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THREE TYPES OF WEAK PULLBACK ATTRACTORS FOR LATTICE PSEUDO-PARABOLIC EQUATIONS DRIVEN BY LOCALLY LIPSCHITZ NOISE

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Abstract. The global well-posedness and long-time mean random dynamics are studied for a high-dimensional non-autonomous stochastic nonlinear lattice pseudo-parabolic equation with *locally* Lipschitz drift and diffusion terms. The existence and uniqueness of three different types of weak pullback mean random attractors as well as their relations are established for the mean random dynamical systems generated by the solution operators. This is the first paper to study the well-posedness and dynamics of the stochastic lattice pseudo-parabolic equation even when the nonlinear noise reduces to the linear one.

1. Introduction.

In this article we study the global well-posedness as well as long-time *mean* random dynamics of the non-autonomous stochastic lattice pseudo-parabolic equation with *locally* Lipschitz white noise defined on the whole N-dimensional integer set $\mathbb{Z}^N$:

$$\begin{aligned}
&du_i(t) + \lambda u_i(t)dt - d\Big(u_{(i_1-1,i_2,\dots,i_N)}(t) + u_{(i_1,i_2-1,\dots,i_N)}(t) + \cdots + u_{(i_1,i_2,\dots,i_N-1)}(t) \\
&- 2Nu_{(i_1,i_2,\dots,i_N)}(t) + u_{(i_1+1,i_2,\dots,i_N)}(t) + u_{(i_1,i_2+1,\dots,i_N)}(t) + \cdots + u_{(i_1,i_2,\dots,i_N+1)}(t)\Big) \\
&- \Big(u_{(i_1-1,i_2,\dots,i_N)}(t) + u_{(i_1,i_2-1,\dots,i_N)}(t) + \cdots + u_{(i_1,i_2,\dots,i_N-1)}(t) \\
&- 2Nu_{(i_1,i_2,\dots,i_N)}(t) + u_{(i_1+1,i_2,\dots,i_N)}(t) + u_{(i_1,i_2+1,\dots,i_N)}(t) + \cdots + u_{(i_1,i_2,\dots,i_N+1)}(t)\Big)dt \\
&= -F_i(u_i(t))dt + g_i(t)dt + \sum_{k=1}^{\infty} \big(h_{k,i}(t) + \delta_{k,i}\hat{\sigma}_{k,i}(u_i(t))\big)dW_k(t), \\
&t > \tau, \quad i = (i_1, i_2, \dots, i_N) \in \mathbb{Z}^N,
\end{aligned} \tag{1.1}$$

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along with initial data:

$$u_i(\tau) = u_{\tau,i}, \quad i = (i_1, i_2, \dots, i_N) \in \mathbb{Z}^N, \tag{1.2}$$

where $N \in \mathbb{N}$, $\tau \in \mathbb{R}$, $\lambda > 0$, $g = (g_i)_{i\in\mathbb{Z}^N}$ and $h = (h_{k,i})_{k\in\mathbb{N},i\in\mathbb{Z}^N}$ are two *random* sequences depending on time, $\delta = (\delta_{k,i})_{k\in\mathbb{N},i\in\mathbb{Z}^N}$ is a given sequence of real numbers, $F_i \in C^1(\mathbb{R}, \mathbb{R})$ and $\hat{\sigma} = (\hat{\sigma}_{k,i})_{k\in\mathbb{N},i\in\mathbb{Z}^N}$ are *locally* Lipschitz continuous nonlinear functions satisfying some conditions, and the sequence of independent two-sided real-valued Wiener process $(W_k)_{k\in\mathbb{N}}$ is defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ satisfying the usual conditions. The last stochastic term in (1.1) is interpreted as an It'o stochastic differential.

The pseudo-parabolic equation (see Xu et al. [26, 27]) is also called a nonclassical diffusion equation (see Kuttler and Aifantis [11]) in the literature. This equation is used to study solid mechanics, non-Newtonian as well as heat conduction, see e.g.,[1]. The blow-up phenomenon of the deterministic pseudo-parabolic equation has been investigated in several interesting papers, see e.g., [26, 27, 25, 32] and the references therein. The long time dynamics by means of random attractors of the stochastic pseudo-parabolic equation was recently discussed in [21, 23, 29]. A contradictory version for the pseudo-parabolic equation is the classical parabolic (also called reaction-diffusion) equation. The large time dynamics in terms of random attractors of the stochastic classical diffusion equation was investigated in [12, 15]. The reader is referred to [5, 6] for the study of attractors of other interesting models.

Lattice equations can be regarded as space discretization version of evolution equations that are widely used for image processing, chemical reaction as well as pattern formation see e.g., [7]. The existence of deterministic and random attractors of lattice equations have been extensively examined in [14] and [2, 4, 9, 22, 28, 31], respectively. In particular, the existence of global attractors of deterministic lattice pseudo-parabolic equation defined on $\mathbb{Z}$ had been investigated in [13, 30]. To the best of our knowledge, there are no documented results reported in the literature on the existence of random attractors of stochastic lattice pseudo-parabolic equation (1.1)-(1.2) defined on $\mathbb{Z}^N$ even in a simple case where $N = 1$ and the diffusion coefficient $\hat{\sigma}_{k,i}(u_i(t))$ is linear in u_i or independent of u_i. Our main task in the paper is to solve this problem in a more general case, that is, we will prove the existence of random attractors of equation (1.1) on $\mathbb{Z}^N$ when the nonlinear diffusion coefficient $\hat{\sigma}_{k,i}(u_i(t))$ is *locally* Lipschitz continuous in $u_i(t)$.

Note that those random attractors in the literature aforementioned were studied under the frameworks of pathwise random dynamical systems [15], and hence those random attractors are also called *pathwise* random attractors. As is well known, a basic but very restrictive condition for investigating the existence of such pathwise random attractors is that we need to transform the stochastic equation into a pathwise one. This transformation can be done for stochastic equations with additive noise or linear multiplicative noise but unavailable for stochastic equations with nonlinear noise. This then introduces many difficulties to prove the existence of *pathwise* random attractors of stochastic systems with nonlinear noise. For the purpose of dealing with the nonlinear noise in (1.1)-(1.2), in the present article, it is unnecessary to transform the stochastic equation (1.1)-(1.2) into a *pathwise* one. In other words, we will alternatively investigate the *mean* random dynamics but not the *pathwise* random dynamics of (1.1)-(1.2) by using the concept of weak pullback mean random attractors (WPMR attractors for short) of mean random dynamical systems, see Wang [16]. By a WPMR attractor in an abstract bochner space $L^p(\Omega,\mathcal{F},X)$, we here mean a minimal, weakly compact and weakly attracting set in $L^p(\Omega,\mathcal{F},X)$, where $p\in(1,\infty)$ and X is a Banach space. The notation of invariant WPMR attractors can be found in Kloeden and Lorenz [10].

In order to study the existence of WPMR attractors for problem (1.1)-(1.2) in $L^2(\Omega,\mathcal{F},\ell^2)$, we must prove the global existence as well as uniqueness of mean square solutions to (1.1)-(1.2) in $L^2(\Omega,\mathcal{F},\ell^2)$. We remark that, since the nonlinear drift and diffusion functions are locally Lipschitz continuous but not globally Lipschitz continuous, we shall find a way to approximate the two functions in here. Our idea to solve the problem is to utilize the stoping time technique as well as the truncate method, see Theorem 3.1.

Based on the global well-posedness of (1.1)-(1.2) in $L^2(\Omega,\mathcal{F},\ell^2)$ we have established, we define a mean random dynamical system Φ via the solutions operators to (1.1)-(1.2), and prove that Φ has a unique WPMR $\mathfrak{D}$-attractor of in $L^2(\Omega,\mathcal{F},\ell^2)$, where $\mathfrak{D}$ is an attracting universe (see (4.6)).

Another important contribution of the paper is to study the backward weak compactness and attraction of WPMR attractors. More precisely, we will introduce another attracting universe $\mathfrak{B}$ (see (4.7), and prove that Φ has a unique backward weakly compact WPMR attractor $\mathcal{A}_{\mathfrak{B}}=\{\mathcal{A}_{\mathfrak{B}}(\tau):\tau\in\mathbb{R}\}\in\mathfrak{B}$ and a unique backward weakly attracting WPMR attractor $\mathcal{U}_{\mathfrak{B}}=\{\mathcal{U}_{\mathfrak{B}}(\tau):\tau\in\mathbb{R}\}\in\mathfrak{B}$, see Theorem 6.4. An interesting thing is that the three types attractors have the relationship $\mathcal{A}_{\mathfrak{D}}=\mathcal{A}_{\mathfrak{B}}\subseteq\mathcal{U}_{\mathfrak{B}}$ even when the attracting universes $\mathfrak{B}$ and $\mathfrak{D}$ are different ($\mathfrak{B}\subseteq\mathfrak{D}$).

We remark that the backward strong compactness of random attractors has been investigated in [3, 20] for stochastic PDEs with *linear* noise. The usual WPMR attractors of stochastic equations with *nonlinear* noise was recently studied in [16, 17, 19, 24, 18]. In this paper we study backward weak compactness and attraction of WPMR attractors for stochastic lattice pseudo-parabolic equations with nonlinear noise.

This paper is organized as bellow. In the next Section we recall the lattice pseudo-parabolic equations with locally Lipschitz noise. In Section 3 we prove the global well-posedness of (1.1)-(1.2). In Section 4 we define a mean random dynamical system. In Section 5 we derive two types of uniform estimates and esyavlish the existence of absorbing sets of two types. In the last Section we prove the existence of WPMR attractors of three types.

2. Lattice pseudo-parabolic equations with locally Lipschitz noise.

In this section we recall the following non-autonomous stochastic lattice pseudo-parabolic equation with locally Lipschitz noise defined on $\mathbb{Z}^N$:

$$\begin{aligned}
&du_i(t)+\lambda u_i(t)dt-d\Big(u_{(i_1-1,i_2,\dots,i_N)}(t)+u_{(i_1,i_2-1,\dots,i_N)}(t)+\cdots+u_{(i_1,i_2,\dots,i_N-1)}(t)\\
&-2Nu_{(i_1,i_2,\dots,i_N)}(t)+u_{(i_1+1,i_2,\dots,i_N)}(t)+u_{(i_1,i_2+1,\dots,i_N)}(t)+\cdots+u_{(i_1,i_2,\dots,i_N+1)}(t)\Big)\\
&-\Big(u_{(i_1-1,i_2,\dots,i_N)}(t)+u_{(i_1,i_2-1,\dots,i_N)}(t)+\cdots+u_{(i_1,i_2,\dots,i_N-1)}(t)\\
&-2Nu_{(i_1,i_2,\dots,i_N)}(t)+u_{(i_1+1,i_2,\dots,i_N)}(t)+u_{(i_1,i_2+1,\dots,i_N)}(t)+\cdots+u_{(i_1,i_2,\dots,i_N+1)}(t)\Big)dt\\
&=-F_i(u_i(t))dt+g_i(t)dt+\sum_{k=1}^{\infty}\big(h_{k,i}(t)+\delta_{k,i}\hat{\sigma}_{k,i}(u_i(t))\big)dW_k(t),\\
&t>\tau,\quad i=(i_1,i_2,\dots,i_N)\in\mathbb{Z}^N,
\end{aligned}\tag{2.1}$$

with initial data:

$$u_i(\tau)=u_{\tau,i},\quad i=(i_1,i_2,\dots,i_N)\in\mathbb{Z}^N,\tag{2.2}$$

where $\tau \in \mathbb{R}$, $\lambda > 0$, the sequence of independent two-sided real-valued Wiener process $(W_j)_{j\in\mathbb{N}}$ is defined on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \ \omega(0) = 0\}$ equipped with the compact-open topology, $\mathcal{F} = \mathfrak{B}(\Omega)$ denotes the Borel sigma-algebra of Ω, $\mathbb{P}$ is the Wiener measure acting on $(\Omega, \mathcal{F})$, and $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing right continuous family of sub-sigma-algebras of $\mathcal{F}$ with all $\mathbb{P}$-null sets.

Consider the Banach space

$$\ell^r = \{u = (u_i)_{i\in\mathbb{Z}^N} : \sum_{i\in\mathbb{Z}^N} |u_i|^r < +\infty\}, \quad \forall r \geq 1,$$

with norm $\|u\|_r = \big(\sum_{i\in\mathbb{Z}^N} |u_i|^r\big)^{\frac{1}{r}}$. In this paper, the norm and the inner product of ℓ^2 are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Assume that F_i is locally Lipschitz continuous from $\mathbb{R}$ to $\mathbb{R}$ uniformly for $i \in \mathbb{Z}^N$, that is, for any compact set $I \subseteq \mathbb{R}$, we can find a constant $c_1 = c_1(I) > 0$ such that

$$|F_i(s_1) - F_i(s_2)| \leq c_1|s_1 - s_2|, \quad \text{for all } s_1, s_2 \in I, \ i \in \mathbb{Z}^N. \tag{2.3}$$

We also assume that

$$F_i(0) = 0 \text{ and } \ F_i'(s) \geq \gamma, \quad \text{for all } i \in \mathbb{Z}^N, \ \ s \in \mathbb{R}, \tag{2.4}$$

where $\gamma \in \mathbb{R}$ is a constant. Assume that $\hat{\sigma}_{k,i}$ are locally Lipschitz continuous form $\mathbb{R}$ to $\mathbb{R}$ uniformly for $k \in \mathbb{N}$ and $i \in \mathbb{Z}^N$, that is, for any compact set $I \subseteq \mathbb{R}$, we can find a constant $c_2 = c_2(I) > 0$ such that

$$|\hat{\sigma}_{k,i}(s_1) - \hat{\sigma}_{k,i}(s_2)| \leq c_2|s_1 - s_2|, \quad \text{for all } s_1, s_2 \in I, \ k \in \mathbb{N}, \ i \in \mathbb{Z}^N. \tag{2.5}$$

Assume that for all $s \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$|\hat{\sigma}_{k,i}(s)| \leq \varphi_{1,i}|s| + \varphi_{2,i}, \quad \varphi_1 = \{\varphi_{1,i}\}_{i\in\mathbb{Z}^N} \in \ell^\infty, \varphi_2 = \{\varphi_{2,i}\}_{i\in\mathbb{Z}^N} \in \ell^2. \tag{2.6}$$

For $\delta = (\delta_{k,i})_{k\in\mathbb{N},i\in\mathbb{Z}^N}$, $g = (g_i)_{i\in\mathbb{Z}^N}$ and $h_k = (h_{k,i})_{i\in\mathbb{Z}^N}$, we assume, for all $\tau \in \mathbb{R}$ and $T > 0$,

$$c_\delta := \sum_{k\in\mathbb{N}} \sum_{i\in\mathbb{Z}^N} |\delta_{k,i}|^2 < \infty, \quad \int_\tau^{\tau+T} \mathbb{E}\Big(\|g(t)\|^2 + \sum_{k\in\mathbb{N}} \|h_k(t)\|^2\Big) dt < \infty. \tag{2.7}$$

Next, we rewrite (2.1)-(2.2) as an abstract one in the sapce ℓ^2. For all $1 \leq j \leq N$, $u = (u_i)_{i\in\mathbb{Z}^N} \in \ell^2$ and $i = (i_1, i_2, \ldots, i_N) \in \mathbb{Z}^N$, we define the operators from ℓ^2 to ℓ^2 by

$$\begin{aligned}
(B_j u)_i &= u_{(i_1,\ldots,i_j+1,\ldots,i_N)} - u_{(i_1,\ldots,i_j,\ldots,i_N)}, \\
(B_j^* u)_i &= u_{(i_1,\ldots,i_j-1,\ldots,i_N)} - u_{(i_1,\ldots,i_j,\ldots,i_N)}, \\
(A_j u)_i &= -u_{(i_1,\ldots,i_j+1,\ldots,i_N)} + 2u_{(i_1,\ldots,i_j,\ldots,i_N)} - u_{(i_1,\ldots,i_j-1,\ldots,i_N)},
\end{aligned}$$

and

$$\begin{aligned}
(Au)_i = &-u_{(i_1-1,i_2,\ldots,i_N)} - u_{(i_1,i_2-1,\ldots,i_N)} - \cdots - u_{(i_1,i_2,\ldots,i_N-1)} \\
&+ 2Nu_{(i_1,i_2,\ldots,i_N)} - u_{(i_1+1,i_2,\ldots,i_N)} - u_{(i_1,i_2+1,\ldots,i_N)} - \cdots - u_{(i_1,i_2,\ldots,i_N+1)}\big).
\end{aligned}$$

For all $1 \leq j \leq N$, $u = (u_i)_{i\in\mathbb{Z}^N} \in \ell^2$ and $v = (v_i)_{i\in\mathbb{Z}^N} \in \ell^2$, we have

$$\|B_j u\| \leq 2\|u\|, \quad (B_j^* u, v) = (u, B_j v), \quad A_j = B_j B_j^* = B_j^* B_j \ \text{ and } \ A = \sum_{j=1}^N A_j. \tag{2.8}$$

For each $i \in \mathbb{Z}^N$, we let $f_i(s) = F_i(s) - \gamma s$ for all $s \in \mathbb{R}$. Then we see from (2.4) that

$$f_i(0) = 0 \text{ and } f_i'(s) \geq 0, \quad \text{for all } i \in \mathbb{Z}^N,\ s \in \mathbb{R}. \tag{2.9}$$

Then we define $F, f, \sigma_k : \ell^2 \to \ell^2$ by

$$F(u) = \big(F_i(u_i)\big)_{i\in\mathbb{Z}^N}, \quad f(u) = \big(f_i(u_i)\big)_{i\in\mathbb{Z}^N}, \quad \sigma_k(u) = \big(\delta_{k,i}\hat{\sigma}_{k,i}(u_i)\big)_{i\in\mathbb{Z}^N}, \ \forall u \in \ell^2.$$

Both F and f are well-defined due to (2.3). In addition, f: $\ell^2 \to \ell^2$ is also locally Lipschitz continuous, that is, for every $n \in \mathbb{N}$, we can find a constant $c_3(n) > 0$ such that for all $u, v \in \ell^2$ with $\|u\| \leq n$ and $\|v\| \leq n$,

$$\|f(u) - f(v)\| \leq c_3(n)\|u - v\|. \tag{2.10}$$

By (2.9), we obtain

$$(f(u) - f(v), u - v) \geq 0, \quad \text{for all } \ u, v \in \ell^2. \tag{2.11}$$

From (2.6)-(2.7), we infer that for all $u \in \ell^2$,

$$\sum_{k\in\mathbb{N}} \|\sigma_k(u)\|^2 \leq 2\sum_{k\in\mathbb{N}}\sum_{i\in\mathbb{Z}^N} |\delta_{k,i}|^2(|\varphi_{1,i}|^2|u_i|^2 + |\psi_{2,i}|^2) \leq 2c_\delta\big(\|\varphi_1\|^2_{\ell^\infty}\|u\|^2 + \|\psi_2\|^2\big). \tag{2.12}$$

Thus we find that σ_k is also well-defined. By (2.5) and (2.7) we deduce that σ_k: $\ell^2 \to \ell^2$ is locally Lipschitz continuous. Then for every $n \in \mathbb{N}$, we can find a constant $c_4(n) > 0$ such that for any $u, v \in \ell^2$ with $\|u\| \leq n$ and $\|v\| \leq n$,

$$\sum_{k\in\mathbb{N}} \|\sigma_k(u) - \sigma_k(v)\|^2 \leq c_4(n)\|u - v\|^2. \tag{2.13}$$

Let $\beta = \lambda + \gamma$, then we are able to rewrite (2.1)-(2.2) as the following system in ℓ^2 for $t > \tau$ with $\tau \in \mathbb{R}$:

$$du(t) + d(Au(t)) + Au(t)dt + \beta u(t)dt \tag{2.14}$$

$$= -f(u(t))dt + g(t)dt + \sum_{k=1}^{\infty}\big(h_k(t) + \sigma_k(u(t))\big)dW_k(t), \tag{2.15}$$

with initial data:

$$u(\tau) = u_\tau \in \ell^2, \tag{2.16}$$

In this article, the solutions to the stochastic (2.14)-(2.16) are understood as bellow.

Definition 2.1. Given $\tau \in \mathbb{R}$ and a $\mathcal{F}_\tau$-measurable $u_\tau \in L^2(\Omega, \ell^2)$, we say a continuous ℓ^2-valued $\mathcal{F}_t$-adapted stochastic process u is a solution of (2.14)-(2.16) if $u \in L^2(\Omega, C([\tau, \tau + T], \ell^2))$ for all $T > 0$, and for all $t \geq \tau$ and almost all $\omega \in \Omega$, we have

$$u(t) + Au(t) = u_\tau + Au_\tau + \int_\tau^t \big(-Au(s) - \beta u(s) - f(u(s)) + g(s)\big)ds$$
$$+ \sum_{k=1}^{\infty}\int_\tau^t \big(h_k(s) + \sigma_k(u(s))\big)dW_k(s), \tag{2.17}$$

in ℓ^2.

3. **Global existence and uniqueness of solutions to (2.14)-(2.16).** In this section we show the existence and uniqueness of solutions to (2.14)-(2.16) in the sense of Definition 2.1. As mentioned before, we must approximate the locally Lipschitz continuous operators f and σ_k. For each $n \in \mathbb{N}$, we define a function $\xi_n : \mathbb{R} \to \mathbb{R}$ by

$$\xi_n(s) = \begin{cases} -n, & \text{for } s \in (-\infty, -n), \\ s, & \text{for } s \in [-n, n], \\ n, & \text{for } s \in (n, +\infty). \end{cases} \tag{3.1}$$

Then

$$\xi_n(0) = 0, \quad |\xi_n(s)| \le n \quad \text{and} \quad |\xi_n(s_1) - \xi_n(s_2)| \le |s_1 - s_2|, \quad \forall s, s_1, s_2 \in \mathbb{R}. \tag{3.2}$$

Let $f^n(u) = \big(f_i(\xi_n(u_i))\big)_{i\in\mathbb{Z}^N}$ and $\sigma_k^n(u) = \big(\delta_{k,i}\hat{\sigma}_{k,i}(\xi_n(u_i))\big)_{i\in\mathbb{Z}^N}$ for $k, n \in \mathbb{N}$ and $u \in \ell^2$, By (2.9), we see

$$(f^n(u) - f^n(v), u - v) \ge 0, \quad \text{for all } n \in \mathbb{N}, u, v \in \ell^2. \tag{3.3}$$

By (2.9), (2.10) and (3.2), for every $n \in \mathbb{N}$, we can find a constant $c_5(n) > 0$ such that

$$\|f^n(u) - f^n(v)\| \le c_5(n)\|u - v\|, \quad \text{for all } u, v \in \ell^2. \tag{3.4}$$

$$\|f^n(u)\| \le c_5(n)\|u\|, \quad \text{for all } u \in \ell^2. \tag{3.5}$$

It follows from (2.5)-(2.7) and (3.2) that there is a $c_6 = c_6(n) > 0$ such that

$$\sum_{k\in\mathbb{N}} \|\sigma_k^n(u) - \sigma_k^n(v)\|^2 \le c_6(n)\|u - v\|^2, \quad \text{for all } u, v \in \ell^2, \tag{3.6}$$

$$\sum_{k\in\mathbb{N}} \|\sigma_k^n(u)\|^2 \le 2c_\delta\big(\|\varphi_1\|^2_{\ell^\infty}\|u\|^2 + \|\varphi_2\|^2\big), \quad \text{for all } u \in \ell^2. \tag{3.7}$$

Given $n \in \mathbb{N}$, we now consider the approximate stochastic system in ℓ^2 for $t > \tau$ with $\tau \in \mathbb{R}$:

$$\begin{aligned} &du_n(t) + d(Au_n(t)) + Au_n(t)dt + \beta u_n(t)dt \\ &= -f^n(u_n(t))dt + g(t)dt + \sum_{k=1}^{\infty} \big(h_k(t) + \sigma_k^n(u_n(t))\big)dW_k(t), \end{aligned} \tag{3.8}$$

with initial data:

$$u_n(\tau) = u_\tau \in \ell^2. \tag{3.9}$$

Following the arguments of showing the well-posedness of stochastic equations in $\mathbb{R}^n$, we can show, under (3.4)-(3.7), that for every $n \in \mathbb{N}$, $\tau \in \mathbb{R}$ and $\mathcal{F}_\tau$-measurable $u_\tau \in L^2(\Omega, \ell^2)$, system (3.8) possesses a unique solution $u_n \in L^2(\Omega, C([\tau,\infty), \ell^2))$ in view of Definition 2.1.

In the next theorem, we will establish the existence of solutions to (2.14)-(2.16) in view of Definition 2.1 by considering the limit of $\{u_n\}_{n=1}^{\infty}$ of solutions to (3.8)-(3.9) as $n \to \infty$.

Theorem 3.1. *Let (2.3)-(2.7) hold. Then for all $\tau \in \mathbb{R}$ and $\mathcal{F}_\tau$-measurable $u_\tau \in L^2(\Omega, \ell^2)$, system (2.14)-(2.16) has a solution u in the sense of Definition 2.1. In addition, u satisfies*

$$\mathbb{E}\Big(\|u\|^2_{C([\tau,\tau+T],\ell^2)}\Big) \le M_1 e^{M_1 T}\bigg(T + \mathbb{E}(\|u_\tau\|^2) + \int_\tau^{\tau+T} \mathbb{E}\Big(\|g(t)\|^2 + \sum_{k\in\mathbb{N}} \|h_k(t)\|^2\Big)dt\bigg), \tag{3.10}$$

where $M_1 > 0$ is a positive constant independent of u_τ, τ and T.

Proof. We first prove that the solutions of (3.8)-(3.9) satisfy that for all $t \geq \tau$ and $n \in \mathbb{N}$,

$$u_{n+1}(t \wedge \varsigma_n) = u_n(t \wedge \varsigma_n) \text{ and } \varsigma_{n+1} \geq \varsigma_n, \text{ a.s.}, \tag{3.11}$$

where the stoping time ς_n is defined by

$$\varsigma_n = \inf\{t \geq \tau : \|u_n\| > n\} \text{ and } \varsigma_n = +\infty \text{ if } \{t \geq \tau : \|u_n\| > n\} = \emptyset. \tag{3.12}$$

By (3.8)-(3.9), we have

$$\begin{aligned} &u_{n+1}(t \wedge \varsigma_n) - u_n(t \wedge \varsigma_n) + Au_{n+1}(t \wedge \varsigma_n) - Au_n(t \wedge \varsigma_n) \\ &+ \int_\tau^{t\wedge\varsigma_n} \big(A(u_{n+1}(s)) - A(u_n(s))\big)ds \\ &+ \beta \int_\tau^{t\wedge\varsigma_n} (u_{n+1}(s) - u_n(s))ds + \int_\tau^{t\wedge\varsigma_n} \big(f^{n+1}(u_{n+1}(s)) - f^n(u_n(s))\big)ds \\ &= \sum_{k=1}^\infty \int_\tau^{t\wedge\varsigma_n} \big(\sigma_k^{n+1}(u_{n+1}(s)) - \sigma_k^n(u_n(s))\big)dW_k(s), \end{aligned} \tag{3.13}$$

Applying Ito's formula to (3.13), we infer that a.s.,

$$\begin{aligned} &\|u_{n+1}(t \wedge \varsigma_n) - u_n(t \wedge \varsigma_n)\|^2 \\ &+ \sum_{j=1}^N \|B_j(u_{n+1}(t \wedge \varsigma_n) - u_n(t \wedge \varsigma_n)\|^2 \\ &+ 2\int_\tau^{t\wedge\varsigma_n} \sum_{j=1}^N \|B_j(u_{n+1}(s) - u_n(s))\|^2 ds + 2\beta \int_\tau^{t\wedge\varsigma_n} \|u_{n+1}(s) - u_n(s)\|^2 ds \\ &+ 2\int_\tau^{t\wedge\varsigma_n} \big(f^{n+1}(u_{n+1}(s)) - f^n(u_n(s)), u_{n+1}(s) - u_n(s)\big)ds \\ &= \sum_{k=1}^\infty \int_\tau^{t\wedge\varsigma_n} \|\sigma_k^{n+1}(u_{n+1}(s)) - \sigma_k^n(u_n(s))\|^2 ds \\ &+ 2\sum_{k=1}^\infty \int_\tau^{t\wedge\varsigma_n} \big(u_{n+1}(s) - u_n(s)\big)\big(\sigma_k^{n+1}(u_{n+1}(s)) - \sigma_k^n(u_n(s))\big)dW_k(s), \end{aligned} \tag{3.14}$$

where we identify $u_{n+1}(s) - u_n(s)$ in the stochastic term with an element in the dual space of ℓ^2 in view of the Riesz representation theorem. By $\|u_n(s)\| \leq n$ for all $s \in [\tau, \varsigma_n)$ we have

$$f^{n+1}(u_n(s)) = f^n(u_n(s)) \text{ and } \sigma_k^{n+1}(u_n(s)) = \sigma_k^n(u_n(s)), \ \forall s \in [\tau, \varsigma_n). \tag{3.15}$$

By (3.3) and (3.15) we find

$$\begin{aligned} &\int_\tau^{t\wedge\varsigma_n} \big(f^{n+1}(u_{n+1}(s)) - f^n(u_n(s)), u_{n+1}(s) - u_n(s)\big)ds \\ &= \int_\tau^{t\wedge\varsigma_n} \big(f^{n+1}(u_{n+1}(s)) - f^{n+1}(u_n(s)), u_{n+1}(s) - u_n(s)\big)ds \geq 0, \end{aligned} \tag{3.16}$$

By (3.6) and (3.15) we get

$$\sum_{k=1}^\infty \int_\tau^{t\wedge\varsigma_n} \|\sigma_k^{n+1}(u_{n+1}(s)) - \sigma_k^n(u_n(s))\|^2 ds$$

$$= \sum_{k=1}^{\infty} \int_{\tau}^{t\wedge\varsigma_n} \|\sigma_k^{n+1}(u_{n+1}(s)) - \sigma_k^{n+1}(u_n(s))\|^2 ds$$
$$\le c_6(n+1) \int_{\tau}^{t\wedge\varsigma_n} \|u_{n+1}(s) - u_n(s)\|^2 ds. \tag{3.17}$$

Then we infer from (3.14)-(3.17) that

$$\mathbb{E}\Big(\sup_{\tau\le s\le t} \|u_{n+1}(s\wedge\varsigma_n) - u_n(s\wedge\varsigma_n)\|^2\Big)$$
$$\le C_0 \int_{\tau}^{t} \mathbb{E}\Big(\sup_{\tau\le s\le r} \|u_{n+1}(s\wedge\varsigma_n) - u_n(s\wedge\varsigma_n)\|^2\Big) dr$$
$$+ 2\mathbb{E}\Bigg(\sup_{\tau\le s\le t\wedge\varsigma_n} \Big|\sum_{k=1}^{\infty} \int_{\tau}^{s} (u_{n+1}(r) - u_n(r))\big(\sigma_k^{n+1}(u_{n+1}(r)) - \sigma_k^{n+1}(u_n(r))\big) dW_k(r)\Big|\Bigg), \tag{3.18}$$

where $C_0 = (2|\beta| + c_6(n+1))$. From the BDG inequality and (3.6), we find a constant $C_1 > 0$ such that

$$2\mathbb{E}\Bigg(\sup_{\tau\le s\le t\wedge\varsigma_n} \Big|\sum_{k=1}^{\infty} \int_{\tau}^{s} (u_{n+1}(r) - u_n(r))(\sigma_k^{n+1}(u_{n+1}(r)) - \sigma_k^{n+1}(u_n(r))) dW_k(r)\Big|\Bigg)$$
$$\le 2C_1\mathbb{E}\Bigg(\Big(\int_{\tau}^{t\wedge\varsigma_n} \Big(\|u_{n+1}(r) - u_n(r)\|^2 \sum_{k=1}^{\infty} \|\sigma_k^{n+1}(u_{n+1}(r)) - \sigma_k^{n+1}(u_n(r))\|^2\Big) dr\Big)^{\frac{1}{2}}\Bigg)$$
$$\le 2C_1\mathbb{E}\Bigg(\sup_{\tau\le r\le t} \|u_{n+1}(r\wedge\varsigma_n) - u_n(r\wedge\varsigma_n)\| \Big(\int_{\tau}^{t\wedge\varsigma_n} \sum_{k=1}^{\infty} \|\sigma_k^{n+1}(u_{n+1}(r)) - \sigma_k^{n+1}(u_n(r))\|^2 dr\Big)^{\frac{1}{2}}\Bigg)$$
$$\le 2\sqrt{c_6(n+1)}C_1\mathbb{E}\Bigg(\sup_{\tau\le r\le t} \|u_{n+1}(r\wedge\varsigma_n) - u_n(r\wedge\varsigma_n)\| \Big(\int_{\tau}^{t\wedge\varsigma_n} \|u_{n+1}(r) - u_n(r)\|^2 dr\Big)^{\frac{1}{2}}\Bigg)$$
$$\le \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\le s\le t} \|u_{n+1}(s\wedge\varsigma_n) - u_n(s\wedge\varsigma_n)\|^2\Big) + C_2 \int_{\tau}^{t} \mathbb{E}\Big(\sup_{\tau\le s\le r} \|u_{n+1}(s\wedge\varsigma_n) - u_n(s\wedge\varsigma_n)\|^2\Big) dr, \tag{3.19}$$

where $C_2 = 2c_6(n+1)C_1^2$. It yields from (3.18)-(3.19) that

$$\mathbb{E}\Big(\sup_{\tau\le s\le t} \|u_{n+1}(s\wedge\varsigma_n) - u_n(s\wedge\varsigma_n)\|^2\Big)$$
$$\le 2(C_0 + C_2) \int_{\tau}^{t} \mathbb{E}\Big(\sup_{\tau\le s\le r} \|u_{n+1}(s\wedge\varsigma_n) - u_n(s\wedge\varsigma_n)\|^2\Big) dr. \tag{3.20}$$

From the Gronwall lemma and (3.20), we find

$$\mathbb{E}\big(\sup_{\tau\le s\le t} \|u_{n+1}(s\wedge\varsigma_n) - u_n(s\wedge\varsigma_n)\|^2\big) = 0, \quad \forall t \ge \tau. \tag{3.21}$$

Then we find that $u_{n+1}(t\wedge\varsigma_n) = u_n(t\wedge\varsigma_n)$ for all $t \ge \tau$ a.s.. From this and (3.12) we find (3.11).

We then prove the stoping time satisfies:

$$\varsigma := \lim_{n\to\infty} \varsigma_n = \sup_{n\in\mathbb{N}} \varsigma_n = \infty, \quad \text{a.s..} \tag{3.22}$$

In order to prove (3.22), we first deduce the following uniform estimates for the solutions u_n to (3.8):

$$\mathbb{E}\big(\|u_n\|^2_{C([\tau,\tau+T],\ell^2)}\big) \le M, \qquad \forall T > 0, \tag{3.23}$$

where

$$M = M_1 e^{M_1 T}\Big(\mathbb{E}(\|u_\tau\|^2) + T + \int_{\tau}^{\tau+T} \mathbb{E}\big(\|g(s)\|^2 + \sum_{k\in\mathbb{N}} \|h_k(s)\|^2\big) ds\Big)$$

and $M_1 > 0$ is a constant independent of u_τ, n, τ and T.

Applying Ito's formula to (3.8), we find

$$\|u_n(t)\|^2 + \sum_{j=1}^{N} \|B_j u_n(t)\|^2 + 2\int_\tau^t \sum_{j=1}^{N} \|B_j u_n(s)\|^2 ds + 2\beta \int_\tau^t \|u_n(s)\|^2 ds$$

$$+ 2\int_\tau^t \big(f^n(u_n(s)), u_n(s)\big) ds$$

$$= \|u_\tau\|^2 + \sum_{j=1}^{N} \|B_j u_\tau\|^2 + 2\int_\tau^t \big(g(s), u_n(s)\big) ds + \sum_{k=1}^{\infty} \int_\tau^t \|h_k(s) + \sigma_k^n(u_n(s))\|^2 ds$$

$$+ 2\sum_{k=1}^{\infty} \int_\tau^t u_n(s)\big(h_k(s) + \sigma_k^n(u_n(s))\big) dW_k(s). \tag{3.24}$$

This together with (3.3) and Young's inequality gives

$$\mathbb{E}\Big(\sup_{\tau \le r \le t} \|u_n(r)\|^2\Big) \le (1+4N)\mathbb{E}(\|u_\tau\|^2) + (1+2|\beta|) \int_\tau^t \mathbb{E}(\|u_n(s)\|^2) ds$$

$$+ \int_\tau^t \mathbb{E}(\|g(s)\|^2) ds + 2\int_\tau^t \mathbb{E}\Big(\sum_{k=1}^{\infty} \|h_k(s)\|^2\Big) ds + 2\sum_{k=1}^{\infty} \int_\tau^t \mathbb{E}\Big(\|\sigma_k^n(u_n(s))\|^2\Big) ds$$

$$+ 2\mathbb{E}\left(\sup_{\tau \le r \le t} \left|\sum_{k=1}^{\infty} \int_\tau^r u_n(s)\big(h_k(s) + \sigma_k^n(u_n(s))\big) dW_k(s)\right|\right). \tag{3.25}$$

It yields from (3.7) that for $t \in [\tau, \tau+T]$,

$$2\sum_{k=1}^{\infty} \int_\tau^t \mathbb{E}\Big(\|\sigma_k^n(u_n(s))\|^2\Big) ds \le 4c_\delta \|\varphi_1\|_{\ell^\infty}^2 \int_\tau^t \mathbb{E}(\|u_n(s)\|^2) ds + 4c_\delta T \|\varphi_2\|^2. \tag{3.26}$$

From the BDG inequality and (3.26) we find that for $t \in [\tau, \tau+T]$,

$$2\mathbb{E}\left(\sup_{\tau \le r \le t} \left|\sum_{k=1}^{\infty} \int_\tau^r u_n(s)\big(h_k(s) + \sigma_k^n(u_n(s))\big) dW_k(s)\right|\right)$$

$$\le 2C_1 \mathbb{E}\left(\int_\tau^t \sum_{k=1}^{\infty} \|u_n(s)\|^2 \|h_k(s) + \sigma_k^n(u_n(s))\|^2 ds\right)^{\frac{1}{2}}$$

$$\le 2\sqrt{2} C_1 \mathbb{E}\left(\sup_{\tau \le s \le t} \|u_n(s)\| \left(\int_\tau^t \sum_{k=1}^{\infty} \Big(\|h_k(s)\|^2 + \|\sigma_k^n(u_n(s))\|^2\Big) ds\right)^{\frac{1}{2}}\right)$$

$$\le \frac{1}{2}\mathbb{E}\Big(\sup_{\tau \le s \le t} \|u_n(s)\|^2\Big) + 4C_1^2 \int_\tau^t \mathbb{E}\Big(\sum_{k=1}^{\infty} \Big(\|h_k(s)\|^2 + \|\sigma_k^n(u_n(s))\|^2\Big) ds$$

$$\le \frac{1}{2}\mathbb{E}\Big(\sup_{\tau \le s \le t} \|u_n(s)\|^2\Big) + 8c_\delta C_1^2 \|\varphi_1\|_{\ell^\infty}^2 \int_\tau^t \mathbb{E}(\|u_n(s)\|^2) ds$$

$$+ 4C_1^2 \int_\tau^t \mathbb{E}\Big(\sum_{k=1}^{\infty} \|h_k(s)\|^2\Big) ds + 8c_\delta T C_1^2 \|\varphi_2\|^2, \tag{3.27}$$

where C_1 is the same number as given in (3.19). Then we find from (3.25)-(3.27) that for $t \in [\tau, \tau+T]$,

$$\mathbb{E}\Big(\sup_{\tau \le r \le t} \|u_n(r)\|^2\Big) \le C_3 \int_\tau^t \mathbb{E}(\sup_{\tau \le r \le s} \|u_n(r)\|^2) ds + C_4. \tag{3.28}$$

where $C_3 = 2 + 4|\beta| + 8c_\delta \|\varphi_1\|^2_{\ell^\infty}(1 + 2C_1^2)$ and C_4 is given by

$$C_4 = 2(1+4N)\mathbb{E}(\|u_\tau\|^2) + 4(1+2C_1^2)\int_\tau^{\tau+T} \mathbb{E}\Big(\|g(s)\|^2 + \sum_{k=1}^{\infty} \|h_k(s)\|^2\Big) ds + 8(1+2C_1^2)c_\delta \|\varphi_2\|^2 T.$$

From the Gronwall lemma and (3.28) we can deduce that

$$\mathbb{E}\Big(\sup_{\tau \le r \le t} \|u_n(r)\|^2\Big) \le C_4 e^{C_3 T}, \quad \forall t \in [\tau, \tau+T]$$

This implies (3.23). In the following, we prove (3.22). Let $T > 0$ be an arbitrary number. By (3.12), we infer that

$$\{\varsigma_n < T\} \subseteq \{\|u_n\|_{C([\tau,\tau+T],\ell^2)} \ge n\}.$$

Then we deduce from Chebychev's inequality and (3.23) that

$$\mathbb{P}\{\varsigma_n < T\} \le \mathbb{P}\Big(\|u_n\|_{C([\tau,\tau+T],\ell^2)} \ge n\} \le \frac{1}{n^2}\mathbb{E}(\|u_n\|^2_{C([\tau,\tau+T],\ell^2)}) \le \frac{M}{n^2},$$

which implies

$$\sum_{n=1}^{\infty} \mathbb{P}\{\varsigma_n < T\} \le M \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Taking $\Omega_T = \bigcap\limits_{m=1}^{\infty} \bigcup\limits_{n=m}^{\infty} \{\varsigma_n < T\}$, we find from the Borel-Cantelli lemma that

$$\mathbb{P}(\Omega_T) = \mathbb{P}\big(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\varsigma_n < T\}\big) = 0.$$

Then for every $\omega \in \Omega \setminus \Omega_T$, we find a $n_0 = n_0(\omega) > 0$ such that $\varsigma_n(\omega) \ge T$ for all $n \ge n_0$, and thus $\varsigma(\omega) \ge T$ for all $\omega \in \Omega \setminus \Omega_T$. Let $\Omega_0 = \bigcup\limits_{T=1}^{\infty} \Omega_T$, then $\mathbb{P}(\Omega_0) = 0$ and $\varsigma(\omega) \ge T$ for all $\omega \in \Omega \setminus \Omega_0$ and $T \in \mathbb{N}$. Then (3.22) yields.

We finally show the existence of solutions to (2.14). By Steps 1-2, we find a $\Omega_1 \subseteq \Omega$ with $\mathbb{P}(\Omega \setminus \Omega_1) = 0$ such that

$$\varsigma(\omega) = \lim_{n\to\infty} \varsigma_n(\omega) = \infty, \quad u_{n+1}(t \wedge \varsigma_n, \omega) = u_n(t \wedge \varsigma_n, \omega), \ \forall n \in \mathbb{N}, \omega \in \Omega_1, t \ge \tau. \tag{3.29}$$

By (3.29), for every $\omega \in \Omega_1$ and $t \ge \tau$, we find a $n_0 = n_0(t, \omega) \ge 1$ such that for all $n \ge n_0$,

$$\varsigma_n(\omega) > t, \text{ and thus } u_n(t,\omega) = u_{n_0}(t,\omega) \tag{3.30}$$

Define a mapping $u : [\tau, \infty) \times \Omega \to \ell^2$ given by

$$u(t,\omega) = \begin{cases} u_n(t,\omega), & \text{if } \omega \in \Omega_1 \text{ and } t \in [\tau, \varsigma_n(\omega)], \\ u_\tau(\omega), & \text{if } \omega \in \Omega \setminus \Omega_1 \text{ and } t \in [\tau, \infty). \end{cases} \tag{3.31}$$

Note that u_n is a continuous ℓ^2-valued process, by (3.31), we infer that u is also continuous for t in ℓ^2 a.s.. By (3.31), we find

$$\lim_{n\to\infty} u_n(t,\omega) = u(t,\omega), \ \forall \omega \in \Omega_1, \ t \ge \tau. \tag{3.32}$$

Since u_n is $\mathcal{F}_t$-adapted, by (3.32) we deduce that u is also $\mathcal{F}_t$-adapted. By (3.32), (3.23) and Fatou's lemma we see

$$\mathbb{E}\Big(\|u\|^2_{C([\tau,\tau+T],\ell^2)}\Big) \le M, \quad \forall T>0.$$

This implies (3.10). By (3.8) we get

$$\begin{aligned} &u_n(t\wedge\varsigma_n)+Au_n(t\wedge\varsigma_n)\\ &= u_\tau + Au_\tau + \int_\tau^{t\wedge\varsigma_n}\big(-Au_n(s)-\beta u_n(s)-f^n(u_n(s))+g(s)\big)ds\\ &+\sum_{k=1}^\infty\int_\tau^{t\wedge\varsigma_n}\big(h_k(s)+\sigma_k^n(u_n(s))\big)dW_k(s), \end{aligned} \tag{3.33}$$

in ℓ^2 for all $t\ge\tau$. By (3.31) we see $u_n(t\wedge\varsigma_n)=u(t\wedge\varsigma_n)$ a.s., which implies a.s.,

$$f^n(u_n(s))=f(u(s)) \ \text{ and } \ \sigma_k^n(u_n(s))=\sigma_k(u(s)), \quad \text{for all } s\in[\tau,\varsigma_n). \tag{3.34}$$

Therefore we see from (3.33)-(3.34) that a.s.,

$$\begin{aligned} u(t\wedge\varsigma_n)+Au(t\wedge\varsigma_n) &= u_\tau + Au_\tau + \int_\tau^{t\wedge\varsigma_n}\big(-Au(s)-\beta u(s)-f(u(s))+g(s)\big)ds\\ &+\sum_{k=1}^\infty\int_\tau^{t\wedge\varsigma_n}\big(h_k(s)+\sigma_k(u(s))\big)dW_k(s), \end{aligned} \tag{3.35}$$

in ℓ^2 for all $t\ge\tau$. Since $\lim_{n\to\infty}\varsigma_n=\infty$ a.s.. Then we find from (3.35) that

$$\begin{aligned} u(t)+Au(t) &= u_\tau + Au_\tau + \int_\tau^{t}\big(-Au(s)-\beta u(s)-f(u(s))+g(s)\big)ds\\ &+\sum_{k=1}^\infty\int_\tau^{t}\big(h_k(s)+\sigma_k(u(s))\big)dW_k(s), \end{aligned}$$

in ℓ^2 for all $t\ge\tau$. Thus u is a solution of (2.14) in view of Definition 2.1. □

Now, we prove the uniqueness of the solutions to (2.14)-(2.16).

Theorem 3.2. *Let* (2.3)-(2.7) *hold. Then the solution to system* (2.14)-(2.16) *is unique.*

Proof. Let u_1 and u_2 be two solutions of (2.14). Given $n\in\mathbb{N}$ and $T>0$, we define a stoping time:

$$T_n=(\tau+T)\wedge\inf\{t\ge\tau : \|u_1(t)\|\ge n \text{ or } \|u_2(t)\|\ge n\}. \tag{3.36}$$

By (2.14)-(2.16), we get

$$\begin{aligned} &u_1(t\wedge T_n)-u_2(t\wedge T_n)+Au_1(t\wedge T_n)-Au_2(t\wedge T_n)+\beta\int_\tau^{t\wedge T_n}(u_1(s)-u_2(s))ds\\ &+\int_\tau^{t\wedge T_n}\big(A(u_1(s))-A(u_2(s))\big)ds+\int_\tau^{t\wedge T_n}\big(f(u_1(s))-f(u_2(s))\big)ds\\ &=u_1(\tau)-u_2(\tau)+Au_1(\tau)-Au_2(\tau)+\sum_{k=1}^\infty\int_\tau^{t\wedge T_n}\big(\sigma_k(u_1(s))-\sigma_k(u_2(s))\big)dW_k(s). \end{aligned} \tag{3.37}$$

From Ito's formula and (3.37), we find that a.s.,

$$\|u_1(t\wedge T_n)-u_2(t\wedge T_n)\|^2+\sum_{j=1}^{N}\|B_j(u_1(t\wedge T_n)-u_2(t\wedge T_n))\|^2$$
$$+2\beta\int_\tau^{t\wedge T_n}\|u_1(s)-u_2(s)\|^2ds$$
$$+2\int_\tau^{t\wedge T_n}\sum_{j=1}^{N}\|B_j(u_1(s)-u_2(s))\|^2ds+2\int_\tau^{t\wedge T_n}\big(f(u_1(s))-f(u_2(s)),u_1(s)-u_2(s)\big)ds$$
$$=\|u_1(\tau)-u_2(\tau)\|^2+\sum_{j=1}^{N}\|B_j(u_1(\tau)-u_2(\tau))\|^2+\sum_{k=1}^{\infty}\int_\tau^{t\wedge T_n}\|\sigma_k(u_1(s))-\sigma_k(u_2(s))\|^2ds$$
$$+2\sum_{k=1}^{\infty}\int_\tau^{t\wedge T_n}\big(u_1(s)-u_2(s)\big)\big(\sigma_k(u_1(s))-\sigma_k(u_2(s))\big)dW_k(s). \tag{3.38}$$

We infer from (2.11) that

$$\int_\tau^{t\wedge T_n}\big(f(u_1(s))-f(u_2(s)),u_1(s)-u_2(s)\big)ds\geq 0. \tag{3.39}$$

By (2.13) and (3.36), we have

$$\sum_{k=1}^{\infty}\int_\tau^{t\wedge T_n}\|\sigma_k(u_1(s))-\sigma_k(u_2(s))\|^2ds\leq c_4(n)\int_\tau^{t\wedge T_n}\|u_1(s)-u_2(s)\|^2ds. \tag{3.40}$$

Then we find from (3.38)-(3.40) that

$$\|u_1(t\wedge T_n)-u_2(t\wedge T_n)\|^2+\|\sum_{j=1}^{N}B_j(u_1(t\wedge T_n)-u_2(t\wedge T_n))\|^2$$
$$+2\int_\tau^{t\wedge T_n}\sum_{j=1}^{N}\|B_j(u_1(s)-u_2(s))\|^2ds$$
$$\leq(1+4N)\|u_1(\tau)-u_2(\tau)\|^2+\big(2|\beta|+c_4(n)\big)\int_\tau^{t\wedge T_n}\|u_1(s)-u_2(s)\|^2ds$$
$$+2\sum_{k=1}^{\infty}\int_\tau^{t\wedge T_n}\big(u_1(s)-u_2(s)\big)\big(\sigma_k(u_1(s))-\sigma_k(u_2(s))\big)dW_k(s).$$

This yields

$$\mathbb{E}\Big(\sup_{\tau\leq s\leq t}\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)$$
$$\leq(1+4N)\mathbb{E}\big(\|u_1(\tau)-u_2(\tau)\|^2\big)+\big(2|\beta|+c_4(n)\big)\int_\tau^t\sup_{\tau\leq s\leq r}\mathbb{E}\Big(\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)dr$$
$$+2\mathbb{E}\left(\sup_{\tau\leq s\leq t\wedge T_n}\left|\sum_{k=1}^{\infty}\int_\tau^s\big(u_1(r)-u_2(r)\big)\big(\sigma_k(u_1(r))-\sigma_k(u_2(r))\big)dW_k(r)\right|\right). \tag{3.41}$$

From the BDG inequality and (3.40), the last term in (3.41) satisfies

$$2\mathbb{E}\left(\sup_{\tau\leq s\leq t\wedge T_n}\left|\sum_{k=1}^{\infty}\int_\tau^s\big(u_1(s)-u_2(s)\big)\big(\sigma_k(u_1(r))-\sigma_k(u_2(r))\big)dW_k(r)\right|\right)$$
$$\leq 2C_1\mathbb{E}\left(\left(\int_\tau^{t\wedge T_n}\left(\|u_1(r)-u_2(r)\|^2\sum_{k=1}^{\infty}\|\sigma_k(u_1(r))-\sigma_k(u_2(r))\|^2\right)dr\right)^{\frac{1}{2}}\right)$$

$$\le 2C_1\mathbb{E}\left(\sup_{\tau\le r\le t}\|u_1(r\wedge T_n)-u_2(r\wedge T_n)\|\left(\int_\tau^{t\wedge T_n}\sum_{k=1}^\infty\|\sigma_k(u_1(r))-\sigma_k(u_n(r))\|^2dr\right)^{\frac12}\right)$$

$$\le 2\sqrt{c_4(n)}C_1\mathbb{E}\left(\sup_{\tau\le r\le t}\|u_1(r\wedge T_n)-u_2(r\wedge T_n)\|\left(\int_\tau^{t\wedge T_n}\|u_1(r)-u_2(r)\|^2dr\right)^{\frac12}\right)$$

$$\le \frac12\mathbb{E}\Big(\sup_{\tau\le s\le t}\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)$$

$$+2c_4(n)C_1^2\int_\tau^t\mathbb{E}\Big(\sup_{\tau\le s\le r}\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)dr. \tag{3.42}$$

Then we find from (3.41)-(3.42) that

$$\mathbb{E}\Big(\sup_{\tau\le s\le t}\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)$$

$$\le 2(1+4N)\mathbb{E}\Big(\|u_1(\tau)-u_2(\tau)\|^2\Big)+C_7\int_\tau^t\sup_{\tau\le s\le r}\mathbb{E}\Big(\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)dr, \tag{3.43}$$

where $C_7=4|\beta|+2c_4(n)+4c_4(n)C_1^2$. From the Gronwall lemma and (3.43) we find

$$\mathbb{E}\Big(\sup_{\tau\le s\le\tau+T}\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)\le 2(1+4N)e^{C_7T}\mathbb{E}\Big(\|u_1(\tau)-u_2(\tau)\|^2\Big). \tag{3.44}$$

For $u_1(\tau)=u_2(\tau)$ in $L^2(\Omega,\ell^2)$, we see from (3.44) that

$$\mathbb{E}\Big(\sup_{\tau\le s\le\tau+T}\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)=0.$$

Then

$$\|u_1(t\wedge T_n)-u_2(t\wedge T_n)\|=0,\quad \text{for all } t\in[\tau,\tau+T] \text{ a.e..}$$

Note that $T_n=\tau+T$ for large enough n thanks to the continuity of u_1 and u_2 in t. Then we find that

$$\|u_1(t)-u_2(t)\|=0,\quad \text{for all } t\in[\tau,\tau+T] \text{ almost surely.}$$

This shows

$$\mathbb{P}\Big(\|u_1(t)-u_2(t)\|^2=0\ \text{ for all }\ t\in[\tau,\tau+T]\Big)=1,\quad \forall T>0.$$

Since T is an arbitrary number, we have

$$\mathbb{P}\Big(\|u_1(t)-u_2(t)\|^2=0\ \text{ for all }\ t\ge\tau\Big)=1.$$

Thus the uniqueness of the solutions yields. □

4. Mean random dynamical systems, attracting universes and conditions.

Now, we rewrite (2.1)-(2.2) as the following stochastic system in ℓ^2 for $t>\tau$ with $\tau\in\mathbb{R}$:

$$du(t)+d(Au(t))+Au(t)dt+\lambda u(t)dt$$

$$=-F(u(t))dt+g(t)dt+\sum_{k=1}^\infty\big(h_k(t)+\sigma_k(u(t))\big)dW_k(t), \tag{4.1}$$

with initial data:

$$u(\tau)=u_\tau\in\ell^2. \tag{4.2}$$

From Theorems 3.1-3.2 we find that for every $\tau \in \mathbb{R}$ and $\mathcal{F}_\tau$-measurable $u_\tau \in L^2(\Omega, \ell^2)$, system (4.1)-(4.2) has a unique solution $u \in C([\tau, \infty), \ell^2)$ $\mathbb{P}$-a.s.. Then by the Lebesgue dominated convergence theorem and the uniform estimates similar to (3.10), we can show $u \in C([\tau, \infty), L^2(\Omega, \ell^2))$. Define a mapping

$$\Phi(t, \tau) : L^2(\Omega, \mathcal{F}_\tau; \ell^2) \to L^2(\Omega, \mathcal{F}_{\tau+t}; \ell^2)$$

by

$$\Phi(t, \tau)u_\tau = u(t + \tau, \tau, u_\tau),\ t \in \mathbb{R}^+,\ \tau \in \mathbb{R},\ u_\tau \in L^2(\Omega, \mathcal{F}_\tau; \ell^2). \tag{4.3}$$

Then we find that Φ is a mean random dynamical system for (4.1)-(4.2) on $L^2(\Omega, \mathcal{F}, \ell^2)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ in view of [16, Def. 2.1], that is, for all $t, s \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$,

1. $\Phi(t, \tau)$ is a mapping from $L^2(\Omega, \mathcal{F}_\tau, \ell^2)$ to $L^2(\Omega, \mathcal{F}_{\tau+t}, \ell^2)$;
2. $\Phi(0, \tau)$ is an identity operator on $L^2(\Omega, \mathcal{F}_\tau, \ell^2)$;
3. $\Phi(t + s, \tau) = \Phi(t, s + \tau) \circ \Phi(s, \tau)$.

Notice that Φ is also said a mean *square* random dynamical system, see e.g., Kloeden and Lorenz [10].

In order to derive several kinds of estimates of solutions to (4.1)-(4.2), we next define two families $\mathcal{D} = \{\mathcal{D}(\tau) \subseteq L^2(\Omega, \mathcal{F}_\tau, \ell^2) : \tau \in \mathbb{R}\}$ and $\mathcal{B} = \{\mathcal{B}(\tau) \subseteq L^2(\Omega, \mathcal{F}_\tau, \ell^2) : \tau \in \mathbb{R}\}$ of bounded nonempty subsets satisfying the following conditions:

$$\lim_{t\to+\infty} e^{-\hat{\lambda}t}\|\mathcal{D}(\tau - t)\|^2_{L^2(\Omega,\mathcal{F}_{\tau-t},\ell^2)} = 0, \quad \tau \in \mathbb{R}, \tag{4.4}$$

$$\lim_{t\to+\infty} e^{-\hat{\lambda}t} \sup_{s\le\tau} \|\mathcal{B}(s - t)\|^2_{L^2(\Omega,\mathcal{F}_{s-t},\ell^2)} = 0, \quad \tau \in \mathbb{R}, \tag{4.5}$$

where $\|\mathcal{D}(\tau - t)\|_{L^2(\Omega,\mathcal{F}_{\tau-t},\ell^2)} = \sup_{u\in\mathcal{D}(\tau-t)} \|u\|_{L^2(\Omega,\mathcal{F}_{\tau-t},\ell^2)}$. Denote by

$$\mathfrak{D} = \left\{\mathcal{D} = \{\mathcal{D}(\tau) \subseteq L^2(\Omega, \mathcal{F}_\tau, \ell^2) : \tau \in \mathbb{R} \text{ and } \mathcal{D}(\tau) \neq \emptyset \text{ is bounded}\} : \mathcal{D} \text{ satisfies (4.4)}\right\}, \tag{4.6}$$

$$\mathfrak{B} = \left\{\mathcal{B} = \{\mathcal{B}(\tau) \subseteq L^2(\Omega, \mathcal{F}_\tau, \ell^2) : \tau \in \mathbb{R} \text{ and } \mathcal{B}(\tau) \neq \emptyset \text{ is bounded}\} : \mathcal{B} \text{ satisfies (4.5)}\right\}. \tag{4.7}$$

To prove our main results, we make the following assumptions:

$$c_\delta\|\varphi_1\|^2_{\ell^\infty} \le \frac{\lambda}{8}, \qquad F_i(s)s \ge 0, \quad \forall s \in \mathbb{R},\ i \in \mathbb{Z}^N, \tag{4.8a}$$

$$\sup_{s\le\tau} \int_{-\infty}^{s} e^{\frac{1}{2}\hat{\lambda}r}\mathbb{E}\Big(\|g(r)\|^2 + \sum_{k\in\mathbb{N}} \|h_k(r)\|^2\Big)dr < \infty, \quad \forall\tau \in \mathbb{R}, \tag{4.8b}$$

where $\hat{\lambda} := 2 \wedge \lambda$.

5. **Two types of uniform estimates and absorbing sets.** In this section we first provide two types of long-time uniform estimates of solutions to problem (4.1)-(4.2).

Lemma 5.1. Let (2.3)-(2.7) and (4.8a)-(4.8b) hold. Then we have the following two types of long-time uniform estimates of solutions to (4.1)-(4.2).

(1) For every $\tau \in \mathbb{R}$ and $\mathcal{D} = \{\mathcal{D}(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{D}$, we can find a $T := T(\tau, \mathcal{D}) > 0$ such that the solutions of (4.1)-(4.2) satisfy

$$\sup_{t\ge T} \sup_{u_{\tau-t}\in\mathcal{D}(\tau-t)} \mathbb{E}(\|u(\tau, \tau - t, u_{\tau-t})\|^2)$$

$$\leq R_{\mathfrak{D}}(\tau) := L + L\int_{-\infty}^{\tau} e^{\hat{\lambda}(r-\tau)}\mathbb{E}\Big(\|g(r)\|^2 + \sum_{k\in\mathbb{N}}\|h_k(r)\|^2\Big)dr. \tag{5.1}$$

(2) For every $\tau \in \mathbb{R}$ and $\mathcal{B} = \{\mathcal{B}(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{B}$, we can find a $T := T(\tau, \mathcal{B}) > 0$ such that the solutions of (4.1)-(4.2) satisfy

$$\sup_{t\geq T}\sup_{s\leq\tau}\sup_{u_{s-t}\in\mathcal{B}(s-t)} \mathbb{E}(\|u(s, s-t, u_{s-t})\|^2)$$
$$\leq R_{\mathfrak{B}}(\tau) := L + L\sup_{s\leq\tau}\int_{-\infty}^{s} e^{\hat{\lambda}(r-s)}\mathbb{E}\Big(\|g(r)\|^2 + \sum_{k\in\mathbb{N}}\|h_k(r)\|^2\Big)dr. \tag{5.2}$$

Here $L > 0$ is a number depending on λ but independent of τ, $\mathcal{D}$ and $\mathcal{B}$.

Proof. Note that the proof of (1) is just a special case of (2) for $s = \tau$, we will only prove (1).

Applying Ito's formula to (4.1)-(4.2), we obtain

$$d(\|u(t)\|^2 + \sum_{j=1}^{N}\|B_j u(t)\|^2) + 2\big(\sum_{j=1}^{N}\|B_j u(t)\|^2 + \lambda\|u(t)\|^2 + (F(u(t))), u(t)\big)dt$$
$$= 2(g(t), u(t))dt + \sum_{k=1}^{\infty}\|h_k(t) + \sigma_k(u(t))\|^2 dt + 2\sum_{k=1}^{\infty} u(t)\big(h_k(t) + \sigma_k(u(t))\big)dW_k(t).$$

This along with (4.8a) implies

$$\frac{d}{dt}\mathbb{E}\big(\|u(t)\|^2 + \sum_{j=1}^{N}\|B_j u(t)\|^2\big) + 2\mathbb{E}\big(\sum_{j=1}^{N}\|B_j u(t)\|^2 + \lambda\|u(t)\|^2\big)$$
$$\leq \frac{1}{2}\lambda\mathbb{E}(\|u(t)\|^2) + \frac{2}{\lambda}\mathbb{E}(\|g(t)\|^2) + 2\sum_{k=1}^{\infty}\mathbb{E}(\|h_k(t)\|^2) + 2\sum_{k=1}^{\infty}\mathbb{E}(\|\sigma_k(u(t))\|^2). \tag{5.3}$$

It follows from (2.12) and (4.8a) that

$$2\sum_{k=1}^{\infty}\mathbb{E}(\|\sigma_k(u(t))\|^2) \leq 4c_\delta\|\varphi_1\|^2_{\ell^\infty}\mathbb{E}(\|u\|^2) + 4c_\delta\|\varphi_2\|^2 \leq \frac{1}{2}\lambda\mathbb{E}(\|u\|^2) + 4c_\delta\|\varphi_2\|^2. \tag{5.4}$$

By (5.3)-(5.4), we get

$$\frac{d}{dt}\mathbb{E}\big(\|u(t)\|^2 + \sum_{j=1}^{N}\|B_j u(t)\|^2\big) + \hat{\lambda}\mathbb{E}\big(\|u(t)\|^2 + \sum_{j=1}^{N}\|B_j u(t)\|^2\big)$$
$$\leq C_8\mathbb{E}(\sum_{k=1}^{\infty}\|h_k(t)\|^2 + \|g(t)\|^2) + 4c_\delta\|\varphi_2\|^2. \tag{5.5}$$

where $\hat{\lambda} := 2 \wedge \lambda$ and $C_8 = \frac{2}{\lambda} + 2$.

For each $\tau \in \mathbb{R}$ and $\mathcal{B} = \{\mathcal{B}(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{B}$, multiplying (5.6) by $e^{\hat{\lambda}t}$ and integrating the result over $(s-t, s)$ for $s \leq \tau$, we see

$$\mathbb{E}\big(\|u(s, s-t, u_{s-t})\|^2 + \sum_{j=1}^{N}\|B_j u(s, s-t, u_{s-t})\|^2\big)$$
$$\leq e^{-\hat{\lambda}t}\mathbb{E}\big(\|u_{s-t}\|^2 + \sum_{j=1}^{N}\|B_j u_{s-t}\|^2\big)$$

$$+ C_8 \int_{\tau-t}^{\tau} e^{\hat{\lambda}(r-\tau)} \mathbb{E}\Big(\sum_{k=1}^{\infty} \|h_k(r)\|^2 + \|g(r)\|^2 \Big) dr + \frac{4c_\delta}{\hat{\lambda}} \|\varphi_2\|^2. \tag{5.6}$$

By $u_{s-t} \in \mathcal{B}(s-t)$ for $s \le \tau$ and $\mathcal{B} \in \mathfrak{B}$, we get, as $t \to +\infty$,

$$e^{-\hat{\lambda}t} \mathbb{E}\big(\|u_{s-t}\|^2 + \sum_{j=1}^{N} \|B_j u_{s-t}\|^2 \big) \le (1+4N) e^{-\hat{\lambda}t} \sup_{s \le \tau} \mathbb{E}(\|\mathcal{B}(s-t)\|^2_{L^2(\Omega, \mathcal{F}_{s-t}, \ell^2)}) \to 0.$$

This along with (5.6) and (4.8b) implies there is a $T := T(\tau, \mathcal{D}) > 0$ such that for all $t \ge T$,

$$\begin{aligned} &\mathbb{E}\big(\|u(s, s-t, u_{s-t})\|^2 + \sum_{j=1}^{N} \|B_j u(s, s-t, u_{s-t})\|^2 \big) \\ &\le 1 + C_8 \sup_{s \le \tau} \int_{-\infty}^{s} e^{\hat{\lambda}(r-s)} \mathbb{E}\Big(\|g(r)\|^2 + \sum_{k \in \mathbb{N}} \|h_k(r)\|^2 \Big) dr + \frac{4c_\delta}{\hat{\lambda}} \|\varphi_2\|^2. \end{aligned}$$

From this we get (5.1). The proof is completed. □

As a direct consequence of Lemma 5.1, we have the existence of two types of absorbing sets for the mean random dynamical system Φ in $L^2(\Omega, \mathcal{F}, \ell^2)$.

Lemma 5.2. Let (2.3)-(2.7) and (4.8a)-(4.8b) hold. Then Φ has two types of absorbing sets in $L^2(\Omega, \mathcal{F}, \ell^2)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$.

(1) Φ has a weakly compact $\mathfrak{D}$-pullback absorbing set $\mathcal{K}_{\mathfrak{D}} = \{\mathcal{K}_{\mathfrak{D}}(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{D}$ in $L^2(\Omega, \mathcal{F}, \ell^2)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$, that is, for every $\tau \in \mathbb{R}$ and $\mathcal{D} \in \mathfrak{D}$, there exists $T = T(\tau, \mathcal{D}) > 0$ such that $\Phi(t, \tau - t)\mathcal{D}(\tau - t) \subseteq \mathcal{K}_{\mathfrak{D}}(\tau)$ for all $t \ge T$, where $\mathcal{K}_{\mathfrak{D}}(\tau) = \{u \in L^2(\Omega, \mathcal{F}_\tau, \ell^2) : \mathbb{E}(\|u\|^2) \le R_{\mathfrak{D}}(\tau)\}$.

(2) Φ has a weakly compact *backward* $\mathfrak{B}$-pullback absorbing set $\mathcal{K}_{\mathfrak{B}} = \{\mathcal{K}_{\mathfrak{B}}(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{B}$ in $L^2(\Omega, \mathcal{F}, \ell^2)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$, that is, for every $\tau \in \mathbb{R}$ and $\mathcal{B} \in \mathfrak{B}$, there exists $T = T(\tau, \mathcal{B}) > 0$ such that $\bigcup_{s \le \tau} \Phi(t, s-t)\mathcal{B}(s-t) \subseteq \mathcal{K}_{\mathfrak{B}}(\tau)$ for all $t \ge T$, where $\mathcal{K}_{\mathfrak{B}}(\tau) = \{u \in L^2(\Omega, \mathcal{F}_\tau, \ell^2) : \mathbb{E}(\|u\|^2) \le R_{\mathfrak{B}}(\tau)\}$.

Proof. The proof of (2) is just a specifical case of (2). Then we only focuss on the proof of (2).

First, by (4.8b) we know that $\mathcal{K}_{\mathfrak{B}}(\tau)$ is a bounded closed convex set in $L^2(\Omega, \mathcal{F}_\tau, \ell^2)$, and so it is a weakly compact subset in $L^2(\Omega, \mathcal{F}_\tau, \ell^2)$. It yields from (2) of Lemma 5.1 that for every $(\tau, \mathcal{B}) \in \mathbb{R} \times \mathfrak{B}$, there exists $T := T(\tau, \mathcal{B}) > 0$ such that

$$\sup_{t \ge T} \sup_{s \le \tau} \sup_{u_{s-t} \in \mathcal{B}(s-t)} \|u(s, s-t, u_{s-t})\|^2_{L^2(\Omega, \mathcal{F}_{s-t}, \ell^2)} \le R_{\mathfrak{B}}(\tau).$$

This implies that for all $t \ge T$,

$$\bigcup_{s \le \tau} \Phi(t, s-t)\mathcal{B}(s-t) \subseteq \mathcal{K}_{\mathfrak{B}}(\tau)$$

Note that for all $s \le \tau$ and $t \ge 0$,

$$\begin{aligned} &\|\mathcal{K}_{\mathfrak{B}}(s-t)\|^2_{L^2(\Omega, \mathcal{F}_{s-t}, \ell^2)} \\ &\le L + L \int_{-\infty}^{s-t} e^{\hat{\lambda}(r-s+t)} \mathbb{E}\Big(\|g(r)\|^2 + \sum_{k=1}^{\infty} \|h_k(r)\|^2 \Big) dr \\ &\le L + L \int_{-\infty}^{s-t} e^{\frac{1}{2}\hat{\lambda}(r-s+t)} \mathbb{E}\Big(\|g(r)\|^2 + \sum_{k=1}^{\infty} \|h_k(r)\|^2 \Big) dr \end{aligned}$$

$$\le L + Le^{\frac{1}{2}\hat{\lambda}t}\int_{-\infty}^{s} e^{\frac{1}{2}\hat{\lambda}(r-s)}\mathbb{E}\Big(\|g(r)\|^2 + \sum_{k=1}^{\infty}\|h_k(r)\|^2\Big)dr.$$

This together with (4.8b) implies, as $t \to +\infty$,

$$e^{-\hat{\lambda}t}\sup_{s\le\tau}\|\mathcal{K}_{\mathfrak{B}}(s-t)\|^2_{L^2(\Omega,\mathcal{F}_{s-t},\ell^2)}$$
$$\le e^{-\hat{\lambda}t}L + Le^{-\frac{1}{2}\hat{\lambda}t}\sup_{s\le\tau}\int_{-\infty}^{s} e^{\frac{1}{2}\hat{\lambda}(r-s)}\mathbb{E}\Big(\|g(r)\|^2 + \sum_{k\in\mathbb{N}}\|h_k(r)\|^2\Big)dr \to 0.$$

Therefore $\mathcal{K}_{\mathfrak{B}} \in \mathfrak{B}$ is a weakly compact backward $\mathfrak{B}$-pullback absorbing set for Φ in $L^2(\Omega,\mathcal{F},\ell^2)$. This concludes the proof. □

6. Three types of weak pullback mean random attractors of (2.1)-(2.2).

Before introduce our main results, we first give some preparations. Given $\tau \in \mathbb{R}$, we note that the weak topology of $L^2(\Omega,\mathcal{F}_\tau,\ell^2)$ has a neighborhood base at a point $\psi_0 \in L^2(\Omega,\mathcal{F}_\tau,\ell^2)$ given by the collection:

$$\Big\{\mathcal{N}^{\varepsilon}_{\phi_1^*,\dots,\phi_n^*}(\psi_0) : \varepsilon > 0, \quad \phi_1^*,\dots,\phi_n^* \in (L^p(\Omega,\mathcal{F}_\tau,X))^*, n = 1,2,\dots,\Big\}. \tag{6.1}$$

Here the set $\mathcal{N}^{\varepsilon}_{\phi_1^*,\dots,\phi_n^*}(\psi_0)$ is given by, for $\varepsilon > 0$ and $\phi_1^*,\dots,\phi_n^* \in (L^2(\Omega,\mathcal{F}_\tau,\ell^2))^*$,

$$\mathcal{N}^{\varepsilon}_{\phi_1^*,\dots,\phi_n^*}(\psi) = \big\{\psi_0 \in L^2(\Omega,\mathcal{F}_\tau,\ell^2) : |\phi_i^*(\psi) - \phi_i^*(\psi_0)| < \varepsilon \ \text{ for all } \ i = 1,\dots,n\big\}. \tag{6.2}$$

If B is a weakly open set containing a point $\psi_0 \in L^2(\Omega,\mathcal{F}_\tau,\ell^2)$, then we say B is a weak neighborhood of the point ψ_0 in $L^2(\Omega,\mathcal{F}_\tau,\ell^2)$. If $\mathcal{B}$ is a weakly open set containing a set $B \subseteq L^2(\Omega,\mathcal{F}_\tau,\ell^2)$, then we say $\mathcal{B}$ is a weak neighborhood of the set B in $L^2(\Omega,\mathcal{F}_\tau,\ell^2)$.

After all preparations established in above sections, we now present the main results on the existence and uniqueness of three types of weak pullback mean random attractors (WPMRAs) for the mean random dynamical system Φ as follows. Here the three types WPMRAs are understood in the following sense.

Definition 6.1. (The usual WPMRA, see [16, Def. 2.4]) A family sets $\mathcal{A} = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{D}$ is called a WPMRA for Φ on $L^2(\Omega,\mathcal{F},\ell^2)$ over $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$ if the following conditions are satisfied.

(i) $\mathcal{A}(\tau)$ is a weakly compact subset of $L^2(\Omega,\mathcal{F}_\tau,\ell^2)$ for every $\tau \in \mathbb{R}$.

(ii) $\mathcal{A}$ is a $\mathfrak{D}$-pullback weakly attracting set of Φ, that is, for each $\tau \in \mathbb{R}$, $\mathcal{D} \in \mathfrak{D}$ and each weak neighborhood $\mathcal{N}^w(\mathcal{A}(\tau))$ of $\mathcal{A}(\tau)$ in $L^2(\Omega,\mathcal{F}_\tau,\ell^2)$, there is a time $T = T(\tau,\mathcal{D},\mathcal{N}^w(\mathcal{A}(\tau))) > 0$ such that

$$\bigcup_{t\ge T}\Phi(t,\tau-t)(\mathcal{D}(\tau-t)) \subseteq \mathcal{N}^w(\mathcal{A}(\tau)).$$

(iii) $\mathcal{A}$ is the minimal element of $\mathfrak{D}$ satisfying (i) as well as (ii).

Definition 6.2. (The backward weakly compact WPMRA) A family sets $\mathcal{A} = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{B}$ is called a *backward weakly compact* WPMRA for Φ on $L^2(\Omega,\mathcal{F},\ell^2)$ over $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$ if the following conditions are satisfied.

(i) The set $\overline{\bigcup_{s\le\tau}\mathcal{A}(s)}^w$ is weakly compact in $L^2(\Omega,\mathcal{F}_\tau,\ell^2)$, where the closure is taken in the sense of the weak topology of $L^2(\Omega,\mathcal{F}_\tau,\ell^2)$.

(ii) $\mathcal{A}$ is a $\mathfrak{B}$-pullback weakly attracting set of Φ in the similar sense of Definition 6.1.

(iii) $\mathcal{A}$ is the minimal element of $\mathfrak{B}$ satisfying (i) as well as (ii).

Definition 6.3. (The backward weakly attracting WPMRA) A family sets $\mathcal{U} = \{\mathcal{U}(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{B}$ is called a *backward weakly attracting* WPMRA for Φ on $L^2(\Omega, \mathcal{F}, \ell^2)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ if the following conditions are satisfied.

(i) $\mathcal{U}(\tau)$ is a weakly compact subset of $L^2(\Omega, \mathcal{F}_\tau, \ell^2)$.

(ii) $\mathcal{U}$ is a *backward* $\mathfrak{B}$-pullback weakly attracting set of Φ in the sense that , for each $\tau \in \mathbb{R}$, $\mathcal{B} \in \mathfrak{B}$ and each weak neighborhood $\mathcal{N}^w(\mathcal{U}(\tau))$ of $\mathcal{U}(\tau)$ in $L^2(\Omega, \mathcal{F}_\tau, \ell^2)$, there is a time $T = T(\tau, \mathcal{B}, \mathcal{N}^w(\mathcal{U}(\tau))) > 0$ such that

$$\bigcup_{t\geq T}\bigcup_{s\leq\tau}\Phi(t, \tau - t)(\mathcal{B}(s - t)) \subseteq \mathcal{N}^w(\mathcal{U}(\tau)).$$

(iii) $\mathcal{U}$ is the minimal element of $\mathfrak{B}$ satisfying (i) as well as (ii).

Note that the notation of backward weakly compact WPMRA and backward weakly attracting WPMRA are strong than the usual WPMRA. The following theorem is concerned with the main results of the paper.

Theorem 6.4. *Let (2.3)-(2.7) and (4.8a)-(4.8b) be satisfied. Then Φ possesses three kinds of WPMRAs.*

(1) Φ has a unique usual *WPMRA $\mathcal{A}_{\mathfrak{D}} = \{\mathcal{A}_{\mathfrak{D}}(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{D}$ in $L^2(\Omega, \mathcal{F}, \ell^2)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ in the sense of Definition 6.1, which is given by*

$$\mathcal{A}_{\mathfrak{D}}(\tau) = \bigcap_{r\geq 0}\overline{\bigcup_{t\geq r}\Phi(t, \tau - t)\mathcal{K}_{\mathfrak{D}}(\tau - t)}^w, \quad \tau \in \mathbb{R}.$$

(2) Φ has a unique backward *weakly compact WPMRA $\mathcal{A}_{\mathfrak{B}} = \{\mathcal{A}_{\mathfrak{B}}(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{B}$ in $L^2(\Omega, \mathcal{F}, \ell^2)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ in the sense of Definition 6.2, which is given by*

$$\mathcal{A}_{\mathfrak{B}}(\tau) = \bigcap_{r\geq 0}\overline{\bigcup_{t\geq r}\Phi(t, \tau - t)\mathcal{K}_{\mathfrak{B}}(\tau - t)}^w.$$

(3) Φ has a unique backward *weakly attracting WPMRA $\mathcal{U}_{\mathfrak{B}} = \{\mathcal{U}_{\mathfrak{B}}(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{B}$ in $L^2(\Omega, \mathcal{F}, \ell^2)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ in the sense of Definition 6.3, which is given by*

$$\mathcal{U}_{\mathfrak{B}}(\tau) = \bigcap_{r\geq 0}\overline{\bigcup_{t\geq r}\bigcup_{s\leq\tau}\Phi(t, s - t)\mathcal{K}_{\mathfrak{B}}(s - t)}^w.$$

(4) The relation of $\mathcal{A}_{\mathfrak{D}}$, $\mathcal{A}_{\mathfrak{B}}$ and $\mathcal{U}_{\mathfrak{B}}$ is $\mathcal{A}_{\mathfrak{D}} = \mathcal{A}_{\mathfrak{B}} \subseteq \mathcal{U}_{\mathfrak{B}}$.

Proof. Proof of (1). By (1) of Lemma 5.2 and the abstract results in [16, Theorem 2.7] we complete the proof of (1) immediately.

Proof of (2). By (2) of Lemma 5.2 and the abstract results in [16, Theorem 2.7] we find that $\mathcal{A}_{\mathfrak{B}} = \{\mathcal{A}_{\mathfrak{B}}(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{B}$ is a WPMRA for Φ in $L^2(\Omega, \mathcal{F}, \ell^2)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ in the sense of Definition 6.1. To show that $\mathcal{A}_{\mathfrak{B}} \in \mathfrak{B}$ is a backward weakly compact WPMRA for Φ in $L^2(\Omega, \mathcal{F}, \ell^2)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ in the sense of Definition 6.2, we only need to show that $\overline{\bigcup_{s\leq\tau}\mathcal{A}_{\mathfrak{B}}(s)}^w$ is weakly compact in $L^2(\Omega, \mathcal{F}_\tau, \ell^2)$. By (2) of Lemma 5.2 we know that $\mathcal{K}_{\mathfrak{B}} \in \mathfrak{B}$ and $\mathcal{K}_{\mathfrak{B}}$ is a *backward* $\mathfrak{B}$-pullback absorbing set for Φ. Then for each $\tau \in \mathbb{R}$, there exists $T_1 = T_1(\tau, \mathcal{K}_{\mathfrak{B}}) > 0$ such that for all $r \geq T_1$,

$$\overline{\bigcup_{t\geq r}\bigcup_{s\leq\tau}\Phi(t, s - t)\mathcal{K}_{\mathfrak{B}}(s - t)}^w \subseteq \overline{\bigcup_{t\geq T_1}\bigcup_{s\leq\tau}\Phi(t, s - t)\mathcal{K}_{\mathfrak{B}}(s - t)}^w \subseteq \overline{\mathcal{K}_{\mathfrak{B}}(\tau)}^w, \quad (6.3)$$

which along with the structure of $\mathcal{A}_{\mathfrak{B}}$ and the weak compactness of $\mathcal{K}_{\mathfrak{B}}(\tau)$ in $L^2(\Omega,\mathcal{F}_\tau,\ell^2)$ yields

$$\begin{aligned}\bigcup_{s\le\tau}\mathcal{A}_{\mathfrak{B}}(s)&=\bigcup_{s\le\tau}\bigcap_{r\ge0}\overline{\bigcup_{t\ge r}\Phi(t,s-t)\mathcal{K}_{\mathfrak{B}}(s-t)}^{w}\\&\subseteq\bigcap_{r\ge T_1}\overline{\bigcup_{t\ge r}\bigcup_{s\le\tau}\Phi(t,s-t)\mathcal{K}_{\mathfrak{B}}(s-t)}^{w}\subseteq\mathcal{K}_{\mathfrak{B}}(\tau).\end{aligned}\tag{6.4}$$

By the weak compactness of $\mathcal{K}_{\mathfrak{B}}(\tau)$ in $L^2(\Omega,\mathcal{F}_\tau,\ell^2)$ we further find that $\overline{\bigcup_{s\le\tau}\mathcal{A}_{\mathfrak{B}}(s)}^{w}$ is weakly compact in $L^2(\Omega,\mathcal{F}_\tau,\ell^2)$. This completes the proof of (2).

Proof of (3). Similar to (6.4) we can prove that $\mathcal{U}_{\mathfrak{B}}(\tau)\subseteq\mathcal{K}_{\mathfrak{B}}(\tau)$, and hence $\mathcal{U}_{\mathfrak{B}}(\tau)$ is a weakly compact set in $L^2(\Omega,\mathcal{F}_\tau,\ell^2)$. By $\mathcal{K}_{\mathfrak{B}}\in\mathfrak{B}$, $\mathcal{U}_{\mathfrak{B}}\subseteq\mathcal{K}_{\mathfrak{B}}$ and the definition of $\mathfrak{B}$ we find $\mathcal{U}_{\mathfrak{B}}\in\mathfrak{B}$. It is easy to check that $\mathcal{A}_{\mathfrak{B}}(\tau)\subseteq\mathcal{U}_{\mathfrak{B}}(\tau)$. This along with the nonemptyness of $\mathcal{A}_{\mathfrak{B}}$ implies the nonemptyness of $\mathcal{U}_{\mathfrak{B}}$.

Next, we show that $\mathcal{U}_{\mathfrak{B}}$ is a backward $\mathfrak{B}$-pullback weakly attracting set in $L^2(\Omega,\mathcal{F},\ell^2)$. First, we need to prove that for every $\tau\in\mathbb{R}$ and every weak neighborhood $\mathcal{N}^w(\mathcal{U}_{\mathfrak{B}}(\tau))$ of $\mathcal{U}_{\mathfrak{B}}(\tau)$ in $L^2(\Omega,\mathcal{F}_\tau,\ell^2)$, there exists $T_2=T_2(\tau,\mathcal{K}_{\mathfrak{B}},N^w(\mathcal{U}_{\mathfrak{B}}(\tau)))\ge T_1$ such that for all $t\ge T_2$,

$$\bigcup_{s\le\tau}\Phi(t,s-t)\mathcal{K}_{\mathfrak{B}}(s-t)\subseteq\mathcal{N}^w(\mathcal{U}_{\mathfrak{B}}(\tau)).\tag{6.5}$$

If (6.5) is incorrect, then we can find $\tau_0\in\mathbb{R}$, $t_n\to+\infty$, $s_n\le\tau_0$, $\psi_n\in\mathcal{K}_{\mathfrak{B}}(s_n-t_n)$ and a weak neighborhood $\mathcal{N}^w(\mathcal{U}_{\mathfrak{B}}(\tau_0))$ of $\mathcal{U}_{\mathfrak{B}}(\tau_0)$ in $L^2(\Omega,\mathcal{F}_{\tau_0},\ell^2)$ such that

$$\Phi(t_n,s_n-t_n)\psi_n\notin\mathcal{N}^w(\mathcal{U}_{\mathfrak{B}}(\tau_0)).\tag{6.6}$$

Since $t_n\to+\infty$, we know, there exists $N_1=N_1(\tau_0,\mathcal{K}_{\mathfrak{B}})>0$ such that $t_n\ge T_2$ for all $n\ge N_1$. And hence by (6.3), we find $\Phi(t_n,s_n-t_n)\psi_n\in\mathcal{K}_{\mathfrak{B}}(\tau_0)$ for all $n\ge N_1$. This along with the weak compactness of $\mathcal{K}_{\mathfrak{B}}(\tau_0)$ in $L^2(\Omega,\mathcal{F}_{\tau_0},\ell^2)$ implies that there exist $\psi_0\in L^2(\Omega,\mathcal{F}_{\tau_0},\ell^2)$ and a subsequence which we do not relabel satisfying

$$\Phi(t_n,s_n-t_n)\psi_n\to\psi_0\ \text{ weakly in } L^2(\Omega,\mathcal{F}_{\tau_0},\ell^2).\tag{6.7}$$

Since $\mathcal{N}^w(\mathcal{U}_{\mathfrak{B}}(\tau_0))$ is weakly open, we know $L^2(\Omega,\mathcal{F}_{\tau_0},\ell^2)\setminus\mathcal{N}^w(\mathcal{U}_{\mathfrak{B}}(\tau_0))$ is weakly closed. In this way we deduce from (6.6) and (6.7) that

$$\psi_0\in L^2(\Omega,\mathcal{F}_{\tau_0},\ell^2)\setminus\mathcal{N}^w(\mathcal{U}_{\mathfrak{B}}(\tau_0)).\tag{6.8}$$

In addition, it follows from (6.7) that for every $\varepsilon>0$ and $\phi_1^*,\dots,\phi_m^*\in(L^2(\Omega,\mathcal{F}_{\tau_0},\ell^2))^*$, there exists $N_2=N_2(\varepsilon,\psi_0,\phi_1^*,\dots,\phi_m^*)\in\mathbb{N}$ such that for all $n\ge N_2$,

$$\Phi(t_n,s_n-t_n)\psi_n\in\mathcal{N}^{\varepsilon}_{\phi_1^*,\dots,\phi_m^*}(\psi_0).\tag{6.9}$$

As an immediate consequence of (6.9) we find $\psi_0\in\mathcal{U}_{\mathfrak{B}}(\tau_0)\subseteq\mathcal{N}^w(\mathcal{U}_{\mathfrak{B}}(\tau_0))$. This is just a contradiction of (6.8). In fact, the details on the proof of $\psi_0\in\mathcal{U}_{\mathfrak{B}}(\tau_0)$ are given bellow. Let $\mathcal{N}^w(\psi_0)$ be an arbitrary weak neighborhood of ψ_0 with respect to the weak topology of $L^2(\Omega,\mathcal{F}_{\tau_0},\ell^2)$. Since the collection in (6.1) is a neighborhood base at ψ_0 in $L^2(\Omega,\mathcal{F}_{\tau_0},\ell^2)$, we can find that there are $\varepsilon>0$ and $\phi_1^*,\dots,\phi_m^*\in(L^2(\Omega,\mathcal{F}_{\tau_0},\ell^2))^*$ so that $\mathcal{N}^{\varepsilon}_{\phi_1^*,\dots,\phi_m^*}(\psi_0)\subseteq\mathcal{N}^w(\psi_0)$. This along with (6.9) implies

$$\Phi(t_n,s_n-t_n)\psi_n\in\mathcal{N}^w(\psi_0),\quad\forall\, n\in\mathbb{N}.\tag{6.10}$$

Given $r \geq 0$, by $t_n \to +\infty$, we know, there exists $N \in \mathbb{N}$ so that $t_n \geq r$ for all $n \geq N$. Then by (6.10) we get, for all $n \geq N$,

$$\Phi(t_n, s_n - t_n)\psi_n \in \mathcal{N}^w(\psi_0) \bigcap \Big(\bigcap_{t \geq r} \bigcap_{s \leq \tau_0} \Phi(t, s-t)\mathcal{K}_{\mathfrak{B}}(s-t) \Big).$$

This implies that $\mathcal{N}^w(\psi_0) \bigcap \Big(\bigcap_{t \geq r} \bigcap_{s \leq \tau_0} \Phi(t, s-t)\mathcal{K}_{\mathfrak{B}}(s-t) \Big)$ is a nonempty set. This shows that ψ_0 is a weak limit point of the set $\bigcup_{t \geq r} \bigcup_{s \leq \tau_0} \Phi(t, s-t)\mathcal{K}_{\mathfrak{B}}(s-t)$ in $L^2(\Omega, \mathcal{F}_{\tau_0}, \ell^2)$, and hence

$$\psi_0 \in \bigcap_{r \geq 0} \overline{\bigcup_{t \geq r} \bigcup_{s \leq \tau_0} \Phi(t, s-t)\mathcal{K}_{\mathfrak{B}}(s-t)}^{w} = \mathcal{U}_{\mathfrak{B}}(\tau_0).$$

On the other hand, by (2) of Lemma 5.2 we know that for every $s \leq \tau$ and $\mathcal{B} \in \mathfrak{B}$, there exists $T_3 = T_3(s, \mathcal{K}_{\mathfrak{B}}, N^w(\mathcal{U}_{\mathfrak{B}}(\tau)), \mathcal{B}) \geq T_2$ such that $\Phi(t, s - T_2 - t)\mathcal{B}(s - T_2 - t) \subseteq \mathcal{K}_{\mathfrak{B}}(s - T_2)$ for all $t \geq T_3$. This along with (6.5) yields, for all $t \geq T_3$,

$$\begin{aligned} &\bigcup_{s \leq \tau} \Phi(t + T_2, s - T_2 - t)\mathcal{B}(s - T_2 - t) \\ &= \bigcup_{s \leq \tau} \Phi(T_2, s - T_2)\Phi(t, s - T_2 - t)\mathcal{B}(s - T_2 - t) \\ &\subseteq \bigcup_{s \leq \tau} \Phi(T_2, s - T_2)\mathcal{K}_{\mathfrak{B}}(s - T_2) \subseteq \mathcal{N}^w(\mathcal{U}_{\mathfrak{B}}(\tau)). \end{aligned}$$

This shows that $\bigcup_{s \leq \tau} \Phi(t, s-t)\mathcal{B}(s-t) \subseteq \mathcal{N}^w(\mathcal{U}_{\mathfrak{B}}(\tau))$ for all $t \geq T_3 + T_2$. Then $\mathcal{U}_{\mathfrak{B}}$ is a backward $\mathfrak{B}$-pullback weakly attracting set for Φ in $L^2(\Omega, \mathcal{F}, \ell^2)$.

According to Definition 6.3, we now only need to prove that $\mathcal{U}_{\mathfrak{B}}$ is the minimal element of $\mathfrak{B}$ with properties (i) and (ii) in Definition 6.3. Let $\mathcal{B} \in \mathfrak{B}$ be a weakly compact backward $\mathfrak{B}$-pullback weakly attracting set for Φ in $L^2(\Omega, \mathcal{F}, \ell^2)$, we next show $\mathcal{U}(\tau) \subseteq \mathcal{B}(\tau)$ for all $\tau \in \mathbb{R}$. If this is not true, then there are $\tau_0 \in \mathbb{R}$ and $\psi_0 \in L^2(\Omega, \mathcal{F}_{\tau_0}, \ell^2)$ satisfying $\psi_0 \in \mathcal{U}(\tau_0) \backslash \mathcal{B}(\tau_0)$. And thereby $\psi_0 \in L^2(\Omega, \mathcal{F}_{\tau_0}, \ell^2) \backslash \mathcal{B}(\tau_0)$. Notice that $L^2(\Omega, \mathcal{F}_{\tau_0}, \ell^2) \setminus \mathcal{B}(\tau_0)$ is weakly open, we know, there exists a weak neighborhood of $\mathcal{N}^w(\psi_0)$ at φ_0 in $L^2(\Omega, \mathcal{F}_{\tau_0}, \ell^2)$ such that $\mathcal{N}^w(\psi_0) \subseteq L^p(\Omega, \mathcal{F}_{\tau_0}, \ell^2) \setminus \mathcal{B}(\tau_0)$. In light of the neighborhood base at $\psi_0 \in L^2(\Omega, \mathcal{F}_{\tau_0}, \ell^2)$ given by (6.1), we find, there exist $\varepsilon > 0$ and $\phi_1^*, \dots, \phi_m^* \in (L^2(\Omega, \mathcal{F}_{\tau_0}, \ell^2))^*$ such that $\mathcal{N}^{\varepsilon}_{\phi_1^*, \dots, \phi_m^*}(\psi_0) \subseteq \mathcal{N}^w(\psi_0)$. As a result, we find

$$\mathcal{N}^{\varepsilon}_{\phi_1^*, \dots, \phi_m^*}(\psi_0) \subseteq L^2(\Omega, \mathcal{F}_{\tau_0}, \ell^2) \setminus \mathcal{B}(\tau_0). \tag{6.11}$$

By $\psi_0 \in \mathcal{U}_{\mathfrak{B}}(\tau_0)$ we find , there are $s_n \leq \tau_0$, $t_n \to +\infty$ and $\psi_n \in \mathcal{B}(s_n - t_n)$ such that

$$\Phi(t_n, s_n - t_n)\psi_n \in \mathcal{N}^{\frac{\varepsilon}{2}}_{\phi_1^*, \dots, \phi_m^*}(\psi_0), \quad \forall\, n \in \mathbb{N}. \tag{6.12}$$

Let

$$\mathcal{N}^{\frac{\varepsilon}{2}}_{\phi_1^*, \dots, \phi_m^*}(\mathcal{B}(\tau)) = \bigcup_{\psi \in \mathcal{B}(\tau)} \mathcal{N}^{\frac{\varepsilon}{2}}_{\phi_1^*, \dots, \phi_m^*}(\psi)$$

be the neighborhood of $\mathcal{B}(\tau)$. Since $t_n \to +\infty$ and $\mathcal{B} \in \mathfrak{B}$ is a backward $\mathfrak{B}$-pullback weakly attracting set for Φ in $L^2(\Omega, \mathcal{F}, \ell^2)$, there exists $N_3 = N_3(\tau_0, \varepsilon, \phi_1^*, \dots, \phi_m^*,$

$\mathcal{K}_{\mathfrak{B}}, \mathcal{B}) \in \mathbb{N}$ such that for all $n \geq N_3$,

$$\bigcup_{s\leq\tau_0} \Phi(t_n, s-t_n)\mathcal{K}_{\mathfrak{B}}(s-t_n) \subseteq \mathcal{N}^{\frac{\varepsilon}{2}}_{\phi_1^*,\ldots,\phi_m^*}(\mathcal{B}(\tau_0)). \tag{6.13}$$

By $s_n \leq \tau_0$, $\psi_n \in \mathcal{D}(s_n - t_n)$ and (6.13) we find $\Phi(t_n, s_n - t_n)\psi_n \in \mathcal{N}^{\frac{\varepsilon}{2}}_{\phi_1^*,\ldots,\phi_m^*}(\mathcal{B}(\tau_0))$ for all $n \geq N_3$. Then, there exists $\widehat{\psi} \in \mathcal{B}(\tau_0)$ such that $\Phi(t_n, s_n - t_n)\psi_n \in \mathcal{N}^{\frac{\varepsilon}{2}}_{\phi_1^*,\ldots,\phi_m^*}(\widehat{\psi})$ for all $n \geq N_3$. This along with (6.12) implies $|\phi_i^*(\widehat{\psi}) - \phi_i^*(\psi_0)| < \varepsilon$ for all $i = 1, \ldots, m$. Then $\widehat{\psi} \in \mathcal{N}^{\varepsilon}_{\phi_1^*,\ldots,\phi_m^*}(\psi_0)$, and hence by (6.11) we find $\widehat{\psi} \in L^2(\Omega, \mathcal{F}_{\tau_0}, \ell^2) \setminus \mathcal{B}(\tau_0)$. This is a contradiction with $\widehat{\psi} \in \mathcal{B}(\tau_0)$, and therefore we have $\mathcal{U}(\tau) \subseteq \mathcal{B}(\tau)$ for all $\tau \in \mathbb{R}$. This completes the proof of (3).

Proof of (4). By Lemma 5.2 we find $\mathcal{K}_{\mathfrak{D}}(\tau) \subseteq \mathcal{K}_{\mathfrak{B}}(\tau)$, and hence $\mathcal{A}_{\mathfrak{D}}(\tau) \subseteq \mathcal{A}_{\mathfrak{B}}(\tau) \subseteq \mathcal{U}_{\mathfrak{B}}(\tau)$. Since $\mathcal{A}_{\mathfrak{D}}$ is a $\mathfrak{D}$-pullback weakly attracting set for Φ in $L^2(\Omega, \mathcal{F}, \ell^2)$, we find, for each $\tau \in \mathbb{R}$, $\mathcal{D} \in \mathfrak{B} \subseteq \mathfrak{D}$ and each weak neighborhood $\mathcal{N}^w(\mathcal{A}_{\mathfrak{D}}(\tau))$ of $\mathcal{A}_{\mathfrak{D}}(\tau)$ in $L^2(\Omega, \mathcal{F}_\tau, \ell^2)$, there exists $T = T(\tau, \mathcal{D}, \mathcal{N}^w(\mathcal{A}_{\mathfrak{D}}(\tau))) > 0$ such that for all $t \geq T$,

$$\Phi(t, \tau - t)\mathcal{D}(\tau - t) \subseteq \mathcal{N}^w(\mathcal{A}_{\mathfrak{D}}(\tau)). \tag{6.14}$$

By $\mathcal{A}_{\mathfrak{D}} \subseteq \mathcal{A}_{\mathfrak{B}}$ and $\mathcal{A}_{\mathfrak{B}} \in \mathfrak{B}$ we also have $\mathcal{A}_{\mathfrak{D}} \in \mathfrak{B}$. Then by (6.14) we know that $\mathcal{A}_{\mathfrak{D}}$ is also a weakly compact $\mathfrak{B}$-pullback weakly attracting set for Φ in $L^2(\Omega, \mathcal{F}, \ell^2)$. Note that $\mathcal{A}_{\mathfrak{B}} \in \mathfrak{B}$ is a WPMRA in the usual sense. Then by the minimality of $\mathcal{A}_{\mathfrak{B}}$ we have $\mathcal{A}_{\mathfrak{B}} \subseteq \mathcal{A}_{\mathfrak{D}}$. The completes the proof of (4). □

Remark 6.5. It maybe interesting to study weak pullback mean random attractors of stochastic equations driven by fractional noise, see [8].

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