ELECTRONIC RESEARCH ARCHIVE
Volume **29**, Number **5**, November **2021**

doi:10.3934/era.2021033
pp. **3193–3203**

RELATIVE MMP WITHOUT $\mathbb{Q}$-FACTORIALITY

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Abstract. We consider the minimal model program for varieties that are not $\mathbb{Q}$-factorial. We show that, in many cases, its steps are simpler than expected. The main applications are to log terminal singularities, removing the earlier $\mathbb{Q}$-factoriality assumption from several theorems of Hacon–Witaszek and de Fernex–Kollár–Xu.

Let (X, Θ) be a dlt pair, projective over a base scheme S, and H an $\mathbb{R}$-divisor that is ample over S. As we run the (X, Θ)-MMP over S with scaling of H as in Definition 1, at the ith step there are 3 possibilities.

- (Divisorial) $X^i \xrightarrow{\phi_i} Z^i = X^{i+1}$,
- (Flipping) $X^i \xrightarrow{\phi_i} Z^i \xleftarrow{\phi_i^+} (X^i)^+ = X^{i+1}$.
- (Mixed) $X^i \xrightarrow{\phi_i} Z^i$, whose exceptional set contains a divisor, followed by a small modification $Z^i \xleftarrow{\psi_i} X^{i+1}$.

Note that the mixed case can occur only if either X^i is not $\mathbb{Q}$-factorial or ϕ_i contracts an extremal face of dimension ≥ 2. In most treatments this is avoided by working with $\mathbb{Q}$-factorial varieties and choosing H sufficiently general.

We can almost always choose the initial X to be nonsingular, but frequently other considerations constrain the choice of H.

Our aim is to discuss a significant special case where the X^i are not $\mathbb{Q}$-factorial and we do contract extremal faces of dimension ≥ 2, but still avoid the mixed case. This has several applications, some of which are discussed in Section 2.

1. Relative MMP with scaling of an exceptional divisor.

Definition 1 (MMP with scaling)**.** Let X, S be Noetherian, normal schemes and $g : X \to S$ a projective morphism. Let Θ be an $\mathbb{R}$-divisor on X and H an $\mathbb{R}$-Cartier, $\mathbb{R}$-divisor on X. Assume that $K_X + \Theta + r_X H$ is g-ample for some r_X.

By the (X, Θ)-*MMP with scaling* of H we mean a sequence of normal, projective schemes $g_j : X^j \to S$ and birational contractions $\tau_j : X^j \dashrightarrow X^{j+1}$, together with real numbers $r_X = r_0 > r_1 \cdots$, that are constructed by the following process.

• We start with $(X^1, \Theta^1, H^1) := (X, \Theta, H)$ and $r_0 = r_X$. If D is any $\mathbb{R}$-divisor on X, we let D^j denote its birational transform on X^j.

2020 *Mathematics Subject Classification.* Primary: 14E30, 14E05, 14J17; Secondary: 14F17, 14J40.

Key words and phrases. Minimal model, dlt singularity, vanishing theorems.

Partial financial support was provided by the NSF under grant number DMS-1901855.

• If X^j, Θ^j, H^j are already defined, we let $r_j < r_{j-1}$ be the unique real number for which $K_{X^j} + \Theta^j + r_j H^j$ is g_j-nef but not g_j-ample. Then the jth step of the MMP is a diagram

$$\begin{array}{ccccc} (X^j, \Theta^j) & \stackrel{\phi_j}{\to} & Z^j & \stackrel{\psi_j}{\leftarrow} & (X^{j+1}, \Theta^{j+1}) \\ g_j & \searrow & \downarrow & \swarrow & g_{j+1} \\ & & S & & \end{array} \qquad (1.1)$$

where

(2) ϕ_j is the contraction defined by $K_{X^j} + \Theta^j + r_j H^j$,
(3) ψ_j is small, and
(4) $K_{X^{j+1}} + \Theta^{j+1} + (r_j - \epsilon) H^{j+1}$ is g_{j+1}-ample for $0 < \epsilon \ll 1$.

Note that (4) implies that H^{j+1} must be $\mathbb{R}$-Cartier.

In general such a diagram need not exist, but if it does, it is unique and then $X^{j+1}, \Theta^{j+1}, H^{j+1}$ satisfy the original assumptions. Thus, as far as the existence of MMP-steps is concerned, we can focus on the 1st step. In this case it is customary to drop the upper indices and write (1.1) as

$$\begin{array}{ccccc} (X, \Theta) & \stackrel{\phi}{\to} & Z & \stackrel{\phi^+}{\leftarrow} & (X^+, \Theta^+) \\ g & \searrow & \downarrow & \swarrow & g^+ \\ & & S & & \end{array} \qquad (1.5)$$

We say that the MMP *terminates* with $g_j : X^j \to S$ if

(6) either $K_{X^j} + \Theta^j$ is g_j-nef, in which case (X^j, Θ^j) is called a *minimal model* of (X, Θ),
(7) or $\phi_j : X^j \to Z^j$ exists and $\dim Z^j < \dim X^j$; then ϕ_j is called a *Fano contraction.*

Warning 1.8. Our terminology is slightly different from [7], where it is assumed that X^j / Z^j has relative Picard number 1, and $r_j = r_{j-1}$ is allowed. In effect, we declare that the composite of all [7]-steps with the same value of r is a single step for us. Thus we sometimes contract an extremal face, not just an extremal ray.

One advantage is that our MMP steps are uniquely determined by the starting data. This makes it possible to extend the theory to algebraic spaces [33].

Theorem 2 is formulated for Noetherian base schemes. We do not prove any new results about the existence of flips, but Theorem 2 says that if the MMP with scaling exists and terminates, then its steps are simpler than expected, and the end result is more controlled than expected.

On the other hand, for 3-dimensional schemes, Theorem 2 can be used to conclude that, in some important cases, the MMP runs and terminates, see Theorem 9.

Theorem 2. *Let Y be a Noetherian, normal scheme and $g : X \to Y$ a projective, birational morphism with reduced exceptional divisor $E = E_1 + \cdots + E_n$. Assume the following (which are frequently easy to achieve, see Paragraphs 7–8).*

(i) (X, Θ) is dlt and the E_i are $\mathbb{Q}$-Cartier.
(ii) $K_X + \Theta \equiv_g E_\Theta$, where $E_\Theta = \sum e_i E_i$.
(iii) $H = \sum h_i E_i$, where $-H$ is effective and $\operatorname{Supp} H = E = \operatorname{Ex}(g)$.
(iv) $K_X + \Theta + r_X H$ is g-ample for some $r_X > 0$.
(v) The h_i are linearly independent over $\mathbb{Q}(e_1, \ldots, e_n)$.

We run the (X, Θ)-MMP with scaling of H. Assume that we reached the jth step as in (1.1). Then the following hold.

(1) $\operatorname{Ex}(\phi_j) \subset \operatorname{Supp}(E^j)$ *and*
  (a) either $\operatorname{Ex}(\phi_j)$ *is an irreducible divisor and* $X^{j+1} = Z^j$,
  (b) or ϕ_j *is small, and there are irreducible components* $E^j_{i_1}, E^j_{i_2}$ *of* E^j *such that* $E^j_{i_1}$ *and* $-E^j_{i_2}$ *are both* ϕ_j*-ample.*

(2) The E^{j+1}_i *are all* $\mathbb{Q}$*-Cartier.*

(3) $E^{j+1}_\Theta + (r_j - \epsilon)H^{j+1}$ *is a* g_{j+1}*-ample* $\mathbb{R}$*-divisor supported on* $\operatorname{Ex}(g_{j+1})$ *for* $0 < \epsilon \ll 1$.

Furthermore, if the MMP terminates with $g_m : X^m \to Y$, *then*

(4) $-E^m_\Theta$ *is effective,* $\operatorname{Supp} E^m_\Theta = g_m^{-1}\big(g_m(\operatorname{Supp} E^m_\Theta)\big)$, *and*

(5) if E^m_Θ *is effective and* $\operatorname{Supp} E_\Theta = E$, *then* $X^m = Y$.

Remark 2.6. In applications the following are the key points:

- (a) We avoided the mixed case.
- (b) In the fipping case we have both ϕ-positive and ϕ-negative divisors.
- (c) In (3) we have an explicit, relatively ample, exceptional $\mathbb{R}$-divisor.
- (d) In case (5) we end with $X^m = Y$ (not with an unknown, small modification of Y).
- (e) In case (5) the last MMP step is a divisorial contraction, giving what [35] calls a *Kollár component;* no further flips needed.

Proof. Assertions (1–3) concern only one MMP-step, so we may as well drop the index j and work with the diagram (1.5). Thus assume that $K_X + \Theta + (r+\epsilon)H$ is g-ample, $K_X + \Theta + rH$ is g-nef and it determines the contraction $\phi : X \to Z$.

Let $N_1(X/Z)$ be the relative cone of curves. The E_i give elements of the dual space $N^1(X/Z)$. If $C \subset X$ is contracted by ϕ then we have a relation

$$\textstyle\sum h_i(E_i \cdot C) = -r^{-1}(E_\Theta \cdot C). \tag{2.7}$$

By Lemma 3 this shows that the E_i are proportional, as functions on $N_1(X/Z)$. Let C' be another contracted curve; set $e := (E_\Theta \cdot C)$ and $e' := (E_\Theta \cdot C')$. Using (2.7) for C and C', we can eliminate r to get that

$$\textstyle\sum h_i\big(e'(E_i \cdot C) - e(E_i \cdot C')\big) = 0. \tag{2.8}$$

By the linear independence of the h_i this implies that $e'(E_i \cdot C) = e(E_i \cdot C')$ for every i. That is, all contracted curves are proportional, as functions on $\langle E_1, \dots, E_n\rangle_{\mathbb{R}} \cong \mathbb{R}^n$. Informally speaking, as far as the E_i are concerned, $N_1(X/Z)$ behaves as if it were 1-dimensional.

Assume first that ϕ contracts some divisor, call it E_1. Then $(E_1 \cdot C) < 0$ for some contracted curve $C \subset E_1$, hence $(E_1 \cdot C') < 0$ for every contracted curve C'. Thus $\operatorname{Ex}(\phi_0) = E_1$. We also know that

$$\phi_*(E_\Theta + rH) = \textstyle\sum_{i>1}(e_i + rh_i)\phi_*(E_i)$$

is $\mathbb{R}$-Cartier on Z and Z/Y-ample, where r is computed by (2.7). So, by Lemma 4, the $\{e_i + rh_i : i > 1\}$ are linearly independent over $\mathbb{Q}$, hence the $\phi_*(E_i)$ are $\mathbb{Q}$-Cartier on Z by Lemma 5. Thus $\phi_*(E_\Theta) = \sum_{i>1} e_i\phi_*(E_i)$ is $\mathbb{R}$-Cartier, hence $X^1 = Z$. This proves (2–3) in the divisorial contraction case.

Otherwise ϕ is small, let C be a contracted curve. Since $(H \cdot C) > 0$, we get that $(E_1 \cdot C) < 0$ for some E_1. So $C \subset E_1$. By [22, 3.39] $E_\Theta + rH$ is anti-effective and

$$g^{-1}\big(g(\operatorname{Supp}(E_\Theta + rH))\big) = \operatorname{Supp}(E_\Theta + rH). \tag{2.9}$$

If E_1 has coefficient 0 in $E_\Theta + rH$ then let $C_1 \subset E_1$ be any curve contracted by g and not contained in the other E_i for $i > 1$. Then C_1 is disjoint from $\operatorname{Supp}(E_\Theta + rH)$ by (2.9), hence $(C_1 \cdot E_\Theta + rH) = 0$. Varying C_1 shows that E_1 is contracted by ϕ, a contradiction.

Thus E_1 appears in $E_\Theta + rH$ with negative coefficient, contributing a positive term to the intersection $\big((E_\Theta + rH) \cdot C\big) = 0$. So there is another divisor $E_2 \subset \operatorname{Ex}(g)$ such that $(E_2 \cdot C) > 0$. This shows (1.b).

Assume next that the flip $\phi^+ : X^+ \to Z$ exists. Since ϕ^+ is small, $\operatorname{Supp}(E_\Theta^+ + rH^+)$ contains all X^+/Y-exceptional divisors. In particular, $E_\Theta^+ + (r-\epsilon)H^+$ is still anti-effective for $0 < \epsilon \ll 1$. By definition $E_\Theta^+ + (r-\epsilon)H^+$ is X^+/Y-ample and its support is the whole X^+/Y-exceptional locus. Thus we also have (2–3) in the flipping case.

Finally, if the MMP terminates with $g_m : X^m \to Y$ then E_Θ^m is a g_m-nef, exceptional $\mathbb{R}$-divisor. Thus $-E_\Theta^m$ is effective and $\operatorname{Supp} E_\Theta^m = g_m^{-1}\big(g_m(\operatorname{Supp} E_\Theta^m)\big)$ by [22, 3.39], proving (4). In case (5) this implies that $\operatorname{Ex}(g_m)$ does not contain any divisor, but, by (3) it supports a g_m-ample divisor. Thus $\dim \operatorname{Ex}(g_m) = 0$, hence $X^m = Y$. □

Lemma 3. *Let V be a K-vectorspace with vectors $v_i \in V$. Let L/K be a field extension and $h_1, \dots, h_n \in L$ linearly independent over K. Assume that*

$$\textstyle\sum_{i=1}^n h_i v_i = \gamma v_0 \quad \text{for some} \quad \gamma \in L.$$

Then $\dim_K \langle v_1, \dots, v_n \rangle \le 1$.

Proof. We may assume that $\dim V = 2$. Choose a basis and write $v_i = (a_i, b_i)$. Then

$$\textstyle\sum_{i=1}^n h_i a_i = \gamma a_0 \quad \text{and} \quad \sum_{i=1}^n h_i b_i = \gamma b_0.$$

This gives that

$$\textstyle\sum_{i=1}^n h_i (b_0 a_i - a_0 b_i) = 0.$$

Since the h_i are linearly independent over K, this implies that $b_0 a_i - a_0 b_i = 0$ for every i. That is $v_i \cdot (b_0, -a_0)^t = 0$ for every i. □

Lemma 4. *Let L/K be a field extension and $h_0, \dots, h_n \in L$ linearly independent over K. Let $\gamma^{-1} = \sum_{i=0}^n r_i h_i$ for some $r_i \in K$ with $r_0 \neq 0$. Then, for any $e_i \in K$, the $e_1 + \gamma h_1, \dots, e_n + \gamma h_n$ are linearly independent over K.*

Proof. Assume that $\sum_{i=1}^n s_i (e_i + \gamma h_i) = 0$, where $s_i \in K$. It rearranges to

$$\textstyle\sum_{i=1}^n s_i h_i = -\big(\sum_{i=1}^n s_i e_i\big) \cdot \sum_{i=0}^n r_i h_i.$$

If $\sum_{i=1}^n s_i e_i = 0$ then the $s_1, \dots, s_n$ are all zero since the h_i are linearly independent over K. Otherwise we get a contradiction since the coefficient of h_0 is nonzero. □

The following is a slight modifications of [3, Lem.1.5.1]; see also [17, 5.3].

Lemma 5. *Let X be a normal scheme, D_i $\mathbb{Q}$-divisors and $d_1, \dots, d_n \in \mathbb{R}$ linearly independent over $\mathbb{Q}$. Then $\sum d_i D_i$ is $\mathbb{R}$-Cartier iff each D_i is $\mathbb{Q}$-Cartier.* □

Comments on $\mathbb{Q}$-factoriality. Theorem 2 may sound unexpected from the MMP point of view, but it is quite natural if one starts with the following conjecture, which is due, in various forms, to Srinivas and myself, cf. [26].

Conjecture 6. *Let X be a normal variety, $x \in X$ a closed point and $\{D_i^X : i \in I\}$ a finite set of prime divisors on X. Then there is a normal variety Y, a closed point $y \in Y$ and prime divisors $\{D_i^Y : i \in I\}$ on Y such that the following hold.*

(1) *The class group of the local ring $\mathcal{O}_{y,Y}$ is generated by K_Y and the D_i^Y.*
(2) *The completion of $(X, \sum D_i^X)$ at x is isomorphic to the completion of $(Y, \sum D_i^Y)$ at y.*

Using [30, Tag 0CAV] one can reformulate (6.2) as a finite type statement:

(3) There are elementary étale morphisms

$$(x, X, \sum D_i^X) \leftarrow (u, U, \sum D_i^U) \rightarrow (y, Y, \sum D_i^Y).$$

Almost all resolution methods commute with étale morphisms, thus if we want to prove something about a resolution of X, it is likely to be equivalent to a statement about resolutions of Y. In particular, if something holds for the $\mathbb{Q}$-factorial case, it should hold in general. This was the reason why I believed that Theorem 2 should work out.

A positive answer to Conjecture 6 (for $I = \emptyset$) is given for isolated complete intersections in [26] and for normal surface singularities in [27].

(Note that [27] uses an even stronger formulation: Every normal, analytic singularity has an algebraization whose class group is generated by the canonical class. This is, however, not true, since not every normal, analytic singularity has an algebraization.)

Existence of certain resolutions.

7 (The assumptions 2.i–v)**.** In most applications of Theorem 2 we start with a normal pair (Y, Δ_Y) where Δ_Y is a boundary, and want to find $g : X \to Y$ and Θ that satisfy the conditions (2.i–v).

Typically we choose a log resolution $g : X \to (Y, \Delta_Y)$. That is, g is birational, X is regular, $\Delta_X := g_*^{-1}\Delta_Y$, $E = \operatorname{Ex}(g)$ and $E + \Delta_X$ is a simple normal crossing divisor. Then we choose $\Delta_X \le \Theta \le E + \Delta_X$; that is, we are free to choose the coefficients of the E_i in $[0, 1]$. Then (2.i) holds and if $K_Y + \Delta_Y$ is $\mathbb{R}$-Cartier then so does (2.ii). There are also situations where one can use the theorem to show that numerical equivalence in (2.ii) implies $\mathbb{R}$-linear equivalence; see [6, 9.12].

We want $K_X + \Theta + rH$ to be g-ample for some r, which is easiest to achieve if H is g-ample. Thus we would like H to be g-ample and g-exceptional for (2.iii–iv) to hold. If X is regular (or at least $\mathbb{Q}$-factorial) then we can wiggle the coefficients of H to achieve (2.v).

The existence of a g-ample and g-exceptional $\mathbb{Q}$-divisor is somewhat subtle, we discuss it next.

8 (Ample, exceptional divisors)**.** Assume that we blow up an ideal sheaf $I \subset \mathcal{O}_Y$ to get $\pi_1 : Y_1 \to Y$. The constant sections of $\mathcal{O}_Y$ give an isomorphism $\mathcal{O}_{Y_1}(1) \cong \mathcal{O}_{Y_1}(-E_1)$ where E_1 is supported on $\pi_1^{-1} \operatorname{Supp}(\mathcal{O}_Y/I)$. Thus, if Y is normal and $\operatorname{Supp}(\mathcal{O}_Y/I)$ has codimension ≥ 2, then E_1 is π_1-ample and π_1-exceptional. A composite of such blow-ups also has an ample, exceptional $\mathbb{Q}$-divisor. Since Hironaka-type resolutions use only such blow-ups, we get the following. (See [32] for the most general case and [18] for an introduction.)

Claim 8.1. *Let Y be a Noetherian, quasi-excellent scheme over a field of characteristic zero. Then any proper, birational $Y' \to Y$ is dominated by a log resolution $g : X \to Y$ that has a g-ample and g-exceptional $\mathbb{Q}$-divisor.* □

Resolution of singularities is also known for 3-dimensional excellent schemes [10], but in its original form it does not guarantee projectivity in general. Nonetheless, combining [6, 2.7] and [23, Cor.3] we get the following.

Claim 8.2. *Let Y be a normal, integral, quasi-excellent scheme of dimension at most three that is separated and of finite type over an affine, quasi-excellent scheme S. Then any proper, birational $Y' \to Y$ is dominated by a log resolution $g : X \to Y$ that has a g-ample and g-exceptional $\mathbb{Q}$-divisor.* □

2. **Applications.** Next we mention some applications. In each case we use Theorem 2 to modify the previous proofs to get more general results. We give only some hints as to how this is done, we refer to the original papers for definitions and details of proofs.

The first two applications are to dlt 3-folds. In both cases Theorem 2 allows us to run MMP in a way that works in every characteristic and also for bases that are not $\mathbb{Q}$-factorial.

Relative MMP for dlt 3-folds.

Theorem 9. *Let (Y, Δ) be a 3-dimensional, normal, Noetherian, excellent pair such that $K_Y + \Delta$ is $\mathbb{R}$-Cartier and Δ is a boundary. Let $g : X \to Y$ be a log resolution with exceptional divisor $E = \sum E_i$. Assume that E supports a g-ample $\mathbb{R}$-divisor H (we can then choose its coefficients sufficiently general).*

Then the MMP over Y, starting with $(X^0, \Theta^0) := (X, E + g_*^{-1}\Delta)$ with scaling of H runs and terminates with a minimal model $g_m : (X^m, \Theta^m) \to Y$. Furthermore,

(1) each step $X^i \dashrightarrow X^{i+1}$ of this MMP is
  (a) either a contraction $\phi_i : X^i \to X^{i+1}$, whose exceptional set is an irreducible component of E^i,
  (b) or a flip $X^i \xrightarrow{\phi_i} Z^i \xleftarrow{\psi_i} (X^i)^+ = X^{i+1}$, and there are irreducible components $E^i_{i_1}, E^i_{i_2}$ such that $E^i_{i_1}$, and $-E^i_{i_2}$ are both ϕ_i-ample,

(2) $\operatorname{Ex}(g_m)$ supports a g_m-ample $\mathbb{R}$-divisor, and

(3) if either (Y, Δ) is plt, or (Y, Δ) is dlt and g is thrifty [20, 2.79], then $X^m = Y$.

Proof. Assume first that the MMP steps exist and the MMP terminates. Note that

$$\begin{aligned} K_X + E + g_*^{-1}\Delta &\sim_{\mathbb{R}} g^*(K_Y + \Delta) + \textstyle\sum_j \big(1 + a(E_j, Y, \Delta)\big) E_j \\ &\sim_{g,\mathbb{R}} \textstyle\sum_j \big(1 + a(E_j, Y, \Delta)\big) E_j =: E_\Theta. \end{aligned}$$

We get from Theorem 2 that (1.a–b) are the possible MMP-steps, and (2–3) from Theorem 2.3–5.

For existence and termination, all details are given in [6, 9.12].

However, I would like to note that we are in a special situation, which can be treated with the methods that are in [29, 1], at least when the closed points of Y have perfect residued fields

The key point is that everything happens inside E. We can thus understand the whole MMP by looking at the 2-dimensional scheme E. This is easiest for termination, which follows from [1, Sec.7].

Contractions for reducible surfaces have been treated in [1, Secs.11-12], see also [12, Chap.6] and [31].

The presence of $E^i_{i_1}, E^i_{i_2}$ means that the flips are rather special; called *1-complemented flips* in [29] and *easy flips* in [1, Sec.20]. I believe that the methods of [29, 1] prove the existence of 1-complemented 3-fold flips in our case; but the details have not been written down.

The short note [34] explains how [15, 3.4] gives 1-complemented 3-fold flips; see [16, 3.1 and 4.3] for stronger results. □

Inversion of adjunction for 3-folds. Using Theorem 9 we can remove the $\mathbb{Q}$-factoriality assumption from [15, Cor.1.5]. The characteristic 0 case, in all dimensions, was proved in [1, 17.6],

Corollary 10. *Let $(X, S+\Delta)$ be a 3-dimensional, normal, Noetherian, excellent pair. Assume that X is normal, S is a reduced divisor, Δ is effective and $K_X+S+\Delta$ is $\mathbb{R}$-Cartier. Let $\bar{S}\to S$ denote the normalization. Then $(\bar{S}, \operatorname{Diff}_{\bar{S}}\Delta)$ is klt iff $(X, S+\Delta)$ is plt near S.* □

This implies that one direction of Reid's classification of terminal singularities using 'general elephants' [28, p.393] works in every characteristic. This could be useful in extending [2] to characteristics ≥ 5.

Corollary 11. *Let (X, S) be a 3-dimensional pair. Assume that X is normal, K_X+S is Cartier, X and S have only isolated singularities, and the normalization $\bar{S}$ of S has canonical singularities. Then X has terminal singularities in a neighborhood of S.* □

Divisor class group of dlt singularities. The divisor class group of a rational surface singularity is finite by [24], and [8] plus an easy argument shows that the divisor class group of a rational 3-dimensional singularity is finitely generated. Thus the divisor class group of a 3-dimensional dlt singularity is finitely generated in characteristic ≥ 7, using [4, Cor.1.3]. Theorem 9 leads—via [21, B.14]—to the following weaker result, which is, however, optimal in characteristics $2, 3, 5$; see [9] for an application.

Proposition 12. [21, B.1] *Let (y, Y, Δ) be a 3-dimensional, Noetherian, excellent, dlt singularity with residue characteristic $p>0$. Then the prime-to-p parts of $\operatorname{Cl}(Y), \operatorname{Cl}(Y^{\mathrm{h}})$ and of $\operatorname{Cl}(\hat{Y})$ are finitely generated, where Y^{h} denotes the henselisation and $\hat{Y}$ the completion.* □

It seems reasonable to conjecture that the same holds in all dimensions, see [21, B.6].

Grauert-Riemenschneider vanishing. One can prove a variant of the Grauert-Riemenschneider (abbreviated as G-R) vanishing theorem [13] by following the steps of the MMP.

Definition 13 (G-R vanishing)**.** Let (Y, Δ_Y) be a pair, Y normal, Δ_Y a boundary (that is, all coefficients are in $[0,1]$) and $g: X\to Y$ a proper, birational morphism with X normal. For an $\mathbb{R}$-divisor F on X let $\operatorname{Ex}(F)$ denote its g-exceptional part. Assume that Y has a dualizing complex. We say that *G-R vanishing* holds for $g: X\to (Y, \Delta_Y)$ if the following is satisfied.

Let D be a $\mathbb{Z}$-divisor and Δ_X an effective $\mathbb{R}$-divisor on X. Assume that

(1) $D\sim_{g,\mathbb{R}} K_X+\Delta_X$, and
(2) $g_*\Delta_X\leq \Delta_Y$, $\lfloor \operatorname{Ex}(\Delta_X)\rfloor = 0$.

Then $R^i g_*\mathcal{O}_X(D)=0$ for $i>0$.

We say that *G-R vanishing* holds over (Y, Δ_Y) if G-R vanishing holds for every log resolution $g: (X, E+g_*^{-1}\Delta_Y)\to (Y, \Delta_Y)$.

By an elementary computation, if X is regular, $W \subset X$ is regular and G-R vanishing holds for $X \to Y$ then it also holds for the blow-up $B_W X \to Y$. This implies that if G-R vanishing holds for one log resolution of (Y, Δ_Y), then it holds for every log resolution; see [5, Sec.1.3].

If Y is essentially of finite type over a field of characteristic 0, then G-R vanishing is a special case of the general Kodaira-type vanishing theorems; see [22, 2.68].

G-R vanishing also holds over 2-dimensional, excellent schemes by [24]; see [20, 10.4]. In particular, if Y is any normal, excellent scheme, then the support of $R^i g_* \mathcal{O}_X(D) = 0$ has codimension ≥ 3 for $i > 0$.

However, G-R vanishing fails for 3-folds in every positive characteristic, as shown by cones over surfaces for which Kodaira's vanishing fails. Thus the following may be the type of G-R vanishing result that one can hope for.

Theorem 14. [5] *Let Y be a 3-dimensional, excellent, dlt pair with a dualizing complex. Assume that closed points of Y have perfect residue fields of characteristic $\neq 2, 3, 5$. Then G-R vanishing holds over Y.*

Proof. Let (Y, Δ_Y) be a 3-dimensional, dlt pair, and $g : X \to Y$ a log resolution. With D as in Definition 13 we need to show that $R^j g_* \mathcal{O}_X(D) = 0$ for $j > 0$. Let $g_i : X^i \to Y$ be the MMP steps as in Theorem 9. The natural idea would be to show that the sheaves $R^j (g_i)_* \mathcal{O}_{X^i}(D^i)$ are independent of i. At the end then we have an isomorphism $g_m : X^m \cong Y$, hence $R^j (g_m)_* \mathcal{O}_{X^m}(D^m) = 0$ for $j > 0$.

A technical problem is that we seem to need various rationality properties of the singularities of the X^i. Therefore, we show instead that, if G-R vanishing holds *over* X^i and X^i satisfies (15.1–3), then G-R vanishing also holds *over* X^{i+1}. Then Theorem 15 gives that X^{i+1} also satisfies (15.1–3), and the induction can go ahead.

For divisorial contractions $X^i \to X^{i+1}$ with exceptional divisor S this is straightfoward, the method of [14, Sec.3] shows that if Kodaira vanishing holds for S then G-R vanishing holds for $X^i \to X^{i+1}$. This is where the char $\neq 2, 3, 5$ assumption is used: Kodaira vanishing can fail for del Pezzo surfaces if char $= 2, 3, 5$; see [4].

For flips $X^i \to Z^i \leftarrow X^{i+1}$ the argument works in any characteristic. First we show as above that G-R vanishing holds over Z^i. Going to X^{i+1} is a spectral sqeuence argument involving $\psi_i : X^{i+1} \to Z^i$. For 3-folds the only nontrivial term is $R^1(\psi_i)_* \mathcal{O}_{X^{i+1}}(D^{i+1})$, and no unexpected cancellations occur; see [5, Lem.21]. □

From G-R vanishing one can derive various rationality properties for all excellent dlt pairs. This can be done by following the method of 2 spectral sequences as in [19] or [20, 7.27]; see [5] for an improved version.

Theorem 15. [5] *Let (X, Δ) be an excellent dlt pair such that G-R vanishing and resolution of singularities hold over (X, Δ). Then*

(1) X has rational singularities.
(2) Every irreducible component of $\lfloor \Delta \rfloor$ is normal and has rational singularities.
(3) Let D be a $\mathbb{Z}$-divisor on X such that $D + \Delta_D$ is $\mathbb{R}$-Cartier for some $0 \leq \Delta_D \leq \Delta$. Then $\mathcal{O}_X(D)$ is CM. □

See [5, 12] for the precise resolution assumptions needed. The conclusions are well known in characteristic 0, see [22, 5.25], [12, Sec.3.13] and [20, 7.27]. For 3-dimensional dlt varieties in char ≥ 7, the first claim was proved in [14, 4].

The next two applications are in characteristic 0.

Dual complex of a resolution. Our results can be used to remove the $\mathbb{Q}$-factoriality assumption from [11, Thm.1.3]. We refer to [11] for the definition of a dual complex and the notion of collapsing of a regular cell complex. We start with the weaker form, Corollary 16, and then state and outline the proof of the stronger version, Theorem 17.

Corollary 16. *Let (Y,Δ) be a dlt variety over field of characteristic 0 and $g : X \to Y$ a thrifty log resolution whose exceptional set supports a g-ample divisor. For a closed point $y \in Y$ let $E_y \subset g^{-1}(y)$ denote the divisorial part. Then $\mathcal{D}(E_y)$ is collapsible to a point (or it is empty).*

Theorem 17. *Let (Y,Δ) be a dlt variety over field of characteristic 0 and $g : X \to Y$ a projective, birational morphism with exceptional set $E = \cup_i E_i$. For $y \in Y$ let $E_y \subset g^{-1}(y)$ denote the divisorial part. Assume that*

(1) $(X, E + g_*^{-1}\Delta)$ is dlt and the E_i are $\mathbb{Q}$-Cartier.
(2) $a(E_i, Y, \Delta) > -1$ for every i.
(3) E supports a g-ample divisor.

Then $\mathcal{D}(E_y)$ is collapsible to a point (or it is empty).

Proof. Fix $y \in Y$. We may assume that (y, Y) is local and, after passing to an elementary étale neighborhood (cf. [30, Tag 02LD]) of $y \in Y$, we may also assume that $g^{-1}(y) \cap E_i$ is connected for every irreducible exceptional divisor E_i (cf. [30, Tag 04HF]).

Let us now run the $(X, E + g_*^{-1}\Delta)$-MMP with scaling of a g-ample $\mathbb{R}$-divisor H that is supported on E and has sufficiently general coefficients. Theorem 2 applies, as we observed during the proof of Theorem 9.

Note that $\mathcal{D}(E_y) \subset \mathcal{D}(E)$ is a *full* subcomplex (that is, a simplex is in $\mathcal{D}(E_y)$ iff all of its vertices are), hence an elementary collapse of $\mathcal{D}(E)$ induces an elementary collapse (or an isomorphism) on $\mathcal{D}(E_y)$. Thus it is enough to show that $\mathcal{D}(E)$ is collapsible to a point (or it is empty).

We claim that each MMP-step as in Theorem 2 induces either a collapse or an isomorphism of $\mathcal{D}(E)$.

By [11, Thm.19] we get an elementary collapse (or an isomorphism) if there is a divisor $E_i^j \subset E^j$ that has positive intersection with the ϕ_j-contracted curves. This takes care of flips by Theorem 2.1.b and most divisorial contractions.

It remains to deal with the case when we contract $E_\ell^j \subset E^j$ and every other E_i^j has 0 intersection number with the contracted curves. Thus $E_i^j \cap E_\ell^j$ is either empty or contains $g_j^{-1}(y) \cap E_\ell^j$. Thus the link of E_ℓ^j in $\mathcal{D}(E^j)$ is a simplex and removing it is a sequence of elementary collapses. □

Dlt modifications of algebraic spaces. By [25], a normal, quasi-projective pair (X,Δ) (over a field of characteristic 0) has both dlt and lc modifications if $K_X + \Delta$ is $\mathbb{R}$-Cartier. (See [20, Sec.1.4] for the definitions.) The lc modification is unique and commutes with étale base change, hence local lc modifications automatically glue to give the same conclusion if X is an algebraic space.

However, dlt modifications are rarely unique, thus it was not obvious that they exist when the base is not quasi-projective. [33] observed that Theorem 2 gives enough uniqueness to allow for gluing. This is not hard when X is a scheme, but needs careful considerations to work for algebraic spaces.

Theorem 18 (Villalobos-Paz)**.** *Let X be a normal algebraic space of finite type over a field of characteristic 0, and Δ a boundary $\mathbb{R}$-divisor on X. Assume that*

$K_X+\Delta$ is $\mathbb{R}$-Cartier. Then (X,Δ) has a modification $g:(X^{\mathrm{dlt}},\Delta^{\mathrm{dlt}})\to(X,\Delta)$ such that

(1) $(X^{\mathrm{dlt}},\Delta^{\mathrm{dlt}})$ is dlt,

(2) $K_{X^{\mathrm{dlt}}}+\Delta^{\mathrm{dlt}}$ is g-nef,

(3) $g_*\Delta^{\mathrm{dlt}}=\Delta$, and

(4) g is projective.

X^{dlt} is not unique, and we can choose

(5) either X^{dlt} to be $\mathbb{Q}$-factorial, or $\mathrm{Ex}(g)$ to support a g-ample $\mathbb{Q}$-divisor.

Acknowledgments. I thank E. Arvidsson, F. Bernasconi, J. Carvajal-Rojas, J. Lacini, A. Stäbler, D. Villalobos-Paz, C. Xu for helpful comments and J. Witaszek for numerous e-mails about flips.

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Received December 2020; 1st revision January 2021; 2nd revision February 2021.

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