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FRACTIONAL p-SUB-LAPLACIAN OPERATOR PROBLEM WITH CONCAVE-CONVEX NONLINEARITIES ON HOMOGENEOUS GROUPS

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Abstract. This study examines the existence and multiplicity of non-negative solutions of the following fractional p-sub-Laplacian problem

$$\begin{cases}(-\Delta_{p,g})^s u = \lambda f(x)|u|^{\alpha-2}u + h(x)|u|^{\beta-2}u & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{G}\setminus\Omega,\end{cases}$$

where Ω is an open bounded in homogeneous Lie group $\mathbb{G}$ with smooth boundary, $p>1$, $s\in(0,1)$, $(-\Delta_{p,g})^s$ is the fractional p-sub-Laplacian operator with respect to the quasi-norm g, $\lambda>0$, $1<\alpha<p<\beta<p_s^*$, $p_s^* := \frac{Qp}{Q-sp}$ is the fractional critical Sobolev exponents, Q is the homogeneous dimensions of the homogeneous Lie group $\mathbb{G}$ with $Q>sp$, and f, h are sign-changing smooth functions. With the help of the Nehari manifold, we prove that the nonlocal problem on homogeneous group has at least two nontrivial solutions when the parameter λ belong to a center subset of $(0,+\infty)$.

1. **Introduction.** We consider the following p-fractional Laplace equation

$$\begin{cases}(-\Delta_{p,g})^s u = \lambda f(x)|u|^{\alpha-2}u + h(x)|u|^{\beta-2}u & \text{in } \Omega\\ \quad u = 0 & \text{in } \mathbb{G}\backslash\Omega\end{cases} \tag{1.1}$$

where Ω is an open bounded domain in homogeneous Lie group $\mathbb{G}$ with smooth boundary, $p>1$, the parameter $\lambda>0$, f and h are sign-changing smooth functions, $1<\alpha<p<\beta<p_s^* := \frac{Qp}{Q-ps}$, p_s^* is the fractional critical Sobolev exponent in this context and $Q>sp$ is the homogeneous dimension of the homogeneous Lie group $\mathbb{G}$. The operator $(-\Delta_{p,g})^s$ is the fractional p-sub-Laplacian operator on $\mathbb{G}$ which is defined by

$$(-\Delta_{p,g})^s u(x) = 2\lim_{\varepsilon\to 0}\int_{\mathbb{G}\backslash\mathcal{B}_g(x,\varepsilon)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{g(y^{-1}\circ x)^{Q+sp}}dy, \quad \forall x\in\mathbb{G},$$

where $\mathcal{B}_g(x,\varepsilon)$ is the quasi-ball of center $x\in\mathbb{G}$ and radius $\varepsilon>0$ with respect to the homogeneous quasi-norm g. In our work, the homogeneous quasi-norm $g:\mathbb{G}\to\mathbb{R}_0^+$ is a continuous function satisfying the following properties:

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(i) $g(x) = 0$ if and only if $x = 0$ for every $x \in \mathbb{G}$;
(ii) $g(x^{-1}) = g(x)$ for every $x \in \mathbb{G}$;
(iii) $g(\delta_\mu(x)) = \mu g(x)$ for every $\mu > 0$ and for every $x \in \mathbb{G}$, where δ_μ is a dilations on homogeneous Lie group $\mathbb{G}$.

Associated with (1.1), we have the energy functional $I_\lambda : E_g^0 \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{p}\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}}dxdy - \frac{\lambda}{\alpha}\int_\Omega f(x)|u|^\alpha dx - \frac{1}{\beta}\int_\Omega h(x)|u|^\beta dx.$$

By a direct calculation, we have that $I_\lambda \in C^1(E_g^0, \mathbb{R})$ and

$$\begin{aligned}\langle I_\lambda'(u), v\rangle = &\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{g(y^{-1}\circ x)^{Q+sp}}dxdy\\ &-\lambda\int_\Omega f(x)|u|^{\alpha-2}uvdx - \int_\Omega h(x)|u|^{\beta-2}uvdx, \quad \forall u, v \in E_g^0,\end{aligned}$$

where E_g^0 is a subspace of E_g defined as $E_g^0 = \{u \in E_g : \ u = 0 \ \text{in} \ \mathbb{G}\backslash\Omega\}$ with the norm

$$\|u\|_{E_g^0} = \left(\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}}dxdy\right)^{\frac{1}{p}}. \tag{1.2}$$

Here $\mathcal{Q} = \mathbb{G}^2\backslash(\mathcal{C}_\Omega \times \mathcal{C}_\Omega)$ and $\mathcal{C}_\Omega = \mathbb{G}\backslash\Omega$. See Section 2 for more details.

Recently, a lot of attention is given to the study of fractional operators of elliptic type due to concrete real world applications in finance, thin obstacle problem, optimization, quasi-geostrophic flow etc. Dirichlet boundary value problem in case of fractional Laplacian with polynomial type nonlinearity using variational methods is recently studied in [4, 6, 21, 20, 19]. For example, Brändle et. al [4] studied the fractional Laplacian operator $(-\Delta)^s$ equation involving concave-convex nonlinearity for the subcritical case in the Euclidean space $\mathbb{R}^N$, they prove that there exists a finite parameter $\Lambda > 0$ such that for each $\lambda \in (0, \Lambda)$ there exist at least two solutions, for $\lambda = \Lambda$ there exists at least one solution and for $\lambda \in (\Lambda, +\infty)$ there is no solution. Barrios et al. [2] studied the non-homogeneous equation involving fractional Laplacian and proved the existence and multiplicity of solutions under suitable conditions of s and q. Zhang, Liu and Jiao [23] studied the fractional equation with critical Sobolev exponent, they proved that the existence and multiplicity of solutions under appropriate conditions on the size of λ. For more other advances on this topic, see [19] for the subcritical, [20] for the critical case, [22] for the supercritical case, and fractional Laplacian equation with Hardy-type potential are shown in [13, 14, 24, 25, 26, 27]. Moreover, for the fractional p-Laplacian equation, eigenvalue problem related to p-fractional Laplacian is studied in [17, 12]. Goyal and Sreenadh [15] studied the fractional p-Laplacian equation involving concave-convex nonlinearities. By using the Nehari manifold and the fibering maps methods, they showed that the problem has at least two non-negative solutions.

In this paper, we present results concerning fractional forms Laplacian operator on homogeneous Lie groups. As usual, the general approach based on homogeneous Lie groups allows one to get insights also in the Abelian case, for example, from the point of view of the possibility of choosing an arbitrary quasi-norm. Moreover, another application of the setting of homogeneous Lie groups is that the results can be equally applied to elliptic and subelliptic problems. We start by discussing fractional Sobolev inequalities on the homogeneous Lie groups. As a consequence of these inequalities, we derive the existence results for the nonlinear problem with fractional p-sub-Laplacian operator and concave-convex nonlinearities and sign-changing weight functions. We also extend this analysis to equations of fractional p-sub-Laplacians and to Riesz type potential operators.

To the best of our knowledge there is no work for fractional p-sub-Laplacian operator with convex-concave type nonlinearity and sign changing weight functions on the homogeneous Lie groups. We have the following existence result.

Theorem 1.1. *Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension Q, and let $s \in (0,1)$, $Q > sp$, $1 < \alpha < p < \beta < p_s^*$ and $f \in L^{\frac{p_s^*}{p_s^*-\alpha}}(\Omega)$, $h \in L^{\frac{p_s^*}{p_s^*-\beta}}(\Omega)$, $f^{\pm} = \max\{\pm f, 0\} \neq 0$, $h^{\pm} = \max\{\pm h, 0\} \neq 0$. Then there exists $\Lambda_* > 0$ such that the equation (1.1) admits at least two non-negative solutions for $\lambda \in (0, \Lambda_*)$.*

The paper is organized as follows: In Section 2, we study the properties of the Sobolev spaces $W_g^{s,p}(\mathbb{G})$ and E_g^0 on homogeneous groups. In Section 3, we introduce Nehari manifold and study the behavior of Nehari manifold by carefully analyzing the associated fibering maps on homogeneous Lie groups. Section 4 contains the existence of nontrivial solutions in $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$.

2. **Functional analytic settings on homogeneous Lie groups.** In this section we discuss nilpotent Lie algebras and groups in the spirt of Folland and Stein's book [11] as well as introducing homogeneous Lie groups. For more analyses and details in this direction we refer to the recent open access book [10] and [1, 5, 3, 7, 8, 9, 18] and references therein.

Let $\mathfrak{g}$ be a real and finite-dimensional Lie algebra, and let $\mathbb{G}$ be the corresponding connected and simply-connected Lie group. The lower central series of $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}_{(1)} := \mathfrak{g}, \qquad \mathfrak{g}_{(j)} := [\mathfrak{g}\,,\, \mathfrak{g}_{(j-1)}].$$

If $\mathfrak{g}_{(s+1)} = \{0\}$ and $\mathfrak{g}_{(s)} \neq \{0\}$, then $\mathfrak{g}$ is said to be nilpotent of step s. A Lie group $\mathbb{G}$ is nilpotent of step s whenever its Lie algebra is nilpotent of step s.

Let $\exp : \mathfrak{g} \to \mathbb{G}$ be a exponential map, and $\mathbb{G}$ be a connected and simply-connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then, exponential map exp is a diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$. Let A be a diagonalisable linear operator on $\mathfrak{g}$ with positive eigenvalues, and $\mu > 0$. Define the mappings are of the form

$$\delta_\mu = \exp(A \log \mu) = \sum_{k=0}^{\infty} (A \log \mu)^k.$$

Then, let us give a family of dilations of a Lie algebra $\mathfrak{g}$ as follow

$$\{\delta_\mu : \mu > 0\},$$

which satisfies:

(i) δ_μ is a morphism of the Lie algebra $\mathfrak{g}$, that is, a linear mapping from $\mathfrak{g}$ to itself which respects to the Lie bracket:

$$[\delta_\mu X, \delta_\mu Y] = \delta_\mu [X, Y], \quad \forall\, X,\, Y \in \mathfrak{g}, \;\; \lambda > 0.$$

(ii) $\delta_{\mu_1 \mu_2} = \delta_{\mu_1} \delta_{\mu_2}$ for all $\mu_i > 0$, $i = 1, 2$. If $k > 0$ and $\{\delta_\mu\}$ is a family of dilations on $\mathfrak{g}$, then so is $\{\widetilde{\delta}_\mu\}$, where $\widetilde{\delta}_\mu = \delta_{\mu^k} = \exp(kA \log \mu)$.

Remark 2.1. *(i)* If a Lie algebra $\mathfrak{g}$ admits a family of dilations, then it is nilpotent, but not all nilpotent Lie algebras admit a dilation structure.

(ii) Since the exponential mapping exp is a global diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$, it induces the corresponding family on $\mathbb{G}$ which we may still call the dilations on $\mathbb{G}$ and denote by δ_μ. Thus, for $x \in \mathbb{G}$ we will write $\delta_\mu(x)$ or abbreviate it writing simply μx, and the origin of $\mathbb{G}$ will be usually denoted by 0.

Definition 2.1. Let δ_μ be a dilations on $\mathbb{G}$. We say that a Lie group $\mathbb{G}$ is a homogeneous Lie group if:

(a) It is a connected and simply-connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra g is endowed with a family of dilations $\{\delta_\mu\}$.
(b) The maps $\exp \circ \delta_\mu \circ \exp^{-1}$ are group automorphism of $\mathbb{G}$.

Now, we give some two examples of homogeneous groups.

Example 2.1. The Euclidean space $\mathbb{R}^N$ is a homogeneous group with dilation given by the scalar multiplication.

Example 2.2. If N is a positive integer, the Heisenberg group $\mathbb{H}^N$ is the group whose underlying manifold is $\mathbb{C}^N \times \mathbb{R}$ and whose multiplication is given by

$$(z,t) \circ (\tilde{z},\tilde{t}) = (z+\tilde{z}, t+\tilde{t}+2Im\langle z,\tilde{z}\rangle),$$

where $(z,t) = (z_1,\cdots,z_N,t) = (x_1,y_1,\cdots,x_N,y_N,t) \in \mathbb{H}^N$, $x \in \mathbb{R}^N$ $y \in \mathbb{R}^N$ and $t \in \mathbb{R}$. The Heisenberg group $\mathbb{H}^N$ is a homogeneous group with dilations

$$\delta_\mu(z_1,\cdots,z_N,t) = (\mu z_1,\cdots,\mu z_N,\mu^2 t).$$

The mappings $\{\delta_\mu\}$ give the dilation structure to an N-dimensional homogeneous Lie group $\mathbb{G}$ with

$$\delta_\mu(x_1,\cdots,x_N) = (\mu^{d_1}x_1,\cdots,\mu^{d_N}x_N),$$

where $(x_1,\cdots,x_N)$ are the exponential coordinates of $x \in \mathbb{G}$, $d_j \in \mathbb{N}$ for every $j = 1,2,\cdots,N$ and $1 = d_1 = \cdots = d_m < d_{m+1} \le \cdots \le d_N$ for $m := \dim(V_1)$. Here the group $\mathbb{G}$ and the algebra $\mathfrak{g}$ are identified through the exponential mapping.

It is customary to denote with $Q := \sum_{i=1}^k i\cdot\dim(V_i)$ the homogeneous dimension of $\mathbb{G}$ which corresponds to the Hausorff dimension of $\mathbb{G}$. From now on Q will always denote the homogneous dimension of $\mathbb{G}$. For example, the homogeneous dimension of Heisenberg group $\mathbb{H}^N$ is $Q := 2N+2$.

Now, we define a homogeneous quasi-norm on a homogeneous Lie group $\mathbb{G}$ to be a continuous function $g : \mathbb{G} \to \mathbb{R}^+$ with the following properties:

(i) $g(x) = 0$ if and only if $x = 0$ for every $x \in \mathbb{G}$;
(ii) $g(x^{-1}) = g(x)$ for every $x \in \mathbb{G}$;
(iii) $g(\delta_\mu(x)) = \mu g(x)$ for every $\mu \in \mathbb{R}^+$ and for every $x \in \mathbb{G}$.

For any measurable set $E \subset \mathbb{G}$, we have $|\delta_\mu(E)| = \mu^Q|E|$, $d(\delta_\mu(x)) = \mu^Q dx$, where $|E|$ denotes the measure of the set E. Let

$$\mathcal{B}_g(x,r) = \{y \in \mathbb{G} : g(x^{-1}\circ y) < r\}$$

be the quasi-ball of radius $r > 0$ about x with respect to the homogeneous quasi-norm g. Then, we have that

$$|\mathcal{B}_g(x,r)| = r^Q, \quad \forall x \in \mathbb{G}.$$

It can be noticed that $\mathcal{B}_g(x,r)$ is the left translate by x of $\mathcal{B}_g(0,r)$, which in turn is the image under δ_r of $\mathcal{B}_g(0,1)$. Moreover, let

$$\mathcal{S}_g(0) := \{x \in \mathbb{G} :\ g(x) = 1\}$$

be the unit sphere with respect to the homogeneous quasi-norm g. Then there is a unique positive Radon measure σ on $\mathcal{S}_g(0)$ such that for all $f \in L^1(\mathbb{G})$, we have

$$\int_{\mathbb{G}} f(x)dx = \int_0^{\infty} \int_{\mathcal{S}_g(0)} f(ry)r^{Q-1}d\sigma(y)dr.$$

Let $\mathbb{G}$ be a homogeneous Lie group, with its basis $X_1, \ . \ . \ . \ , X_N$, generating its Lie algebra $\mathfrak{g}$ through their commutators. Then, the sub-Laplacian operator is defined as

$$\mathcal{L} := X_1^2 + \cdots + X_N^2.$$

In the sequel, we use the following notations for the horizontal gradient

$$\nabla_{\mathbb{G}} := (X_1, X_2, \cdots, X_N),$$

and for the horizontal divergence

$$\mathrm{div}_{\mathbb{G}} v := \nabla_{\mathbb{G}} u \cdot v.$$

Using the Green's first and second formulae, we can define the p-sub-Laplacian on homogeneous groups $\mathbb{G}$ as

$$\mathcal{L}_p u = \mathrm{div}_{\mathbb{G}}(|\nabla_{\mathbb{G}} u|^{p-2} \nabla_{\mathbb{G}} u), \quad p \in (1, +\infty).$$

Recently, a great deal of attention has been focused on studying of equations or systems involving fractional Laplacian and corresponding nonlocal problems, both for their interesting theoretical structure and for their concrete applications, see [6, 17, 19, 20] and references therein. The fractional p-Laplacian operator $(-\Delta_p)^s$, $s \in (0, 1)$, is defined as

$$(-\Delta_p)^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+ps}} dy, \ \forall x \in \mathbb{R}^n.$$

This type of operator arises in a quite natural way in many different contexts, such as, the thin obstacle problem, finance, phase transitions, anomalous diffusion, flame propagation and many others.

Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension Q, $p > 1$ and $s \in (0, 1)$. Compared to the fractional p-Laplacian problem, the fractional p-sub-Laplacian $(-\Delta_{p,g})^s$ on $\mathbb{G}$ can be defined as

$$(-\Delta_{p,g})^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{G} \setminus \mathcal{B}_g(x,\varepsilon)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{g(y^{-1} \circ x)^{Q+sp}} dy, \ \forall x \in \mathbb{G},$$

where g is a quasi-norm on $\mathbb{G}$ and $\mathcal{B}_g(x, \varepsilon)$ is a quasi-ball with respect to g, with radius ε centered at $x \in \mathbb{G}$.

Now we recall the definitions of the fractional Sobolev spaces on homogeneous Lie groups $\mathbb{G}$. For a measurable function $u : \mathbb{G} \to \mathbb{R}$ we define the Gagliardo quasi-seminorm by

$$[u]_{s,p,g} = \left(\int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u(x) - u(y)|^p}{g(y^{-1} \circ x)^{Q+sp}} dxdy \right)^{\frac{1}{p}}.$$

For $p > 1$ and $s \in (0, 1)$, we introduce the the functional Sobolev space on homogeneous Lie groups $\mathbb{G}$ by

$$W_g^{s,p}(\mathbb{G}) = \{u \in L^p(\mathbb{G}) \,|\, u \text{ is measurable and } [u]_{s,p,g} < +\infty\},$$

and endowed with the norm

$$\|u\|_{W_g^{s,p}(\mathbb{G})} = \|u\|_{L^p(\mathbb{G})} + [u]_{s,p,g}.$$

Similarly, if $\Omega \subset \mathbb{G}$ is a Haar measurable set, we define the Sobolev space

$$W_g^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \begin{array}{l} u \text{ is measurable and} \\ \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy \right)^{\frac{1}{p}} < +\infty \end{array} \right\},$$

endowed with norm

$$\|u\|_{W_g^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy \right)^{\frac{1}{p}}. \tag{2.1}$$

In [16, Theorem 2], Kassymov and Suragan given the following analogue of the fractional Sobolev inequality on homogeneous groups $\mathbb{G}$.

Theorem 2.1. *Let $\mathbb{G}$ be a homogeneous group with homogeneous dimension Q. Assume that $p > 1$, $s \in (0,1)$, $Q > sp$ and g denotes a quasi-norm on $\mathbb{G}$. Then, for any measurable and compactly supported function $u : \mathbb{G} \to \mathbb{R}$, there exists a positive constant $C = C(Q,p,s) > 0$ such that*

$$\|u\|^p_{L^{p_s^*}(\mathbb{G})} \le C\,[u]^p_{s,p,g} = C \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy,$$

where $p_s^* := p^*(Q,s,p) = \frac{Qp}{Q-sp}$ is the fractional critical Sobolev exponents on homogeneous group.

Let $\mathcal{Q} = \mathbb{G}^2 \backslash (\mathcal{C}_\Omega \times \mathcal{C}_\Omega)$ and $\mathcal{C}_\Omega = \mathbb{G}\backslash\Omega$. We define the Sobolev space

$$E_g = \left\{ u \;\middle|\; \begin{array}{l} u : \mathbb{G} \to \mathbb{R} \text{ is measurable, } u|_\Omega \in L^p(\Omega) \\ \text{and } \iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy < +\infty \end{array} \right\}$$

endowed with the norm as following

$$\|u\|_{E_g} = \|u\|_{L^p(\Omega)} + \left(\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy \right)^{\frac{1}{p}}. \tag{2.2}$$

Indeed, if $\|u\|_{E_g} = 0$, we get that $\|u\|_{L^p(\Omega)} = 0$ and $\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy = 0$. Then, the above equalities imply that $u = 0$ a.e. in Ω and $u(x) = u(y) = const.$ a.e. in $\mathcal{Q}$. So, we can get that $u = 0$ a.e. in $\mathbb{G}$ and $\|\cdot\|_{E_g}$ is a norm on E_g.

Let $E_g^0 = \{u \in E_g : u = 0 \text{ in } \mathbb{G}\backslash\Omega\}$ be a subspace of E_g. Then, for any $p > 1$, E_g^0 is a Banach space and have the following properties.

Lemma 2.1. *The following hold.*

(i) If $u \in E_g$, then $u \in W_g^{s,p}(\Omega)$ and $\|u\|_{W_g^{s,p}(\Omega)} \le \|u\|_{E_g}$.
(ii) If $u \in E_g^0$, then $u \in W_g^{s,p}(\mathbb{G})$ and $\|u\|_{W_g^{s,p}(\Omega)} \le \|u\|_{W_g^{s,p}(\mathbb{G})} = \|u\|_{E_g}$.

Proof. (i) Let $u \in E_g$, since $\Omega \times \Omega$ is strictly contained in $\mathcal{Q}$, we have

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy \le \iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy < \infty. \tag{2.3}$$

Thus $\|u\|_{W_g^{s,p}(\Omega)} \le \|u\|_{E_g}$ and deduced the result (i).

(ii) For each $u \in E_g^0$, we get $u = 0$ on $\mathbb{G} \setminus \Omega$. Hence, $\|u\|_{L^2(\mathbb{G})} = \|u\|_{L^2(\Omega)}$ and

$$\int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy = \iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy < +\infty, \tag{2.4}$$

which and (2.3) yield the result (ii). $\square$

Theorem 2.2. *Let $\mathbb{G}$ be a homogeneous group with homogeneous dimension Q. Assume that $p>1$, $s\in(0,1)$, $Q>sp$ and g be a quasi-norm on $\mathbb{G}$. Then for every $u\in E_g^0$ there exists a positive constant $c=c(Q,s)>0$ depending on Q and s such that*

$$\|u\|^p_{L^{p_s^*}(\Omega)}=\|u\|^p_{L^{p_s^*}(\mathbb{G})}\le c\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}}dxdy.$$

Proof. For any $u\in E_g^0$, by Lemma 2.1 (ii) and Theorem 2.1, we know that $u\in W_g^{s,p}(\mathbb{G})$ and $W_g^{s,p}(\mathbb{G})\hookrightarrow L^{p_s^*}(\mathbb{G})$. Then, we have

$$\|u\|^p_{L^{p_s^*}(\Omega)}=\|u\|^p_{L^{p_s^*}(\mathbb{G})}\le c\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}}dxdy,$$

and completes the proof of Theorem 2.2. □

Lemma 2.2. *The space $(E_g^0,\|\cdot\|_{E_g^0})$ is a reflexive Banach space.*

Proof. Let $\{u_k\}_k\subset E_g^0$ be a Cauchy sequence. By Lemma 2.1 and Theorem 2.2, $\{u_k\}_k$ is Cauchy sequence in $L^p(\Omega)$ and so $\{u_k\}_k$ has a convergent subsequence. We assume $u_k\to u$ in $L^p(\Omega)$. Since $u_k=0$ in $\mathbb{G}\setminus\Omega$, we define $u=0$ in $\mathbb{G}\setminus\Omega$ and then $u_k\to u$ strongly in $L^p(\mathbb{G})$ as $k\to\infty$. So, there exists a subsequence of $\{u_k\}_k$, still denoted by $\{u_k\}_k$, such that $u_k\to u$ a.e. in $\mathbb{G}$. By Fatou's Lemma and using the fact that $\{u_k\}$ is a Cauchy sequence, we get that $u\in E_g^0$ and $\|u_k-u\|_{E_g}\to 0$ as $k\to+\infty$. Hence E_g^0 is a Banach space. Reflexivity of E_g^0 follows from the fact that E_g^0 is a closed subspace of reflexive Banach space $W_g^{s,p}(\mathbb{G})$. □

From above results, we can defined the following scalar product

$$\langle u,v\rangle_{E_g^0}=\iint_{\mathcal{Q}}\frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}}dxdy, \tag{2.5}$$

and norm

$$\|u\|_{E_g^0}=\left(\iint_{\mathcal{Q}}\frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}}dxdy\right)^{\frac{1}{p}} \tag{2.6}$$

for the reflexive Banach space E_g^0. Since $u=0$ a.e. in $\mathbb{G}\backslash\Omega$, we note that the (2.5) and (2.6) can be extended to all $\mathbb{G}$. Moreover, for any $u\in E_g$, by Theorem 2.2 and the embedding $L^{p_s^*}(\Omega)\hookrightarrow L^p(\Omega)$, there exist C_1 and $C_2>0$ such that

$$C_1\iint_{\mathcal{Q}}\frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}}dxdy\le\|u\|^p_{E_g}\le C_2\iint_{\mathcal{Q}}\frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}}dxdy. \tag{2.7}$$

This imply that the norm $\|\cdot\|_{E_g^0}$ on E_g^0 is equivalent to the norm $\|\cdot\|_{E_g}$ on E_g, and the norm $\|\cdot\|_{E_g^0}$ involves the interaction between Ω and $\mathbb{G}\setminus\Omega$. But the norms in (2.1) and (2.2) are not same because $\Omega\times\Omega$ is strictly contained in $\mathcal{Q}$.

Lemma 2.3. *Let $\{u_k\}_k$ be a bounded sequence in E_g^0. Then, there exists $u\in L^q(\mathbb{G})$ such that $u_k\to u$ in $L^q(\mathbb{G})$ as $k\to\infty$ for any $q\in[1,p_s^*)$.*

Proof. Let $\{u_k\}_k$ is bounded in E_g^0, by Lemmas 2.1 and (2.7), $\{u_k\}_k$ is bounded in $W_g^{s,p}(\Omega)$ and also in $L^p(\Omega)$. Then by assumption on Ω and [4, Corollary 7.2], there exists $u\in L^q(\Omega)$ such that up to a subsequence $u_k\to u$ in $L^q(\Omega)$ as $k\to\infty$ for any $q\in[1,p_s^*)$. Since $u_k=0$ on $\mathbb{G}\setminus\Omega$, we can define $u:=0$ in $\mathbb{G}\setminus\Omega$ and we get $u_k\to u$ in $L^q(\mathbb{G})$. □

From Theorem 2.2, Lemma 2.1 and Lemma 2.3, we have that the embedding $E_g^0 \hookrightarrow L^q(\Omega)$ is continuous for any $q \in [1, p_s^*]$ and compact whenever $q \in [1, p_s^*)$. Let $S_{p_s^*}$ be the best constant for the embedding of $E_g^0 \hookrightarrow L^{p_s^*}(\Omega)$ defined by

$$S_{p_s^*} = \inf_{u \in E_g^0 \setminus \{0\}} \frac{\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy}{(\int_\Omega |u|^{p_s^*} dx)^{\frac{p}{p_s^*}}}.$$

3. **The fibering properties.** Since the energy functional I_λ is not bounded below on the space E_g^0, we consider the Nehari minimization problem: for $\lambda > 0$,

$$c_\lambda = \inf\{I_\lambda(u) : \ u \in \mathcal{N}_\lambda\},$$

where

$$\mathcal{N}_\lambda := \left\{u \in E_g^0 \setminus \{0\} : \langle I'_\lambda(u), u\rangle = 0\right\}.$$

For any $u \in \mathcal{N}_\lambda$, the following equality hold

$$\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy = \lambda \int_\Omega f(x)|u|^\alpha dx + \int_\Omega h(x)|u|^\beta dx, \tag{3.1}$$

which implies that $\mathcal{N}_\lambda$ contains all nonzero solutions of equation (1.1). Moreover, we have the following result.

Lemma 3.1. *I_λ is coercive and bounded below on $\mathcal{N}_\lambda$ for all $\lambda > 0$.*

Proof. Let $\lambda > 0$ and for all $u \in \mathcal{N}_\lambda$, from (3.1) and Theorem 2.2 there holds

$$\begin{aligned} I_\lambda(u) &= \frac{1}{p}\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy - \frac{\lambda}{\alpha}\int_\Omega f(x)|u|^\alpha dx - \frac{1}{\beta}\int_\Omega h(x)|u|^\beta dx \\ &= \Big(\frac{1}{p} - \frac{1}{\beta}\Big)\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy - \lambda\Big(\frac{1}{\alpha} - \frac{1}{\beta}\Big)\int_\Omega f(x)|u|^\alpha dx \\ &\geq \frac{\beta - p}{p\beta}\|u\|_{E_g^0}^p - \frac{\lambda(\beta-\alpha)}{\alpha\beta}\|f\|_{\frac{p_s^*}{p_s^*-\alpha}} S_{p_s^*}^{-\frac{\alpha}{p}}\|u\|_{E_g^0}^\alpha, \end{aligned} \tag{3.2}$$

which yields that I_λ is bounded below and coercive on $\mathcal{N}_\lambda$ since $\beta > p > \alpha$ and $S_{\alpha_s^*} > 0$. This completes the proof of Lemma 3.1. □

Define

$$\Phi_\lambda(u) = \langle I'_\lambda(u), u\rangle, \quad \forall\, u \in E_g^0 \setminus \{0\}.$$

Then, we see that $\Phi_\lambda \in C^1(E_g^0, \mathbb{R})$, $\mathcal{N}_\lambda = \Phi_\lambda^{-1}(0)\setminus\{0\}$, and for all $u \in \mathcal{N}_\lambda$ we get that

$$\begin{aligned} \langle \Phi'_\lambda(u), u\rangle &= p\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{g(y^{-1}\circ x)^{Q+sp}} dxdy - \lambda\alpha\int_\Omega f(x)|u|^\alpha dx - \beta\int_\Omega h(x)|u|^\beta dx \\ &= (p-\alpha)\|u\|_{E_g^0}^p - (\beta-\alpha)\int_\Omega h(x)|u|^\beta dx \\ &= (p-\beta)\|u\|_{E_g^0}^p + (\beta-\alpha)\lambda\int_\Omega f(x)|u|^\alpha dx. \end{aligned} \tag{3.3}$$

We split $\mathcal{N}_\lambda$ into three parts

$$\begin{aligned} \mathcal{N}_\lambda^+ &:= \{u \in \mathcal{N}_\lambda : \langle \Phi'_\lambda(u), u\rangle > 0\}; \\ \mathcal{N}_\lambda^0 &:= \{u \in \mathcal{N}_\lambda : \langle \Phi'_\lambda(u), u\rangle = 0\}; \\ \mathcal{N}_\lambda^- &:= \{u \in \mathcal{N}_\lambda : \langle \Phi'_\lambda(u), u\rangle < 0\}. \end{aligned} \tag{3.4}$$

On $\mathcal{N}_\lambda^0$, the following result hold.

Lemma 3.2. *For each $\lambda > 0$, let u_0 be a local minimizer for I_λ on $\mathcal{N}_\lambda \backslash \mathcal{N}_\lambda^0$, then u_0 is a critical point of I_λ.*

Proof. Since u_0 is a local minimizer for I_λ on $\mathcal{N}_\lambda$, that is, u_0 is a solution of the optimization problem

$$\min\{I_\lambda(u) : u \in \mathcal{N}_\lambda\} = \min\{I_\lambda(u) : \Phi_\lambda(u) = 0\}.$$

Then, by the theory of Lagrange multipliers, there exists a constant $\theta \in \mathbb{R}$ such that

$$\langle I'_\lambda(u_0), u_0\rangle = \theta\langle \Phi'_\lambda(u_0), u_0\rangle.$$

Since $u_0 \notin \mathcal{N}_\lambda^0$, we have $\langle \Phi'_\lambda(u_0), u_0\rangle \neq 0$, thus $\theta = 0$, this completes the proof. $\square$

Remark 3.1. Lemmas 3.1 and 3.2 imply that the functional I_λ is bounded below on an appropriate subset of E_g^0 and the minimizers of functional I_λ on subsets $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^-$ giving raise to solutions of (1.1).

Now, for $t > 0$, define the fibering maps $\phi_u : t \mapsto I_\lambda(tu)$ as

$$\phi_u(t) = \frac{t^p}{p} \iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^p}{g(y^{-1} \circ x)^{Q+sp}} dxdy - \frac{\lambda t^\alpha}{\alpha} \int_\Omega f(x)|u|^\alpha dx - \frac{t^\beta}{\beta} \int_\Omega h(x)|u|^\beta dx. \tag{3.5}$$

We note that

$$\begin{aligned}
\phi'_u(t) &= t^{p-1} \iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^p}{g(y^{-1} \circ x)^{Q+sp}} dxdy - \lambda t^{\alpha-1} \int_\Omega f(x)|u|^\alpha dx - t^{\beta-1} \int_\Omega h(x)|u|^\beta dx \\
&= \frac{1}{t}\left(\iint_{\mathcal{Q}} \frac{|tu(x) - tu(y)|^p}{g(y^{-1} \circ x)^{Q+sp}} dxdy - \lambda \int_\Omega f(x)|tu|^\alpha dx - \int_\Omega h(x)|tu|^\beta dx \right) \\
&= \frac{1}{t}\langle I'_\lambda(tu), tu\rangle.
\end{aligned}$$

This gives that $tu \in \mathcal{N}_\lambda$ if and only if $\phi'_u(t) = 0$ and in particular, $u \in \mathcal{N}_\lambda$ if and only if $\phi'_u(1) = 0$. Moreover, for each $u \in \mathcal{N}_\lambda$, from (3.1), (3.3) and (3.5) we get that

$$\begin{aligned}
\phi''_u(1) &= (p-1) \iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^p}{g(y^{-1} \circ x)^{Q+sp}} dxdy - \lambda(\alpha - 1) \int_\Omega f(x)|u|^\alpha dx \\
&\quad - (\beta - 1) \int_\Omega h(x)|u|^\beta dx \\
&= \langle \Phi'_\lambda(u), u\rangle - \langle I'_\lambda(u), u\rangle = \langle \Phi'_\lambda(u), u\rangle,
\end{aligned}$$

which and (3.4) yield that $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^0$ are corresponding to local minima, local maxima and points of inflection of $\phi_u(t)$, namely

$$\begin{aligned}
\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : \phi''_u(1) > 0\}, \\
\mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : \phi''_u(1) < 0\}, \\
\mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : \phi''_u(1) = 0\}.
\end{aligned}$$

In order to understand the Nehari manifold and the fibering maps, we consider the function $m_u : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$m_u(t) = t^{p-\alpha} \iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^p}{g(y^{-1} \circ x)^{Q+sp}} dxdy - t^{\beta-\alpha} \int_\Omega h(x)|u|^\beta dx.$$

Clearly, for any $t > 0$,

$$\phi'_u(t) = t^{\alpha-1}\Big(m_u(t) - \lambda \int_\Omega f(x)|u|^\alpha dx\Big). \tag{3.6}$$

Thus, we obtain that

$$\begin{aligned} tu \in \mathcal{N}_\lambda &\Longleftrightarrow \phi_u'(t) = 0 \\ &\Longleftrightarrow t \text{ is a solution of equation } m_u(t) = \lambda \int_\Omega f(x)|u|^\alpha dx. \end{aligned} \tag{3.7}$$

From the expression (3.5) of ϕ_u, we see that the behavior of the fibering maps ϕ_u according to the sign of $\int_\Omega f(x)|u|^\alpha dx$ and $\int_\Omega h(x)|u|^\beta dx$, then we will study the following four cases.

Case 1: $\int_\Omega f(x)|u|^\alpha dx < 0$ and $\int_\Omega h(x)|u|^\beta dx < 0$. In this case, we have that $m_u(t) > 0$ for all $t > 0$, and the equation

$$m_u(t) = \lambda \int_\Omega f(x)|u|^\alpha dx \tag{3.8}$$

has no solution for all $\lambda > 0$. See Fig.1.

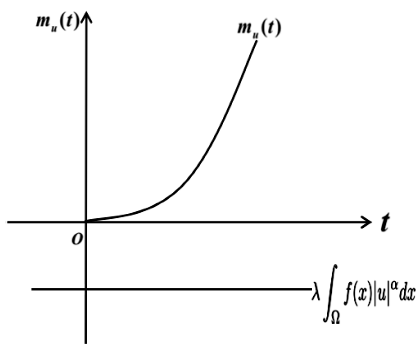

Fig. 1

Case 2: $\int_\Omega f(x)|u|^\alpha dx > 0$ and $\int_\Omega h(x)|u|^\beta dx < 0$. Since $\int_\Omega h(x)|u|^\beta dx < 0$, we have $m_u(t) > 0$ for all $t > 0$, and $\lim\limits_{t\to+\infty} m_u(t) \to +\infty$. Moreover, for all $t > 0$,

$$m_u'(t) = (p-\alpha)t^{p-\alpha-1}\|u\|^p_{E_g^0} - (\beta-\alpha)t^{\beta-\alpha-1}\int_\Omega h(x)|u|^\beta dx > 0.$$

This gives that m_u is strictly increasing on $(0, +\infty)$. Then, there exists a unique $t_1 = t_1(u) > 0$ such that $m_u(t_1) = \lambda \int_\Omega f(x)|u|^\alpha dx$, see Fig.2. Moreover, for all $t \in (0, t_1)$, $m_u(t) < \lambda \int_\Omega f(x)|u|^\alpha dx$, and for all $t \in (t_1, +\infty)$, $m_u(t) > \lambda \int_\Omega f(x)|u|^\alpha dx$. So, together with (3.6) we have that $\phi_u(t)$ is decreasing on $(0, t_1)$, increasing on (t_1, ∞) and $\phi_u'(t_1) = 0$. Those imply that the function ϕ_u has exactly one critical point $t_1 > 0$ such that $t_1 u \in \mathcal{N}_\lambda^+$, that is, $t_1 > 0$ is a global minimum point of function ϕ_u, see Fig.3.

Case 3: $\int_\Omega f(x)|u|^\alpha dx < 0$ and $\int_\Omega h(x)|u|^\beta dx > 0$. Since $\beta - \alpha > p - \alpha$ and $\int_\Omega h(x)|u|^\beta dx > 0$, we have $m_u(t) \to -\infty$ as $t \to +\infty$. Moreover, from $m_u(0) = 0$ and $\lim_{t\to 0^+} m_u'(t) > 0$, we get that the problem (3.8) has a unique solution $t_2 = t_2(u) > 0$ for each u, see Fig. 4. Similarly as Case 2, we obtain that $\phi_u'(t) > 0$ for all $t \in (0, t_2)$, and $\phi_u'(t) < 0$ for all $t \in (t_2, +\infty)$, which yields that ϕ_u is increasing on $(0, t_2)$, decreasing on (t_2, ∞). So, $t_2 > 0$ is the global maximum point of ϕ_u such that $t_2 u \in \mathcal{N}_\lambda^-$, see Fig.5.

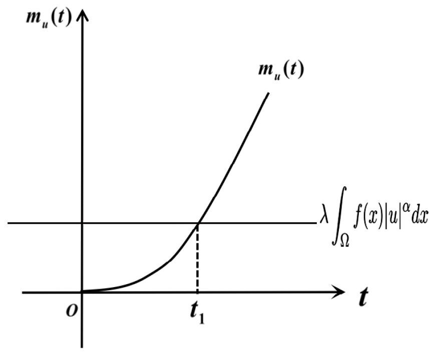

Fig. 2

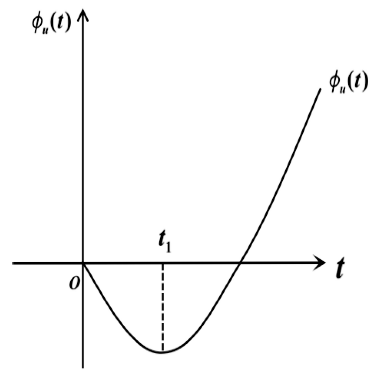

Fig. 3

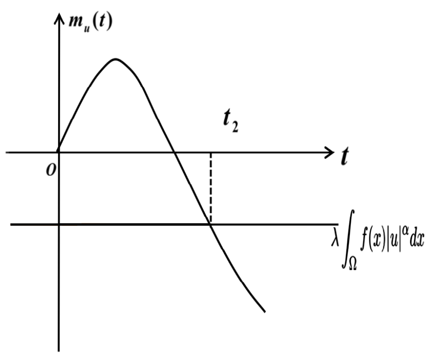

Fig. 4

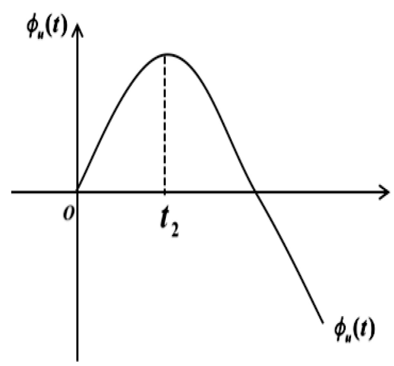

Fig.5

Case 4: $\int_\Omega f(x)|u|^\alpha dx > 0$ and $\int_\Omega h(x)|u|^\beta dx > 0$. Similarly as Case 3, we get that $m_u(0) = 0$, $m_u(t) \to -\infty$ as $t \to +\infty$ and there exists

$$t_{\max} = \left(\frac{(p-\alpha)\|u\|^p_{E^0_g}}{(\beta-\alpha)\int_\Omega h(x)|u|^\beta dx} \right)^{\frac{1}{\beta-p}}$$

such that m_u is increasing on $(0, t_{\max})$, decreasing on $(t_{\max}, \infty)$ and the function m_u attains its maximum value at $t_{\max}$. See Fig.6. Moreover,

$$m_u(t_{\max}) = \left[\left(\frac{p-\alpha}{\beta-\alpha}\right)^{\frac{p-\alpha}{\beta-p}} - \left(\frac{p-\alpha}{\beta-\alpha}\right)^{\frac{\beta-\alpha}{\beta-p}} \right] \frac{\|u\|^{\frac{p(\beta-\alpha)}{\beta-p}}_{E^0_g}}{(\int_\Omega h(x)|u|^\beta dx)^{\frac{p-\alpha}{\beta-p}}}.$$

If $0 < \lambda \int_\Omega f(x)|u|^\alpha dx < m_u(t_{\max})$, there exist $t_3 := t_3(u)$, $t_4 := t_4(u) > 0$ such that $t_3 < t_{\max} < t_4$ and

$$m_u(t_3) = \lambda \int_\Omega f(x)|u|^\alpha dx = m_u(t_4).$$

Then, $\phi'_u(t) < 0$ for all $t \in (0, t_3)$, $\phi'_u(t) > 0$ for all $t \in (t_3, t_4)$, and $\phi'_u(t) < 0$ for all $t \in (t_4, +\infty)$, that is, ϕ_u is decreasing on $(0, t_3)$, increasing on (t_3, t_4), and decreasing on $(t_4, +\infty)$. So, under the condition of $0 < \lambda \int_\Omega f(x)|u|^\alpha dx < m_u(t_{\max})$, there exist $t_3 < t_{\max} < t_4$ that ϕ_u has a local minimum point $t_3 > 0$ and a local maximum point $t_4 > 0$ with $t_3 u \in \mathcal{N}_\lambda^+$ and $t_4 u \in \mathcal{N}_\lambda^-$. See Fig.7.

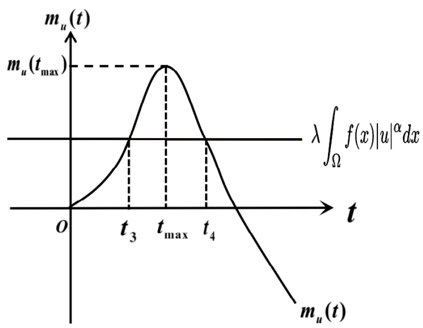

Fig. 6

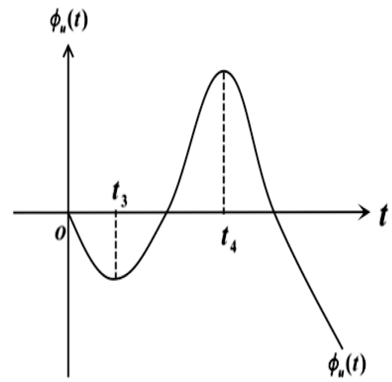

Fig. 7

From the above proof of Case 4, we have the following result.

Lemma 3.3. *For some $u \in E_g^0\backslash\{0\}$ with $\int_\Omega f(x)|u|^\alpha dx > 0$ and $\int_\Omega h(x)|u|^\beta dx > 0$. Then, there exists a positive constant $\lambda_0 > 0$ such that for any $0 < \lambda < \lambda_0$, there are unique τ_1, $\tau_2 > 0$ such that $\tau_1 < t_{max} < \tau_2$ and $\tau_1 u \in \mathcal{N}_\lambda^+$ and $\tau_2 u \in \mathcal{N}_\lambda^-$. Moreover,*

$$I_\lambda(\tau_1 u) = \min_{0\le t\le t_{max}(u)} I_\lambda(u), \quad I_\lambda(\tau_2 u) = \max_{t\ge 0} I_\lambda(u).$$

Proof. From the proof of Case 4, we only need to prove that there exists $\lambda_0 > 0$ such that, for any $\lambda \in (0, \lambda_0)$, the following inequality holds

$$\lambda \int_\Omega f(x)|u|^\alpha dx < m_u(t_{\max}). \tag{3.9}$$

Indeed, since

$$\begin{aligned}
&\frac{m_u(t_{\max})}{\int_\Omega f(x)|u|^\alpha dx} \\
&= \left[\left(\frac{p-\alpha}{\beta-\alpha}\right)^{\frac{p-\alpha}{\beta-p}} - \left(\frac{p-\alpha}{\beta-\alpha}\right)^{\frac{\beta-\alpha}{\beta-p}}\right] \frac{\|u\|_{E_g^0}^{\frac{p(\beta-\alpha)}{\beta-p}}}{(\int_\Omega h(x)|u|^\beta dx)^{\frac{p-\alpha}{\beta-p}} \int_\Omega f(x)|u|^\alpha dx} \\
&\ge \left[\left(\frac{p-\alpha}{\beta-\alpha}\right)^{\frac{p-\alpha}{\beta-p}} - \left(\frac{p-\alpha}{\beta-\alpha}\right)^{\frac{\beta-\alpha}{\beta-p}}\right] \frac{\|u\|_{E_g^0}^{\frac{p(\beta-\alpha)}{\beta-p}}}{(\|h\|_{\frac{p_s^*}{p_s^*-\beta}} S_{p_s^*}^{-\frac{\beta}{p}} \|u\|_{E_g^0}^\beta)^{\frac{p-\alpha}{\beta-p}} \|f\|_{\frac{p_s^*}{p_s^*-\alpha}} S_{p_s^*}^{-\frac{\alpha}{p}} \|u\|_{E_g^0}^\alpha} \\
&= \left[\left(\frac{p-\alpha}{\beta-\alpha}\right)^{\frac{p-\alpha}{\beta-p}} - \left(\frac{p-\alpha}{\beta-\alpha}\right)^{\frac{\beta-\alpha}{\beta-p}}\right] \frac{S_{p_s^*}^{\frac{\beta-\alpha}{\beta-p}}}{\|f\|_{\frac{p_s^*}{p_s^*-\alpha}} \|h\|_{\frac{p_s^*}{p_s^*-\beta}}^{\frac{p-\alpha}{\beta-p}}}.
\end{aligned}$$

Thus, taking $\lambda_0 := \left[\left(\frac{p-\alpha}{\beta-\alpha}\right)^{\frac{p-\alpha}{\beta-p}} - \left(\frac{p-\alpha}{\beta-\alpha}\right)^{\frac{\beta-\alpha}{\beta-p}}\right] \frac{S_{p_s^*}^{\frac{\beta-\alpha}{\beta-p}}}{\|f\|_{\frac{p_s^*}{p_s^*-\alpha}} \|h\|_{\frac{p_s^*}{p_s^*-\beta}}^{\frac{p-\alpha}{\beta-p}}}$, the inequality (3.9) holds for all $\lambda \in (0, \lambda_0)$ and completes the proof of Lemma 3.3. $\square$

4. **Existence of solutions on $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$.** In this section, we use the results in Section 3 to prove the existence of a nontrivial solution on $\mathcal{N}_\lambda^+$, as well as on $\mathcal{N}_\lambda^-$. First, we state the following result. Let

$$\Lambda_0 = \frac{\beta-\alpha}{p-\alpha}\left[\frac{p-\alpha}{\beta-\alpha}\|h\|^{-1}_{\frac{p_s^*}{p_s^*-\beta}}\right]^{\frac{p-\alpha}{\beta-p}}\|f\|^{-1}_{\frac{p_s^*}{p_s^*-\alpha}}S_{p_s^*}^{\frac{\beta-\alpha}{\beta-p}}.$$

Lemma 4.1. *For each $0<\lambda<\Lambda_0$, we have $\mathcal{N}_\lambda^0=\emptyset$.*

Proof. We consider the following two cases.

Case I: $u\in\mathcal{N}_\lambda$ and $\int_\Omega h(x)|u|^\beta dx\le 0$. We have

$$\lambda\int_\Omega f(x)|u|^\alpha dx = \|u\|^p_{E_g^0}-\int_\Omega h(x)|u|^\beta dx>0.$$

Thus, $\langle\Phi'_\lambda(u),u\rangle=(p-\alpha)\|u\|^p_{E_g^0}-(\beta-\alpha)\int_\Omega h(x)|u|^\beta dx>0$ and so $u\notin\mathcal{N}_\lambda^0$.

Case II: $u\in\mathcal{N}_\lambda$ and $\int_\Omega h(x)|u|^\beta dx>0$. Suppose that $\mathcal{N}_\lambda^0\neq\emptyset$ for all $\lambda\in(0,\Lambda_0)$. By (3.3), for each $u\in\mathcal{N}_\lambda^0$ we have

$$\|u\|^p_{E_g^0}=\frac{(\beta-\alpha)}{p-\alpha}\int_\Omega h(x)|u|^\beta dx, \tag{4.1}$$

and

$$\|u\|^p_{E_g^0}=\lambda\frac{\beta-\alpha}{\beta-p}\int_\Omega f(x)|u|^\alpha dx. \tag{4.2}$$

Thus, (4.1), (4.2) and the Sobolev inequality imply that

$$\|u\|_{E_g^0}\ge\left[\frac{p-\alpha}{\beta-\alpha}\|h\|^{-1}_{\frac{p_s^*}{p_s^*-\beta}}S_{p_s^*}^{\frac{\beta}{p}}\right]^{\frac{1}{\beta-p}}, \tag{4.3}$$

and

$$\|u\|_{E_g^0}\le\left(\lambda\frac{p-\alpha}{\beta-\alpha}\|f\|_{\frac{p_s^*}{p_s^*-\alpha}}S_{p_s^*}^{-\frac{\alpha}{p}}\right)^{\frac{1}{p-\alpha}}. \tag{4.4}$$

where $S_{p_s^*}>0$ is the best constant of Sobolev embedding. So, (4.3) and (4.4) imply that

$$\lambda\ge\left[\frac{p-\alpha}{\beta-\alpha}\|h\|^{-1}_{\frac{p_s^*}{p_s^*-\beta}}\right]^{\frac{p-\alpha}{\beta-p}}\frac{\beta-\alpha}{p-\alpha}\|f\|^{-1}_{\frac{p_s^*}{p_s^*-\alpha}}S_{p_s^*}^{\frac{\beta-\alpha}{\beta-p}},$$

contradicting with the assumption. □

Let

$$\Lambda_*=\min\{\,\lambda_0\ ,\ \Lambda_0\,\}.$$

By Lemmas 3.1, 3.2 and 4.1, for each $\lambda\in(0,\Lambda_*)$, we get $\mathcal{N}_\lambda^+\cap\mathcal{N}_\lambda^-=\emptyset$, $\mathcal{N}_\lambda^+\cup\mathcal{N}_\lambda^-=\mathcal{N}_\lambda$, and the energy functional I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$, $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$. Then, the Ekeland variational principle implies that I_λ has a minimizing sequence on each manifold of $\mathcal{N}_\lambda$, $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$. Define

$$c_\lambda=\inf\{I_\lambda(u):\ u\in\mathcal{N}_\lambda\};$$
$$c_\lambda^+=\inf\{I_\lambda(u):\ u\in\mathcal{N}_\lambda^+\};$$
$$c_\lambda^-=\inf\{I_\lambda(u):\ u\in\mathcal{N}_\lambda^-\}.$$

Lemma 4.2. *The following facts hold.*

(i) *If $\lambda\in(0,\Lambda_*)$, then we have $c_\lambda\le c_\lambda^+<0$.*

(ii) *If* $\lambda \in (0, \Lambda_*)$, *then we have* $c_\lambda^- > c_0$ *for some positive constant* c_0 *depending on* $\lambda, p, s, \alpha, \beta$ *and* $S_{p_s^*}$.

Proof. (i) Let $u \in \mathcal{N}_\lambda^+$, by (3.3) we have

$$\frac{p-\alpha}{\beta-\alpha}\|u\|_{E_g^0}^p > \int_\Omega h(x)|u|^\beta dx.$$

Hence

$$\begin{aligned} I_\lambda(u) &= \left(\frac{1}{p}-\frac{1}{\alpha}\right)\|u\|_{E_g^0}^p + \left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\int_\Omega h(x)|u|^\beta dx \\ &\le \left[\left(\frac{1}{p}-\frac{1}{\alpha}\right) + \left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{p-\alpha}{\beta-\alpha}\right]\|u\|_{E_g^0}^p \\ &\le \frac{(\alpha-p)(\beta-p)}{p\alpha\beta}\|u\|_{E_g^0}^p < 0. \end{aligned}$$

From this inequality and the definition of c_λ, c_λ^+, we deduce that $c_\lambda \le c_\lambda^+ < 0$.

(ii) Let $u \in \mathcal{N}_\lambda^-$, from (3.3) we have

$$\frac{p-\alpha}{\beta-\alpha}\|u\|_{E_g^0}^p < \int_\Omega h(x)|u|^\beta dx. \tag{4.5}$$

Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\int_\Omega h(x)|u|^\beta dx \le \|h\|_{\frac{p_s^*}{p_s^*-\beta}} S_{p_s^*}^{-\frac{\beta}{p}}\|u\|_{E_g^0}^\beta,$$

which and (4.5) imply that

$$\|u\|_{E_g^0} > \left(\frac{p-\alpha}{\beta-\alpha}\|g\|^{-1}_{\frac{p_s^*}{p_s^*-\beta}} S_{p_s^*}^{\frac{\beta}{p}}\right)^{\frac{1}{\beta-p}}, \quad \forall u \in \mathcal{N}_\lambda^-. \tag{4.6}$$

Putting together (3.2) and (4.6), we have

$$\begin{aligned} I_\lambda(u) &= \frac{\beta-p}{p\beta}\|u\|_{E_g^0}^p - \frac{\beta-\alpha}{\alpha\beta}\int_\Omega \lambda f(x)|u|^\alpha dx \\ &\ge \|u\|_{E_g^0}^\alpha\left[\frac{\beta-p}{p\beta}\|u\|_{E_g^0}^{p-\alpha} - \frac{\beta-\alpha}{\alpha\beta}\lambda\|f\|_{\frac{p_s^*}{p_s^*-\alpha}} S_{p_s^*}^{-\frac{\alpha}{p}}\right] \\ &> \left(\frac{p-\alpha}{\beta-\alpha}\|h\|^{-1}_{\frac{p_s^*}{p_s^*-\beta}} S_{p_s^*}^{\frac{\beta}{p}}\right)^{\frac{\alpha}{\beta-p}} \\ &\quad \times\left(-\frac{\beta-\alpha}{\alpha\beta}\lambda\|f\|_{\frac{p_s^*}{p_s^*-\alpha}} S_{p_s^*}^{-\frac{\alpha}{p}} + \frac{\beta-p}{p\beta}\left(\frac{p-\alpha}{\beta-\alpha}\|h\|^{-1}_{\frac{p_s^*}{p_s^*-\beta}} S_{p_s^*}^{\frac{\beta}{p}}\right)^{\frac{p-\alpha}{\beta-p}}\right). \end{aligned}$$

Thus, if $\lambda \in (0, \Lambda_*)$, we get that $I_\lambda(u) > c_0 > 0$ for all $u \in \mathcal{N}_\lambda^-$ for some positive constant $c_0 = c_0(\lambda, p, s, \alpha, \beta, S_{p_s^*})$, and completes the proof. □

Theorem 4.1. *Let* $\mathbb{G}$ *be a homogeneous Lie group with homogeneous dimension* Q. *Assume that* $p > 1$, $s \in (0,1)$, $Q > sp$ *and* $\lambda \in (0, \Lambda_*)$ *hold. Then* I_λ *has a local minimizer* u_λ^+ *in* $\mathcal{N}_\lambda^+$ *satisfying* $I_\lambda(u_\lambda^+) = c_\lambda^+ = c_\lambda$ *and* u_λ^+ *is a nontrivial nonnegative solution of* (1.1).

Proof. Since I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$ and so on $\mathcal{N}_\lambda^+$, by the Ekeland variational principle, there exists a minimizing sequence $\{u_k\}_k \subset \mathcal{N}_\lambda^+$ such that

$$I_\lambda(u_k) \to c_\lambda^+ \quad \text{and} \quad I'_\lambda(u_k) \to 0 \quad \text{as} \quad k \to +\infty. \tag{4.7}$$

From Lemma 3.3 and Case 4, we known that $c_\lambda^+ < 0$.

As I_λ is coercive on $\mathcal{N}_\lambda$, $\{u_k\}_k$ is a bounded sequence in E_g^0. Therefore, by Theorem 2.2, there is a subsequence, still denoted by $\{u_k\}_k$, and $u_\lambda^+ \in E_g^0$ such that

$$\begin{cases} u_k \rightharpoonup u_\lambda^+ \ \text{ in } \ E_g^0, \\ u_k \to u_\lambda^+ \ \text{ in } \ L^q(\Omega) \ \text{ for all } \ 1 \le q < p_s^*. \end{cases} \tag{4.8}$$

By the Hölder inequality and Dominated convergence theorem and (4.8), we obtain

$$\lim_{k\to\infty} \int_\Omega f(x)|u_k|^\alpha dx = \int_\Omega f(x)|u_\lambda^+|^\alpha dx, \tag{4.9}$$

and

$$\lim_{k\to\infty} \int_\Omega h(x)|u_k|^\beta dx = \int_\Omega h(x)|u_\lambda^+|^\beta dx. \tag{4.10}$$

First, we claim that $\int_\Omega f(x)|u_\lambda^+|^\alpha dx \neq 0$, we argue by contradiction, then we have $\int_\Omega f(x)|u_k|^\alpha dx \to 0$ as $k \to \infty$. Thus

$$\|u_k\|_{E_g^0}^p = \int_\Omega h(x)|u_k|^\beta dx + o_k(1),$$

and

$$I_\lambda(u_k) = \frac{1}{p}\|u_k\|_{E_g^0}^p - \frac{1}{\beta}\int_\Omega h(x)|u_k|^\beta dx = \left(\frac{1}{p} - \frac{1}{\beta}\right)\|u_k\|_{E_g^0}^p + o_k(1).$$

This contradicts $I_\lambda(u_k) \to c_\lambda < 0$ as $k \to \infty$.

From (4.8), (4.9) and (4.10), we know u_λ^+ is a weak solution of (1.1). We now claim u_λ^+ is a nontrivial solution of (1.1). Since $u_k \in \mathcal{N}_\lambda$, we have

$$I_\lambda(u_k) = \frac{\beta - p}{p\beta}\|u_k\|_{E_g^0}^p - \frac{\beta-\alpha}{\alpha\beta}\lambda\int_\Omega f(x)|u_k|^\alpha dx, \tag{4.11}$$

which implies that

$$\lambda\int_\Omega f(x)|u_k|^\alpha dx = \frac{\alpha(\beta-p)}{p(\beta-\alpha)}\|u_k\|_{E_g^0}^p - \frac{\alpha\beta}{\beta-\alpha}I_\lambda(u_k) \ge -\frac{\alpha\beta}{\beta-\alpha}I_\lambda(u_k). \tag{4.12}$$

Let $k \to \infty$ in (4.12), by (4.7), (4.8) and $c_\lambda^+ < 0$, we have that

$$\lambda\int_\Omega f(x)|u_\lambda^+|^\alpha dx \ge -\frac{\alpha\beta}{\beta-\alpha}c_\lambda^+ > 0.$$

Using this inequality, we get that u_λ^+ is a nontrivial solution of (1.1).

Next, we show that $u_k \to u_\lambda^+$ in E_g^0 and $I_\lambda(u_\lambda^+) = c_\lambda^+$. Since $u_\lambda^+ \in \mathcal{N}_\lambda$ and (4.11), we obtain

$$\begin{aligned} c_{\lambda^+} &\le I_\lambda(u_\lambda^+) \\ &= \frac{\beta-p}{p\beta}\|u_\lambda^+\|_{E_g^0}^p - \frac{\beta-\alpha}{\alpha\beta}\int_\Omega \lambda f(x)|u_\lambda^+|^\alpha dx \\ &\le \liminf_{k\to\infty}\left(\frac{\beta-p}{p\beta}\|u_k\|_{E_g^0}^p - \frac{\beta-\alpha}{\alpha\beta}\int_\Omega \lambda f(x)|u_k|^\alpha dx\right) \\ &\le \lim_{k\to\infty}\left(\frac{\beta-p}{p\beta}\|u_k\|_{E_g^0}^p - \frac{\beta-\alpha}{\alpha\beta}\int_\Omega \lambda f(x)|u_k|^\alpha dx\right) \\ &\le \lim_{k\to\infty} I_\lambda(u_k) = c_\lambda^+. \end{aligned} \tag{4.13}$$

Hence, thanks to (4.13) we obtain $I_\lambda(u_\lambda^+) = c_\lambda^+$ and $\lim_{k\to\infty} \|u_k\|_{E_g^0}^p = \|u_\lambda^+\|_{E_g^0}^p$, this implies that $u_k \to u_\lambda^+$ in E_g^0.

Finally, we claim that $u_\lambda^+ \in \mathcal{N}_\lambda^+$. Assume by contradiction that $u_\lambda^+ \in \mathcal{N}_\lambda^-$, then by Lemma 3.3, there exist unique τ_λ^+ and $\tau_\lambda^- > 0$ such that $\tau_\lambda^+ u_\lambda^+ \in \mathcal{N}_\lambda^+$ and $\tau_\lambda^- u_\lambda^+ \in \mathcal{N}_\lambda^-$. In particular, we have $\tau_\lambda^+ < \tau_\lambda^- = 1$. Since

$$\frac{d}{dt} I_\lambda(\tau_\lambda^+ u_\lambda^+) = 0 \text{ and } \frac{d^2}{dt^2} I_\lambda(\tau_\lambda^+ u_\lambda^+) > 0,$$

there exists $t_* \in (\tau_\lambda^+, \tau_\lambda^-)$ such that $I_\lambda(\tau_\lambda^+ u_\lambda^+) < I_\lambda(t_* u_\lambda^+)$, from which and Lemma 3.3, we have

$$I_\lambda(\tau_\lambda^+ u_\lambda^+) < I_\lambda(t_* u_\lambda^+) \le I_\lambda(\tau_\lambda^- u_\lambda^+) = I_\lambda(u_\lambda^+),$$

a contraction. Note that $I_\lambda(u_\lambda^+) = I_\lambda(|u_\lambda^+|)$ and $|u_\lambda^+| \in \mathcal{N}_\lambda$, so by Lemma 3.2, we obtain that u_λ^+ is a nontrivial nonnegative solution of (1.1) and conclude the proof. $\square$

Next, we establish the existence of a local minimum for I_λ on $\mathcal{N}_\lambda^-$.

Theorem 4.2. *Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension Q, and let $p > 1$, $s \in (0,1)$, $Q > sp$ and $\lambda \in (0, \Lambda_*)$. Then I_λ has a local minimizer u_λ^- in $\mathcal{N}_\lambda^-$ satisfying $I_\lambda(u_\lambda^-) = c_\lambda^-$ and u_λ^- is a nontrivial nonnegative solution of* (1.1).

Proof. From Lemma 4.2, we get $I_\lambda(u) \ge c_0 > 0$ for any $u \in \mathcal{N}_\lambda^-$. Hence $c_\lambda^- \ge c_0 > 0$ and there is a minimizing sequence $\{u_k\}_k \subset \mathcal{N}_\lambda^-$ such that

$$I_\lambda(u_k) \to c_\lambda^- \text{ and } I_\lambda'(u_k) \to 0 \text{ as } k \to \infty. \tag{4.14}$$

Hence, (4.14) and the coerciveness of I_λ yield that $\{u_k\}_k$ is a bounded in E_g^0. Without loss of generality, we can suppose that $u_\lambda^- \in \mathcal{N}_\lambda^-$ such that $u_k \rightharpoonup u_\lambda^-$ in E_g^0, and $u_k \to u_\lambda^-$ in $L^q(\Omega)$ for any $q \in [1, p_s^*)$. Using this results, as the proof of Theorem 4.1, we get that

$$\lim_{k\to\infty} \int_\Omega f(x)|u_k|^\alpha dx = \int_\Omega \lambda f(x)|u_\lambda^-|^\alpha dx,$$

and

$$\lim_{k\to\infty} \int_\Omega h(x)|u_k|^\beta dx = \int_\Omega h(x)|u_\lambda^-|^\beta dx.$$

Now we claim that $\int_\Omega f(x)|u_\lambda^-|^\alpha dx \ne 0$. To see this, by contradiction, we suppose that $\int_\Omega f(x)|u_k|^\alpha dx \to 0$ as $k \to \infty$, from which we have

$$\|u_k\|_{E_g^0}^p = \int_\Omega h(x)|u_k|^\beta dx + o_k(1)$$

and

$$\begin{aligned} I_\lambda(u_k) &= \frac{1}{p}\|u_k\|_{E_g^0}^p - \frac{1}{\beta}\int_\Omega h(x)|u_k|^\beta dx + o_k(1) \\ &= \left(\frac{1}{p} - \frac{1}{\beta}\right)\|u_k\|_{E_g^0}^p + o_k(1) \ge 0. \end{aligned}$$

This contradicts $I_\lambda(u_k) \to c_\lambda < 0$ as $k \to \infty$.

Finally we prove $u_k \to u_\lambda^-$ in E_g^0. Other case, we have

$$\begin{aligned}
&\|u_\lambda^-\|_{E_g^0}^p - \lambda \int_\Omega f(x)|u_\lambda^-|^\alpha dx - \int_\Omega h(x)|u_\lambda^-|^\beta dx \\
&\le \liminf_{k\to\infty} \left(\|u_k\|^p - \lambda \int_\Omega f(x)|u_k|^\alpha dx - \int_\Omega h(x)|u_k|^\beta dx \right) \\
&\le \lim_{k\to\infty} \left(\|u_k\|^p - \lambda \int_\Omega f(x)|u_k|^\alpha dx - \int_\Omega h(x)|u_k|^\beta dx \right) = 0.
\end{aligned}$$

Which contradicts $u_\lambda^- \in \mathcal{N}_\lambda^-$. Hence $u_k \to u_\lambda^-$ in E_g^0 as $k \to \infty$. Similar to Theorem 4.1, the proof can be completed. □

Now, we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. For $0 < \lambda < \Lambda_*$, by Theorems 4.1 and 4.2, there are $u_\lambda^+ \in \mathcal{N}_\lambda^+$ and $u_\lambda^- \in \mathcal{N}_\lambda^-$ such that

$$I_\lambda(u_\lambda^+) = c_\lambda^+, \quad I_\lambda(u_\lambda^-) = c_\lambda^-.$$

So, the equation (1.1) admits at least two nontrivial nonnegative solutions u_λ^+ and u_λ^-. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, it results u_λ^+ and u_λ^- are distinct nontrivial nonnegative solutions of problem (1.1) and the thesis is proved.

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