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ON INNER POISSON STRUCTURES OF A QUANTUM CLUSTER ALGEBRA WITHOUT COEFFICIENTS

Fang Li and Jie Pan*

Department of Mathematics, Zhejiang University (Yuquan Campus)
Hangzhou, Zhejiang 310027, China

Abstract. The main aim of this article is to characterize inner Poisson structure on a quantum cluster algebra without coefficients. Mainly, we prove that inner Poisson structure on a quantum cluster algebra without coefficients is always a standard Poisson structure. We introduce the concept of so-called locally inner Poisson structure on a quantum cluster algebra and then show it is equivalent to locally standard Poisson structure in the case without coefficients. Based on the result from [7] we obtain finally the equivalence between locally inner Poisson structure and compatible Poisson structure in this case.

1. **Introduction and preliminaries.** The introduction of quantum cluster algebras in [1] is an important development of the theory of cluster algebras, which establishes a connection between cluster theory and the theory of quantum groups, see [2] and [5]. This is closely related to compatible Poisson structures on cluster algebras, see [3], [6]. Moreover, in [7] we studied compatible Poisson structures on quantum cluster algebras and the theory of second quantization related to such Poisson structure.

In this article, we focus on another special kind of Poisson structures, i.e, inner Poisson structure on a quantum cluster algebra without coefficients. It is found that an inner Poisson structure is in fact a standard Poisson structure and we use the result in [7] to connect it to a compatible Poisson structure on the same algebra.

We know from [8] that an inner Poisson structure on the path algebra of a finite connected quiver without oriented cycles is always piecewise standard, see Example 1.4. Together with our result, it shows that in some sense, non-commutativity for multiplication and non-trivial inner Poisson structures cannot exist simultaneously in an associative algebra, at least in these two examples.

First, we introduce some related notations and definitions.

For $n \in \mathbb{N}$, the n**-regular tree** $\mathbb{T}_n$ is a tree (i.e. a connected undirected acyclic graph) whose each vertex $t \in \mathbb{T}_n$ is incident to n edges labeled $1, \cdots, n$ respectively.

Definition 1.1. Fix $n \leq m \in \mathbb{N}$.

(a) A **quantum seed** at vertex $t \in \mathbb{T}_n$ is a triple $\Sigma = (\tilde{X}(t), \tilde{B}(t), \Lambda(t))$ such that

• $\tilde{B}(t)$ is an $m \times n$ integer matrix such that the principal part is skew-symmetrizable, i.e. there is a positive diagonal matrix D satisfying $DB(t)$ is skew-symmetric, where

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* Corresponding author.

$B(t)$ is the first n rows of $\tilde{B}(t)$.

• $\Lambda(t)$ is an $m\times m$ skew-symmetric integer matrix and $(\tilde{B}(t),\Lambda(t))$ is a **compatible pair**, i.e,

$$\tilde{B}(t)^{\top}\Lambda(t)=\begin{pmatrix}D & O\end{pmatrix}. \tag{1}$$

• The **(extended) cluster** $\tilde{X}(t)=(X_t^{e_1},X_t^{e_2},\cdots,X_t^{e_n},X^{e_{n+1}},\cdots,X^{e_m})$ at t is an m-tuple satisfying

$$X_t^eX_t^f=q^{\frac{1}{2}e\Lambda(t)f^{\top}}X_t^{e+f},\quad \forall e,f\in\mathbb{Z}^m,$$

where $\{e_i\}_{i=1}^m$ is the standard basis of $\mathbb{Z}^m$. $X_t^{e_i},i\in[1,n]$ are called **cluster variables** at t while $X^{e_i},i\in[n+1,m]$ are called **frozen variables**.

(b) For any $k\in[1,n]$, define the **mutation** μ_k at direction k to be $\mu_k(\Sigma)=\Sigma'=(\tilde{X}',\tilde{B}',\Lambda')$ such that

$$\mu_k(X_t^{e_k})=X_t^{-e_k+[b_k(t)]_+}+X_t^{-e_k+[-b_k(t)]_+}$$

where $b_k(t)$ is the k-th column of $\tilde{B}(t)$, $[a]_+=max\{a,0\}$ choosing the larger one of a and 0 for any $a\in\mathbb{Z}$ and $[\alpha]_+=(max\{a_1,0\},\cdots,max\{a_s,0\})^T$ for any vector $\alpha=(a_1,\cdots,a_s)^T$.

$$\tilde{X}'=(\tilde{X}(t))\backslash\{X_t^{e_k}\})\bigcup\{\mu_k(X_t^{e_k})\}.$$

$$\tilde{B}'=\mu_k(\tilde{B}(t))=(b'_{ij})_{m\times n}$$

satisfying that

$$b'_{ij}=\begin{cases}-b_{ij}(t) & if\quad i=k\quad or\quad j=k\\ b_{ij}(t)+sgn(b_{ik}(t))[b_{ik}(t)b_{kj}(t)]_+ & otherwise\end{cases} \tag{2}$$

And $\Lambda'=\mu_k(\Lambda(t))=(\lambda'_{ij})_{m\times m}$ satisfying

$$\lambda'_{ij}=\begin{cases}-\lambda_{kj}(t)+\sum\limits_{l=1}^{m}[b_{lk}(t)]_+\lambda_{lj}(t) & if\quad i=k\neq j\\ -\lambda_{ik}(t)+\sum\limits_{l=1}^{m}[b_{lk}(t)]_+\lambda_{il}(t) & if\quad j=k\neq i\\ \lambda_{ij}(t) & otherwise\end{cases} \tag{3}$$

Note that (1) requests $\tilde{B}(t)$ and $\Lambda(t)$ to be of full column rank. It can be verified that $\mu_k(\Sigma)$ is also a quantum seed and μ_k is an involution.

For the Laurent polynomial ring $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ with a formal variable q, define the **quantum torus** T_t at t to be a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra generated by $\tilde{X}(t)$. Denoted by $\mathcal{F}_q$ the skew-field of fractions of T_t. It does not depend on the choice of t.

Definition 1.2. Given seeds $\Sigma(t)=(\tilde{X}(t),\tilde{B}(t),\Lambda(t))$ at $t\in\mathbb{T}_n$ so that $\Sigma(t')=\mu_k(\Sigma(t))$ for any $t-t'$ in $\mathbb{T}_n$ connected by an edge labeled $k\in[1,n]$, then the $\mathbb{Z}[q^{\pm\frac{1}{2}}][X^{\pm e_1},\cdots,X^{\pm e_m}]$-subalgebra of $\mathcal{F}_q$ generated by all variables in $\bigcup\limits_{t\in\mathbb{T}_n}X(t)$ is called the **quantum cluster algebra** $A_q(\Sigma)$ (or simply A_q) associated with Σ.

A **Poisson structure** on an associative k-algebra $\mathcal{A}$ means a triple $(\mathcal{A},\cdot,\{-,-\})$ where $(\mathcal{A},\{-,-\})$ is a Lie k-algebra i.e. satisfying Jacobi identity such that the Leibniz rule holds: for any $a,b,c\in\mathcal{A}$,

$$\{a,bc\}=\{a,b\}c+b\{a,c\}.$$

Algebra $\mathcal{A}$ together with a Poisson structure on it is called a **Poisson algebra**. Denote the **Hamiltonian** of $a\in\mathcal{A}$ by

$$ham(a)=\{a,-\}\in End_k(\mathcal{A}).$$

Then the Leibniz rule is equivalent to that $ham(a)$ is a derivation of $\mathcal{A}$ as an associative algebra for any $a \in \mathcal{A}$.

Definition 1.3. Let $\mathcal{A}$ be an associative algebra. $[a, b] = ab - ba$ is called the **commutator** of a and b, for any $a, b \in \mathcal{A}$. For any $\lambda \in k$, $(\mathcal{A}, \cdot, \lambda[-,-])$ is a Poisson algebra called a **standard Poisson structure** on $(\mathcal{A}, \cdot)$.

A Poisson algebra $(\mathcal{A}, \cdot, \{-,-\})$ is said to be **inner** if for any $a \in \mathcal{A}$, $ham(a) = [a', -]$ for some $a' \in \mathcal{A}$, i.e. it is an inner derivation.

As a natural generalization of standard Poisson algebras, inner Poisson structures often arise. The following two properties indicate what they should be like for some associative algebra $\mathcal{A}$:
(a) ([4]) If the first Hochschild cohomology of $(\mathcal{A}, \cdot)$ vanishes, then any Poisson structure on $\mathcal{A}$ is inner.
(b) ([8]) A Poisson bracket $\{-,-\}$ on $(\mathcal{A}, \cdot)$ is an inner Poisson bracket if and only if there is a k-linear transformation g of $\mathcal{A}$ satisfying $ham(a) = [g(a), -]$ for any $a \in \mathcal{A}$ and

$$[g(x), y] = [x, g(y)], \forall x, y \in \mathcal{A}, \tag{4}$$

$$[g(x), g(y)] - g([g(x), y]) \in Z(\mathcal{A}), \forall x, y \in \mathcal{A}, \tag{5}$$

where $Z(\mathcal{A})$ is the center of the Lie bracket $[-,-]$. Furthermore, for any inner Poisson bracket $\{-,-\}$ on $(\mathcal{A}, \cdot)$, we can always find a k-linear transformation g_0 of $\mathcal{A}$ satisfying the above equations and meantime,

$$Z(\mathcal{A}) \subseteq Ker(g_0). \tag{6}$$

Moreover, it is proved in [8] that

Example 1.4. For a finite connected quiver Q without oriented cycles,

$$kQ = k \cdot 1 \oplus \bigoplus_{1 \leq i \leq m} I_i,$$

is a decomposition into indecomposable ideals of the Lie algebra $(kQ, [-,-])$. Furthermore, if $\{-,-\}$ is an inner Poisson structure on the path algebra kQ, then there is a unique vector $(\lambda_1, \cdots, \lambda_m) \in k^m$ such that

$$ham(a) = \lambda_i[a, -], \text{for any } a \in I_i, 1 \leq i \leq m. \tag{7}$$

Conversely, for any vector $(\lambda_1, \cdots, \lambda_m) \in k^m$, there is a unique inner Poisson structure on kQ (up to a Poisson algebra isomorphism) satisfying (7).

From now, let $\mathcal{P}(\mathcal{A})$ be the set of the k-linear transformations g of $\mathcal{A}$ satisfying (4), (5). Define an equivalence relation $\sim$ on $\mathcal{P}(\mathcal{A}) : g \sim g'$ if and only if there exists $\tau \in Aut(\mathcal{A}, \cdot)$ such that $\mathrm{Im}(\tau g \tau^{-1} - g') \subseteq Z(\mathcal{A})$. Denote by $[g]$ the equivalence class of g.

Two Poisson structures on $(\mathcal{A}, \cdot)$ are called **isomorphic** as Poisson algebras if there exists an associative algebra automorphism τ of $(\mathcal{A}, \cdot)$ such that it is also a Lie algebra homomorphism. Denote by $[(\mathcal{A}, \cdot, \{-,-\})]$ the iso-class of $(\mathcal{A}, \cdot, \{-,-\})$.

The paper is organized as follows. In Section 2, we discuss the inner Poisson structures on a quantum cluster algebra without coefficients and prove the main theorem.

Theorem 1.5. *(Theorem 2.5) Let A_q be a quantum cluster algebra without coefficients, any inner Poisson structure on A_q must be a standard Poisson structure.*

Then in Section 3, we generalize the definition to locally inner Poisson structures and find following equivalence.

Theorem 1.6. *(Theorem 3.5) Let A_q be a quantum cluster algebra without coefficients and $\{-,-\}$ a Poisson structure on A_q. The following statements are equivalent:*
(*a*) *$\{-,-\}$ is locally standard.*
(*b*) *$\{-,-\}$ is locally inner.*
(*c*) *$\{-,-\}$ is compatible with A_q.*

2. Proof of the main theorem.

The following theorem from [8] gives a correspondence between inner Poisson brackets and k-linear transformations.

Theorem 2.1 ([8])**.** *Let $(\mathcal{A},\cdot)$ be an associative algebras. Then the map*

$$\{\textit{equivalence classes of } \mathcal{P}(\mathcal{A})\} \to \{\textit{isoclasses of inner Poisson structures on } (\mathcal{A},\cdot)\}$$

given by

$$[g] \mapsto [(\mathcal{A},\cdot,\{-,-\})],\ \textit{where } ham(a) = [g(a),-], \forall a \in \mathcal{A}$$

is bijective.

Because of the above theorem, we can focus on the k-linear transformations when studying inner Poisson structures. In this section, we study the inner Poisson structures of a quantum cluster algebra A_q with deformation matrix Λ.

Because g is k-linear, we only need to think about its action on Laurent monomials in A_q. In this section when we say Laurent monomials, we actually mean Laurent monomials in the initial cluster.

Lemma 2.2. *For a quantum cluster algebra A_q, if $g \in \mathcal{P}(A_q)$, then for any $h \in [1,m]$ and any cluster $\tilde{X} = \{X_1,\cdots,X_n,X_{n+1},\cdots,X_m\}$, we have*

$$g(X_h) = k_1^h X_h + \sum_{i=2}^{l_h} k_i^h X_1^{a_{i1}^h} X_2^{a_{i2}^h} \cdots X_m^{a_{im}^h},$$

which is expanded in a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-linearly independent form, with $l_h \in \mathbb{N}$, $a_{i1}^h,\cdots,a_{im}^h \in \mathbb{Z}$ and $k_1^h, k_i^h \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ for $2 \le i \le l_h$, satisfying that

$$(a_{i1}^h, a_{i2}^h, \ldots, a_{im}^h)\Lambda = (\lambda_{h1}^i, \lambda_{h2}^i, \ldots, \lambda_{hm}^i) \tag{8}$$

where $\lambda_{hp}^i = 0$ or λ_{hp} for $1 \le p \le m$.

Proof. Assume

$$g(X_1) = \sum_{i=1}^{l_1} k_i X_1^{a_{i1}} X_2^{a_{i2}} \cdots X_m^{a_{im}}, k_i \neq 0$$

and

$$g(X_2) = \sum_{i=1}^{l_2} p_i X_1^{b_{i1}} X_2^{b_{i2}} \cdots X_m^{b_{im}}, p_i \neq 0$$

are expanded in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-linearly independent forms. And assume $\{-,-\}$ is the inner Poisson bracket corresponding to g, i.e. $\{X,Y\} = [g(X),Y]$ for any $X,Y \in A_q$.

Because $0=\{X_1,X_1\}=[g(X_1),X_1]=\sum\limits_{i=1}^{l_1}(q^{\sum\limits_{t=1}^{m}a_{it}\lambda_{t1}}-1)X_1^{a_{i1}+1}X_2^{a_{i2}}\cdots X_m^{a_{im}}$, we have

$$\sum_{t=1}^{m}a_{it}\lambda_{t1}=0 \quad \text{for any} \quad 1<i<l_1.$$

Similarly,

$$\sum_{t=1}^{m}b_{it}\lambda_{t2}=0 \quad \text{for any} \quad 1<i<l_2.$$

Moreover, according to (4), we then obtain that

$$\{X_1,X_2\}=[g(X_1),X_2]=\sum_{i=1}^{l_1}k_i(q^{\sum\limits_{t=2}^{m}a_{it}\lambda_{t2}}-q^{a_{i1}\lambda_{21}})X_1^{a_{i1}}X_2^{a_{i2}+1}X_3^{a_{i3}}\cdots X_m^{a_{im}}; \quad (9)$$

$$\{X_1,X_2\}=[X_1,g(X_2)]=\sum_{i=1}^{l_2}p_i(1-q^{\sum\limits_{t=1}^{m}b_{it}\lambda_{t1}})X_1^{b_{i1}+1}X_2^{b_{i2}}X_3^{b_{i3}}\cdots X_m^{b_{im}}. \quad (10)$$

Trivially, the expansions of the right-sides of (9) and (10) are also in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-linearly independent forms, which are the same due to the algebraic independence of $\{X_1,X_2,\cdots,X_m\}$. Hence there exists $l_0\leq l_1,l_2$ such that there are l_0 monomials with non-zero-coefficients in the expansions of the right-sides of (9) and (10) respectively and the coefficients of other monomials are all zeros.

Without loss of generality, suppose these l_0 monomials with non-zero-coefficients are just the first l_0 ones in the expansions of the right-sides of (9) and (10) respectively. We may assume they are in one-by-one correspondence indexed by $i=1,2,\cdots,l_0$. Hence due to the above discussion, we obtain that

Case 1. For i satisfying $1\leq i\leq l_0$,

$$\begin{cases} a_{i1}=b_{i1}+1\\ a_{i2}+1=b_{i2}\\ a_{it}=b_{it}, \text{ for } 3\leq t\leq m\\ k_i(q^{\sum\limits_{t=2}^{m}a_{it}\lambda_{t2}}-q^{a_{i1}\lambda_{21}})=p_i(1-q^{\sum\limits_{t=1}^{m}b_{it}\lambda_{t1}})\neq 0\\ \sum\limits_{t=1}^{m}a_{it}\lambda_{t1}=\sum\limits_{t=1}^{m}b_{it}\lambda_{t2}=0 \end{cases}$$

Case 2. For i,j satisfying $l_0<i\leq l_1$, $l_0<j\leq l_2$, we have

$$q^{\sum\limits_{t=2}^{m}a_{it}\lambda_{t2}}-q^{a_{i1}\lambda_{21}}=1-q^{\sum\limits_{t=1}^{m}b_{jt}\lambda_{t1}}=\sum_{t=1}^{m}a_{it}\lambda_{t1}=\sum_{t=1}^{m}b_{jt}\lambda_{t1}=0.$$

From Case 1, we get that for $1\leq i\leq l_0$,

$$\begin{cases} \sum\limits_{t=1}^{m}a_{it}\lambda_{t1}=0\\ \sum\limits_{t=1}^{m}a_{it}\lambda_{t2}=\sum\limits_{t=1}^{m}b_{it}\lambda_{t2}+\lambda_{12}-\lambda_{22}=\lambda_{12}. \end{cases} \quad (11)$$

From Case 2, we have that for $l_0<i\leq l_1$,

$$\begin{cases} \sum\limits_{t=1}^{m}a_{it}\lambda_{t1}=0\\ \sum\limits_{t=1}^{m}a_{it}\lambda_{t2}=0. \end{cases} \quad (12)$$

In the above discussion, replacing X_2 by other X_p for $p \neq 1, 2$, we get similarly that:

$$\sum_{t=1}^{m} a_{it}\lambda_{tp} = \lambda_{1p}^{i} \tag{13}$$

where $\lambda_{1p}^{i} = \lambda_{1p}$ or 0 for any $3 \leq p \leq m, 1 \leq i \leq l_1$.

In summary from (11), (12) and (13), we have

$$(a_{i1}, a_{i2}, \cdots, a_{im})\Lambda = (\lambda_{11}^{i}, \lambda_{12}^{i}, \cdots, \lambda_{1m}^{i})$$

for any $1 \leq i \leq l_1$, i.e, in the expansion of $g(X_1)$ any term $k_i X_1^{a_{i1}} X_2^{a_{i2}} \cdots X_m^{a_{im}}$ with $k_i \neq 0$ must have $(a_{i1}, \cdots, a_{im})$ as a solution of above equations.

When $(a_{11}, a_{12}, \cdots, a_{1m}) = (1, 0, \cdots, 0)$, (8) is satisfied for $\lambda_{1p}^{1} = \lambda_{1p}$ for any p. So in the expansion of $g(X_1)$, we may consider the monomial $k_1 X_1$ as the first term, i.e. $i = 1$. Note that it maybe not exist if its coefficient k_1 is zero.

Then we have the expansion of $g(X_1)$ as follows:

$$g(X_1) = k_1 X_1 + \sum_{i=2}^{l_1} k_i X_1^{a_{i1}} X_2^{a_{i2}} \cdots X_m^{a_{im}},$$

and

$$(a_{i1}, a_{i2}, \ldots, a_{im})\Lambda = (\lambda_{11}^{i}, \lambda_{12}^{i}, \ldots, \lambda_{1m}^{i})$$

where $\lambda_{1k}^{i} = 0$ or λ_{1k}. It implies this lemma holds for $h = 1$.

The similar discussion for any $X_h, h \in [1, m]$ can be given to complete the proof. □

In the rest of this section we will always assume A_q is a quantum cluster algebra without coefficients, i.e, $m = n$. Then (1) becomes

$$B^{\top}\Lambda = D$$

Following this, B and Λ are both of rank n and invertible. So $n > 1$ since $B = 0$ when $n = 1$. And in this case $(a_{i1}, a_{i2}, \ldots, a_{im}) = (\lambda_{11}^{i}, \lambda_{12}^{i}, \ldots, \lambda_{1m}^{i})\Lambda^{-1}$.

Lemma 2.3. *Let A_q be a quantum cluster algebra without coefficients. If $g \in \mathcal{P}(A_q)$ satisfies that $g(X) = k_X X$ for any Laurent monomial X in A_q with $k_X \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$, then there is a scalar $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-transformation $g' \in \mathcal{P}(A_q)$ such that $g' \sim g$.*

Proof. For any Laurent monomial $X = pX_1^{m_1} X_2^{m_2} \cdots X_n^{m_n} \in A_q$, X communicates with X_i if and only if $\sum_j m_j \lambda_{ji} = 0$. Therefore $X \in Z(A_q)$ the center of A_q if and only if $(m_1, m_2, \cdots, m_n)\Lambda = 0$. Because Λ is invertible, we have $Z(A_q) = \mathbb{Z}[q^{\pm\frac{1}{2}}]$.

Therefore for any non-constant Laurent monomial $X \in A_q$, we can find a Laurent monomial $Y_0 \in A_q$ such that $[X, Y_0] \neq 0$.

For two non-constant Laurent monomials $X, Y \in A_q$, we claim $k_X = k_Y$.

Case 1. Assume $XY \neq YX$.

Denote the Poisson bracket associated to g as $\{-, -\}$. Then, first, we have

$$\{X, Y\} = [g(X), Y] = k_X XY - k_X YX.$$

On the other hand, according to (4) we have also

$$\{X, Y\} = [X, g(Y)] = k_Y XY - k_Y YX.$$

Thus since $XY \neq YX$, we obtain $k_X = k_Y$.

Case 2. Assume $XY = YX$.

Since $X, Y \notin Z(A_q)$, there are Laurent monomials M, N in A_q such that $XM \neq MX, YN \neq NY$. Then from Case 1, we have $k_X = k_M$, $k_Y = k_N$.

If either $YM \neq MY$ or $XN \neq NX$, then $k_Y = k_M$ or $k_X = k_N$. It follows that $k_X = k_Y$.

Otherwise, $YM = MY$ and $XN = NX$. It is easy to see that $X(MN) \neq (MN)X, Y(MN) \neq (MN)Y$. So, from Case 1, $k_X = k_{MN} = k_Y$.

Then, there exists a fixed element $k_0 \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ such that $k_0 = k_X$ for any non-constant Laurent monomial $X \in A_q$. It follows that for any such X,

$$g(X) = k_0 X. \tag{14}$$

For any constant $a \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ and any $W \in A_q$, we have

$$[g(a), W] = \{a, W\} = [a, g(W)] = 0.$$

Therefore $g(a) \in Z(A_q) = \mathbb{Z}[q^{\pm\frac{1}{2}}]$, that is, $g(a)$ is a constant.

Let g' be the k_0-scalar linear transformation of A_q, that is, for any $W \in A_q$, define $g'(W) = k_0 W$. Trivially, $g' \in \mathcal{P}(A_q)$.

By (14) and since $g(a)$ is a constant for any $a \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$, we have $\mathrm{Im}(g - g') \subseteq Z(A_q) = \mathbb{Z}[q^{\pm\frac{1}{2}}]$. It means that $g \sim g'$. □

Lemma 2.4. *Let A_q be a quantum cluster algebra without coefficients. Then for any $g \in \mathcal{P}(A_q)$,*

(a) for any Laurent monomial X in A_q, $g(X) = k_X X + k'_X$, where $k_X, k'_X \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$;

(b) there is a scalar $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-transformation $g_0 \in \mathcal{P}(A_q)$ such that $g_0 \sim g$.

Proof. (a) According to Lemma 2.2,

$$g(X_h) = k_1^h X_h + \sum_{i=2}^{l_h} k_i^h X_1^{a_{i1}^h} X_2^{a_{i2}^h} \cdots X_n^{a_{in}^h}, \tag{15}$$

where

$$(a_{i1}^h, a_{i2}^h, \ldots, a_{in}^h)\Lambda = (\lambda_{h1}^i, \lambda_{h2}^i, \ldots, \lambda_{hn}^i)$$

and $\lambda_{hp}^i = 0$ or λ_{hp} for $1 \leq p \leq n$. For $m_1, \cdots, m_n \in \mathbb{Z}$, assume

$$g(X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n}) = \sum_{j=1}^{l} f_j X_1^{c_{j1}} X_2^{c_{j2}} \cdots X_n^{c_{jn}}, \tag{16}$$

satisfying $c_{1t} = m_t$ for $t \in [1, n]$, as a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-linearly independent expansion except that f_1 may be zero. Let $\{-,-\}$ be the Poisson structure correspond to g.

According to (4), we have:

$$\begin{aligned}
& \{X_1, X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n}\} \\
= \; & [g(X_1), X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n}] \\
= \; & k_1^1 X_1^{m_1+1} X_2^{m_2} \cdots X_n^{m_n} - k_1^1 X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n} X_1 \\
& + \sum_{i=2}^{l_1} k_i^1 X_1^{a_{i1}1} \cdots X_n^{a_{in}^1} X_1^{m_1} \cdots X_n^{m_n} - \sum_{i=2}^{l_1} k_i^1 X_1^{m_1} \cdots X_n^{m_n} X_1^{a_{i1}^1} \cdots X_n^{a_{in}^1} \\
= \; & k_1^1 (1 - q^{\sum_{t=1}^{n} m_t \lambda_{t1}}) X_1^{m_1+1} X_2^{m_2} \cdots X_n^{m_n}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=2}^{l_1} k_i^1 (q^{\sum\limits_{r>s} a_{ir} m_s \lambda_{rs}} - q^{\sum\limits_{r<s} a_{ir} m_s \lambda_{sr}}) X_1^{m_1+a_{i1}^1} \cdots X_n^{m_n+a_{in}^1} \\
= \;& k_1^1 (1 - q^{\sum\limits_{t=1}^{n} m_t \lambda_{t1}}) X_1^{m_1+1} X_2^{m_2} \cdots X_n^{m_n} \\
&+\sum_{i=2}^{l_1} k_i^1 q^{\sum\limits_{r>s} a_{ir}^1 m_s \lambda_{rs}} (1 - q^{\sum\limits_{r,s=1}^{n} a_{ir}^1 m_s \lambda_{sr}}) X_1^{m_1+a_{i1}^1} \cdots X_n^{m_n+a_{in}^1};
\end{aligned}
$$

on the other hand,

$$
\begin{aligned}
\{X_1, X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n}\} = \;& [X_1, g(X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n})] \\
= \;& \sum_{j=1}^{l} f_j (X_1^{c_{j1}+1} X_2^{c_{j2}} \cdots X_n^{c_{jn}} - X_1^{c_{j1}} X_2^{c_{j2}} \cdots X_n^{c_{jn}X_1}) \\
= \;& \sum_{j=1}^{l} f_j (1 - q^{\sum\limits_{t=1}^{n} c_{jt} \lambda_{t1}}) X_1^{c_{j1}+1} X_2^{c_{j2}} \cdots X_n^{c_{jn}}
\end{aligned}
$$

Note that in the last step of the first expansion of $\{X_1, X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n}\}$, we have

$$
\begin{aligned}
\sum_{r,s=1}^{n} a_{ir}^1 m_s \lambda_{sr} = -\sum_{s=1}^{n} (\sum_{r=1}^{n} a_{ir}^1 \lambda_{rs}) m_s \;&= -(a_{i1}^1 \cdots a_{in}^1) \Lambda (m_1 \cdots m_n)^\top \\
&= -(\lambda_{11}^i \cdots \lambda_{1n}^i)(m_1 \cdots m_n)^\top.
\end{aligned} \tag{17}
$$

The last steps of the two kinds of expansions of $\{X_1, X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n}\}$ are both in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-linearly independent forms, which are the same due to the algebraic independence of $\{X_1, X_2, \cdots, X_m\}$. Hence, for some $l_0 \le l_1, l$, there are $l_0 - 1$ monomials with non-zero-coefficients in the last steps of two kinds of expansions above respectively and the coefficients of other monomials are all zeros, besides the first terms in these two expansion which maybe be zero or non-zero in the various cases.

Without loss of generality, suppose the $l_0 - 1$ monomials with non-zero-coefficients are just those ones whose indexes are with $2 \le i \le l_0$ and $2 \le j \le l_0$ respectively in the last steps of two kinds of expansions above, that is, we assume they are in one-by-one correspondence indexed by $i = 2, \cdots, l_0$.

Thus, due to the above discussion, from comparation of coefficients, we obtain that

$$
A_1 = k_1^1 (1 - q^{\sum\limits_{t=1}^{n} m_t \lambda_{t1}}) X_1^{m_1+1} X_2^{m_2} \cdots X_n^{m_n} = f_1 (1 - q^{\sum\limits_{t=1}^{n} c_{1t} \lambda_{t1}}) X_1^{c_{11}+1} X_2^{c_{12}} \cdots X_n^{c_{1n}}. \tag{18}
$$

When $2 \le i \le l_0$,

$$
\begin{aligned}
0 \;&\ne k_i^1 q^{\sum\limits_{r>s} a_{ir}^1 m_s \lambda_{rs}} (1 - q^{\sum\limits_{r,s=1}^{n} a_{ir}^1 m_s \lambda_{sr}}) X_1^{m_1+a_{i1}^1} \cdots X_n^{m_n+a_{in}^1} \\
&= f_i (1 - q^{\sum\limits_{t=1}^{n} c_{it} \lambda_{t1}}) X_1^{c_{i1}+1} X_2^{c_{i2}} \cdots X_n^{c_{in}}.
\end{aligned} \tag{19}
$$

When $l_0 < i \le l_h, l_0 < j \le l$,

$$
\begin{aligned}
0 \;&= k_i^1 q^{\sum\limits_{r>s} a_{ir}^1 m_s \lambda_{rs}} (1 - q^{\sum\limits_{r,s=1}^{n} a_{ir}^1 m_s \lambda_{sr}}) X_1^{m_1+a_{i1}^1} \cdots X_n^{m_n+a_{in}^1} \\
&= f_j (1 - q^{\sum\limits_{t=1}^{n} c_{jt} \lambda_{t1}}) X_1^{c_{j1}+1} X_2^{c_{j2}} \cdots X_n^{c_{jn}}.
\end{aligned} \tag{20}
$$

In (18), we have that $A_1 = 0$ if and only if $\sum\limits_{t=1}^{n} m_t \lambda_{t1} = 0$; otherwise, $A_1 \ne 0$ then $f_1 = k_1^1$.

From (19) and (17), we obtain that for $2 \leq i \leq l_0$,

$$\begin{cases} c_{i1} = m_1 + a^1_{i1} - 1 \\ c_{ip} = m_p + a^1_{ip}, \text{ for } 2 \leq p \leq n \\ 0 \neq \sum\limits_{r,s=1}^{n} a_{ir} m_s \lambda_{sr} = -\sum\limits_{s=1}^{n} \lambda^i_{1s} m_s \\ 0 \neq \sum\limits_{t=1}^{n} c_{it}\lambda_{t1} = \sum\limits_{t=1}^{n} (m_t + a^1_{it})\lambda_{t1} = \sum\limits_{t=1}^{n} m_t\lambda_{t1} + \lambda^i_{11} = \sum\limits_{t=1}^{n} m_t\lambda_{t1}. \end{cases}$$

From (20) and (17), we obtain that for $i, j > l_0$,

$$\sum_{s=1}^{n} \lambda^i_{1s} m_s = \sum_{p=1}^{n} c_{jp}\lambda_{p1} = 0. \tag{21}$$

In conclusion, $(c_{j1}, \cdots, c_{jn})$ with $f_j \neq 0$ must satisfy one of (18), (19) and (20) for any $j = 1, 2, \cdots, l_0$.

In the same way, replacing X_1 by X_h, $h \in [1, n]$, we will also obtain three equalities similar to (18), (19) and (20) such that $(c_{j1}, \cdots, c_{jn})$ with $f_j \neq 0$ satisfies one of three equalities.

According to our assumption, we always have $(c_{11}, \cdots, c_{1n}) = (m_1, \cdots, m_n)$.

Now we want to prove by contradiction that $(c_{j1}, \cdots, c_{jn})$ can only be $(0, \cdots, 0)$ for $2 \leq j \leq l$. Hence, we first assume that $(c_{j1}, \cdots, c_{jn}) \neq (0, \cdots, 0)$ in this case.

We can choose some special $m^o_1, \cdots, m^o_n \in \mathbb{Z}^n_{\geq 0}$ such that

$$\begin{cases} \sum\limits_{t=1}^{n} m^o_t \lambda_{th} \neq \lambda_{ih} \text{ for any } i, h \\ \sum\limits_{t=1}^{n} m^o_t \lambda_{th} \neq 0 \text{ for any } i, h \\ m^o_t \geq 0 \text{ for any } t \end{cases} \tag{22}$$

For any X_h, $h \in [1, n]$, we first claim that under the condition (22), $(c_{j1}, \cdots, c_{jn})$ with $f_j \neq 0$ does not satisfy the equality similar to (19).

In fact, because Λ is invertible, so since $(c_{j1}, \cdots, c_{jn}) \neq (0, \cdots, 0)$, we have $(c_{j1}, \cdots, c_{jn})\Lambda \neq (0, \cdots, 0)$. Therefore $(c_{j1}, \cdots, c_{jn})$ can not satisfy an equality similar to (20) for all $h \in [1, n]$, i.e. it must satisfy some equations similar to (18) or (19) for some h. Therefore all of the possible $(c_{j1}, \cdots, c_{jn}) \neq (0, \cdots, 0)$ are $(m^o_1 + a^h_{i1}, \cdots, m^o_h + a^h_{ih} - 1, \cdots, m^o_n + a^h_{in})$ for some i and h. Hence for any h, by (22),

$$\begin{aligned} \sum_{t=1}^{n} c_{jt}\lambda_{th} = \sum_{t=1}^{n} (m^o_t + a^p_{it})\lambda_{th} - \lambda_{ph} &= \sum_{t=1}^{n} m^o_t\lambda_{th} + \lambda^i_{ph} - \lambda_{ph} \\ &= \begin{cases} \sum\limits_{t=1}^{n} m^o_t\lambda_{th} \neq 0, & \text{if } \lambda^i_{ph} = \lambda_{ph} \\ \sum\limits_{t=1}^{n} m^o_t\lambda_{th} - \lambda_{ph} \neq 0, & \text{if } \lambda^i_{ph} = 0. \end{cases} \end{aligned}$$

So for any h, $(0, \cdots, 0)$ is the only $(c_{j1}, \cdots, c_{jn})$ satisfying the equalities similar to (20), $(m^o_1, \cdots, m^o_n)$ is the only $(c_{11}, \cdots, c_{1n})$ satisfying the equalities similar to (18), while all of that

$$(m^o_1 + a^p_{i1}, \cdots, m^o_p + a^p_{ip} - 1, \cdots, m^o_n + a^p_{in})$$

satisfy the equalities similar to (19).

Hence for any $(c_{j1},\cdots,c_{jn})\neq(0,\cdots,0)$ with $2\leq j\leq l_0$, we have

$$\begin{array}{llll} c_{j1}=m_1^o+a_{j1}^1-1= & \cdots & = m_1^o+a_{j1}^n \\ \vdots \quad\quad \vdots & \ddots & \vdots \\ c_{jn}=m_n^o+a_{jn}^1 & =\cdots & = m_1^o+a_{jn}^n-1. \end{array} \tag{23}$$

Then for any $h_1\neq h_2$,

$$\begin{cases} a_{jh_1}^{h_1}-1=a_{jh_1}^{h_2} \\ a_{jh_2}^{h_1}=a_{jh_2}^{h_2}-1 \\ a_{jh}^{h_1}=a_{jh}^{h_2}, \quad \text{for any } h\neq h_1,h_2 \end{cases}$$

Therefore,

$$\sum_{h=1}^{n}a_{jh}^{h_1}\lambda_{hh_1}=\lambda_{h_1h_1}^{j}=0=\lambda_{h_1h_1}$$

and

$$\sum_{h=1}^{n}a_{jh}^{h_1}\lambda_{hh_2}=\sum_{h=1}^{n}a_{jh}^{h_2}\lambda_{hh_2}+\lambda_{h_1h_2}=\lambda_{h_2h_2}^{j}+\lambda_{h_1h_2}=\lambda_{h_1h_2},$$

for any h_1,h_2. Hence $(a_{j1}^{h_1},\cdots,a_{jn}^{h_1})\Lambda=(\lambda_{h_11},\cdots,\lambda_{h_1n})$. Again because Λ is invertible, we have $(a_{j1}^{h_1},\cdots,a_{jn}^{h_1})=e_{h_1}$. Therefore by (23) we get

$$(c_{j1},\cdots,c_{jn})=(m_1^o,\cdots,m_n^o)+(a_{j1}^{h_1},\cdots,a_{jn}^{h_1})-e_{h_1}=(m_1^o,\cdots,m_n^o),$$

which contradicts to our assumption as $j\geq 2$. Thus in conclusion, under the condition (22), $(c_{j1},\cdots,c_{jn})$ with $f_j\neq 0$ does not satisfy the equality similar to (19) for any h.

Hence, indeed, the case satisfying the equality similar to (19) would not happen under the assumption of (22). It means for any h, we only have (20) to hold for $2\leq i\leq l_h$. Therefore, $\sum\limits_{s=1}^{n}\lambda_{hs}^{i}m_s^o=0$ for any h and $2\leq i\leq l_h$ according to (21). Define sets:

$S_{ih}=\{(m_1,\cdots,m_n)\in\mathbb{Z}_{\geq 0}^n \mid (m_1,\cdots,m_n)\Lambda=(t_1,\cdots,t_{h-1},\lambda_{ih},t_{h+1},\cdots,t_n)$ $\forall t_1,\cdots,t_{h-1},t_{h+1},\cdots,t_n\in\mathbb{Z}\}$ for any $i,h\in[1,n]$;

$T_h=\{(m_1,\cdots,m_n)\in\mathbb{Z}_{\geq 0}^n \mid (m_1,\cdots,m_n)\Lambda=(t_1,\cdots,t_{h-1},0,t_{h+1},\cdots,t_n)$ $\forall t_1,\cdots,t_{h-1},t_{h+1},\cdots,t_n\in\mathbb{Z}\}$ for any $h\in[1,n]$.

Then the set of positive integer vectors $(m_1,\cdots,m_n)$ satisfying (22) is equal to the set

$$\mathbb{Z}_{\geq 0}^n\backslash\bigcup_{i,h\in[1,n]}(S_{ih}\cup T_i).$$

For any i,h, the sets S_{ih},T_i lie discretely in their corresponding $(n-1)$-dimensional nonnegative cones C_{ih},D_i in $\mathbb{Q}^n$ respectively. All of C_{ih},D_i are contained in the n-dimensional nonnegative cone (or say, the first quadrant) $\mathbb{Q}_{\geq 0}^n$ of $\mathbb{Q}^n$. Let $C'=\bigcup\limits_{i,h\in[1,n]}(C_{ih}\cup D_i)$.

It is easy to see that every l-dimensional nonnegative cone included in $\mathbb{Q}_{\geq 0}^n$ can be seen uniquely as an intersection of an l-dimensional linear subspace and $\mathbb{Q}_{\geq 0}^n$ for any $l\leq n$. Denote by P_{ih} the $(n-1)$-dimensional linear subspace such that $C_{ih}=P_{ih}\bigcap\mathbb{Q}_{\geq 0}^n$ and by Q_i the $(n-1)$-dimensional linear subspace such that $D_i=Q_i\bigcap\mathbb{Q}_{\geq 0}^n$ for $i,h\in[1,n]$.

Let $P' = \bigcup_{i,h\in[1,n]} P_{ih} \cup Q_i$. Then, $C' \subseteq P'$.

Assume there is at most $p(< n)$ linearly independent vectors in $\mathbb{Q}^n_{\geq 0} \setminus C'$. Let P_0 be the subspace spanned by these p linearly independent vectors. Then $\mathbb{Q}^n_{\geq 0} \subseteq P_0 \bigcup C' \subseteq P_0 \bigcup P'$. But the standard basis $\{e_1, \cdots, e_n\} \subseteq \mathbb{Q}^n_{\geq 0}$. It follows that $\mathbb{Q}^n \subseteq P_0 \bigcup P'$, which contradicts to the well-known fact that every finite n-dimensional linear space can not be contained in a union of finitely many subspaces with dimensions less than n.

Hence, we can find n linearly independent vectors in $\mathbb{Q}^n_{\geq 0} \setminus C'$, say $v_1, \cdots, v_n \in \mathbb{Q}^n_{\geq 0}$, whose coordinates satisfy respectively the condition (22).

Now, we can find an $a \in \mathbb{Z}_+$ such that $av_i \in \mathbb{Z}^n_{\geq 0}$. Without loss of generality, we may think for each $av_i = (m^o_{1i}, \cdots, m^o_{ni})$ $(i = 1, \cdots, n)$, the condition (22) still is satisfied. Otherwise, the only possibility is that the first condition in (22) is not satisfied, then we can always replace a by ra for certain $r \in \mathbb{Z}_+$ such that the first condition in (22) is satisfied, too.

In summary, we can obtain $\mathbb{Q}$-linearly independent vectors $av_i = (m^o_{1i}, \cdots, m^o_{ni}) \in \mathbb{Z}^n_{\geq 0}$ $(i = 1, \cdots, n)$ satisfying (22).

And as we discussed above, the following equation is satisfied:

$$\begin{pmatrix} \lambda^i_{h1} & \cdots & \lambda^i_{hn} \end{pmatrix} \begin{pmatrix} m^o_{11} & \cdots & m^o_{1n} \\ \vdots & \ddots & \vdots \\ m^o_{n1} & \cdots & m^o_{nn} \end{pmatrix} = 0$$

So $(\lambda^i_{h1}, \cdots, \lambda^i_{hn})$ can only be $(0, \cdots, 0)$ for any h, $2 \leq i \leq l_1$. Then it follows from Lemma 2.2 that for any h and $2 \leq i \leq l_1$,

$$(a^h_{i1}, \cdots, a^h_{in}) = (0, \cdots, 0). \tag{24}$$

Then, we have $g(X_h) = k^h_1 X_h + k^h_2$ where $k^h_2 \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$.

For general $(m_1, \cdots, m_n) \in \mathbb{Z}^n$ such that $X_1^{m_1} \cdots X_n^{m_n}$ is a Laurent monomial in A_q. According to our above discussion and by (24), we have

$$\begin{aligned} k^1_1(1 - q^{\sum_{t=1}^{n} m_t\lambda_{t1}})X_1^{m_1+1}X_2^{m_2}\cdots X_n^{m_n} &= \{X_1, X_1^{m_1}\cdots X_n^{m_n}\} \\ &= \sum_j f_j(1 - q^{\sum_{t=1}^{n} c_{jt}\lambda_{t1}})X_1^{c_{j1}+1}X_2^{c_{j2}}\cdots X_n^{c_{jn}} \end{aligned}$$

So we have $\sum_{t=1}^{n} c_{jt}\lambda_{t1} = 0$ for any $j \geq 2$. Replacing X_1 by X_h, $h \in [1, n]$, we obtain that $(c_{j1}, \cdots, c_{jn})\Lambda = 0$ for any $j \geq 2$. However, it contradicts to that Λ is invertible since we have assumed $(c_{j1}, \cdots, c_{jn}) \neq (0, \cdots, 0)$.

Hence $(c_{j1}, \cdots, c_{jn}) = (0, \cdots, 0)$ for any $j \geq 2$.

Then by (16), we get $g(X_1^{m_1} \cdots X_n^{m_n}) = f_1 X_1^{m_1} \cdots X_n^{m_n} + f_2$, where $f_1, f_2 \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$. That is, for any cluster Laurent monomial X in A_q,

$$g(X) = k_X X + k'_X, \tag{25}$$

where $k_X, k'_X \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$.

(b) For any $g \in \mathcal{P}(A_q)$, by (25), we define g' to be the map satisfying $g'(X) = k_X X$ for any cluster Laurent monomial $X \in A_q$ and $g'(a) = 0$ for any $a \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$. Trivially, $g' \in \mathcal{P}(A_q)$. Since $\mathrm{Im}(g - g') \subseteq Z(A_q)$, we have $g \sim g'$.

By Lemma 2.3, there is a scalar $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-transformation $g_0 \in \mathcal{P}(A_q)$ such that $g_0 \sim g'$. It follows that $g_0 \sim g$. □

Combining Lemma 2.3 and Lemma 2.4, we get our main result on inner Poisson structures.

Theorem 2.5. *Let A_q be a quantum cluster algebra without coefficients, any inner Poisson structure on A_q must be a standard Poisson structure.*

Proof. According to Theorem 2.1, any inner Poisson bracket on A_q corresponds to a linear transformation $g \in \mathcal{P}(A_q)$ up to isomorphism. By Lemma 2.3 and Lemma 2.4, we can choose a scalar linear transformation g' in the iso-class of g, that is, $g'(W) = k_0 W$ for a fixed element $k_0 \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ and for any $W \in A_q$. It follows that $ham(W) = k_0[W, -]$ for any $W \in A_q$, which means the Poisson structure is standard. □

3. **On locally inner Poisson structures.** First we introduce the following definitions and results in [7].

Definition 3.1. (a) For a quantum cluster algebra A_q, one of its extended cluster $\tilde{X}(t) = (X_1, \cdots, X_m)$ at $t \in \mathbb{T}_n$ is said to be **log-canonical** with respect to a Poisson structure $(A_q, \cdot, \{-,-\})$ if $\{X_i, X_j\} = \omega_{ij} X^{e_i+e_j}$, where $\omega_{ij} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ for any $i, j \in [1, m]$.

(b) A Poisson structure $\{-,-\}$ on a quantum cluster algebra A_q is called **compatible** with A_q if all clusters in A_q are log-canonical with respect to $\{-,-\}$.

In [7], we define the cluster decomposition of a quantum cluster algebra as following. Let A_{q,I_1}, A_{q,I_2} be two quantum cluster algebras with initial seeds $(\tilde{X}_{I_1}, \tilde{B}_{I_1}, \Lambda_{I_1})$ and $(\tilde{X}_{I_2}, \tilde{B}_{I_2}, \Lambda_{I_2})$ respectively, and let Θ be an $|I_1| \times |I_2|$ integer matrix satisfying

$$\begin{cases} \tilde{B}_{I_1}^{\top}\Theta &= O \\ \Theta \tilde{B}_{I_2} &= O. \end{cases} \tag{26}$$

Define $A_{q,I_1} \bigsqcup_{\Theta} A_{q,I_2}$ to be the algebra equivalent to $A_{q,I_1} \bigotimes_{\mathbb{Z}[q^{\pm\frac{1}{2}}]} A_{q,I_2}$ as a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-module with twist multiplication:

$$(a \otimes b)(c \otimes d) = \sum_{i,j} k_i l_j q^{\frac{1}{2}\bar{r}_i^{\top}\Theta \bar{s}_j} a \tilde{X}_{I_1}^{\bar{s}_j} \otimes \tilde{X}_{I_2}^{\bar{r}_i} d \tag{27}$$

for $b = \sum\limits_i k_i \tilde{X}_{I_2}^{\bar{r}_i}, c = \sum\limits_j l_j \tilde{X}_{I_1}^{\bar{s}_j}$, where $\bar{r}_i, \bar{s}_j$ are exponential column vectors.

Let A_{q,I_i} be quantum cluster algebras with initial seeds $(\tilde{X}_{I_i}, \tilde{B}_{I_i}, \Lambda_{I_i})$ for $i \in [1, r]$. $\bigsqcup$ is associative in the sense that

$$(A_{q,I_1} \bigsqcup\nolimits_{\Theta_1} A_{q,I_2}) \bigsqcup\nolimits_{\Theta'} A_{q,I_3} = A_{q,I_1} \bigsqcup\nolimits_{\Theta''} (A_{q,I_2} \bigsqcup\nolimits_{\Theta_3} A_{q,I_3}),$$

where

$$\Theta' = \begin{pmatrix} \Theta_2 \\ \Theta_3 \end{pmatrix}, \quad \Theta'' = \begin{pmatrix} \Theta_1 & \Theta_2 \end{pmatrix}.$$

Theorem 3.2. [7] *Let A_q be a quantum cluster algebra with initial seed $(\tilde{X}, \tilde{B}, \Lambda)$ and $\{-,-\}$ a compatible Poisson bracket on A_q. Assume Ω is the Poisson matrix of the initial cluster with respect to $\{-,-\}$, $\tilde{B}$ has the decomposition $\tilde{B} = \bigoplus\limits_{i=1}^{r} \tilde{B}_{I_i}$ with indecomposables $\tilde{B}_{I_i}$ for $i \in [1, r]$, and A_{q,I_i} is the quantum cluster indecomposable subalgebra of A_q determined by $(\tilde{B}_{I_i}, \Lambda_{I_i})$. Then $A_q \cong \bigsqcup\limits_i A_{q,I_i}$.*

We call $A_q \cong \bigsqcup_{i=1}^{r} A_{q,I_i}$ a **cluster decomposition** of A_q.

In particular, when A_q is a quantum cluster algebra without coefficients, B is invertible. Hence by (26), we have $\Theta = O$. So, from (27), we also obtain $(a\otimes b)(c\otimes d) = ac\otimes bd$ in this case, which means the cluster decomposition is exactly tensor decomposition $A_q = \bigotimes_{i=1}^{r} A_{q,I_i}$.

We generalize inner Poisson structures to locally inner structures in the sense of cluster decomposition.

Definition 3.3. Let A_q be a quantum cluster algebra with the cluster decomposition $A_q = \bigsqcup_{i=1}^{r} A_{q,I_i}$.

(a) A Poisson structure $\{-,-\}$ on A_q is said to be **locally inner** if for any $a\in A_q$ and $i\in[1,r]$, there is $a_i\in A_{q,I_i}$ such that $ham(a)|_{A_{q,I_i}} = [a_i,-]$.

(b) A Poisson structure $\{-,-\}$ on A_q is a **locally standard Poisson structure** if $\{X_i,X_j\}=0$ when i and j are from different I_r and $\{-,-\}$ is of standard poisson structure on each X_{I_r}, i.e, $\{X_i,X_j\} = a_r[X_i,X_j]$, where $i,j\in I_r, a_r\in\mathbb{Z}[q^{\pm\frac{1}{2}}]$.

Proposition 1. *Let A_q be a quantum cluster algebra without coefficients, any locally inner Poisson structure on A_q is locally standard.*

Proof. Assume A_q has the cluster decomposition $A_q = \bigotimes_{i=1}^{r} A_{q,I_i}$ and $\{-,-\}$ is a locally inner Poisson bracket on A_q. According to the definition, $\{-,-\}$ is inner when restricted on each A_{q,I_i}. Hence by Theorem 2.5, $ham(a)\mid_{A_{q,I_i}} = \lambda_i[a,-]$ for some $\lambda_i\in\mathbb{Z}[q^{\pm\frac{1}{2}}]$ and any $a\in A_{q,I_i}$. Moreover, for any $a\in A_{q,I_i}, a'\in A_{q,I_j}$ and $i\neq j$,

$$\{a,a'\} = [a_j,a']\in A_{q,I_j}, \quad\text{and}\quad \{a,a'\} = [a,a'_i]\in A_{q,I_i}.$$

So, $\{a,a'\}\in A_{q,I_i}\bigcap A_{q,I_j} = \mathbb{Z}[q^{\pm\frac{1}{2}}]$ according to the independence of cluster variables and $I_i\cap I_j=\emptyset$. And, note that the expansions of $[a_j,a']$ and $[a,a'_i]$ will not contain non-zero constant terms in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ due to the definitions of the operation $[\ ,\ ]$ and quantum torus. Thus, $\{a,a'\}=0$.

Therefore, for any $a\in A_{q,I_i}$,

$$ham(a)\mid_{A_{q,I_j}} = \begin{cases} \lambda_i[a,-] & i=j;\\ 0 & i\neq j.\end{cases}$$

Then the Poisson structure is exactly locally standard. □

Theorem 3.4. [7] *Let A_q be a quantum cluster algebra without coefficients. Then a Poisson structure $\{-,-\}$ on A_q is compatible with A_q if and only if it is locally standard on A_q.*

Since a locally standard Poisson structure is evidently locally inner, combining Proposition 1 and Theorem 3.4, we have the final conclusion:

Theorem 3.5. *Let A_q be a quantum cluster algebra without coefficients and $\{-,-\}$ a Poisson structure on A_q. The following statements are equivalent:*

(*a*) *$\{-,-\}$ is locally standard.*
(*b*) *$\{-,-\}$ is locally inner.*
(*c*) *$\{-,-\}$ is compatible with A_q.*

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E-mail address: fangli@zju.edu.cn
E-mail address: panjie_zhejiang@qq.com