GLOBAL DYNAMICS OF THE SOLUTION FOR A BISTABLE REACTION DIFFUSION EQUATION WITH NONLOCAL EFFECT

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Abstract. This paper is devoted to studying the Cauchy problem corresponding to the nonlocal bistable reaction diffusion equation. It is the first attempt to use the method of comparison principle to study the well-posedness for the nonlocal bistable reaction-diffusion equation. We show that the problem has a unique solution for any non-negative bounded initial value by using Gronwall's inequality. Moreover, the boundedness of the solution is obtained by means of the auxiliary problem. Finally, in the case that the initial data with compactly supported, we analyze the asymptotic behavior of the solution.

1. **Introduction.** In this paper, we will consider the following Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ku^2(1 - \phi * u) - bu, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (1)$$

where

$$(\phi * u)(x,t) := \int_{\mathbb{R}} \phi(x - y)u(y,t)dy,$$

and $\phi(x)$ satisfies the following assumptions

$$\phi(x) \geq 0, \ \phi(0) > 0, \ \phi(x) = \phi(-x), \ \int_{\mathbb{R}} \phi(x)dx = 1, \ \int_{\mathbb{R}} x^2\phi(x)dx < \infty.$$

The model (1) is described as the population dynamics with nonlocal consumption of resources. Where $u(x,t)$ represents the density of population at the position x and time t. The coefficients k, b are positive. The term $ku^2(1 - \phi * u)$ is interpreted as the reproduction of the population, which is proportional to the square of density u, and to available resources $1 - \phi * u$. The convolution term $\phi * u$ represents the nonlocal consumption of resources, specific is that the consumption of resources at the space point x is determined by the individuals located at $y \in (-\infty, \infty)$. The last term $-bu$ represents mortality of the population. Moreover, the nonlocal model (1) can be used to describe the emergence and evolution of biological species and the process of speciation [7, 12].

When $\phi(x)$ tends to be a δ-function, the model (1) can be changed to the classical reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ku^2(1 - u) - bu, \tag{2}$$

if $k > 4b$, the equation (2) has three zeros

$$u_+ = 0, \ u_0 = \frac{k - \sqrt{k^2 - 4kb}}{2k}, \ u_- = \frac{k + \sqrt{k^2 - 4kb}}{2k},$$

there are two stable steady-states u_- and u_+, u_0 is unstable point, which implies that the model (2) is bistable. The traveling wave solution, asymptotic speed of propagation and Cauchy problem of the equation (2) have been studied extensively, see [22]. Since the consumption of resources with nonlocal makes more reasonable for describing the behavior of various biological phenomena. Apreutesei et al. [5] introduced the nonlocal consumption of resources into equation (2), that is the equation (1), they explored the Fredholm operator and used it to prove the existence of travelling waves in case that the integral is sufficiently small. For more results of (1) can refer to [2, 7, 22]. When the support of the function ϕ is not small, Demin and Volpert [9] and Alfaro et al. [4] proved the existence of traveling wave solution when the nonlinearities with the form of $u(\phi * u)(1 - u) - \alpha u$ and $u(u - \theta)(1 - \phi * u)$, respectively. Moreover, Li et al. [19] and Han et al. [14] obtained the general results for the problem (1) when the nonlocal term without limit by using monotone iteration method and Leray-Schauder degree theory, respectively. But it is only partially solved and much work in this area remains open questions. It is worth noting that two points should be pointed out through the above researches, one is that the difficulties caused by different nonlocal locations are different, and the other is that the difficulties caused by different nonlocal strength are also different. Therefore, the methods of studying the problem are quite different.

Actually, most of the research on the nonlocal reaction-diffusion equation is focused on monostable case (the dynamic behavior of the solution is relatively simple), such as Hamel and Ryzhik [13], Ai [1], Faye and Holzer [11] and others have done a vest of researches, and have obtained plenty of meaningful results in the traveling wave solution [3, 20, 21], the asymptotic propagation speed of the solution [1, 6, 13], the spatial dynamic behavior of the solution corresponding to the Cauchy problem [8], the bifurcation [11].

Especially in recent decades, there are many results about the spatial dynamics behavior of the Cauchy problem [10, 15, 16, 17, 18, 19, 23]. Deng and Wu [10] analyzed the global stability for Cauchy problem

$$\begin{cases} u_t - \triangle u = u\left[f(u) - \alpha \int_{\mathbb{R}^n} g(x - y)u(y, t)dy \right], \\ u(x, 0) = u_0(x), \end{cases}$$

and shown the existence and uniqueness of the solution by establishing comparison principle and constructing monotone sequences. Han and Yang [17] further considered the nonlocal reaction-diffusion-mutation model

$$\begin{cases} u_t = \theta u_{xx} + d u_{xx} + u\{1 + \alpha u - \beta u^2 - (1 + \alpha - \beta) \\ \quad \times \int_{\mathbb{R}} \int_{\Theta} k(x - y, \theta - \theta')u(y, \theta', t)d\theta'dy\}, \\ u(x, \theta, 0) = u_0(x, \theta), \\ \frac{\partial u}{\partial \theta}(x, \theta, t) = 0, \end{cases}$$

and obtained the well-posedness of solutions, including the existence, uniqueness and global stability. From the above researches, it can be seen that the study of the spatial dynamics of the solution has a great enlightenment and guidance on understanding the nonlocal effect. To date, however, the research on the Cauchy problem of the nonlocal bistable reaction-diffusion equation is still blank. So a very natural question, what is the solution of the nonlocal bistable reaction-diffusion equation corresponding to the Cauchy problem?

Inspired by [4, 5, 10, 17], we will try to solve (or partially solve) the global dynamics of the solution of problem (1) in this paper. The main difficulty is that due to the introduction of nonlocal term, the maximum principle of problem (1) is not valid and the maximum modulus estimate of solution cannot be obtained. For such a difficulty, we will define suitable super- and sub-solutions and construct monotonic iterative sequences to obtain the existence of the solution. Furthermore, the uniqueness is given by using fundamental solution and Gronwall's inequality. Finally, we obtain the uniform boundedness of the solution by means of auxiliary function.

This paper is organized as follows. In section 2, some preparations including the notion of super- and sub-solutions, as well as the comparison principle will be given. Then we consider the existence and uniqueness of the global solution of the Cauchy problem (1) in Section 3. The results of the asymptotic behavior of solution will be obtained in Section 4.

2. **Preliminaries.** In this section, we will do the preparation works. First, we will review the notion of the super- and sub-solutions of the problem (1), and then give the order of the super- and sub-solutions. For convenience, we define $I_T = \mathbb{R} \times (0, T)$ and $B_T = \mathbb{R} \times [0, T)$. We now make the following definition, which will be used frequently in our paper.

Definition 2.1. Assume $\overline{u}(x,t)$, $\underline{u}(x,t) \in C^{2,1}(I_T) \cap C_B(B_T)$, where C_B is a bounded continuous space. They are called super- and sub-solutions of Cauchy problem (1), respectively, if $\overline{u}(x,t)$ and $\underline{u}(x,t)$ satisfy

$$\begin{cases} \overline{u}_t \geq \overline{u}_{xx} + k\overline{u}^2 \left(1 - \int_{\mathbb{R}} \phi(x-y)\underline{u}(y)dy\right) - b\overline{u}, & (x,t) \in I_T, \\ \overline{u}(x,0) \geq u_0(x), & x \in \mathbb{R}, \end{cases} \tag{3}$$

and

$$\begin{cases} \underline{u}_t \leq \underline{u}_{xx} + k\underline{u}^2 \left(1 - \int_{\mathbb{R}} \phi(x-y)\overline{u}(y)dy\right) - b\underline{u}, & (x,t) \in I_T, \\ \underline{u}(x,0) \leq u_0(x), & x \in \mathbb{R}. \end{cases} \tag{4}$$

Under the definition of super- and sub-solutions, the following result is obtained.

Lemma 2.2. *Suppose that $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are super- and sub-solutions of problem (1), respectively, where $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are nonnegative bounded functions. Then,*

$$\overline{u}(x,t) \geq \underline{u}(x,t), \ (x,t) \in B_T.$$

Proof. Let $u(x,t) = \overline{u}(x,t) - \underline{u}(x,t)$. Then for any $(x,t) \in I_T$, we have

$$u_t - u_{xx} \geq k\overline{u}^2(1 - \phi * \underline{u}) - b\overline{u} - k\underline{u}^2(1 - \phi * \overline{u}) + b\underline{u}$$

$$= -bu + k\overline{u}^2(1 - \phi * \underline{u}) - k\underline{u}^2(1 - \phi * \overline{u})$$
$$= -bu + ku\left(\overline{u} + \underline{u}\right) - k\overline{u}^2(\phi * \underline{u}) + k\underline{u}^2(\phi * \overline{u})$$
$$= -bu + ku(\overline{u} + \underline{u}) - ku(\overline{u} + \underline{u})(\phi * \overline{u}) + k\overline{u}^2(\phi * u)$$
$$= u\left(-b + k(\overline{u} + \underline{u}) - k(\overline{u} + \underline{u})(\phi * \overline{u})\right) + k\overline{u}^2(\phi * u).$$

This yields that

$$\begin{cases} u_t - u_{xx} + d_1(x,t)u \geq k\overline{u}^2(\phi * u), & (x,t) \in I_T, \\ u(x,0) \geq 0, & x \in \mathbb{R}, \end{cases} \tag{5}$$

where

$$d_1(x,t) := -\left(-b + k(\overline{u} + \underline{u}) - k(\overline{u} + \underline{u})(\phi * \overline{u})\right).$$

Take $\sigma > 0$ large enough such that

$$d_2(x,t) := \sigma - \left(-b + k(\overline{u} + \underline{u}) - k(\overline{u} + \underline{u})(\phi * \overline{u})\right) \geq 0, \quad \forall(x,t) \in I_T.$$

Denote $\widetilde{u} = e^{-\sigma t}u$, from (5) we can see that

$$\begin{cases} \widetilde{u}_t - \widetilde{u}_{xx} + d_2(x,t)\widetilde{u} \geq k\overline{u}^2\left(\phi * \widetilde{u}\right), & (x,t) \in I_T, \\ \widetilde{u}(x,0) \geq 0, & x \in \mathbb{R}. \end{cases}$$

Since the functions $\overline{u}$, $\underline{u}$ are nonnegative bounded in B_T, then, there exists $M > 0$ such that

$$0 \leq \overline{u}, \ \underline{u} \leq M \quad \text{for } (x,t) \in B_T.$$

Further, we know that $d_2(x,t)$ is nonnegative bounded in B_T. To complete our proof, it suffices to show that

$$\widetilde{u} \geq 0 \quad \text{in } I_{T_0},$$

where $T_0 = \min\left\{T, \frac{s}{kM^2}\right\}$ and s is positive which will be determined by (6). For contradiction we assume that $\widetilde{u} < 0$ at some points in I_{T_0}, then, we get

$$\widetilde{u}_{inf} = \inf_{(x,t)\in I_{T_0}} \widetilde{u}(x,t) < 0,$$

since $\widetilde{u}$ is bounded, then there exists a positive constant s and a point $(x^*, t^*) \in I_{T_0}$, such that

$$\widetilde{u}\left(x^*, t^*\right) \leq 0,$$

and

$$\widetilde{u}\left(x^*, t^*\right) \leq s\widetilde{u}_{inf}. \tag{6}$$

We now define

$$w = \frac{\widetilde{u}}{1 + x^2 + \zeta t},$$

where ζ is a positive constant. Then, we obtain

$$(\zeta - 2)w - 4xw_x + \left(1 + x^2 + \zeta t\right)(w_t - w_{xx}) + d_2\left(1 + x^2 + \zeta t\right)w$$
$$\geq k\overline{u}^2(\phi * \widetilde{u}),$$

thus,

$$\begin{cases} \left(1 + x^2 + \zeta t\right)(w_t - w_{xx} + d_2 w) + (\zeta - 2)w - 4xw_x \\ \geq k\overline{u}^2\left(\phi * \widetilde{u}\right), & (x,t) \in I_T, \\ w(x,0) \geq 0, & x \in \mathbb{R}. \end{cases} \tag{7}$$

By the definition of $w(x,t)$ that $\lim_{|x|\to+\infty} w(x,t) = 0$ and together with (7), hence there exists a point $(\widetilde{x}, \widetilde{t}) \in I_{T_0}$, such that w reaches the negative minimum w_{min}, one has,

$$w_{min} = \min_{(x,t)\in I_{T_0}} \frac{\widetilde{u}(x,t)}{1 + x^2 + \zeta t} \leq \frac{\widetilde{u}(x^*, t^*)}{1 + (x^*)^2 + \zeta t^*}.$$

Combining with (6), we have

$$w_{min} \leq \frac{s\widetilde{u}_{inf}}{1 + (x^*)^2 + \zeta t^*},$$

equivalent to

$$\widetilde{u}_{inf} \geq \frac{\left(1 + (x^*)^2 + \zeta t^*\right) w_{min}}{s}. \tag{8}$$

Due to $w_t \leq 0, w_x = 0$ and $w_{xx} \geq 0$ at $(\widetilde{x}, \widetilde{t})$, together with (7), we get

$$d_2 \left(1 + \widetilde{x}^2 + \zeta\widetilde{t}\right) w_{min} + (\zeta - 2)w_{min} \geq k\overline{u}^2\widetilde{u}_{inf}.$$

Furthermore, we deduce that

$$(\zeta - 2)w_{min} \geq k\overline{u}^2\widetilde{u}_{inf},$$

together with (8), one has

$$(\zeta - 2)w_{min} \geq k\overline{u}^2 \frac{\left(1 + (x^*)^2 + \zeta t^*\right) w_{min}}{s},$$

then,

$$(\zeta - 2) \leq \frac{k\overline{u}^2}{s}\left(1 + (x^*)^2 + \zeta t^*\right),$$

further,

$$\left(1 - \frac{k\overline{u}^2}{s}t^*\right)\zeta \leq \frac{k\overline{u}^2}{s}\left(1 + (x^*)^2\right) + 2,$$

clearly

$$\left(1 - \frac{kM^2}{s}T_0\right)\zeta \leq \frac{k\overline{u}^2}{s}\left(1 + (x^*)^2\right) + 2, \tag{9}$$

since x^* is independent of ζ, if choosing ζ large enough that the inequality (9) doesn't hold, which is a contradiction, this yields

$$\widetilde{u}(x,t) \geq 0 \quad \text{in } B_{T_0}.$$

If $T > T_0$, we repeat the above process with the initial time $t = T_0$, thus

$$\widetilde{u}(x,t) \geq 0 \quad \text{in } B_T.$$

This implies that $u \geq 0$, that is, $\overline{u}(x,t) \geq \underline{u}(x,t)$ in B_T. This completes the proof. $\square$

3. **Well-posedness of the solution about (1.1).** In this section, we study the well-posedness of problem (1). The existence and uniqueness of global solution is based on the comparison principle. Firstly, we investigate the existence of solution for the problem (1) by constructing two monotone sequences. Later, we prove the uniqueness of the solution for the problem (1) by using Gronwall's inequality and fundamental-solution. Finally, we will study the uniform boundedness of the solution for the problem (1).

Theorem 3.1. *Suppose that $\overline{u}$, $\underline{u}$ are nonnegative functions and $\overline{u}$, $\underline{u}$ are super- and sub-solutions of problem (1) in I_T, respectively. Then the problem (1) has a solution $u(x,t)$ in I_T and satisfies*

$$\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t) \quad for \ (x,t) \in B_T.$$

Proof. Since $\overline{u}$, $\underline{u}$ are nonnegative and bounded functions, then there exists $N > 0$ such that

$$0 \le \overline{u}(x,t), \ \underline{u}(x,t) \le N \quad \text{in } B_T.$$

Moreover, we choose $L > 0$ large enough, such that

$$L > \left\{ b - 2\theta k, b - 2\tilde{\theta}k, b - \hat{\theta}k \right\},$$

where θ, $\tilde{\theta}$, $\hat{\theta}$ will be determined later.

Denote $\overline{u}^{(0)} = \overline{u}$ and $\underline{u}^{(0)} = \underline{u}$, there are two sequences $\left\{ \overline{u}^{(m)} \right\}_{m=0}^{\infty}$ and $\left\{ \underline{u}^{(m)} \right\}_{m=0}^{\infty}$ through the following iteration format for $m = 1, 2, \cdots$,

$$\begin{cases} \overline{u}_t^{(m)} - \overline{u}_{xx}^{(m)} + L\overline{u}^{(m)} = k\left(\overline{u}^{(m-1)}\right)^2 - k\left(\overline{u}^{(m)}\right)^2\left(\phi * \underline{u}^{(m-1)}\right) - b\overline{u}^{(m-1)} \\ \quad + L\overline{u}^{(m-1)}, & (x,t) \in I_T, \\ \overline{u}^{(m)}(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

$$(10)$$

and

$$\begin{cases} \underline{u}_t^{(m)} - \underline{u}_{xx}^{(m)} + L\underline{u}^{(m)} = k\left(\underline{u}^{(m-1)}\right)^2 - k\left(\underline{u}^{(m)}\right)^2\left(\phi * \overline{u}^{(m-1)}\right) - b\underline{u}^{(m-1)} \\ \quad + L\underline{u}^{(m-1)}, & (x,t) \in I_T, \\ \underline{u}^{(m)}(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

$$(11)$$

First of all, we claim that

$$\underline{u} \le \underline{u}^{(1)} \le \overline{u}^{(1)} \le \overline{u} \quad \text{in } B_T. \tag{12}$$

Let $\widehat{u} = \underline{u}^{(1)} - \underline{u}$, then from (4) and (11), we have

$$\begin{cases} \widehat{u}_t - \widehat{u}_{xx} \ge -k\left(\underline{u}^{(1)}\right)^2(\phi * \overline{u}) + k\underline{u}^2(\phi * \overline{u}) + L\left(\underline{u} - \underline{u}^{(1)}\right) \\ \quad = -\left(k\left(\underline{u}^{(1)} + \underline{u}\right)(\phi * \overline{u}) + L\right)\widehat{u}, & (x,t) \in I_T, \\ \widehat{u}(x,0) \ge 0, & x \in \mathbb{R}, \end{cases}$$

by applying the comparison principle, we know that $\widehat{u} \ge 0$, namely, $\underline{u}^{(1)} \ge \underline{u}$ in B_T. Similarly, let $\widetilde{u} = \overline{u} - \overline{u}^{(1)}$, from (3) and (10) we obtain

$$\begin{cases} \widetilde{u}_t - \widetilde{u}_{xx} \ge -k\overline{u}^2(\phi * \underline{u}) + k\left(\overline{u}^{(1)}\right)^2(\phi * \underline{u}) - L\left(\overline{u} - \overline{u}^{(1)}\right) \\ \quad = -\left(k(\overline{u} + \overline{u}^{(1)})(\phi * \underline{u}) + L\right)\widetilde{u}, & (x,t) \in I_T, \\ \widetilde{u}(x,0) \ge 0, & x \in \mathbb{R}. \end{cases}$$

Using the comparison principle, one has $\widetilde{u} \geq 0$, which means $\overline{u} \geq \overline{u}^{(1)}$ in B_T. Finally, we prove $\overline{u}^{(1)} \geq \underline{u}^{(1)}$, let $\overline{\overline{\widetilde{u}}} = \overline{u}^{(1)} - \underline{u}^{(1)}$, then from (10) and (11), $\overline{\overline{\widetilde{u}}}$ satisfies

$$\overline{\overline{\widetilde{u}}}_t - \overline{\overline{\widetilde{u}}}_{xx} = k\overline{u}^2 - k\underline{u}^2 - k\left(\overline{u}^{(1)}\right)^2 (\phi * \underline{u}) + k\left(\underline{u}^{(1)}\right)^2 (\phi * \overline{u}) - b\overline{u} + b\underline{u} - L\left(\overline{u}^{(1)} - \overline{u}\right)$$

$$+ L\left(\underline{u}^{(1)} - \underline{u}\right)$$

$$= -k\left(\overline{u}^{(1)}\right)^2 (\phi * \underline{u}) + k\left(\underline{u}^{(1)}\right)^2 (\phi * \underline{u}) + k\left(\underline{u}^{(1)}\right)^2 (\phi * \overline{u}) - k\left(\underline{u}^{(1)}\right)^2 (\phi * \underline{u})$$

$$- b(\overline{u} - \underline{u}) - L\left(\overline{u}^{(1)} - \underline{u}^{(1)}\right) + L\left(\overline{u} - \underline{u}\right) + k(\overline{u} + \underline{u})(\overline{u} - \underline{u})$$

$$= -\left(k\left(\overline{u}^{(1)} + \underline{u}^{(1)}\right)(\phi * \underline{u}) + L\right)\overline{\overline{\widetilde{u}}} + k\left(\underline{u}^{(1)}\right)^2 (\phi * (\overline{u} - \underline{u}))$$

$$+ (-b + L + 2\theta k)(\overline{u} - \underline{u})$$

$$\geq -\left(k\left(\overline{u}^{(1)} + \underline{u}^{(1)}\right)(\phi * \underline{u}) + L\right)\overline{\overline{\widetilde{u}}},$$

where $\theta \in (\underline{u}, \overline{u})$, then we have

$$\begin{cases} \overline{\overline{\widetilde{u}}}_t - \overline{\overline{\widetilde{u}}}_{xx} \geq -\left(k\left(\overline{u}^{(1)} + \underline{u}^{(1)}\right)(\phi * \underline{u}) + L\right)\overline{\overline{\widetilde{u}}}, & (x,t) \in I_T, \\ \overline{\overline{\widetilde{u}}}(x,0) \geq 0, & x \in \mathbb{R}. \end{cases}$$

We obtain $\overline{\overline{\widetilde{u}}} \geq 0$ by means of the comparison principle, that is, $\overline{u}^{(1)} \geq \underline{u}^{(1)}$ in B_T. Thus, $\underline{u} \leq \underline{u}^{(1)} \leq \overline{u}^{(1)} \leq \overline{u}$ in B_T.

Next, we prove that $\overline{u}^{(1)}$ and $\underline{u}^{(1)}$ is a pair of super- and sub-solutions for problem (1). From (10) $\overline{u}^{(1)}$ satisfies

$$\overline{u}_t^{(1)} - \overline{u}_{xx}^{(1)} = k\overline{u}^2 - k\left(\overline{u}^{(1)}\right)^2 (\phi * \underline{u}) - b\overline{u} - L\left(\overline{u}^{(1)} - \overline{u}\right)$$

$$\geq k\overline{u}^2 - k\left(\overline{u}^{(1)}\right)^2 \left(\phi * \underline{u}^{(1)}\right) - b\overline{u} - L\left(\overline{u}^{(1)} - \overline{u}\right)$$

$$= k\left(\overline{u}^{(1)}\right)^2 - k\left(\overline{u}^{(1)}\right)^2 \left(\phi * \underline{u}^{(1)}\right) - b\overline{u}^{(1)} - k\left(\overline{u}^{(1)}\right)^2 + b\overline{u}^{(1)} + k\overline{u}^2 - b\overline{u}$$

$$- L\left(\overline{u}^{(1)} - \overline{u}\right)$$

$$= k\left(\overline{u}^{(1)}\right)^2 - k\left(\overline{u}^{(1)}\right)^2 \left(\phi * \underline{u}^{(1)}\right) - b\overline{u}^{(1)} + \left(2\tilde{\theta}k - b + L\right)\left(\overline{u} - \overline{u}^{(1)}\right)$$

$$\geq k\left(\overline{u}^{(1)}\right)^2 - k\left(\overline{u}^{(1)}\right)^2 \left(\phi * \underline{u}^{(1)}\right) - b\overline{u}^{(1)}, \tag{13}$$

where $\tilde{\theta} \in \left(\overline{u}^{(1)}, \overline{u}\right)$. On the other hand, from (11) $\underline{u}^{(1)}$ satisfies

$$\underline{u}_t^{(1)} - \underline{v}_{xx}^{(1)} = k\underline{u}^2 - k\left(\underline{u}^{(1)}\right)^2 (\phi * \overline{u}) - b\underline{u} - L\left(\underline{u}^{(1)} - \underline{u}\right)$$

$$\leq k\underline{u}^2 - k\left(\underline{u}^{(1)}\right)^2 \left(\phi * \overline{u}^{(1)}\right) - b\underline{u} - L\left(\underline{u}^{(1)} - \underline{u}\right)$$

$$= k\left(\underline{u}^{(1)}\right)^2 - k\left(\underline{u}^{(1)}\right)^2 \left(\phi * \overline{u}^{(1)}\right) - b\underline{u}^{(1)} - k\left(\underline{u}^{(1)}\right)^2 + b\underline{u}^{(1)} + k\underline{u}^2 - b\underline{u}$$

$$- L\left(\underline{u}^{(1)} - \underline{u}\right)$$

$$= k\left(\underline{u}^{(1)}\right)^2 - k\left(\underline{u}^{(1)}\right)^2 \left(\phi * \overline{u}^{(1)}\right) - b\underline{u}^{(1)} - \left(2\hat{\theta}k - b + L\right)\left(\overline{u}^{(1)} - \overline{u}\right)$$

$$\leq k\left(\underline{u}^{(1)}\right)^2 - k\left(\underline{u}^{(1)}\right)^2 \left(\phi * \overline{u}^{(1)}\right) - b\underline{u}^{(1)}, \tag{14}$$

where $\hat{\theta} \in \left(\underline{u}, \underline{u}^{(1)}\right)$. Thanks to the Definition 2.1, together with (13) and (14), we obtain that $\overline{u}^{(1)}$ and $\underline{u}^{(1)}$ are super- and sub-solutions of (1) respectively.

We assume $\overline{u}^{(m)}$ and $\underline{u}^{(m)}$ are super- and sub-solutions of the problem (1) respectively. By repeating above processes, one has

$$\underline{u}^{(m)} \leq \underline{u}^{(m+1)} \leq \overline{u}^{(m+1)} \leq \overline{u}^{(m)} \quad \text{in } B_T,$$

and

$$\underline{v}^{(m)} \leq \underline{v}^{(m+1)} \leq \overline{v}^{(m+1)} \leq \overline{v}^{(m)} \quad \text{in } B_T.$$

Likewise, $\overline{u}^{(m+1)}$ and $\underline{u}^{(m+1)}$ are super- and sub-solutions of (1) respectively. Thus, combine with (12), we have that, for $m = 0, 1, 2, \cdots,$

$$\underline{u} \leq \underline{u}^{(1)} \leq \underline{u}^{(2)} \leq \cdots \leq \underline{u}^{(m)} \leq \overline{u}^{(m)} \leq \cdots \leq \overline{u}^{(2)} \leq \overline{u}^{(1)} \leq \overline{u} \quad \text{in } B_T.$$

Therefore, there exist u and $\mathfrak{u}$ such that

$$\lim_{m \to +\infty} \underline{u}^{(m)} = u \quad \text{and} \quad \lim_{m \to +\infty} \overline{u}^{(m)} = \mathfrak{u}.$$

Additionally, It is clearly that $\mathfrak{u}$ and u are super- and sub-solutions of (1), respectively, and there is $u = \mathfrak{u}$, thus u is the bounded solution of (1). This completes the proof. $\square$

Thanks to Theorem 3.1, we further have the following result immediately.

Lemma 3.2. *For any nonnegative bounded initial $u_0(x)$, the solution of the problem* (1) *exists.*

Proof. From Theorem 3.1, if $\overline{u}$ and $\underline{u}$ are super- and sub-solutions respectively, one has $\overline{u}(x,0) \geq u_0(x) \geq \underline{u}(x,0)$. From the Definition 2.1, one may choose 0 as the sub-solution of problem (1). Next, we construct the super-solution as follows.

Since $u_0(x)$ is nonnegative bounded function, then, we choose $M > 0$ sufficiently enough such that

$$\max\left\{\|u_0(x)\|_{L^\infty}, 1\right\} \leq M, \quad \forall x \in \mathbb{R}.$$

We know that 0 and M is a pair of the super- and sub-solutions. Therefore, we obtain our conclusion from Theorem 3.1. This completes the proof. $\square$

Moreover, the uniqueness of the solutions about the problem (1) will be given as follows.

Theorem 3.3. *The problem* (1) *admits a unique bounded solution for $(x,t) \in B_T$.*

Proof. According to Theorem 3.1, there exist solutions for the problem (1). To get our conclusion, we suppose that u_1 and u_2 are bounded solutions of (1) in I_T. By direct computations, u_i (i=1, 2) satisfies

$$u_i(x,t) = \int_{\mathbb{R}} \Phi(x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}} \Phi(x-y, t-s)\Big[ku_i^2(y,s)\big(1$$
$$- \int_{\mathbb{R}} \phi(y-z)u_i(z,s)dz - bu_i(y,s)\big)\Big]dyds,$$

where $\Phi(x,t)$ is the fundamental solution of the heat equation. Let $\widetilde{u} = u_1 - u_2$, one has

$$\widetilde{u}(x,t) = \int_0^t \int_{\mathbb{R}} \Phi(x-y, t-s)\Big[\widetilde{u}(y,s)(2k\check{\theta} - 2k\check{\theta}(\phi * u_2) - b)\Big]dyds$$
$$- \int_0^t \int_{\mathbb{R}} \Phi(x-y, t-s)ku_1^2(y,s)(\phi * \widetilde{u})dyds. \tag{15}$$

where $\check{\theta} \in (u_1, u_2)$.

From Lemma 3.2, we know that u_1, u_2 are all nonnegative and bounded functions in B_T. Therefore, there exists $N > 0$ such that

$$0 \leq u_1,\ u_2 \leq N \quad \text{in } B_T.$$

Define

$$M_1 := 2k\check{\theta} - 2k\check{\theta}N - b, \quad M_2 := kN^2.$$

Furthermore, from (15), we deduce

$$\|\widetilde{u}(.,t)\|_{L^\infty(\mathbb{R})} \leq \int_0^t M_1 \|\widetilde{u}(.,s)\|_{L^\infty(\mathbb{R})} ds + \int_0^t M_2 \|\widetilde{u}(.,s)\|_{L^\infty(\mathbb{R})} ds$$

$$= (M_1 + M_2) \int_0^t \|\widetilde{u}(.,s)\|_{L^\infty(\mathbb{R})} ds \quad \text{for } t \in (0,T).$$

By Gronwall's inequality, we obtain

$$\|\widetilde{u}\|_{L^\infty} = 0 \quad \text{for } t \in (0,T).$$

Since $\widetilde{u}$ is continuous, we have $\widetilde{u} \equiv 0$, that is, $u_1 \equiv u_2$ in I_T. This completes the proof. $\qquad\square$

Finally, we prove the uniform boundedness of the solution of problem (1).

Theorem 3.4. *The nonnegative solution of the problem* (1) *is uniformly bounded, i.e. there exists a positive constant $M > 0$, such that*

$$0 \leq u(x,t) \leq M \quad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+.$$

Proof. Since u satisfies

$$\begin{cases} u_t = u_{xx} + ku^2(1 - \phi * u) - bu \\ \quad \leq u_{xx} + ku^2 - bu, & (x,t) \in B_T, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

we can get u less than the solution of the ODE problem

$$\begin{cases} \frac{dz}{dt} = kz^2 - bz, \\ z(0) = \|u_0\|_{L^\infty}. \end{cases}$$

By a simple computation, we have

$$z \leq \max\left\{ \|u_0\|_{L^\infty},\ \frac{b}{k} + C \right\},$$

where C is a constant, then, there exists a positive constant $M > 0$ such that $u(x,t) \leq M$. From the conclusion which implies that the solution of problem (1) blow-up is impossible. This completes the proof. $\qquad\square$

4. **The asymptotic behavior of Cauchy problem** (1)**.** In this section, we establish the following result for the solution of the Cauchy problem (1) with compactly supported initial data.

Theorem 4.1. *Let u be the solution of the Cauchy problem* (1) *with a non-negative initial condition $u_0(x) \in L^\infty(\mathbb{R})$, such that $u_0(x) \not\equiv 0$, then*

$$\liminf_{t \to +\infty} \left(\min_{|x| < 2\sqrt{kM}t} u(x,t) \right) > 0. \tag{16}$$

Further, if $u_0(x)$ is compactly supported, then

$$\lim_{t \to +\infty} \left(\max_{|x| \geq 2\sqrt{kM}t} u(x,t) \right) = 0. \tag{17}$$

where M is an upper bound of u.

Proof. We first consider (17), since $u_0(x)$ is compactly supported. Hence, there exists $R > 0$ such that $u_0(x) = 0$ for $|x| \geq R$. Since $u(x,t) \geq 0$ for all $t \geq 0$ and $x \in \mathbb{R}$, one has

$$ku^2(1 - \phi * u) - bu \leq ku^2 - bu, \ (x,t) \in \mathbb{R} \times (0,\infty).$$

Let v denote the solution of the following Cauchy problem

$$\begin{cases} v_t = v_{xx} + kv^2 - bv, \\ v(x,0) = u_0(x). \end{cases}$$

It follows from the comparison principle that $0 \leq u(x,t) \leq v(x,t)$ for all $t > 0$ and $x \in \mathbb{R}$. By an elementary calculation, we have

$$0 \leq u(x,t) \leq \frac{bu_0(x)}{ku_0(x) - (ku_0(x) - b)}.$$

Choose $c > 0$ such that $c \geq 2\sqrt{kM}$. For $t \geq \frac{R}{c}$ and $|x| \geq ct$, one has

$$0 \leq u(x,t) \leq \frac{b\|u_0(x)\|}{k\|u_0(x)\| - (k\|u_0(x)\| - b)} = 0,$$

which immediately yields (17).

We now verify (16). Since u is non-negative, for contradiction we assume that there exists $\tilde{c}_1 \geq 0$ and two sequences $(t_n)_{n \in \mathbb{N}}$ in $(0, +\infty)$ and $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ such that

$$0 \leq \tilde{c}_1 < 2\sqrt{kM},$$

and

$$\begin{cases} |x_n| \leq \tilde{c}_1 t_n, \ \text{for all } n \in \mathbb{N}, \\ t_n \to +\infty \text{ and } u(x_n, t_n) \to 0 \text{ as } n \to +\infty. \end{cases}$$

We introduce

$$c_n := \frac{x_n}{t_n} \in [-\tilde{c}_1, \tilde{c}_1]. \tag{18}$$

Up to extraction of a subsequence, one can assume that $c_n \to c_\infty \in [-\tilde{c}_1, \tilde{c}_1]$ as $n \to +\infty$.

For every $n \in \mathbb{N}$ and $(x,t) \in \mathbb{R} \times (-t_n, \infty)$, we define the shifted function

$$u_n(x,t) = u(x + x_n, t + t_n).$$

From Theorem 3.4, we know that $\left(\|u_n\|_{L^\infty(\mathbb{R} \times (-t_n, +\infty))} \right)_{n \in \mathbb{N}}$ is bounded. Therefore, from the standard parabolic estimate which implies that the function u_n converge in $C_{loc}^{2,1}(\mathbb{R} \times \mathbb{R})$, Up to extraction of a subsequence, to a classical bounded solution u_∞ of

$$(u_\infty)_t = (u_\infty)_{xx} + ku_\infty^2(1 - \phi * u_\infty) - bu_\infty$$

$$= (u_\infty)_{xx} + ku_\infty \left(u_\infty - u_\infty(\phi * u_\infty) - \frac{b}{k} \right) \text{ in } \mathbb{R} \times \mathbb{R},$$

such that $u_\infty \geq 0$ and $u_\infty(0,0) = 0$. By viewing $\left(u_\infty - u_\infty(\phi * u_\infty) - \frac{b}{k}\right)$ as a coefficient in $L^\infty(\mathbb{R} \times \mathbb{R})$, thanks to the strong parabolic maximum principle and uniqueness of the solutions of the Cauchy problem which implies $u_\infty(x,t) = 0$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$ (since the limit u_∞ is unique, one can deduce that the sequence $(u_n)_{n \in \mathbb{N}}$ converges to 0 in $C_{loc}^{2,1}(\mathbb{R} \times \mathbb{R})$). One can define the functions $\tilde{u}_n$ in $\mathbb{R} \times (-t_n, +\infty)$ by

$$\begin{aligned}
\tilde{u}_n(x,t) &= u_n(x + c_n t, t) \\
&= u(x + c_n(t + t_n), t + t_n),
\end{aligned}$$

and $\tilde{u}_n(x,t)$ converge to 0 locally and uniformly in $\mathbb{R} \times \mathbb{R}$, due to the fact that the boundedness of c_n defined in (18). Hence the non-negative functions $\phi * \tilde{u}_n$ also converge to 0 locally uniformly in $\mathbb{R} \times \mathbb{R}$ since the sequence $\left(\|\tilde{u}_n\|_{L^\infty(\mathbb{R} \times (-t_n, +\infty))}\right)_{n \in \mathbb{N}}$ is bounded.

We now fix some parameters which are independent of n. First, we choose $\delta > 0$ satisfying

$$k\left(M - \frac{b}{k} - M\delta\right) \geq \frac{c^2}{4} + \delta, \tag{19}$$

where M is the upper bound of u and v, also let $R > 0$ such that

$$\frac{\pi^2}{4R^2} \leq \delta. \tag{20}$$

Since $u(.,1)$ is continuous from parabolic regularity and is positive in $\mathbb{R}$ from the strong parabolic maximum principle, one can take $\eta > 0$ such that

$$u(.,1) \geq \eta > 0 \text{ for all } |x| \leq R + c.$$

Without loss of generality, we assume that $t_n > 1$ for every $n \in \mathbb{N}$. Since $\phi * \tilde{u}_n \to 0$ locally uniformly in $\mathbb{R} \times \mathbb{R}$ as $n \to +\infty$, we may define

$$t_n^* = \inf\left\{t \in [-t_n + 1, 0]; \phi * \tilde{u}_n \leq \delta \text{ in } [-R, R] \times [t, 0]\right\}, \ n \geq N,$$

where δ and R are as in (19) and (20), and assume that $t_n^* < 0$. Furthermore, for every $n \geq N$, by the continuity of $\phi * \tilde{u}_n$ in $\mathbb{R} \times (-t_n, \infty)$, we then conclude that the infimum is the minimum in the definition of t_n^* and

$$0 \leq \phi * \tilde{u}_n \leq \delta \text{ in } [-R, R] \times [t_n^*, 0]. \tag{21}$$

On the other hand, we have

$$\tilde{u}_n(x, -t_n + 1) = u(x + c_n, 1) \geq \eta \text{ for all } |x| \leq R,$$

for all $n \in \mathbb{N}$. According to the above analysis $\phi * \tilde{u}_n$ in $\mathbb{R} \times (-t_n, +\infty)$ and by minimality of t_n^*, for each $n \geq N$, we have the following dichotomy:

$$\begin{cases} \text{either} & t_n^* > -t_n + 1 \text{ and } \max_{[-R,R]}(\phi * u_n)(., t_n^*) \geq \delta, \\ \text{or} & t_n^* = -t_n + 1 \text{ and } \min_{[-R,R]} u_n(., t_n^*) \geq \eta. \end{cases} \tag{22}$$

Next, we assert that there exists $\rho > 0$ such that

$$\min_{[-R,R]} \tilde{u}_n(., t_n^*) \geq \rho > 0 \text{ for all } n \geq N. \tag{23}$$

This assertion is clearly if the second assertion of (22) always holds. Notice that

$$\min_{[-R,R]} \tilde{u}_n(., t_n^*) > 0 \text{ for each fixed } n \geq N.$$

For contradiction, we assume that (23) is not hold, then up to extraction of a subsequence, there exists a sequence of $(y_n)_{n \geq N}$ in $[-R, R]$ such that

$$\tilde{u}_n(y_n, t_n^*) \to 0 \text{ and}$$

$$y_n \to y_\infty \in [-R, R] \text{ as } n \to +\infty.$$

We use the transformation defined by

$$w_n(x, t) = \tilde{u}_n(x, t + t_n^*),$$

for all $n \geq N$ and $(x, t) \in \mathbb{R} \times (-t_n - t_n^*, +\infty)$. Observe that $\tilde{u}_n$ satisfies

$$(\tilde{u}_n)_t = (\tilde{u}_n)_{xx} + c_n(\tilde{u}_n)_x + k\tilde{u}_n \left(\tilde{u}_n - \tilde{u}_n(\phi * \tilde{u}_n) - \frac{b}{k} \right) \text{ in } \mathbb{R} \times (-t_n, +\infty), \quad (24)$$

then the functions w_n also satisfies (24) in $\mathbb{R} \times (-t_n - t_n^*, +\infty)$. Note that $-t_n - t_{n^*} \leq -1$ for all $n \geq N$, that $c_n \to c_\infty$ as $n \to +\infty$, that the functions w_n are all non-negative and that the sequence $\left(\|w_n\|_{L^\infty(\mathbb{R} \times (-t_n - t_n^*, +\infty))} \right)_{n \geq N}$ is bounded. Using the standard parabolic estimates, we obtain w_n converge in $C_{loc}^{2,1}(\mathbb{R} \times (-1, +\infty))$, up to extraction of a subsequence, to a classical bounded solution w_∞ of the following equation

$$(w_\infty)_t = (w_\infty)_{xx} + c_\infty(w_\infty)_x + kw_\infty^2(1 - \phi * w_\infty) - bw_\infty$$

$$= (w_\infty)_{xx} + c_\infty(w_\infty)_x + kw_\infty \left(w_\infty - w_\infty(\phi * w_\infty) - \frac{b}{k} \right) \text{ in } \mathbb{R} \times (-1, +\infty),$$

such that

$$w_\infty(x, t) \geq 0 \text{ for all } (x, t) \in \mathbb{R} \times (-1, +\infty),$$

and

$$w_\infty(y_\infty, 0) = 0.$$

Thanks to the strong maximum principle and the uniqueness of the solution for the Cauchy problem, we see that

$$w_\infty(x, t) = 0 \text{ for all } (x, t) \in \mathbb{R} \times (-1, +\infty),$$

one has that $w_n \to 0$ as $n \to +\infty$ (at least) are uniform in $\mathbb{R} \times (-1, +\infty)$, it is clearly that

$$\phi * w_n \to 0 \text{ as } n \to +\infty,$$

locally uniform in $\mathbb{R} \times (-1, +\infty)$ by boundedness of the sequence $\left(\|w_n\|_{L^\infty(\mathbb{R} \times (-1, +\infty))} \right)_{n \geq N}$. We then have

$$\tilde{u}_n(., t_n^*) \to 0 \text{ and } (\phi * \tilde{u}_n)(., t_n^*) \to 0,$$

locally uniform in $\mathbb{R}$ as $n \to +\infty$. This is a contradiction to (22) due to the fact that both δ and η are positive. Therefore, (23) holds.

Now, in view of (21), (23) and (24), we have the following situation: for each $n \geq N$, one has $-t_n + 1 \leq t_n^* < 0$ and in the box $[-R, R] \times [t_n^*, 0]$, the non-negative function $\tilde{u}_n$ satisfies

$$\begin{cases} (\tilde{u}_n)_t = (\tilde{u}_n)_{xx} + c_n(\tilde{u}_n)_x + k\tilde{u}_n^2(1 - \phi * \tilde{u}_n) - b\tilde{u}_n \\ \geq (\tilde{u}_n)_{xx} + c_n(\tilde{u}_n)_x + k\tilde{u}_n \left(M - M\delta - \frac{b}{k} \right) \\ \text{in } [-R, R] \times [t_n^*, 0], \\ \tilde{u}_n(\pm R, t) \geq 0 \text{ for all } t \in [t_n^*, 0], \\ \tilde{u}_n(x, t_n^*) \geq \rho \text{ for all } x \in [-R, R]. \end{cases} \quad (25)$$

On the other hand, for every $n \geq N$, the function ψ_n defined in $[-R, R]$ by

$$\psi_n(x) = \rho e^{-c_n x/2 - cR/2} \cos\left(\frac{\pi x}{2R}\right).$$

Then one has $0 \leq \psi_n \leq \rho$ in $[-R, R]$ from $\psi_n(\pm R) = 0$ and

$$\psi_n'' + c_n \psi_n' + k\left(M - M\delta - \frac{b}{k}\right)\psi_n$$
$$= \left(k\left(M - M\delta - \frac{b}{k}\right) - \frac{c_n^2}{4} - \frac{\pi^2}{4R^2}\right)\psi_n$$
$$\geq 0 \text{ in } [-R, R].$$

Notice that the time-independent function ψ_n is a sub-solution for the problem (25). It follows from the parabolic maximum principle that

$$\tilde{u}_n(x, t) \geq \psi_n(x) \text{ for all } (x, t) \in [-R, R] \times [t_n^*, 0],$$

for all $n \geq N$. In particular

$$u(x_n, t_n) = \tilde{u}_n(0, 0) \geq \psi_n(0)$$
$$= \rho e^{-cR/2} \text{ for all } n \geq N.$$

However, the assumption means that

$$u(x_n, t_n) \to 0 \text{ as } n \to +\infty.$$

Since $\rho e^{-cR/2} > 0$ from the positivity of ρ in (23) which reaches a contradiction. This completes the proof. $\qquad \square$

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