aligned}$$

Thus also from (20) and $B_2 = 0$, we have $\varphi_u(\bar{s},t) \le 0$ for all $t \in [0,\infty)$. Hence, $(s_u,t_u) \notin \{\bar{s}\} \times [0,\infty)$. That is, (s_u,t_u) is an inner maximizer of φ_u in $[0,\bar{s}) \times [0,\infty)$. Hence by using (19), we obtain

$$\begin{aligned}c_k \ge& J_k(s_u u^+ + t_u u^-) + \frac{s_u^2}{2}A_1 - \frac{s_u^{2^*_\alpha}}{2^*_\alpha}B_1 + \frac{t_u^2}{2}A_2 - \frac{t_u^{2^*_\alpha}}{2^*_\alpha}B_2 \\ &+ \frac{bs_u^4}{2}A_3\|u^+\|_K^2 + \frac{bs_u^4}{4}A_3^2 + \frac{bt_u^4}{2}A_4\|u^-\|_K^2 + \frac{bt_u^4}{4}A_4^2 \\ >& J_k(s_u u^+ + t_u u^-) \ge c_k,\end{aligned}$$

which is a contradiction.

Since $u^\pm \ne 0$, by Lemma 2.1, there are $s_u, t_u > 0$ such that $\widetilde{u} := s_u u^+ + t_u u^- \in \mathcal{M}_k$. On the other hand, Fatou's Lemma follows that

$$\begin{aligned}\langle J_k'(u), u^\pm\rangle \le& \liminf_{n\to\infty} \|u_n^\pm\|^2 + b\liminf_{n\to\infty}\|u_n\|_K^2 (u_n, u_n^\pm)_K \\ &- \lim_{n\to\infty}\int_{\mathbb{R}^3} f(x,u_n^\pm)u_n^\pm dx - \lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n^\pm|^{2^*_\alpha} + \liminf_{n\to\infty} L(u_n) \\ \le& \lim_{n\to\infty}\langle J_k'(u_n), u_n^\pm\rangle = 0.\end{aligned}$$

By Lemma 2.1, we know $0 < s_u, t_u \le 1$. Since $u_n \in \mathcal{M}_k$ and $B_1 = B_2 = 0$, we have

$$\begin{aligned}c_k \le& J_k(\widetilde{u}) - \frac{1}{4}\langle J_k'(\widetilde{u}), \widetilde{u}\rangle \\ =& \frac{1}{4}(\|s_u u^+\|^2 + \|t_u u^-\|^2) + (\frac{1}{4} - \frac{1}{2^*_\alpha})(|s_u u^+|_{2^*_\alpha}^{2^*_\alpha} + |t_u u^-|_{2^*_\alpha}^{2^*_\alpha}) \\ &+ \frac{k}{4}\int_{\mathbb{R}^3}[f(x,s_u u^+)(s_u u^+) - 4F(x,s_u u^+)]dx \\ &+ \frac{k}{4}\int_{\mathbb{R}^3}[f(x,t_u u^-)(t_u u^-) - 4F(x,t_u u^-)]dx \\ \le& \liminf_{n\to\infty}[J_k(u_n) - \frac{1}{4}\langle J_k'(u_n), u_n\rangle] = c_k.\end{aligned}$$

So, we have completed proof of Lemma 2.2. □

3. **The proof of main results.** In this section, we will prove main results.

3.1 The proof of Theorem 1.1.

Proof. Since $u_k \in \mathcal{M}_k$ and $J_k(u_k^+ + u_k^-) = c_k$, by Lemma 2.1, for $(s,t) \in (\mathbb{R}_+ \times \mathbb{R}_+)\backslash(1,1)$, we have

$$J_k(su_k^+ + tu_k^-) < c_k. \tag{21}$$

If $J_k'(u_k) \neq 0$, then there exist $\delta > 0$ and $\theta > 0$ such that

$$\|J_k'(v)\| \geq \theta, \text{ for all } \|v - u_k\| \leq 3\delta.$$

We know by the result (15), if $u \in \mathcal{M}_k$, there exists $L > 0$ such that $\|u^\pm\| > L$ and we can assume $6\delta < L$. Let $Q := (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$, and $g(s,t) = su_k^+ + tu_k^-$, $(s,t) \in Q$. In view of (21), it is easy to see that

$$\overline{c}_k := \max_{\partial Q} I \circ g < c_k. \tag{22}$$

Let $\varepsilon := min\{(c_k - \overline{c}_k)/4, \theta\delta/8\}$ and $S_\delta := B(u_k, \delta)$, there exists a deformation $\eta \in C([0,1] \times E, E)$ satisfying

(a) $\eta(t,v) = v$ if $t = 0$, or $v \notin (J_k)^{-1}([c_k - 2\varepsilon, c_k + 2\varepsilon]) \cap S_{2\delta}$;

(b) $\eta(1, (J_k)^{c_k+\varepsilon} \cap S_\delta) \subset (J_k)^{c_k - \varepsilon}$;

(c) $J_k(\eta(1,v)) \leq J_k(v)$ for all $v \in E$;

(d) $J_k(\eta(\cdot, v))$ is non increasing for every $v \in E$.

To finish the proof of Theorem 1.1, one of the key points is to prove that

$$\max_{(s,t)\in\bar{Q}} J_k(\eta(1, g(s,t))) < c_k. \tag{23}$$

The other is to prove that $\eta(1, g(Q)) \cap \mathcal{M}_k \neq \emptyset$. Let us begin this work. In fact, it follows from Lemma 2.1 that $g(s,t) \in (J_k)^{c_k+\varepsilon}$. On the other hand, from (a) and (d), we get

$$J_k(\eta(1,v)) \leq J_k(\eta(0,v)) = J_k(v), \ \ \forall v \in E. \tag{24}$$

For $(s,t) \in Q$, when $s \neq 1$ or $t \neq 1$, according to (21) and (24),

$$J_k(\eta(1, g(s,t))) \leq J_k(g(s,t)) < c_k.$$

If $s = 1$ and $t = 1$, that is, $g(1,1) = u_k$, so that it holds $g(1,1) \in J_k^{c_k+\varepsilon} \cap S_\delta$, then by (b)

$$J_k(\eta(1, g(1,1))) \leq c_k - \varepsilon < c_k.$$

Thus (23) holds. Then, let $\varphi(s,t) := \eta(1, g(s,t))$ and

$$\Upsilon(s,t) := (\frac{1}{s}\langle J_k'(\varphi(s,t)), (\varphi(s,t))^+\rangle, \frac{1}{t}\langle J_k'(\varphi(s,t)), (\varphi(s,t))^-\rangle).$$

The claim holds if there exists $(s_0, t_0) \in Q$ such that $\Upsilon(s_0, t_0) = (0,0)$. Since

$$\begin{aligned}\|g(s,t) - u_k\|^2 =& \|(s-1)u_k^+ + (t-1)u_k^-\|^2 \\ \geq& |s-1|^2\|u_k^+\|^2 > |s-1|^2(6\delta)^2,\end{aligned}$$

and $|s-1|^2(6\delta)^2 > 4\delta^2 \Leftrightarrow s < 2/3$ or $s > 4/3$, using (i) and the range of s, for $s = \frac{1}{2}$ and for every $t \in [\frac{1}{2}, \frac{3}{2}]$ we have $g(\frac{1}{2}, t) \notin S_{2\delta}$, so from (a), we have $\varphi(\frac{1}{2}, t) = g(\frac{1}{2}, t)$. Thus

$$\Upsilon(\frac{1}{2}, t) = (2\langle J_k'(\frac{1}{2}u_k^+ + tu_k^-), \frac{1}{2}u_k^+\rangle, \frac{1}{t}\langle J_k'(\frac{1}{2}u_k^+ + tu_k^-), tu^-\rangle).$$

On the other hand, from (9) and $u \in \mathcal{M}_k$, we have that

$$H(t,t) = (t^2 - t^4)(\|u^+\|^2 + D(u)) + (t^4 - t^{2^*_\alpha}) \int_{\mathbb{R}^3} |u^+|^{2^*_\alpha} dx + kt^4 \int_{\mathbb{R}^3} \left(f(x,u^+) - \frac{f(x,tu^+)}{t^3} \right) u^+ dx.$$

According to (f_3), when $0 < t < 1$, $H(t,t) > 0$, while in the case $t > 1$, $H(t,t) < 0$. Similarly, it is easy to get $G(t,t) > 0$ for $t \in (0,1)$, $G(t,t) < 0$ for $t > 1$. By above discussions, due to (f_3) and $2^*_\alpha > 4$, we have

$$\begin{aligned} H(\frac{1}{2},t) =& \|\frac{1}{2}u_k^+\|^2 + \frac{t}{2}D(u_k) \\ &+ b\big(\frac{1}{4}\|u_k^+\|_K^2 + t^2\|u_k^-\|_K^2 + \frac{t}{a}D(u_k)\big)\big(\frac{1}{4}\|u_k^+\|_K^2 + \frac{t}{2a}D(u_k)\big) \\ &- (\frac{1}{2})^{2^*_\alpha}\int_{\mathbb{R}^3} |u_k^+|^{2^*_\alpha}dx - k\int_{\mathbb{R}^3} f(x,\frac{1}{2}u_k^+)\frac{1}{2}u_k^+dx \geq H(\frac{1}{2},\frac{1}{2}) > 0, \end{aligned}$$

which implies that

$$H(\frac{1}{2},t) > 0, \quad \forall t \in [\frac{1}{2},\frac{3}{2}]. \tag{25}$$

Analogously, $\varphi(\frac{3}{2},t) = g(\frac{3}{2},t)$ implies that

$$\begin{aligned} H(\frac{3}{2},t) =& \|\frac{3}{2}u_k^+\|^2 + \frac{3t}{2}D(u_k) + b\big(\frac{9}{4}\|u_k^+\|_K^2 + t^2\|u_k^-\|_K^2 \\ &+ \frac{3t}{a}D(u_k)\big)\big(\frac{9}{4}\|u_k^+\|_K^2 + \frac{3t}{2a}D(u_k)\big) \\ &- (\frac{3}{2})^{2^*_\alpha}\int_{\mathbb{R}^3} |u_k^+|^{2^*_\alpha}dx - k\int_{\mathbb{R}^3} f(x,\frac{3}{2}u_k^+)\frac{3}{2}u_k^+dx \leq H(\frac{3}{2},\frac{3}{2}) < 0, \end{aligned}$$

that is,

$$H(\frac{3}{2},t) < 0, \quad \forall t \in [\frac{1}{2},\frac{3}{2}]. \tag{26}$$

By the same way,

$$G(s,\frac{1}{2}) > 0, \quad \forall s \in [\frac{1}{2},\frac{3}{2}], \quad \text{and} \quad G(s,\frac{3}{2}) > 0, \quad \forall s \in [\frac{1}{2},\frac{3}{2}]. \tag{27}$$

From (25)-(27), the assumptions of Miranda's Theorem (see Lemma 2.4 in [17]) are satisfied. Thus, there exists $(s_0,t_0) \in Q$ such taht $\Upsilon(s_0,t_0) = 0$, i.e. $\eta(1,g(s_0,t_0)) \in \mathcal{M}_k$. Comparing to (23), u_k is a ground state nodal solution of problem (1). □

3.2 The proof of Theorem 1.2.

Proof. Recall that $\beta = \frac{\alpha}{3}S^{\frac{3}{2\alpha}}$, where $S := \inf_{u\in E\setminus\{0\}} \frac{\|u\|^2}{(\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx)^{\frac{2}{2^*_\alpha}}}$. Similar to the proof of Lemma 2.2, we claim that there exists $k_1^\star > 0$ such that for all $k \geq k_1^\star$, there exists $v_k \in \mathcal{N}_k$ such that $J_k(v_k) = c^* > 0$ and there is $k^\star > 0$ such that $c^* < \beta$ for all $k \geq k^\star$. By standard processes, we can assume $v_n \rightharpoonup v_k \in E$. Therefore, $\liminf_{n\to\infty} J_k(tv_n) \geq c^* \geq J_k(tv_k) + \frac{t^2}{2}A - \frac{t^{2^*_\alpha}}{2^*_\alpha}B + \frac{bt^4}{4}(C^2 + 2C\|v_k\|_K^2)$, where $A = \lim_{n\to\infty}\|v_n - v_k\|^2$, $B = \lim_{n\to\infty}|v_n - v_k|_{2^*_\alpha}^{2^*_\alpha}$ and $C = \|v_n - v_k\|_K^2$.

Firstly, we prove that $v_k \neq 0$. By contradiction, we suppose $v_k = 0$. The Case $B = 0$ is contained in above equality. So we only consider the case $B > 0$. The fact $\beta \leq \frac{\alpha}{3}\left(\frac{A_1}{(B_1)^{\frac{2}{2^*_\alpha}}}\right)^{\frac{3}{2\alpha}}$ follows that

$$\begin{aligned}\beta \leq & \frac{\tilde{t}^2}{2}A - \frac{\tilde{t}^{2^*_\alpha}}{2^*_\alpha}B := \max_{t\geq 0}\{\frac{t^2}{2}A - \frac{t^{2^*_\alpha}}{2^*_\alpha}B\} \\ \leq & \max_{t\geq 0}\{\frac{t^2}{2}A - \frac{t^{2^*_\alpha}}{2^*_\alpha}B + \frac{bt^4}{4}(C^2 + 2C\|v_k\|_K^2)\} \leq c^* < \beta.\end{aligned}$$

Which is a contradiction.

Then, we prove that $B = 0$ and c^* *is achieved by* v_k. Indirectly, we suppose that $B > 0$. Firstly, we can maximize $\varphi_{v_k}(t) = J_k(tv_k)$ in $[0, \infty)$ at t_v, which be an inner maximizer. So we get $\frac{t_v^2}{2}A - \frac{t_v^{2^*_\alpha}}{2^*_\alpha}B + \frac{bt_v^4}{4}(C^2 + 2C\|v_k\|_K^2) > 0$. Thus from $t_v v_k \in \mathcal{N}_b^k$ we get a contradiction by

$$c^* \leq J_k(t_v v_k) < J_k(t_v v_k) + \frac{t_v^2}{2}A - \frac{t_v^{2^*_\alpha}}{2^*_\alpha}B + \frac{bt_v^4}{4}(C^2 + 2C\|v_k\|_K^2) \leq c^*.$$

From the above arguments we know $\widetilde{v} := t_v v_k \in \mathcal{N}_k$. Furthermore, we have

$$\begin{aligned}c^* \leq & J_k(\tilde{v}) - \frac{1}{4}\langle J_k'(\tilde{v}), \tilde{v}\rangle \leq \frac{1}{4}\|v_k\|^2 \\ & + (\frac{1}{4} - \frac{1}{2^*_\alpha})|v_k|_{2^*_\alpha}^{2^*_\alpha} + \frac{k}{4}\int_{\mathbb{R}^3}[f(x, v_k)v_k - 4F(x, v_k)]dx \\ = & \liminf_{n\to\infty}[J_k(v_n) - \frac{1}{4}\langle J_k'(v_n), v_n\rangle] = c^*.\end{aligned}$$

Therefore, $t_v = 1$, and c^* is achieved by $v_k \in \mathcal{N}_k$.

By standard arguments, we can find $v_k \in E$, a ground state solution of problem (1). For all $k \geq k^\star$, the problem (1) also has a ground state nodal solution u_k. Let $k^{\star\star} = \max\{k^\star, k_1^\star\}$. Suppose that $u_k = u^+ + u^-$, we let $s_{u^+}, t_{u^-} \in (0, 1)$ such that

$$s_{u^+}u^+ \in \mathcal{N}_k, \quad t_{u^-}u^- \in \mathcal{N}_k.$$

Thus, the above fact follows that

$$2c^* \leq J_k(s_{u^+}u^+) + J_k(t_{u^-}u^-) \leq J_k(s_{u^+}u^+ + t_{u^-}u^-) < J_k(u^+ + u^-) = c_k.$$

□

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