

GLOBAL BOUNDEDNESS FOR A CHEMOTAXIS-COMPETITION SYSTEM WITH SIGNAL DEPENDENT SENSITIVITY AND LOOP

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ABSTRACT. In this work, the fully parabolic chemotaxis-competition system with loop

$$\left\{ \begin{array}{l} \partial_t u_1 = d_1 \Delta u_1 - \nabla \cdot (u_1 \chi_{11}(v_1) \nabla v_1) \\ \quad - \nabla \cdot (u_1 \chi_{12}(v_2) \nabla v_2) + \mu_1 u_1 (1 - u_1 - a_1 u_2), \\ \partial_t u_2 = d_2 \Delta u_2 - \nabla \cdot (u_2 \chi_{21}(v_1) \nabla v_1) \\ \quad - \nabla \cdot (u_2 \chi_{22}(v_2) \nabla v_2) + \mu_2 u_2 (1 - u_2 - a_2 u_1), \\ \partial_t v_1 = d_3 \Delta v_1 - \lambda_1 v_1 + h_1(u_1, u_2), \\ \partial_t v_2 = d_4 \Delta v_2 - \lambda_2 v_2 + h_2(u_1, u_2) \end{array} \right.$$

is considered under the homogeneous Neumann boundary condition, where $x \in \Omega, t > 0, \Omega \subset \mathbb{R}^n (n \leq 3)$ is a bounded domain with smooth boundary. For any regular nonnegative initial data, it is proved that if the parameters μ_1, μ_2 are sufficiently large, then the system possesses a unique and global classical solution for $n \leq 3$. Specifically, when $n = 2$, the global boundedness can be attained without any constraints on μ_1, μ_2 .

1. Introduction. In this paper, we consider the following initial boundary value problem

$$\left\{ \begin{array}{l} \partial_t u_1 = d_1 \Delta u_1 - \nabla \cdot (u_1 \chi_{11}(v_1) \nabla v_1) \\ \quad - \nabla \cdot (u_1 \chi_{12}(v_2) \nabla v_2) + \mu_1 u_1 (1 - u_1 - a_1 u_2), \quad x \in \Omega, t > 0, \\ \partial_t u_2 = d_2 \Delta u_2 - \nabla \cdot (u_2 \chi_{21}(v_1) \nabla v_1) \\ \quad - \nabla \cdot (u_2 \chi_{22}(v_2) \nabla v_2) + \mu_2 u_2 (1 - u_2 - a_2 u_1), \quad x \in \Omega, t > 0, \\ \partial_t v_1 = d_3 \Delta v_1 - \lambda_1 v_1 + h_1(u_1, u_2), \quad x \in \Omega, t > 0, \\ \partial_t v_2 = d_4 \Delta v_2 - \lambda_2 v_2 + h_2(u_1, u_2), \quad x \in \Omega, t > 0, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial v_1}{\partial \nu} = \frac{\partial v_2}{\partial \nu} = 0, \quad x \in \Omega, t > 0, \\ u_1(x, 0) = u_{10}(x), u_2(x, 0) = u_{20}(x), v_1(x, 0) \\ \quad = v_{10}(x), v_2(x, 0) = v_{20}(x), \quad x \in \Omega \end{array} \right. \quad (1)$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, where $\frac{\partial}{\partial \nu}$ represents differentiation with respect to the outward normal on $\partial\Omega$, $d_1, d_2, d_3, d_4, \mu_1, \mu_2, \lambda_1, \lambda_2, a_1, a_2$ are positive constants. This model comes from [5], it describes the chemotactical communication named EGF/CSF-1 paracrine invasion loop, which might be a target to control or prevent metastasis with therapeutic methods. u_1, u_2 represent

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the densities of macrophages and tumor cells, $v_i (i = 1, 2)$ denote the concentration of the chemical. The chemotactic sensitivity functions $\chi_{ij} (i, j = 1, 2)$ are smooth and positive. Let the initial data u_{10}, u_{20}, v_{10} and v_{20} satisfy

$$0 \leq u_{10} \in C^0(\bar{\Omega}), 0 \leq u_{20} \in C^0(\bar{\Omega}), 0 \leq v_{10} \in W^{1,\infty}(\bar{\Omega}), 0 \leq v_{20} \in W^{1,\infty}(\bar{\Omega}). \quad (2)$$

For the case $\chi_{12} = \chi_{21} = 0$, $h_1 = h_1(u_2)$, $h_2 = h_2(u_1)$, which describes the situation that tumor cells and macrophages mutually attract each other through chemotactic signals. In such a case, the global solvability, boundedness and asymptotic behavior have been investigated intensively, for instance, Wang et al. detected the boundedness of solutions for $n \leq 3$ in [17], also, they explored the asymptotic behavior of solutions for any $n \geq 1$. Choosing $h_1(u_2) = u_2, h_2(u_1) = u_1$, χ_{11}, χ_{22} are two constants, when $\mu_1 = \mu_2 = 0$, the global boundedness and blow-up of solutions have been considered in [6, 11, 20]. When $\mu_1, \mu_2 \neq 0$, for the fully parabolic case, the global boundedness and large time behavior for $n \leq 2$ and $n = 3$ were detected in [3] and [8] respectively; as for the parabolic-elliptic case, for all $n \geq 1$, the global boundedness and asymptotic behavior were obtained in [21, 12]; afterwards, the results in [21, 12] were partially improved by Wang et al. in [18].

When $\chi_{ij} (i, j = 1, 2)$ are constants, $h_1 = \alpha_1 u_1 + \beta_1 u_2$, $h_2 = \alpha_2 u_1 + \beta_2 u_2$. Without respect to the kinetic terms, Espejo et al. derived the simultaneous blow-up phenomenon in [4] for the parabolic-elliptic case of (1) in the whole space \mathbb{R}^2 . Considering the Lotka-Volterra-type competition, whether the parabolic-elliptic case or the fully parabolic case of (1), the global dynamics of solutions were detected, it was found that the solution of (1) is globally bounded without any requirement on the size of the parameters for the fully parabolic case in the lower dimensions $n \leq 2$ [14], while the largeness of parameters μ_1, μ_2 is needed to guarantee the global solvability of (1) for $n = 3$ [15], and the global solution of this system exponentially approaches to a steady state for all $n \geq 1$ [14], specifically, the system was shown to exhibit the large population densities phenomenon in [16], that is, the solution exhibits unbounded peculiarity for the proper choice of initial data. As for the parabolic elliptic case, in [13], the global boundedness result were established for $n \geq 2$ under the condition that $\frac{\chi_{11}}{\mu_1}, \frac{\chi_{12}}{\mu_1}, \frac{\chi_{21}}{\mu_2}$ and $\frac{\chi_{22}}{\mu_2}$ are suitably small, moreover, the large time behavior of solution was derived.

In summary, for the two-species and two-stimuli chemotaxis system, most of the results are focusing on the case that the chemotactic sensitivity functions are constants and the signal production is linear. Therefore, the objective in the present study is to investigate the global boundedness of solutions for (1) when $\chi_{ij} (i, j = 1, 2), h_1, h_2$ are general functions. Our work is motivated by the method in [17], but in contrast, the existence of the chemical signalling loop in our model makes the computations and analysis fairly subtle.

We shall suppose throughout this paper that the functions $\chi_{ij}(s), h_i(s, \tau) (i, j = 1, 2)$ satisfy the following conditions:

(H1) $\chi_{ij}(s) \in C^{1+\theta}([0, \infty))$, $i, j = 1, 2$, for some $\theta > 0$.

(H2) $0 < \chi_{11}(s), \chi_{12}(s) \leq K_1$ for some $K_1 > 0$; $0 < \chi_{21}(s), \chi_{22}(s) \leq K_2$ for some $K_2 > 0$.

(H3) $h_i(s, \tau) \in C^{1+\theta}([0, \infty) \times [0, \infty))$, $i = 1, 2$, for some $\theta > 0$.

(H4) $h_i(0, 0) = 0$ and $0 < \frac{\partial h_i(s, \tau)}{\partial s}, \frac{\partial h_i(s, \tau)}{\partial \tau} < C_{h_i}$, with $C_{h_i} > 0$, $i = 1, 2$.

From the above (H3) and (H4), a straight calculation yields

$$h_i(s, \tau) - h_i(0, 0) = C_{h_i}(s + \tau) \text{ for } i = 1, 2. \quad (3)$$

Now we state our main results as follows.

Theorem 1.1. *Let $\Omega \subset R^n (n \leq 3)$ be a smoothly bounded domain, and let $d_1, d_2, d_3, d_4, \mu_1, \mu_2, \lambda_1, \lambda_2, a_1, a_2$ be positive constants. Assume that $\chi_{ij}, h_i (i, j = 1, 2)$ satisfy (H1)-(H4), μ_1, μ_2 satisfy*

$$\mu_1 \geq \max \left\{ \frac{2K_1^2}{d_1} \eta_1 + (2n + 8) \frac{C_{h_1}^2}{d_3} + \frac{2K_1^2}{d_3} + \frac{2K_1^2}{d_4} + 2, \frac{2K_1^2}{d_1} \eta_1 + (2n + 8) \frac{C_{h_2}^2}{d_4} + \frac{2K_1^2}{d_3} + \frac{2K_1^2}{d_4} + 2 \right\}, \tag{4}$$

$$\mu_2 \geq \max \left\{ \frac{2K_2^2}{d_2} \eta_2 + (2n + 8) \frac{C_{h_1}^2}{d_3} + \frac{2K_2^2}{d_3} + \frac{2K_2^2}{d_4} + 2, \frac{2K_2^2}{d_2} \eta_2 + (2n + 8) \frac{C_{h_2}^2}{d_4} + \frac{2K_2^2}{d_3} + \frac{2K_2^2}{d_4} + 2 \right\} \tag{5}$$

with

$$\begin{aligned} \eta_1 &= \frac{2(d_1 + d_3)^2}{d_1 d_3} + \frac{2(d_1 + d_4)^2}{d_1 d_4} + \frac{2C_{h_1}^2 + 2C_{h_2}^2 + 1}{d_1}, \\ \eta_2 &= \frac{2(d_2 + d_3)^2}{d_2 d_3} + \frac{2(d_2 + d_4)^2}{d_2 d_4} + \frac{2C_{h_1}^2 + 2C_{h_2}^2 + 1}{d_2}. \end{aligned} \tag{6}$$

Then for all u_{10}, u_{20}, v_{10} and v_{20} satisfying (2), the classical solution (u_1, u_2, v_1, v_2) of (1) is unique and globally bounded in the sense that

$$\|u_1(\cdot, t)\|_{L^\infty(\Omega)} + \|v_1(\cdot, t)\|_{L^\infty(\Omega)} + \|u_2(\cdot, t)\|_{L^\infty(\Omega)} + \|v_2(\cdot, t)\|_{L^\infty(\Omega)} \leq C$$

for all $t \geq 0$, with some constant $C > 0$ that is independent of t .

Corollary 1. *Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain, and let $d_1, d_2, d_3, d_4, \mu_1, \mu_2, \lambda_1, \lambda_2, a_1, a_2$ be positive constants. Assume that $\chi_{ij}, h_i (i, j = 1, 2)$ satisfy (H1)-(H4), and $\chi'_{ij} (i, j = 1, 2)$ fulfill*

$$|\chi'_{ij}(s)| \leq L \text{ for all } s \geq 0 \tag{7}$$

with some $L > 0$. Then for all u_{10}, u_{20}, v_{10} and v_{20} satisfying (2), the classical solution (u_1, u_2, v_1, v_2) of (1) is globally bounded.

Remark 1. It is obvious that there exist functions $\chi_{ij}, h_i (i, j = 1, 2)$ which satisfy (H1)-(H4), such as, we can choose the standard chemotactic sensitivity functions $\chi_{ij}(s) = \frac{c_0}{(1+cs)^2}$ with $c_0, c > 0$, and choose $h_i = c_1 u_1 + c_2 u_2$ with $c_1, c_2 > 0$.

In this paper, we deal with the quasilinear chemotaxis-competition system with loop. First, we give the local existence and some properties to prepare for the later work. Next, under the condition that μ_1, μ_2 are sufficiently large, we establish the global boundedness result when $n \leq 3$. At last, for the case $n = 2$, we obtain the boundedness result without any requirement on the size of μ_1, μ_2 .

2. Preliminary. As a preliminary, we first give the local existence and some important estimates of solutions for (1).

Lemma 2.1. *Let $\Omega \subset R^n (n \geq 1)$ be a smoothly bounded domain, and let $\chi_{ij}, h_i (i, j = 1, 2)$ satisfy (H1)-(H4). Assume that the initial data $u_{10}, u_{20}, v_{10}, v_{20}$ satisfy*

(2). Then there exists a maximal $T_{\max} \in (0, \infty]$ such that the system (1) has a unique nonnegative classical solution (u_1, u_2, v_1, v_2)

$$\begin{aligned} u_1, u_2 &\in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ v_1, v_2 &\in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \end{aligned}$$

which satisfies

$$\text{either } T_{\max} = \infty, \text{ or } \|u_1(\cdot, t)\|_{L^\infty(\Omega)} + \|u_2(\cdot, t)\|_{L^\infty(\Omega)} \rightarrow \infty \text{ as } t \rightarrow T_{\max}.$$

Besides, the solution fulfills

$$\int_{\Omega} u_1(x, t) dx \leq m_1 := \max \left\{ \int_{\Omega} u_{10}(x) dx, |\Omega| \right\} \text{ for all } t \in (0, T_{\max}) \quad (8)$$

and

$$\int_{\Omega} u_2(x, t) dx \leq m_2 := \max \left\{ \int_{\Omega} u_{20}(x) dx, |\Omega| \right\} \text{ for all } t \in (0, T_{\max}) \quad (9)$$

as well as

$$\int_t^{t+\tau} \int_{\Omega} u_1^2(x, t) dx ds \leq \kappa_1 := m_1 + \frac{m_1}{\mu_1} \text{ for all } t \in [0, T_{\max} - \tau) \quad (10)$$

and

$$\int_t^{t+\tau} \int_{\Omega} u_2^2(x, t) dx ds \leq \kappa_2 := m_2 + \frac{m_2}{\mu_2} \text{ for all } t \in [0, T_{\max} - \tau), \quad (11)$$

where $\tau = \min \left\{ 1, \frac{T_{\max}}{2} \right\}$.

Proof. The local existence of classical solution to (1) can be shown by using well-established methods for chemotaxis problems in [19]. And the relation (8)-(11) can be directly derived by a similar method in [15]. \square

Next, we recall the following lemma (see Lemma 3.4 in [9] or Lemma 2.3 in [1]), which is significant for our latter proof.

Lemma 2.2. Let $T > 0$, $0 \leq f \in L^1_{loc}([0, T])$, $y(t)$ be a nonnegative absolutely continuous function on $[0, T)$. Assume that there exist $a > 0, b > 0$ such that

$$\int_t^{t+\tau} f(s) ds \leq b \text{ for all } t \in [0, T - \tau)$$

and

$$y'(t) + ay(t) \leq f(t) \text{ for almost all } t \in (0, T),$$

then

$$y(t) \leq \max \left\{ y(0) + b, \frac{b}{a} + 2b \right\} \text{ for all } t \in (0, T),$$

where $\tau = \min \left\{ 1, \frac{T}{2} \right\}$.

Based on Lemma 2.1 and Lemma 2.2, we can now derive some basic properties of v_1, v_2 .

Lemma 2.3. Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a smooth and bounded domain, $\lambda_i > 0, a_i > 0, \mu_i > 0, d_j > 0 (i = 1, 2, j = 1, 2, 3, 4)$. Assume that h_1, h_2 satisfy (H3)-(H4). Then there exist $M_1, M_2, N_1, N_2 > 0$ such that the solution of (1) satisfies

$$\int_{\Omega} |\nabla v_1(x, t)|^2 dx \leq M_1, \quad \int_{\Omega} |\nabla v_2(x, t)|^2 dx \leq M_2 \text{ for all } t \in [0, T_{\max}), \quad (12)$$

$$\int_t^{t+\tau} \int_{\Omega} |\Delta v_1(x, s)|^2 dx ds \leq N_1, \int_t^{t+\tau} \int_{\Omega} |\Delta v_2(x, s)|^2 dx ds \leq N_2, \tag{13}$$

for all $t \in [0, T_{max} - \tau)$, where $\tau = \min \{1, \frac{T_{max}}{2}\}$.

Proof. Multiplying the third equation in (1) by $-\Delta v_1$ and integrating the result equation over Ω , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v_1|^2 dx + d_3 \int_{\Omega} |\Delta v_1|^2 dx + \lambda_1 \int_{\Omega} |\nabla v_1|^2 dx = - \int_{\Omega} h_1(u_1, u_2) \Delta v_1 dx \\ & \leq \frac{d_3}{2} \int_{\Omega} |\Delta v_1|^2 dx + \frac{C_{h_1}^2}{d_3} \int_{\Omega} u_1^2 dx + \frac{C_{h_1}^2}{d_3} \int_{\Omega} u_2^2 dx \text{ for all } t \in (0, T_{max}), \end{aligned}$$

therefore,

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\nabla v_1|^2 dx + d_3 \int_{\Omega} |\Delta v_1|^2 dx + 2\lambda_1 \int_{\Omega} |\nabla v_1|^2 dx \\ & \leq \frac{2C_{h_1}^2}{d_3} \int_{\Omega} u_1^2 dx + \frac{2C_{h_1}^2}{d_3} \int_{\Omega} u_2^2 dx \end{aligned} \tag{14}$$

for all $t \in (0, T_{max})$. By denoting

$$\begin{aligned} y_1(t) & := \int_{\Omega} |\nabla v_1(x, t)|^2 dx, \quad f_1(t) \\ & = \frac{2C_{h_1}^2}{d_3} \int_{\Omega} u_1^2 dx + \frac{2C_{h_1}^2}{d_3} \int_{\Omega} u_2^2 dx \text{ for all } t \in (0, T_{max}), \end{aligned}$$

we obtain

$$y_1'(t) + 2\lambda_1 y_1(t) + d_3 \int_{\Omega} |\Delta v_1(x, t)|^2 dx \leq f_1(t) \text{ for all } t \in (0, T_{max}). \tag{15}$$

Lemma 2.1 and the definition of f_1 entail that

$$\int_t^{t+\tau} f_1(s) ds \leq c_1 \tag{16}$$

with some $c_1 > 0$, then it follows from Lemma 2.2 that

$$y_1(t) = \int_{\Omega} |\nabla v_1(x, t)|^2 dx \leq c_2 \tag{17}$$

with some $c_2 > 0$. Whence by an integration of (15) over $(t, t + \tau)$ we find

$$\begin{aligned} & y_1(t + \tau) + 2\lambda_1 \int_t^{t+\tau} y_1(s) ds + d_3 \int_t^{t+\tau} \int_{\Omega} |\Delta v_1(x, s)|^2 dx ds \\ & \leq y_1(t) + \int_t^{t+\tau} f_1(s) ds \leq c_1 + c_2 \text{ for all } t \in [0, T_{max} - \tau), \end{aligned}$$

along with the nonnegativity of y_1 , we conclude (13). The other inequalities in (12) and (13) can be obtained by the similar method. □

To improve the condition that warrants the global boundedness of solution for (1) when $n = 2$, we recall the following generalization of Gagliardo-Nirenberg inequality which is given in Lemma A.5 of [10].

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^2$ be a smooth and bounded domain. Then for all $\varphi \in W^{1,2}(\Omega)$, one can find $C > 0$ such that for any $\epsilon > 0$ there exists $C_{\epsilon} > 0$ with the property that*

$$\|\varphi\|_{L^3(\Omega)}^3 \leq \epsilon \|\nabla \varphi\|_{L^2(\Omega)}^2 \|\varphi \ln |\varphi|\|_{L^1(\Omega)} + C \|\varphi\|_{L^1(\Omega)}^3 + C_{\epsilon}, \tag{18}$$

The following lemma plays an important role in the proof of Corollary 1, and the proof is similar to Lemma 2.5 in [14].

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a smooth and bounded domain, (u_1, u_2, v_1, v_2) be the classical solution of (1). Assume that $\chi_{ij}, h_i (i = 1, 2, j = 1, 2, 3, 4)$ satisfy (H1)-(H4). Let $p \geq 1$ be such that $\frac{n}{2} < p \leq n$ and*

$$\sup_{t \in (0, T_{\max})} (\|u_1(\cdot, t)\|_{L^p(\Omega)} + \|u_2(\cdot, t)\|_{L^p(\Omega)}) < \infty. \quad (19)$$

Then $T_{\max} = \infty$, and

$$\sup_{t > 0} (\|u_1(\cdot, t)\|_{L^\infty(\Omega)} + \|u_2(\cdot, t)\|_{L^\infty(\Omega)} + \|v_1(\cdot, t)\|_{L^\infty(\Omega)} + \|v_2(\cdot, t)\|_{L^\infty(\Omega)}) < \infty. \quad (20)$$

3. The global boundedness of solutions. In this section, we first prove the global boundedness of solutions for $n \leq 3$ under the condition that μ_1, μ_2 are sufficiently large; next, we remove the requirement on the largeness of parameters μ_1, μ_2 when $n = 2$.

3.1. Proof of Theorem 1.1. To prepare our analysis, we establish several differential inequalities in the following two lemmas.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a smooth and bounded domain, $\lambda_i > 0, \mu_i > 0, d_j > 0 (i = 1, 2, j = 1, 2, 3, 4)$. Assume that $\chi_{ij}, h_i (i = 1, 2, j = 1, 2, 3, 4)$ satisfy (H1)-(H4). Then for any classical solution (u_1, u_2, v_1, v_2) of (1) we have*

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u_1^2 dx + d_1 \int_{\Omega} |\nabla u_1|^2 dx &\leq \frac{2K_1^2}{d_1} \int_{\Omega} u_1^2 |\nabla v_1|^2 dx + \frac{2K_1^2}{d_1} \int_{\Omega} u_1^2 |\nabla v_2|^2 dx \\ &\quad + 2\mu_1 \int_{\Omega} u_1^2 (1 - u_1) dx, \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u_2^2 dx + d_2 \int_{\Omega} |\nabla u_2|^2 dx &\leq \frac{2K_2^2}{d_2} \int_{\Omega} u_2^2 |\nabla v_1|^2 dx + \frac{2K_2^2}{d_2} \int_{\Omega} u_2^2 |\nabla v_2|^2 dx \\ &\quad + 2\mu_2 \int_{\Omega} u_2^2 (1 - u_2) dx \end{aligned} \quad (22)$$

for all $t \in (0, T_{\max})$. And

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} |\nabla v_1|^4 dx + 4\lambda_1 \int_{\Omega} |\nabla v_1|^4 dx + d_3 \int_{\Omega} |\nabla |\nabla v_1|^2|^2 dx \\ &\leq (2n + 8) \frac{C_{h_1}^2}{d_3} \int_{\Omega} (u_1^2 + u_2^2) |\nabla v_1|^2 dx \\ &\quad + 2d_3 \int_{\partial\Omega} |\nabla v_1|^2 \frac{\partial |\nabla v_1|^2}{\partial \nu} dS \text{ for all } t \in (0, T_{\max}) \end{aligned} \quad (23)$$

as well as

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} |\nabla v_2|^4 dx + 4\lambda_2 \int_{\Omega} |\nabla v_2|^4 dx + d_4 \int_{\Omega} |\nabla |\nabla v_2|^2|^2 dx \\ &\leq (2n + 8) \frac{C_{h_2}^2}{d_4} \int_{\Omega} (u_1^2 + u_2^2) |\nabla v_2|^2 dx \\ &\quad + 2d_4 \int_{\partial\Omega} |\nabla v_2|^2 \frac{\partial |\nabla v_2|^2}{\partial \nu} dS \text{ for all } t \in (0, T_{\max}). \end{aligned} \quad (24)$$

Proof. Multiplying the first equation in (1) by u_1 and integrating by parts over Ω , in light of (H1)-(H2) and the Young inequality, we can see that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} u_1^2 dx + d_1 \int_{\Omega} |\nabla u_1|^2 dx \\ &= \int_{\Omega} u_1 \chi_{11}(v_1) \nabla v_1 \cdot \nabla u_1 dx + \int_{\Omega} u_1 \chi_{12}(v_2) \nabla v_2 \cdot \nabla u_1 dx \\ & \quad + \mu_1 \int_{\Omega} u_1^2 (1 - u_1 - a_1 u_2) dx \\ & \leq K_1 \int_{\Omega} u_1 \nabla v_1 \cdot \nabla u_1 dx + K_1 \int_{\Omega} u_1 \nabla v_2 \cdot \nabla u_1 dx + \mu_1 \int_{\Omega} u_1^2 (1 - u_1 - a_1 u_2) dx \\ & \leq \frac{d_1}{2} \int_{\Omega} |\nabla u_1|^2 dx + \frac{K_1^2}{d_1} \int_{\Omega} u_1^2 |\nabla v_1|^2 dx + \frac{K_1^2}{d_1} \int_{\Omega} u_1^2 |\nabla v_2|^2 dx + \mu_1 \int_{\Omega} u_1^2 (1 - u_1) dx \end{aligned}$$

for all $t \in (0, T_{max})$, which directly yields (21). Similarly, we can derive (22).

To derive (23), in light of the third equation in (1) and the identity $2\nabla v_1 \cdot \nabla \Delta v_1 = \Delta |\nabla v_1|^2 - 2|D^2 v_1|^2$, it follows that

$$\begin{aligned} & \frac{1}{4} \frac{d}{dt} \int_{\Omega} |\nabla v_1|^4 dx = \int_{\Omega} (\nabla v_1)^3 \cdot \nabla v_{1t} dx \\ &= d_3 \int_{\Omega} |\nabla v_1|^2 \left(\frac{1}{2} \Delta |\nabla v_1|^2 - |D^2 v_1|^2 \right) dx - \lambda_1 \int_{\Omega} |\nabla v_1|^4 dx \\ & \quad - \int_{\Omega} h_1(u_1, u_2) \nabla \cdot (|\nabla v_1|^2 \nabla v_1) dx \\ & \leq \frac{d_3}{2} \int_{\partial\Omega} |\nabla v_1|^2 \frac{\partial |\nabla v_1|^2}{\partial \nu} dS - \frac{d_3}{2} \int_{\Omega} |\nabla |\nabla v_1|^2|^2 dx - d_3 \int_{\Omega} |\nabla v_1|^2 |D^2 v_1|^2 dx \\ & \quad - \lambda_1 \int_{\Omega} |\nabla v_1|^4 dx - \int_{\Omega} h_1(u_1, u_2) |\nabla v_1|^2 \Delta v_1 dx - \int_{\Omega} h_1(u_1, u_2) \nabla v_1 \cdot \nabla |\nabla v_1|^2 dx \end{aligned} \tag{25}$$

for all $t \in (0, T_{max})$, here, in view of (H4) and the relation $|\Delta v_1|^2 \leq n|D^2 v_1|^2$, one obtains

$$\begin{aligned} & - \int_{\Omega} h_1(u_1, u_2) |\nabla v_1|^2 \Delta v_1 dx \leq \sqrt{n} C_{h_1} \int_{\Omega} (u_1 + u_2) |\nabla v_1|^2 |D^2 v_1| dx \\ & \leq d_3 \int_{\Omega} |\nabla v_1|^2 |D^2 v_1|^2 dx + \frac{n C_{h_1}^2}{2 d_3} (u_1^2 + u_2^2) |\nabla v_1|^2 dx \text{ for all } t \in (0, T_{max}), \end{aligned} \tag{26}$$

and

$$\begin{aligned} & - \int_{\Omega} h_1(u_1, u_2) \nabla v_1 \cdot \nabla |\nabla v_1|^2 dx \leq C_{h_1} \int_{\Omega} (u_1 + u_2) |\nabla v_1| \cdot \nabla |\nabla v_1|^2 dx \\ & \leq \frac{d_3}{4} \int_{\Omega} |\nabla |\nabla v_1|^2|^2 dx + \frac{C_{h_1}^2}{d_3} (2u_1^2 + 2u_2^2) |\nabla v_1|^2 dx \text{ for all } t \in (0, T_{max}). \end{aligned} \tag{27}$$

Consequently, plugging (26) and (27) into (25), we arrive at (23). In addition, (24) can be established in a same manner. □

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a smooth and bounded domain, $\lambda_i > 0, \mu_i > 0, d_j > 0 (i = 1, 2, j = 1, 2, 3, 4)$. Assume that $\chi_{ij}, h_i (i = 1, 2, j = 1, 2, 3, 4)$ satisfy*

(H1)-(H4). Then for any classical solution (u_1, u_2, v_1, v_2) of (1) we have

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} u_1 |\nabla v_1|^2 dx + \int_{\Omega} u_1 |\nabla v_1|^2 dx \\
& \leq \frac{3d_3}{8} \int_{\Omega} |\nabla |\nabla v_1|^2|^2 dx + \left(\frac{2(d_1 + d_3)^2}{d_3} + C_{h_1}^2 \right) \int_{\Omega} |\nabla u_1|^2 dx + C_{h_1}^2 \int_{\Omega} |\nabla u_2|^2 dx \\
& \quad + (\mu_1 - 2\lambda_1 + 1) \int_{\Omega} u_1 |\nabla v_1|^2 dx + \left(\frac{2K_1^2}{d_3} - \mu_1 + 2 \right) \int_{\Omega} u_1^2 |\nabla v_1|^2 dx \\
& \quad + \frac{2K_1^2}{d_3} \int_{\Omega} u_1^2 |\nabla v_2|^2 dx + d_3 \int_{\partial\Omega} u_1 \frac{\partial |\nabla v_1|^2}{\partial\nu} dS \text{ for all } t \in (0, T_{max}),
\end{aligned} \tag{28}$$

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} u_1 |\nabla v_2|^2 dx + \int_{\Omega} u_1 |\nabla v_2|^2 dx \\
& \leq \frac{3d_4}{8} \int_{\Omega} |\nabla |\nabla v_2|^2|^2 dx + \left(\frac{2(d_1 + d_4)^2}{d_4} + C_{h_2}^2 \right) \int_{\Omega} |\nabla u_1|^2 dx + C_{h_2}^2 \int_{\Omega} |\nabla u_2|^2 dx \\
& \quad + (\mu_1 - 2\lambda_2 + 1) \int_{\Omega} u_1 |\nabla v_2|^2 dx + \left(\frac{2K_1^2}{d_4} - \mu_1 + 2 \right) \int_{\Omega} u_1^2 |\nabla v_2|^2 dx \\
& \quad + \frac{2K_1^2}{d_4} \int_{\Omega} u_1^2 |\nabla v_1|^2 dx + d_4 \int_{\partial\Omega} u_1 \frac{\partial |\nabla v_2|^2}{\partial\nu} dS \text{ for all } t \in (0, T_{max}),
\end{aligned} \tag{29}$$

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} u_2 |\nabla v_1|^2 dx + \int_{\Omega} u_2 |\nabla v_1|^2 dx \\
& \leq \frac{3d_3}{8} \int_{\Omega} |\nabla |\nabla v_1|^2|^2 dx + \left(\frac{2(d_2 + d_3)^2}{d_3} + C_{h_1}^2 \right) \int_{\Omega} |\nabla u_2|^2 dx + C_{h_1}^2 \int_{\Omega} |\nabla u_1|^2 dx \\
& \quad + (\mu_2 - 2\lambda_1 + 1) \int_{\Omega} u_2 |\nabla v_1|^2 dx + \left(\frac{2K_2^2}{d_3} - \mu_2 + 2 \right) \int_{\Omega} u_2^2 |\nabla v_1|^2 dx \\
& \quad + \frac{2K_2^2}{d_3} \int_{\Omega} u_2^2 |\nabla v_2|^2 dx + d_3 \int_{\partial\Omega} u_2 \frac{\partial |\nabla v_1|^2}{\partial\nu} dS \text{ for all } t \in (0, T_{max}),
\end{aligned} \tag{30}$$

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} u_2 |\nabla v_2|^2 dx + \int_{\Omega} u_2 |\nabla v_2|^2 dx \\
& \leq \frac{3d_4}{8} \int_{\Omega} |\nabla |\nabla v_2|^2|^2 dx + \left(\frac{2(d_2 + d_4)^2}{d_4} + C_{h_2}^2 \right) \int_{\Omega} |\nabla u_2|^2 dx + C_{h_2}^2 \int_{\Omega} |\nabla u_1|^2 dx \\
& \quad + (\mu_2 - 2\lambda_2 + 1) \int_{\Omega} u_2 |\nabla v_2|^2 dx + \left(\frac{2K_2^2}{d_4} - \mu_2 + 2 \right) \int_{\Omega} u_2^2 |\nabla v_2|^2 dx \\
& \quad + \frac{2K_2^2}{d_4} \int_{\Omega} u_2^2 |\nabla v_1|^2 dx + d_4 \int_{\partial\Omega} u_2 \frac{\partial |\nabla v_2|^2}{\partial\nu} dS \text{ for all } t \in (0, T_{max}).
\end{aligned} \tag{31}$$

Proof. Notice that the estimates for (28)-(31) are similar, thereupon, we only consider the priori estimates for (28). Utilizing the first equation and the third equation in (1), one finds

$$\begin{aligned}
 & \frac{d}{dt} \int_{\Omega} u_1 |\nabla v_1|^2 dx = \int_{\Omega} |\nabla v_1|^2 u_{1t} dx + 2 \int_{\Omega} u_1 \nabla v_1 \cdot \nabla v_{1t} \\
 & = -d_1 \int_{\Omega} \nabla u_1 \cdot \nabla |\nabla v_1|^2 dx + \int_{\Omega} u_1 \chi_{11}(v_1) \nabla v_1 \cdot \nabla |\nabla v_1|^2 dx \\
 & \quad + \int_{\Omega} u_1 \chi_{12}(v_2) \nabla v_2 \cdot \nabla |\nabla v_1|^2 dx + \mu_1 \int_{\Omega} u_1 |\nabla v_1|^2 (1 - u_1 - a_1 u_2) dx \\
 & \quad + d_3 \int_{\Omega} u_1 (\Delta |\nabla v_1|^2 - 2|D^2 v_1|^2) dx - 2\lambda_1 \int_{\Omega} u_1 |\nabla v_1|^2 dx \\
 & \quad + 2 \int_{\Omega} u_1 \left(\frac{\partial h_1(u_1, u_2)}{\partial u_1} \nabla u_1 + \frac{\partial h_1(u_1, u_2)}{\partial u_2} \nabla u_2 \right) \cdot \nabla v_1 dx \\
 & \leq -(d_1 + d_3) \int_{\Omega} \nabla u_1 \cdot \nabla |\nabla v_1|^2 dx + \int_{\Omega} u_1 \chi_{11}(v_1) \nabla v_1 \cdot \nabla |\nabla v_1|^2 dx \\
 & \quad + \int_{\Omega} u_1 \chi_{12}(v_2) \nabla v_2 \cdot \nabla |\nabla v_1|^2 dx + (\mu_1 - 2\lambda_1) \int_{\Omega} u_1 |\nabla v_1|^2 dx - \mu_1 \int_{\Omega} u_1^2 |\nabla v_1|^2 dx \\
 & \quad + d_3 \int_{\partial\Omega} u_1 \frac{\partial |\nabla v_1|^2}{\partial \nu} dS + 2 \int_{\Omega} \frac{\partial h_1(u_1, u_2)}{\partial u_1} u_1 \nabla u_1 \cdot \nabla v_1 dx \\
 & \quad + 2 \int_{\Omega} \frac{\partial h_1(u_1, u_2)}{\partial u_2} u_1 \nabla u_2 \cdot \nabla v_1 dx
 \end{aligned} \tag{32}$$

for all $t \in (0, T_{max})$, where we have used the identity $2\nabla v_1 \cdot \nabla \Delta v_1 = \Delta |\nabla v_1|^2 - 2|D^2 v_1|^2$. To estimate the right side term of (32), we apply the Young inequality and the conditions (H2), (H4) to obtain

$$\begin{aligned}
 & -(d_1 + d_3) \int_{\Omega} \nabla u_1 \cdot \nabla |\nabla v_1|^2 dx + \int_{\Omega} u_1 \chi_{11}(v_1) \nabla v_1 \cdot \nabla |\nabla v_1|^2 dx \\
 & + \int_{\Omega} u_1 \chi_{12}(v_2) \nabla v_2 \cdot \nabla |\nabla v_1|^2 dx \leq \frac{d_3}{8} \int_{\Omega} |\nabla |\nabla v_1|^2|^2 dx + \frac{2(d_1 + d_3)^2}{d_3} \int_{\Omega} |\nabla u_1|^2 dx \\
 & + \frac{d_3}{8} \int_{\Omega} |\nabla |\nabla v_1|^2|^2 dx + \frac{2K_1^2}{d_3} \int_{\Omega} u_1^2 |\nabla v_1|^2 dx \\
 & + \frac{d_3}{8} \int_{\Omega} |\nabla |\nabla v_1|^2|^2 dx + \frac{2K_1^2}{d_3} \int_{\Omega} u_1^2 |\nabla v_2|^2 dx \text{ for all } t \in (0, T_{max})
 \end{aligned} \tag{33}$$

and

$$\begin{aligned}
 & 2 \int_{\Omega} \frac{\partial h_1(u_1, u_2)}{\partial u_1} u_1 \nabla u_1 \cdot \nabla v_1 dx + 2 \int_{\Omega} \frac{\partial h_1(u_1, u_2)}{\partial u_2} u_1 \nabla u_2 \cdot \nabla v_1 dx \\
 & \leq C_{h_1}^2 \int_{\Omega} |\nabla u_1|^2 dx + \int_{\Omega} u_1^2 |\nabla v_1|^2 dx + C_{h_1}^2 \int_{\Omega} |\nabla u_2|^2 dx + \int_{\Omega} u_1^2 |\nabla v_1|^2 dx
 \end{aligned} \tag{34}$$

for all $t \in (0, T_{max})$. Inserting (33) and (34) into (32), one can immediately get (28). Analogously, we can obtain (29)-(31). \square

With Lemma 3.1 and Lemma 3.2 at hand, now, relying on a series of estimates, under an additional largeness assumption on μ_1, μ_2 , we can attain the boundedness of $\int_{\Omega} u_1^2 dx, \int_{\Omega} u_2^2 dx, \int_{\Omega} |\nabla v_1|^4 dx$ and $\int_{\Omega} |\nabla v_2|^4 dx$.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a smooth and bounded domain, $\lambda_i > 0, \mu_i > 0, d_j > 0 (i = 1, 2, j=1, 2, 3, 4)$. Assume that $\chi_{ij}, h_i (i = 1, 2, j = 1, 2, 3, 4)$ satisfy*

(H1)-(H4), and (u_1, u_2, v_1, v_2) is a classical solution of (1). Then if μ_1, μ_2 satisfy (4) and (5) in Theorem 1.1, one can find $C_1 > 0$ such that

$$\int_{\Omega} u_1^2 dx + \int_{\Omega} u_2^2 dx + \int_{\Omega} |\nabla v_1|^4 dx + \int_{\Omega} |\nabla v_2|^4 dx \leq C_1 \text{ for all } t \in (0, T_{max}). \quad (35)$$

Proof. It follows from Lemma 3.1 and Lemma 3.2 that

$$\begin{aligned} & \frac{d}{dt} \left(\eta_1 \int_{\Omega} u_1^2 + \eta_2 \int_{\Omega} u_2^2 + \int_{\Omega} |\nabla v_1|^4 + \int_{\Omega} |\nabla v_2|^4 + \int_{\Omega} u_1 |\nabla v_1|^2 + \int_{\Omega} u_1 |\nabla v_2|^2 \right. \\ & \left. + \int_{\Omega} u_2 |\nabla v_1|^2 + \int_{\Omega} u_2 |\nabla v_2|^2 \right) + \int_{\Omega} u_1^2 + \int_{\Omega} u_2^2 + \int_{\Omega} |\nabla u_1|^2 + \int_{\Omega} |\nabla u_2|^2 \\ & + 4\lambda_1 \int_{\Omega} |\nabla v_1|^4 + 4\lambda_2 \int_{\Omega} |\nabla v_2|^4 + \frac{d_3}{4} \int_{\Omega} |\nabla |\nabla v_1|^2|^2 + \frac{d_4}{4} \int_{\Omega} |\nabla |\nabla v_2|^2|^2 \\ & + \int_{\Omega} u_1 |\nabla v_1|^2 + \int_{\Omega} u_1 |\nabla v_2|^2 + \int_{\Omega} u_2 |\nabla v_1|^2 + \int_{\Omega} u_2 |\nabla v_2|^2 \\ & \leq (2\eta_1 \mu_1 + 1) \int_{\Omega} u_1^2 - 2\eta_1 \mu_1 \int_{\Omega} u_1^3 + (2\eta_2 \mu_2 + 1) \int_{\Omega} u_2^2 - 2\eta_2 \mu_2 \int_{\Omega} u_2^3 \\ & + (\mu_1 - 2\lambda_1 + 1) \int_{\Omega} u_1 |\nabla v_1|^2 + (\mu_1 - 2\lambda_2 + 1) \int_{\Omega} u_1 |\nabla v_2|^2 \\ & + (\mu_2 - 2\lambda_1 + 1) \int_{\Omega} u_2 |\nabla v_1|^2 + (\mu_2 - 2\lambda_2 + 1) \int_{\Omega} u_2 |\nabla v_2|^2 \\ & + 2d_3 \int_{\partial\Omega} |\nabla v_1|^2 \frac{\partial |\nabla v_1|^2}{\partial \nu} + 2d_4 \int_{\partial\Omega} |\nabla v_2|^2 \frac{\partial |\nabla v_2|^2}{\partial \nu} \\ & + d_3 \int_{\partial\Omega} u_1 \frac{\partial |\nabla v_1|^2}{\partial \nu} + d_4 \int_{\partial\Omega} u_1 \frac{\partial |\nabla v_2|^2}{\partial \nu} + d_3 \int_{\partial\Omega} u_2 \frac{\partial |\nabla v_1|^2}{\partial \nu} + d_4 \int_{\partial\Omega} u_2 \frac{\partial |\nabla v_2|^2}{\partial \nu} \\ & + \left(\frac{2K_1^2}{d_1} \eta_1 + (2n+8) \frac{C_{h_1}^2}{d_3} + \frac{2K_1^2}{d_3} + \frac{2K_1^2}{d_4} + 2 - \mu_1 \right) \int_{\Omega} u_1^2 |\nabla v_1|^2 \\ & + \left(\frac{2K_1^2}{d_1} \eta_1 + (2n+8) \frac{C_{h_2}^2}{d_4} + \frac{2K_1^2}{d_3} + \frac{2K_1^2}{d_4} + 2 - \mu_1 \right) \int_{\Omega} u_1^2 |\nabla v_2|^2 \\ & + \left(\frac{2K_2^2}{d_2} \eta_2 + (2n+8) \frac{C_{h_1}^2}{d_3} + \frac{2K_2^2}{d_3} + \frac{2K_2^2}{d_4} + 2 - \mu_2 \right) \int_{\Omega} u_2^2 |\nabla v_1|^2 \\ & + \left(\frac{2K_2^2}{d_2} \eta_2 + (2n+8) \frac{C_{h_2}^2}{d_4} + \frac{2K_2^2}{d_3} + \frac{2K_2^2}{d_4} + 2 - \mu_2 \right) \int_{\Omega} u_2^2 |\nabla v_2|^2. \end{aligned} \quad (36)$$

Making use of the Young inequality, for any $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in (0, 1)$, it is clear that

$$\begin{aligned} & (\mu_1 - 2\lambda_1 + 1) \int_{\Omega} u_1 |\nabla v_1|^2 dx \leq \epsilon_1 \int_{\Omega} |\nabla v_1|^4 dx + \frac{(\mu_1 - 2\lambda_1 + 1)^2}{4\epsilon_1} \int_{\Omega} u_1^2 dx, \\ & (\mu_1 - 2\lambda_2 + 1) \int_{\Omega} u_1 |\nabla v_2|^2 dx \leq \epsilon_2 \int_{\Omega} |\nabla v_2|^4 dx + \frac{(\mu_1 - 2\lambda_2 + 1)^2}{4\epsilon_2} \int_{\Omega} u_1^2 dx, \\ & (\mu_2 - 2\lambda_1 + 1) \int_{\Omega} u_2 |\nabla v_1|^2 dx \leq \epsilon_3 \int_{\Omega} |\nabla v_1|^4 dx + \frac{(\mu_2 - 2\lambda_1 + 1)^2}{4\epsilon_3} \int_{\Omega} u_2^2 dx, \\ & (\mu_2 - 2\lambda_2 + 1) \int_{\Omega} u_2 |\nabla v_2|^2 dx \leq \epsilon_4 \int_{\Omega} |\nabla v_2|^4 dx + \frac{(\mu_2 - 2\lambda_2 + 1)^2}{4\epsilon_4} \int_{\Omega} u_2^2 dx. \end{aligned} \quad (37)$$

According to Lemma 4.2 in [7], there exists $\mathcal{C} > 0$ such that

$$\frac{\partial |\nabla v_i|^2}{\partial \nu} \leq \mathcal{C} |\nabla v_i|^2, \quad i = 1, 2, \text{ for all } t \in (0, T_{max}), x \in \partial\Omega. \tag{38}$$

And thanks to the boundary trace embedding:

$$W^{1,2}(\Omega) \hookrightarrow W^{\frac{1}{2},2}(\Omega) \hookrightarrow L^2(\partial\Omega), \tag{39}$$

which warrants that for any $\epsilon_5 > 0$, one can find $C(\epsilon_5) > 0$ such that

$$\int_{\partial\Omega} \phi^2 dS \leq \epsilon_5 \int_{\Omega} |\nabla \phi|^2 dx + C(\epsilon_5) \left(\int_{\Omega} |\phi| dx \right)^2 \tag{40}$$

holds for all $\phi \in W^{1,2}(\Omega)$. Thereupon, invoking Young’s inequality, (38) and (40), we can see that

$$\begin{aligned} & 2d_3 \int_{\partial\Omega} |\nabla v_1|^2 \frac{\partial |\nabla v_1|^2}{\partial \nu} + 2d_4 \int_{\partial\Omega} |\nabla v_2|^2 \frac{\partial |\nabla v_2|^2}{\partial \nu} + d_3 \int_{\partial\Omega} (u_1 + u_2) \frac{\partial |\nabla v_1|^2}{\partial \nu} \\ & + d_4 \int_{\partial\Omega} (u_1 + u_2) \frac{\partial |\nabla v_2|^2}{\partial \nu} \\ & \leq 2d_3 \mathcal{C} \int_{\partial\Omega} |\nabla v_1|^4 + 2d_4 \mathcal{C} \int_{\partial\Omega} |\nabla v_2|^4 + d_3 \mathcal{C} \int_{\partial\Omega} (u_1 + u_2) |\nabla v_1|^2 \\ & + d_4 \mathcal{C} \int_{\partial\Omega} (u_1 + u_2) |\nabla v_2|^2 \\ & \leq \mathcal{C}(2d_3 + 2d_3^2) \int_{\partial\Omega} |\nabla v_1|^4 + \mathcal{C}(2d_4 + 2d_4^2) \int_{\partial\Omega} |\nabla v_2|^4 + 2\mathcal{C} \int_{\partial\Omega} u_1^2 + 2\mathcal{C} \int_{\partial\Omega} u_2^2 \\ & \leq \hat{\epsilon} \int_{\Omega} |\nabla |\nabla v_1|^2|^2 + C_1(\hat{\epsilon}) \left(\int_{\Omega} |\nabla v_1|^2 \right)^2 + \hat{\epsilon} \int_{\Omega} |\nabla |\nabla v_2|^2|^2 + C_2(\hat{\epsilon}) \left(\int_{\Omega} |\nabla v_2|^2 \right)^2 \\ & + \hat{\epsilon} \int_{\Omega} |\nabla u_1|^2 + C_3(\hat{\epsilon}) \left(\int_{\Omega} u_1 \right)^2 + \hat{\epsilon} \int_{\Omega} |\nabla u_2|^2 + C_4(\hat{\epsilon}) \left(\int_{\Omega} u_2 \right)^2, \end{aligned} \tag{41}$$

where $0 < \hat{\epsilon} < \min \{ \frac{d_3}{4}, \frac{d_4}{4}, 1 \}$. Inserting (37) and (41) into (36), utilizing (4) and (5), we can see that

$$\begin{aligned} & \frac{d}{dt} \left(\eta_1 \int_{\Omega} u_1^2 + \eta_2 \int_{\Omega} u_2^2 + \int_{\Omega} |\nabla v_1|^4 + \int_{\Omega} |\nabla v_2|^4 + \int_{\Omega} u_1 |\nabla v_1|^2 + \int_{\Omega} u_1 |\nabla v_2|^2 \right. \\ & \left. + \int_{\Omega} u_2 |\nabla v_1|^2 + \int_{\Omega} u_2 |\nabla v_2|^2 \right) + \int_{\Omega} u_1^2 dx + \int_{\Omega} u_2^2 + (1 - \hat{\epsilon}) \int_{\Omega} |\nabla u_1|^2 \\ & + (1 - \hat{\epsilon}) \int_{\Omega} |\nabla u_2|^2 + (4\lambda_1 - \epsilon_1 - \epsilon_3) \int_{\Omega} |\nabla v_1|^4 + (4\lambda_2 - \epsilon_2 - \epsilon_4) \int_{\Omega} |\nabla v_2|^4 \\ & + \left(\frac{d_3}{4} - \hat{\epsilon} \right) \int_{\Omega} |\nabla |\nabla v_1|^2|^2 + \left(\frac{d_4}{4} - \hat{\epsilon} \right) \int_{\Omega} |\nabla |\nabla v_2|^2|^2 \\ & + \int_{\Omega} (u_1 + u_2) (|\nabla v_1|^2 + |\nabla v_2|^2) \\ & \leq \left(2\eta_1 \mu_1 + 1 + \frac{(\mu_1 - 2\lambda_1 + 1)^2}{4\epsilon_1} + \frac{(\mu_1 - 2\lambda_2 + 1)^2}{4\epsilon_2} \right) \int_{\Omega} u_1^2 - 2\eta_1 \mu_1 \int_{\Omega} u_1^3 \\ & + \left(2\eta_2 \mu_2 + 1 + \frac{(\mu_2 - 2\lambda_1 + 1)^2}{4\epsilon_3} + \frac{(\mu_2 - 2\lambda_2 + 1)^2}{4\epsilon_4} \right) \int_{\Omega} u_2^2 - 2\eta_2 \mu_2 \int_{\Omega} u_2^3 \end{aligned} \tag{42}$$

$$\begin{aligned}
& + C_1(\hat{\epsilon}) \left(\int_{\Omega} |\nabla v_1|^2 \right)^2 + C_2(\hat{\epsilon}) \left(\int_{\Omega} |\nabla v_2|^2 \right)^2 + C_3(\hat{\epsilon}) \left(\int_{\Omega} u_1 \right)^2 + C_4(\hat{\epsilon}) \left(\int_{\Omega} u_2 \right)^2 \\
& \text{for all } t \in (0, T_{max}) \text{ and } \epsilon_i \in (0, 1), (i = 1, 2, 3, 4). \text{ Let} \\
& Y(t) := \eta_1 \int_{\Omega} u_1^2 dx + \eta_2 \int_{\Omega} u_2^2 dx + \int_{\Omega} |\nabla v_1|^4 dx + \int_{\Omega} |\nabla v_2|^4 dx + \int_{\Omega} u_1 |\nabla v_1|^2 dx \\
& \quad + \int_{\Omega} u_1 |\nabla v_2|^2 dx + \int_{\Omega} u_2 |\nabla v_1|^2 dx + \int_{\Omega} u_2 |\nabla v_2|^2 dx,
\end{aligned} \tag{43}$$

selecting $\epsilon_i, (i = 1, 2, 3, 4)$ satisfying

$$\epsilon_1 + \epsilon_3 < 2\lambda_1, \quad \epsilon_2 + \epsilon_4 < 2\lambda_2,$$

applying Young's inequality, Lemma 2.1, Lemma 2.3 and (42), we can find $\delta > 0$ and $C_1 > 0$ such that

$$\frac{d}{dt} Y(t) + \delta Y(t) \leq C_1 \text{ for all } t \in (0, T_{max}), \tag{44}$$

which with the ODE comparison principle means that (35) holds for all $t \in (0, T_{max})$. \square

On the basis of the L^2 bound for u_1, u_2 , we can now establish the L^p estimates for u_1, u_2 .

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n (n \leq 3)$ be a smooth and bounded domain, $\lambda_i > 0, \mu_i > 0, d_j > 0 (i = 1, 2, j = 1, 2, 3, 4)$. Assume that $\chi_{ij}, h_i (i = 1, 2, j = 1, 2, 3, 4)$ satisfy (H1)-(H4), and (u_1, u_2, v_1, v_2) is a classical solution of (1). If there exists $C_2 > 0$ such that*

$$\int_{\Omega} u_1^2 dx + \int_{\Omega} u_2^2 dx + \int_{\Omega} |\nabla v_1|^4 dx + \int_{\Omega} |\nabla v_2|^4 dx \leq C_2 \text{ for all } t \in (0, T_{max}). \tag{45}$$

Then for all $p > 1$, one can find a positive constant C_3 such that

$$\|u_1\|_{L^p(\Omega)} + \|u_2\|_{L^p(\Omega)} \leq C_3 \text{ for all } t \in (0, T_{max}). \tag{46}$$

Proof. Testing the first equation of (1) by u_1^{p-1} to obtain

$$\begin{aligned}
& \frac{1}{p} \frac{d}{dt} \int_{\Omega} u_1^p \\
& = -d_1(p-1) \int_{\Omega} u_1^{p-2} |\nabla u_1|^2 + (p-1) \int_{\Omega} u_1^{p-1} \chi_{11}(v_1) \nabla v_1 \cdot \nabla u_1 \\
& \quad + (p-1) \int_{\Omega} u_1^{p-1} \chi_{12}(v_2) \nabla v_2 \cdot \nabla u_1 + \mu_1 \int_{\Omega} u_1^p (1 - u_1 - a_1 u_2) \\
& \leq -d_1(p-1) \int_{\Omega} u_1^{p-2} |\nabla u_1|^2 + \frac{d_1(p-1)}{4} \int_{\Omega} u_1^{p-2} |\nabla u_1|^2 + \frac{p-1}{d_1} K_1^2 \int_{\Omega} u_1^p |\nabla v_1|^2 \\
& \quad + \frac{d_1(p-1)}{4} \int_{\Omega} u_1^{p-2} |\nabla u_1|^2 + \frac{p-1}{d_1} K_1^2 \int_{\Omega} u_1^p |\nabla v_2|^2 + \mu_1 \int_{\Omega} u_1^p (1 - u_1) \\
& \leq -\frac{d_1(p-1)}{2} \int_{\Omega} u_1^{p-2} |\nabla u_1|^2 \\
& \quad + \frac{p-1}{d_1} K_1^2 \int_{\Omega} u_1^p (|\nabla v_1|^2 + |\nabla v_2|^2) + \mu_1 \int_{\Omega} u_1^p (1 - u_1)
\end{aligned} \tag{47}$$

for all $t \in (0, T_{max})$. Since (45) entails that

$$\int_{\Omega} |\nabla v_i|^4 dx \leq C_2 \text{ for all } t \in (0, T_{max}), i = 1, 2, \tag{48}$$

with $C_2 > 0$, and thus we apply the Hölder inequality to obtain

$$\begin{aligned} & \frac{p-1}{d_1} K_1^2 \int_{\Omega} u_1^p (|\nabla v_1|^2 + |\nabla v_2|^2) dx \\ & \leq \frac{p-1}{d_1} K_1^2 \left(\int_{\Omega} u_1^{2p} dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v_1|^4 dx \right)^{\frac{1}{2}} \\ & \quad + \frac{p-1}{d_1} K_1^2 \left(\int_{\Omega} u_1^{2p} dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v_2|^4 dx \right)^{\frac{1}{2}} \\ & \leq C_3 \left(\int_{\Omega} u_1^{2p} dx \right)^{\frac{1}{2}} \text{ for all } t \in (0, T_{max}) \end{aligned} \tag{49}$$

with $C_3 > 0$, to estimate the term $\left(\int_{\Omega} u_1^{2p} dx \right)^{\frac{1}{2}}$, fixing

$$\hat{\lambda} := \frac{\frac{pn}{2} - \frac{n}{4}}{1 - \frac{n}{2} + \frac{pn}{2}},$$

due to the fact that $n \leq 3$, this guarantees that $\hat{\lambda} \in (0, 1)$, then we can employ the Gagliardo-Nirenberg inequality, Lemma 2.1 and Young’s inequality to get

$$\begin{aligned} C_3 \left(\int_{\Omega} u_1^{2p} dx \right)^{\frac{1}{2}} &= C_3 \|u_1^{\frac{p}{2}}\|_{L^4(\Omega)}^2 \leq C_4 \left(\|\nabla u_1^{\frac{p}{2}}\|_{L^2(\Omega)}^{2\hat{\lambda}} \|u_1^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{2(1-\hat{\lambda})} + \|u_1^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^2 \right) \\ &\leq C_5 \left(\|\nabla u_1^{\frac{p}{2}}\|_{L^2(\Omega)}^{2\hat{\lambda}} + 1 \right) \\ &\leq \frac{d_1(p-1)}{2} \int_{\Omega} u_1^{p-2} |\nabla u_1|^2 dx + C_6 \end{aligned} \tag{50}$$

with $C_4, C_5, C_6 > 0$. A combination of (47), (49) and (50) yields

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u_1^p dx + \int_{\Omega} u_1^p dx \leq \mu_1 \int_{\Omega} u_1^p (1 - u_1) dx + \int_{\Omega} u_1^p dx + C_6 \leq C_7 \tag{51}$$

with $C_7 > 0$. Therefore, we apply a comparison argument to establish the boundedness of $\|u_1\|_{L^p(\Omega)}$. Applying the same arguments as above, we can easily obtain the boundedness of $\|u_2\|_{L^p(\Omega)}$. This completes the proof of this lemma. \square

Proof of Theorem 1.1. In light of Lemma 3.4, it follows from the well known Moser-type iterations and Lemma 2.1 that Theorem 1.1 holds. \square

3.2. The improvement of boundedness for $n = 2$: proof of Corollary 1.1.

In the above section, the global boundedness of solution is derived under the condition that μ_1, μ_2 are sufficiently large. In this section, motivated by the method in [2, 3, 17], we remove the restriction on μ_1, μ_2 when $n = 2$. The key point of the proof is to establish the boundedness of $\int_{\Omega} |\nabla v_1|^4 dx$ and $\int_{\Omega} |\nabla v_2|^4 dx$, to this end, we first establish the boundedness of $\int_t^{t+\tau} \int_{\Omega} |\nabla v_i|^4 dx, i = 1, 2$ in the following lemma.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n (n \leq 2)$ be a smooth and bounded domain, $\lambda_i > 0, \mu_i > 0, d_j > 0 (i = 1, 2, j=1, 2, 3, 4)$. Assume that $h_i (i = 1, 2)$ satisfy (H3)-(H4), and (u_1, u_2, v_1, v_2) is a classical solution of (1). Then one can find a constant $C_4 > 0$ such that*

$$\int_t^{t+\tau} \int_{\Omega} |\nabla v_1|^4 dx ds + \int_t^{t+\tau} \int_{\Omega} |\nabla v_2|^4 dx ds \leq C_4 \text{ for all } t \in (0, T_{max}), \tag{52}$$

where $\tau = \min \{1, \frac{T_{max}}{2}\}$.

Proof. Since $n \leq 2$, we can apply the Gagliardo-Nirenberg inequality to obtain

$$\begin{aligned} \int_{\Omega} |\nabla v_1|^4 dx &= \|\nabla v_1\|_{L^4(\Omega)}^4 \leq C(GN)^4 \left(\|\Delta v_1\|_{L^2(\Omega)}^{\bar{\lambda}} \|\nabla v_1\|_{L^2(\Omega)}^{1-\bar{\lambda}} + \|\nabla v_1\|_{L^2(\Omega)} \right)^4 \\ &\leq C_8 \left(\|\Delta v_1\|_{L^2(\Omega)}^2 + 1 \right) \end{aligned} \tag{53}$$

with $C_8 > 0, \bar{\lambda} = \frac{n}{4} \in (0, \frac{1}{2}]$, where we have used the boundedness result of $\|\nabla v_1\|_{L^2(\Omega)}$ in Lemma 2.3. Integrating (53) over $(t, t + \tau)$, and by virtue of Lemma 2.3, we can compute

$$\int_t^{t+\tau} \int_{\Omega} |\nabla v_1|^4 dx ds \leq C_9 \text{ for all } t \in (0, T_{max}), \tag{54}$$

where $C_9 > 0$. Similarly, we can derive

$$\int_t^{t+\tau} \int_{\Omega} |\nabla v_1|^4 dx ds \leq C_{10} \text{ for all } t \in (0, T_{max}) \tag{55}$$

with $C_{10} > 0$, this readily yields (52). □

In the second step, we derive the boundedness of $\int_{\Omega} |u_i \ln u_i| dx, i = 1, 2$.

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^n (n \leq 2)$ be a smooth and bounded domain, $\lambda_i > 0, \mu_i > 0, d_j > 0 (i = 1, 2, j=1, 2, 3, 4)$. Assume that $\chi_{ij} (i = 1, 2, j = 1, 2)$ satisfy*

$$0 < \chi'_{ij}(s) \leq L \text{ for all } s \geq 0 \tag{56}$$

with $L > 0$, and (u_1, u_2, v_1, v_2) is a classical solution of (1). Then there exists $C_4 > 0$ such that

$$\int_{\Omega} |u_1 \ln u_1| dx \leq C_4, \int_{\Omega} |u_2 \ln u_2| dx \leq C_4 \text{ for all } t \in (0, T_{max}). \tag{57}$$

Proof. Multiplying the first equation in (1) by $(1 + \ln u_1)$ and integrating it over Ω , one obtains

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} u_1 \ln u_1 dx \\ &= -d_1 \int_{\Omega} \frac{|\nabla u_1|^2}{u_1} dx + \int_{\Omega} \chi_{11}(v_1) \nabla v_1 \cdot \nabla u_1 dx + \int_{\Omega} \chi_{12}(v_2) \nabla v_2 \cdot \nabla u_1 dx \\ &\quad + \mu_1 \int_{\Omega} u_1 (1 - u_1 - a_1 u_2) (1 + \ln u_1) \\ &= -d_1 \int_{\Omega} \frac{|\nabla u_1|^2}{u_1} dx + \int_{\Omega} \nabla \left(\int_1^{v_1} \chi_{11}(s) ds \right) \cdot \nabla u_1 dx \\ &\quad + \int_{\Omega} \nabla \left(\int_1^{v_2} \chi_{12}(s) ds \right) \cdot \nabla u_1 dx \end{aligned} \tag{58}$$

$$\begin{aligned}
 & + \mu_1 \int_{\Omega} u_1(1 - u_1 - a_1 u_2)(1 + \ln u_1) \\
 & = -d_1 \int_{\Omega} \frac{|\nabla u_1|^2}{u_1} dx - \int_{\Omega} \Delta \left(\int_1^{v_1} \chi_{11}(s) ds \right) u_1 dx - \int_{\Omega} \Delta \left(\int_1^{v_2} \chi_{12}(s) ds \right) u_1 dx \\
 & + \mu_1 \int_{\Omega} u_1(1 - u_1 - a_1 u_2)(1 + \ln u_1) \text{ for all } t \in (0, T_{max}).
 \end{aligned}$$

To estimate $-\int_{\Omega} \Delta \left(\int_1^{v_1} \chi_{11}(s) ds \right) u_1 dx$, utilizing the Young inequality and (56), it follows

$$\begin{aligned}
 - \int_{\Omega} \Delta \left(\int_1^{v_1} \chi_{11}(s) ds \right) u_1 dx & \leq \frac{1}{2} \int_{\Omega} u_1^2 dx + \frac{1}{2} \int_{\Omega} \left| \Delta \left(\int_1^{v_1} \chi_{11}(s) ds \right) \right|^2 dx \\
 & = \frac{1}{2} \|u_1\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\chi'_{11}(v_1) |\nabla v_1|^2 + \chi_{11}(v_1) \Delta v_1\|_{L^2(\Omega)}^2 \\
 & \leq \frac{1}{2} \|u_1\|_{L^2(\Omega)}^2 + L^2 \|\nabla v_1\|_{L^4(\Omega)}^4 + K_1^2 \|\Delta v_1\|_{L^2(\Omega)}^2.
 \end{aligned} \tag{59}$$

Like wise, we have

$$- \int_{\Omega} \Delta \left(\int_1^{v_2} \chi_{12}(s) ds \right) u_1 dx \leq \frac{1}{2} \|u_1\|_{L^2(\Omega)}^2 + L^2 \|\nabla v_2\|_{L^4(\Omega)}^4 + K_1^2 \|\Delta v_2\|_{L^2(\Omega)}^2. \tag{60}$$

As for $\mu_1 \int_{\Omega} u_1(1 - u_1 - a_1 u_2)(1 + \ln u_1)$, in view of the boundedness of $\int_{\Omega} u_i dx (i = 1, 2)$ in Lemma 2.1, and the inequalities $\rho(1 - \rho) \leq \frac{1}{4}$, $\rho(1 - \rho) \ln \rho \leq 0$ as well as $-\rho \ln \rho \leq \frac{1}{e}$ for all $\rho > 0$, it holds that

$$\begin{aligned}
 & \mu_1 \int_{\Omega} u_1(1 - u_1 - a_1 u_2)(1 + \ln u_1) \\
 & \leq \mu_1 \int_{\Omega} u_1(1 - u_1) dx + \mu_1 \int_{\Omega} u_1(1 - u_1) \ln u_1 dx - a_1 \mu_1 \int_{\Omega} u_1 u_2 \ln u_1 dx \tag{61} \\
 & \leq \frac{\mu_1 |\Omega|}{4} + \frac{a_1 \mu_1 m_2}{e} \text{ for all } t \in (0, T_{max}).
 \end{aligned}$$

Plugging (59)-(61) into (58), we can see that

$$\begin{aligned}
 \frac{d}{dt} \int_{\Omega} u_1 \ln u_1 dx + d_1 \int_{\Omega} \frac{|\nabla u_1|^2}{u_1} dx & \leq \|u_1\|_{L^2(\Omega)}^2 + (L + K_1)^2 \left(\|\nabla v_1\|_{L^4(\Omega)}^4 \right. \\
 & \left. + \|\Delta v_1\|_{L^2(\Omega)}^2 + \|\nabla v_2\|_{L^4(\Omega)}^4 + \|\Delta v_2\|_{L^2(\Omega)}^2 \right) + \frac{\mu_1 |\Omega|}{4} + \frac{a_1 \mu_1 m_2}{e}
 \end{aligned} \tag{62}$$

for all $t \in (0, T_{max})$. According to $n \leq 2$, we can utilize the Gagliardo-Nirenberg inequality to derive

$$\begin{aligned}
 \int_{\Omega} u_1 \ln u_1 dx \leq \int_{\Omega} u_1^2 dx = \|u_1^{\frac{1}{2}}\|_{L^4(\Omega)}^4 & \leq C_{11} \|\nabla u_1^{\frac{1}{2}}\|_{L^2(\Omega)}^{4\bar{\lambda}} \|u_1^{\frac{1}{2}}\|_{L^2(\Omega)}^{4(1-\bar{\lambda})} + C_{11} \|u_1^{\frac{1}{2}}\|_{L^2(\Omega)}^4 \\
 & \leq C_{12} \left(\int_{\Omega} \frac{|\nabla u_1|^2}{u_1} + 1 \right)
 \end{aligned} \tag{63}$$

where $\bar{\lambda} = \frac{n}{4} \in (0, \frac{1}{2}]$, $C_{12} > 0$. Together with (62) and (63), we arrive at

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} u_1 \ln u_1 dx + \frac{d_1}{C_{12}} \int_{\Omega} u_1 \ln u_1 dx \\ & \leq \|u_1\|_{L^2(\Omega)}^2 + (L + K_1)^2 \left(\|\nabla v_1\|_{L^4(\Omega)}^4 + \|\Delta v_1\|_{L^2(\Omega)}^2 + \|\nabla v_2\|_{L^4(\Omega)}^4 + \|\Delta v_2\|_{L^2(\Omega)}^2 \right) \\ & \quad + \frac{\mu_1|\Omega|}{4} + \frac{a_1\mu_1m_2}{e} + d_1 \end{aligned} \tag{64}$$

for all $t \in (0, T_{max})$. Thus, we infer from (10), (13), (52) and Lemma 2.2 that $\int_{\Omega} |u_1 \ln u_1| dx$ is bounded. Equally, we can establish the boundedness for $\int_{\Omega} |u_2 \ln u_2| dx$. Whereby the proof is completed. \square

In what follows, we proceed to show that $\int_{\Omega} u_1^2 dx + \int_{\Omega} u_2^2 dx + \int_{\Omega} |\nabla v_1|^4 dx + \int_{\Omega} |\nabla v_2|^4 dx$ is bounded for all $\mu_1, \mu_2 > 0$.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^2$ be a smooth and bounded domain, $\lambda_i > 0$, $\mu_i > 0$, $d_j > 0$ ($i = 1, 2, j = 1, 2, 3, 4$). Assume that the conditions in Lemma 3.6 are satisfied, and (u_1, u_2, v_1, v_2) is a classical solution of (1). Then there exists $C_5 > 0$ such that*

$$\int_{\Omega} u_1^2 dx + \int_{\Omega} u_2^2 dx + \int_{\Omega} |\nabla v_1|^4 dx + \int_{\Omega} |\nabla v_2|^4 dx \leq C_5 \text{ for all } t \in (0, T_{max}). \tag{65}$$

Proof. From (21)-(24), we deduce that

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\Omega} u_1^2 dx + \int_{\Omega} u_2^2 dx + \int_{\Omega} |\nabla v_1|^4 dx + \int_{\Omega} |\nabla v_2|^4 dx \right) + d_1 \int_{\Omega} |\nabla u_1|^2 dx \\ & \quad + d_2 \int_{\Omega} |\nabla u_2|^2 dx + 4\lambda_1 \int_{\Omega} |\nabla v_1|^4 dx + d_3 \int_{\Omega} |\nabla |\nabla v_1|^2|^2 dx + 4\lambda_2 \int_{\Omega} |\nabla v_2|^4 dx \\ & \quad + d_4 \int_{\Omega} |\nabla |\nabla v_2|^2|^2 dx + \int_{\Omega} (u_1^2 + u_2^2) dx \\ & \leq \left((2n + 8) \frac{C_{h_1}^2}{d_3} + \frac{2K_1^2}{d_1} \right) \int_{\Omega} u_1^2 |\nabla v_1|^2 dx + \left((2n + 8) \frac{C_{h_1}^2}{d_3} + \frac{2K_2^2}{d_2} \right) \int_{\Omega} u_2^2 |\nabla v_1|^2 dx \\ & \quad + \left((2n + 8) \frac{C_{h_2}^2}{d_4} + \frac{2K_1^2}{d_1} \right) \int_{\Omega} u_1^2 |\nabla v_2|^2 dx + \left((2n + 8) \frac{C_{h_2}^2}{d_4} + \frac{2K_2^2}{d_2} \right) \int_{\Omega} u_2^2 |\nabla v_2|^2 dx \\ & \quad + 2d_3 \int_{\partial\Omega} |\nabla v_1|^2 \frac{\partial |\nabla v_1|^2}{\partial \nu} dS + 2d_4 \int_{\partial\Omega} |\nabla v_2|^2 \frac{\partial |\nabla v_2|^2}{\partial \nu} dS + \int_{\Omega} (u_1^2 + u_2^2) dx \\ & \quad + 2\mu_1 \int_{\Omega} u_1^2 (1 - u_1) dx + 2\mu_2 \int_{\Omega} u_2^2 (1 - u_2) dx \text{ for all } t \in (0, T_{max}). \end{aligned} \tag{66}$$

To handle the first term on the right side of (66), we notice that $n = 2$, accordingly, in view of (18), the following inequality

$$\|u_1\|_{L^3(\Omega)}^3 \leq \epsilon \|\nabla u_1\|_{L^2(\Omega)}^2 \|u_1 \ln |u_1|\|_{L^1(\Omega)} + C \|u_1\|_{L^1(\Omega)}^3 + C_{\epsilon} \tag{67}$$

holds for all $\epsilon > 0$, then, the boundedness of $\|u_1 \ln |u_1|\|_{L^1(\Omega)}$ and $\|u_1\|_{L^1(\Omega)}$ in Lemma 3.6 and Lemma 2.1 guarantees that

$$\|u_1\|_{L^3(\Omega)}^3 \leq C_{13} \epsilon \|\nabla u_1\|_{L^2(\Omega)}^2 + C_{13} \text{ for all } t \in (0, T_{max}) \text{ and } \epsilon > 0 \tag{68}$$

with $C_{13} > 0$. Therefore, utilizing the Hölder inequality, the Gagliardo-Nirenberg inequality and the Young inequality, it follows from (12), (67) as well as (68) that

$$\begin{aligned}
 & \left((2n + 8) \frac{C_{h_1}^2}{d_3} + \frac{2K_1^2}{d_1} \right) \int_{\Omega} u_1^2 |\nabla v_1|^2 dx \\
 & \leq \left((2n + 8) \frac{C_{h_1}^2}{d_3} + \frac{2K_1^2}{d_1} \right) \|u_1\|_{L^3(\Omega)}^2 \|\nabla v_1\|_{L^3(\Omega)}^2 \\
 & \leq \left((2n + 8) \frac{C_{h_1}^2}{d_3} + \frac{2K_1^2}{d_1} \right) \\
 & \quad \|u_1\|_{L^3(\Omega)}^2 \left(C_{GN} \|\nabla |\nabla v_1|^2\|_{L^2(\Omega)}^{\frac{2}{3}} \|\nabla v_1\|_{L^1(\Omega)}^{\frac{1}{3}} + C_{GN} \|\nabla v_1\|_{L^1(\Omega)} \right) \\
 & \leq \left((2n + 8) \frac{C_{h_1}^2}{d_3} + \frac{2K_1^2}{d_1} \right) \|u_1\|_{L^3(\Omega)}^2 \left(C_{14} \|\nabla |\nabla v_1|^2\|_{L^2(\Omega)}^{\frac{2}{3}} + C_{14} \right) \\
 & \leq \frac{d_3}{4} \|\nabla |\nabla v_1|^2\|_{L^2(\Omega)}^2 + C_{15} \|u_1\|_{L^3(\Omega)}^3 + C_{15} \\
 & \leq \frac{d_3}{4} \|\nabla |\nabla v_1|^2\|_{L^2(\Omega)}^2 + \frac{d_1}{2} \|\nabla u_1\|_{L^2(\Omega)}^2 + C_{16} \text{ for all } t \in (0, T_{max})
 \end{aligned} \tag{69}$$

with $C_{14}, C_{15}, C_{16} > 0$, where we have fixed $\epsilon = \frac{d_1}{2C_{13}C_{15}}$ when using (68).

In a similar way, we can find $C_{17}, C_{18}, C_{19} > 0$ such that

$$\left((2n + 8) \frac{C_{h_1}^2}{d_3} + \frac{2K_2^2}{d_2} \right) \int_{\Omega} u_2^2 |\nabla v_1|^2 dx \leq \frac{d_3}{4} \|\nabla |\nabla v_1|^2\|_{L^2(\Omega)}^2 + \frac{d_2}{2} \|\nabla u_2\|_{L^2(\Omega)}^2 + C_{17} \tag{70}$$

and

$$\left((2n + 8) \frac{C_{h_2}^2}{d_4} + \frac{2K_1^2}{d_1} \right) \int_{\Omega} u_1^2 |\nabla v_2|^2 dx \leq \frac{d_4}{4} \|\nabla |\nabla v_2|^2\|_{L^2(\Omega)}^2 + \frac{d_1}{2} \|\nabla u_1\|_{L^2(\Omega)}^2 + C_{18} \tag{71}$$

as well as

$$\left((2n + 8) \frac{C_{h_2}^2}{d_4} + \frac{2K_2^2}{d_2} \right) \int_{\Omega} u_2^2 |\nabla v_2|^2 dx \leq \frac{d_4}{4} \|\nabla |\nabla v_2|^2\|_{L^2(\Omega)}^2 + \frac{d_2}{2} \|\nabla u_2\|_{L^2(\Omega)}^2 + C_{19} \tag{72}$$

for all $t \in (0, T_{max})$.

For the term $2d_3 \int_{\partial\Omega} |\nabla v_1|^2 \frac{\partial |\nabla v_1|^2}{\partial \nu} dS + 2d_4 \int_{\partial\Omega} |\nabla v_2|^2 \frac{\partial |\nabla v_2|^2}{\partial \nu} dS$, (12) and (38)-(40) enable us to see that

$$\begin{aligned}
 & 2d_3 \int_{\partial\Omega} |\nabla v_1|^2 \frac{\partial |\nabla v_1|^2}{\partial \nu} dS + 2d_4 \int_{\partial\Omega} |\nabla v_2|^2 \frac{\partial |\nabla v_2|^2}{\partial \nu} dS \\
 & \leq \frac{d_3}{2} \|\nabla |\nabla v_1|^2\|_{L^2(\Omega)}^2 + \frac{d_4}{2} \|\nabla |\nabla v_2|^2\|_{L^2(\Omega)}^2 + C_{20} \text{ for all } t \in (0, T_{max})
 \end{aligned} \tag{73}$$

with $C_{20} > 0$.

What's more, making use of the identity $s^2(1 - s) \leq \frac{4}{27}$ for all $s > 0$, the last two terms in (66) can be bounded as

$$2\mu_1 \int_{\Omega} u_1^2(1 - u_1) dx + 2\mu_2 \int_{\Omega} u_2^2(1 - u_2) dx \leq \frac{8|\Omega|}{27}(\mu_1 + \mu_2) \text{ for all } t \in (0, T_{max}). \tag{74}$$

Thus, plugging (69)-(74) into (66), we can conclude that

$$\frac{d}{dt}\tilde{Y}(t) + \delta\tilde{Y}(t) \leq \int_{\Omega} (u_1^2 + u_2^2)dx + C_{21} \text{ for all } t \in (0, T_{max}), \quad (75)$$

where $\tilde{Y}(t) := \int_{\Omega} u_1^2 dx + \int_{\Omega} u_2^2 dx + \int_{\Omega} |\nabla v_1|^4 dx + \int_{\Omega} |\nabla v_2|^4 dx$, $\delta := \min\{4\lambda_1, 4\lambda_2, 1\}$, $C_{21} > 0$, which combined with (10), (11) and Lemma 2.2 show (65). \square

Proof of Corollary 1. A combination of Lemma 2.5 and Lemma 3.7 directly yields Corollary 1. \square

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