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STRUCTURE OF SYMPATHETIC LIE SUPERALGEBRAS

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Abstract. Sympathetic Lie superalgebras are defined and some classical properties of sympathetic Lie superalgebras are given. Among the main results, we prove that any Lie superalgebra L contains a maximal sympathetic graded ideal and we obtain some properties about sympathetic decomposition. More specifically, we study a general sympathetic Lie superalgebra L with graded ideals I, J and S such that $L = I \oplus J$ and L/S is a sympathetic Lie superalgebra, and we obtain some properties of L/S. Furthermore, under certain assumptions on L we prove that the derivation algebra $\mathrm{Der}(L)$ is sympathetic and that if in addition L is indecomposable, then $\mathrm{Der}(L)$ is simply sympathetic.

1. **Introduction.** In the seventies, Kac gave a comprehensive presentation of Lie superalgebra theory in [7]. The development of Lie superalgebra theory largely depends on Lie algebra theory. Note that every semi-simple Lie algebra L satisfies $[L, L] = L$, $\mathrm{Der}(L) = \mathrm{ad}(L)$ and $C(L) = \{0\}$. In 1970s, Flato raised the question whether three properties characterize semi-simple Lie algebras. Later, Angelopouos gave a negative answer by constructing a class of counterexamples, where the minimal dimension is 35 with a Levi subalgebra isomorphic to $sl(2)$. After that, the concept of sympathetic Lie algebras was proposed by Benayadi. Furthermore, Benayadi gave more classical properties of semi-simple Lie algebras which are still valid for sympathetic Lie algebras in [2, 3]. Subsequently, this concept attracted more and more scholars' attention.

This article aims to study what we call sympathetic Lie superalgebras, which are Lie superalgebras satisfying the three aforementioned properties.

This paper is organized as follows: In Section 2, we introduce the concepts and some simple properties needed in this article. In Section 3, we get some conclusions related to sympathetic Lie superalgebras through the graded ideals and show that some properties in [2] are still valid for sympathetic Lie superalgebras. In Section 4, we study perfect Lie superalgebras. Applying to sympathetic ones, we show that every sympathetic Lie superalgebra can be uniquely decomposed to a direct sum of irreducible sympathetic ideals. In Section 5, we focus on each sympathetic Lie superalgebra that has a maximal sympathetic ideal and a decomposition of direct sum of the maximal sympathetic ideal and a characteristic ideal, which is called sympathetic decomposition. After that, we give the concept of simply sympathetic and some properties. In Section 6, we focus on graded ideals such that L/I is a sympathetic Lie superalgebra and then research some properties. Meanwhile, we

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also give that the $\mathrm{Der}(L)$ is sympathetic, and if L is indecomposable, then $\mathrm{Der}(L)$ is simply sympathetic.

The field involved in this article is algebraically closed of characteristic 0.

2. **Preliminaries.**

Throughout this paper, L always denotes a finite-dimension Lie superalgebra over a commutative ring R with 1. A Lie superalgebra L is called perfect if the derived algebra $[L, L] = L$. The center of L is denoted by $C(L)$. L is called centerless if $C(L) = \{0\}$. $\mathrm{Der}(L)$ is the derivation algebra of L.

Definition 2.1. A Lie superalgebra L is called sympathetic if $C(L) = \{0\}$, $[L, L] = L$, $\mathrm{Der}(L) = \mathrm{ad}(L)$.

Example 1. Recall the classical simple Lie superalgebras $osp(n, 2r)(n, r \geq 1)$, $spl(n, m)(n \neq m)$, $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, Γ_2 and Γ_3. From [9], if they are perfect, then they are sympathetic.

Definition 2.2. Let L be a Lie superalgebra, I be a graded ideal of L. Then I is said to be a direct factor if there exists a graded ideal J of L such that $L = I \oplus J$.

Definition 2.3. [4] Let L be a Lie superalgebra, I be a subspace of L. Then I is called characteristic ideal if for every $D \in \mathrm{Der}(L)$, $D(I) \subseteq I$.

Lemma 2.4. *Let L be a Lie superalgebra, I be a graded ideal of L. If I is perfect, then I is a characteristic ideal of L.*

Proof. With I a graded ideal of L, for each $D \in \mathrm{Der}(L)$, $x_i, y_i \in I$, we have

$$D([x_i, y_i]) = [D(x_i), y_i] + (-1)^{|x_i||D|}[x_i, D(y_i)] \in I.$$

Since I is perfect, it then follows that $D(I) \subseteq I$. Thus I is a characteristic ideal. $\square$

Proposition 1. *Suppose that L is a perfect Lie superalgebra, I is a graded ideal of L. If I is a direct factor of L, then I is perfect.*

Proof. Since I is a direct factor of L, there exists a graded ideal J of L such that $L = I \oplus J$, in particular, $[I, J] = \{0\}$. It follows that $[L, L] = [I, I] \oplus [J, J]$, which implies that both I and J are perfect. $\square$

Proposition 2. *Let L be a Lie superalgebra with trivial center, I be a direct factor of L. Then $C(I) = \{0\}$.*

3. **The sympathetic Lie superalgebras.**

Lemma 3.1. *If I is a sympathetic graded ideal of L, then there exists a graded ideal J such that $L = I \oplus J$.*

Proof. Let $J = C_L(I)$. By [4] of Lemma 2.3, we have $C_L(I)$ is a graded ideal of L. For any $x \in L$, $\mathrm{ad}x \in \mathrm{Der}(I)$ since $I \lhd L$. By virtue of $\mathrm{Der}(I) = \mathrm{ad}(I)$, there exists a derivation D in $\mathrm{Der}(I)$ such that $\mathrm{ad}x = D$. Thereby there exists $y \in I$ such that $D(z) = [x, z] = [y, z]$ for $\forall z \in I$. Then $[x - y, z] = 0$ and $x - y \in C_L(I) = J$. Hence $x = j + y$ for some $j \in J$. $I \cap J = I \cap C_L(I) = C(I) = \{0\}$ since I is sympathetic. Thus $L = I \oplus J$. $\square$

Lemma 3.2. *Let L be a sympathetic Lie superalgebra, I be a graded ideal of L. Then I is a direct factor of L if and only if I is sympathetic.*

Proof. On account of I is a direct factor of L, there exists a graded ideal J of L such that $L = I \oplus J$. Then $L = [L, L] = [I, I] \oplus [J, J]$. Since $L = I \oplus J$, we can get $I = [I, I]$. Suppose $D \in \mathrm{Der}(I)$. In the following we define $d : L \to L$ to be a linear map by

$$d|_I = D, \ d|_J = 0.$$

An easy calculation shows that D belongs to $\mathrm{Der}(L)$. Then there exists $u \in L$ such that $d = \mathrm{ad}u$. Because of $L = I \oplus J$, and $u = u_I + u_J$, for $u_I \in I$ and $u_J \in J$, therefore $D = \mathrm{ad}u_I$. Hence we can obtain $\mathrm{Der}(I) = \mathrm{ad}(I)$. Thus, we conclude that $C(I) \subset C(L) = \{0\}$. It follows that I is sympathetic.

Conversely, Let $J = \{x \in L \mid [x, I] = \{0\}\}$, $\mathrm{ad}x|_I \in \mathrm{Der}(I)$. Then there exists y of I, such that $\mathrm{ad}x|_I = \mathrm{ad}y$. Hence $[x - y, z] = 0$, for any $z \in I$, which proves that $j = x - y \in J$, then $L = I + J$. Suppose $a \in I \cap J$, $b \in L$. Consider $b = b_I + b_J$, $b_I \in I$, $b_J \in J$. Therefore, we have $[a, b] = [a, b_I] + [a, b_J] = 0$. Thus, $a \in C(L) = \{0\}$. As a result, $L = I \oplus J$. □

In view of Lemma 3.2, we immediately have the following consequences.

Corollary 1. *Let L be a Lie superalgebra, and I be a sympathetic graded ideal of L. Then I is a direct factor of L.*

Corollary 2. *If $\mathrm{Der}(L) = \mathrm{ad}(L)$, then $\mathrm{Der}(I) = \mathrm{ad}(I)$.*

Proposition 3. *Let L be a Lie superalgebra, I, J be two graded ideals of L and $I \cap J = \{0\}$. If I and J are sympathetic, then $S = I \oplus J$ is also sympathetic.*

Proof. On account of $I \cap J = \{0\}$, we can get

$$[S, S] = [I, I] \oplus [J, J] = I \oplus J = S.$$

Suppose that $D \in \mathrm{Der}(S)$. For any $D \in \mathrm{Der}(L)$, $x_i, y_i \in I$, we can get

$$D([x_i, y_i]) = [D(x_i), y_i] + (-1)^{|x_i||D|}[x_i, D(y_i)] \in I.$$

Since I is a graded ideal of L, thus we obtain $D(I) \subseteq I$. Then we have I is a characteristic ideal of S, the same for J. Therefore $D|_I \in \mathrm{Der}(I)$, $D|_J \in \mathrm{Der}(J)$. Hence, there exist $x \in I$ and $y \in J$ such that

$$D|_I = \mathrm{ad}x \ , \ D|_J = \mathrm{ad}y.$$

Assume that $D_1 = \mathrm{ad}(x + y)$. In the following, we shall prove $D = D_1$.

For any $s \in S$, there exist $u \in I$, $v \in J$ such that $s = u + v$. Hence

$$\begin{aligned} D_1(s) = D_1(u + v) &= [x + y, u + v] = [x, u] + [y, v] = D|_I(u) + D|_J(v) \\ &= D(u) + D(v) = D(u + v) = D(s), \end{aligned}$$

it follows that $D = \mathrm{ad}(x, y)$, clearly, $\mathrm{Der}(S) = \mathrm{ad}(S)$.

Set $c \in C(S)$, $c = c_I + c_J$ with $c_I \in I$ and $c_J \in J$. Thus,

$$[c_I, I] = [c, I] = \{0\} = [c_J, J] = [c, J],$$

we can obtain $c_I = 0$ and $c_J = 0$, hence $c = 0$, it means that $C(S) = \{0\}$, consequently, the proposition holds. □

Corollary 3. *Let $L_i (i = 1, \cdots, n)$ be sympathetic Lie superalgebras, then $L_1 \times L_2 \times \cdots \times L_n$ is also sympathetic.*

Corollary 4. *Any extension of sympathetic Lie superalgebras is also sympathetic and trivial.*

Proof. Set L_1, L_2 be two sympathetic Lie superalgebras, L be the extension. By Corollary 1, there exists a surjective morphism $\mu : L \to L_1$ such that $\mathrm{Ker}(\mu) = L_2$, therefore L_2 is a direct factor of L, hence the extension L is trivial, that is to say $L \cong L_1 \times L_2$. Following, according to Corollary 3, L is sympathetic . □

Proposition 4. *Let L be a Lie superalgebra over a field $\mathbb{F}$ and T be an extension of $\mathbb{F}$. Then $L \otimes_{\mathbb{F}} T$ is sympathetic if and only if L is sympathetic.*

Proof. By a result of [8], $\mathrm{Der}(L \otimes_{\mathbb{F}} T) = \mathrm{Der}(L) \otimes_{\mathbb{F}} T$. □

Lemma 3.3. [9] *Let L be a Lie superalgebra over a field $\mathbb{F}$. Each of the following statements is strictly stronger than the foregoing one.*

(1) L does not contain non-zero solvable graded ideals.
(2) L is the direct product of finitely many simple Lie superalgebras.
(3) The killing form of L is nondegenerate.
(4) All finite-dimensional graded representations of L are completely reducible.

Theorem 3.4. *Let L be a perfect Lie superalgebra. If $L = L_0 \oplus L_1$ has the nondegenerate Killing form over a field $\mathbb{F}$, then L_0 and L are sympathetic.*

Proof. By virtue of L has the nondegenerate Killing form, $\mathrm{Der}(L) = \mathrm{ad}(L)$ by [9]. Since $C(L)$ is abelian, $C(L)$ is solvable. Then $C(L) = \{0\}$ by Lemma 3.3. Thus L is sympathetic. By the hypothesis of this theorem, we have L is sympathetic. Set K be the nondegenerate Killing form of L. Then it restriction to L_0, $K\mid_{L_0 \times L_0}$ is the Killing form of L_0, which is nondegenerate. Thereby L_0 is a semisimple Lie algebra and $\mathrm{Der}(L) = \mathrm{ad}(L)$ by [5]. By virtue of L_0 has no nonzero solvable ideal and $C(L_0) = \{0\}$, L_0 is sympathetic. □

Definition 3.5. [4] Let L be a Lie superalgebra and $h(L) = L \oplus \mathrm{Der}(L)$. Define the bracket in $h(L)$ by $[x + D, y + E] = [x, y] + Dy(-1)^{\alpha\beta}Ex + [D, E]$, where $E \in (\mathrm{Der}(L))_\alpha$, $x \in L_\beta$. Then $h(L)$ is a Lie superalgebra. We call $h(L)$ a holomorph Lie superalgebra.

Remark 1. [4]

(1) If $C(L) = \{0\}$, then $C(\mathrm{Der}(L)) = \{0\}$, since $C(\mathrm{Der}(L)) \subseteq \mathrm{Der}(L)$ and $C(\mathrm{Der}(L)) \subseteq C_{\mathrm{Der}(L)}(\mathrm{ad}(L)) = \{0\}$.
(2) $L \lhd h(L)$ and $h(L)/g \simeq \mathrm{Der}(L)$.
(3) $L \cap C_{h(L)}(L) = C(L)$.

Theorem 3.6. *Let L be a perfect Lie superalgebra. The following conditions are equivalent:*

(1) L is a sympathetic Lie superalgebra.
(2) Any splitting extension $\mathfrak{e}$ by L is a trivial extension and $\mathfrak{e} = L \oplus C_{\mathfrak{e}}(L)$.
(3) $h(L) = L \oplus C_{h(L)}(L)$.

Proof. (1) ⇒ (2) Let $\mathfrak{e}$ be a splitting extension by L. Then $L \lhd \mathfrak{e}$ and $C_{\mathfrak{e}}(L) \lhd \mathfrak{e}$. By virtue of $C(L) = \{0\}$ by (1), $L \cap C_{\mathfrak{e}}(L) = \{0\}$. Let $x \in \mathfrak{e}$. Since $L \lhd \mathfrak{e}$, $\mathrm{ad}x \in L$. Therefore the restriction $\mathrm{ad}x|_L$ on L is a derivation of L. But L is sympathetic, thus $\mathrm{ad}x|_L$ is an inner derivation of L. We define θ by $\theta(x) = \mathrm{ad}x|_L$ for $\forall x \in \mathfrak{e}$. Since $\mathrm{Der}(L) = \mathrm{ad}(L) \simeq L$, the map θ is a homomorphism from $\mathfrak{e}$ onto $\mathrm{Der}(L)$ with kernal $C_{\mathfrak{e}}(L)$. Hence $\mathrm{dim}\mathfrak{e} = \mathrm{dim}L + \mathrm{dim}C_{\mathfrak{e}}(L)$.

(2) ⇒ (3) It is clear by setting $\mathfrak{e} = h(L)$.

(3) $\Rightarrow$ (1) According to Remark 1 $C(L) = L \cap C_{h(L)}(L) = \{0\}$. $C_{h(L)}(L) \simeq h(L)/L$ by (3) and $h(L)/L \simeq \mathrm{Der}(L) \simeq C_{h(L)}(L) \simeq L$ from Remark 1. By virtue of $C(L) = \{0\}$, $L \simeq \mathrm{ad}(L)$. Hence $\mathrm{Der}(L) \simeq \mathrm{ad}(L)$. □

4. **Decomposition of perfect Lie superalgebras.**

Definition 4.1. A Lie superalgebra L is called irreducible if it has no non-zero direct factor.

Lemma 4.2. *Let L be a Lie superalgebra. Then $L = I_1 \oplus \cdots \oplus I_n$, with every $I_i (i = 1, \ldots, n)$ is an irreducible ideal of L.*

Proof. Induction on the dimension of L. If $\dim L = 0$ or 1, then the lemma is true. It follows that we suppose that the lemma is true if $\dim L < n$. Set L be a Lie superalgebra with $\dim L = n$. If L is reducible, then there exist two proper ideals I, J of L such that $L = I \oplus J$.

According to inductive hypothesis, we can get $I = I_1 \oplus \cdots \oplus I_n$ with every I_i is an irreducible graded ideal of I and $J = J_1 \oplus \cdots \oplus J_m$ with every J_i is an irreducible graded ideal of J, consequently, it's sufficient to see that every graded ideal of I(resp. J) is a graded ideal of L. □

Lemma 4.3. *Suppose that L is a Lie superalgebra, I is a perfect graded ideal of L. If $L = I_1 \oplus \cdots \oplus I_n$ with every I_i is a graded ideal of L, then*

$$I = I \cap I_1 \oplus \cdots \oplus I \cap I_n.$$

Proof. By the assumption we have

$$I = [I, I] \subseteq [I, L] \subseteq I,$$

then we can obtain

$$I = [I, I_1] \oplus \cdots \oplus [I, I_n] \subseteq I \cap I_1 \oplus \cdots \oplus I \cap I_n \subseteq I,$$

hence the lemma holds. □

Theorem 4.4. *Suppose that L is a perfect Lie superalgebra. Then $L = s_1 \oplus \cdots \oplus s_n$ where every s_i is an irreducible perfect graded ideal of L. Moreover, this decomposition is unique.*

Proof. According to Lemmas 3.2 and 4.2, we know that the decomposition exists. Let

$$L = s_1 \oplus \cdots \oplus s_n = s_1^{'} \oplus \cdots \oplus s_m^{'}$$

be two decompositions of L into irreducible perfect graded ideals. By Lemma 4.3, we have

$$s_i = s_i \cap s_1^{'} \oplus \cdots \oplus s_i \cap s_m^{'},$$

where $i \in \{1, \cdots, n\}$. Hence there exists $\alpha(i) \in \{1, \cdots, m\}$ such that $s_i = s_i \cap s_{\alpha(i)}^{'}$ since s_i is irreducible, we can obtain $s_i \subseteq s_{\alpha(i)}^{'}$. Again according to Lemma 4.3, we have

$$s_{\alpha(i)}^{'} = s_{\alpha(i)}^{'} \cap s_1 \oplus \cdots \oplus s_{\alpha(i)}^{'} \cap s_n,$$

there exists a $j \in \{1, \cdots, n\}$ such that

$$s_{\alpha(i)}^{'} = s_{\alpha(i)}^{'} \cap s_j,$$

thus $s_{\alpha(i)}^{'} \subseteq s_j$, which implies that

$$s_i \subseteq s_{\alpha(i)}^{'} \subseteq s_j.$$

Therefore $i = j$ and $s_i = s'_{\alpha(i)}$. It follows that $n = m$ and there exists a permutation α of $\{1, \cdots, n\}$ such that $s_i = s'_{\alpha(i)}$ for any $i \in \{1, \cdots, n\}$, hence the decomposition is unique. □

Corollary 5. *Suppose that L is a perfect Lie superalgebra and I is a proper ideal of L. Set $L = s_1 \oplus \cdots \oplus s_n$ be the unique decomposition of L into irreducible perfect graded ideals, then the following statements hold*

(1) I is an irreducible direct factor of L if and only if there exists an $i \in \{1, \cdots, n\}$ such that $I = s_i$.
(2) I is a direct factor of L if and only if there exist $\{i_1, \cdots, i_m\} \subseteq \{1, \cdots, n\}$ such that $I = s_{i_1} \oplus \cdots \oplus s_{i_m}$.

Proof. (1) Set I be an irreducible direct factor of L, there exists an $i \in \{1, \cdots, n\}$ such that $I = I \cap s_i$ by Lemmas 3.2 and 4.3, so that $I \subseteq s_i$. Since I is a direct factor of L, there exists a graded ideal J of L such that $L = I \oplus J$ and by Lemma 4.3, $s_i = s_i \cap I$. According to Lemma 4.3, we have

$$s_i = s_i \cap I \oplus s_i \cap J.$$

On account of s_i is irreducible, then $s_i = s_i \cap I$ or $s_i = s_i \cap J$. If $s_i = s_i \cap J$, then $I \subseteq s_i \subseteq J$, contradiction. Consequently, $s_i \subseteq I$ and $I = s_i$.

(2) Assume that I is a direct factor of L. Then according to Lemma 3.2, we have I is perfect. Therefore, there exist $I_1, \cdots, I_m$ irreducible prefect graded ideals of I such that $I = I_1 \oplus \cdots \oplus I_m$ by Theorem 4.4. Since I is a direct factor of L, it is easy to see that every I_j is an irreducible direct factor of L . Apply (1), there exists an $i_j \in \{1, \cdots, n\}$ such that $I_j = s_{i_j}$ for every I_j. Hence, $I = s_{i_1} \oplus \cdots \oplus s_{i_m}$. □

Proposition 5. *Let L be a perfect Lie superalgebra such that $\mathrm{Der}(L) = \mathrm{ad}(L)$ (resp. $C(L) = \{0\}$). Then*

$$L = s_1 \oplus \cdots \oplus s_n$$

where each s_i is an irreducible perfect graded ideal of L such that

$$\mathrm{Der}(s_i) = \mathrm{ad}(s_i)\ (\text{resp.}\, C(s_i) = \{0\}).$$

Proposition 6. *Let L be a sympathetic Lie superalgebra. Then*

$$L = s_1 \oplus \cdots \oplus s_n$$

where each s_i is an irreducible sympathetic graded ideal of L. Moreover, the decomposition is unique.

According to Lemma 3.2, Corollary 5 and Proposition 5, it is straightforward to show the following corollary.

Corollary 6. *Let L be a sympathetic Lie superalgebra.*

(1) L has a finite number of irreducible sympathetic graded ideals $s_1, \cdots, s_n$ and $L = s_1 \oplus \cdots \oplus s_n$.
(2) A graded ideal I of L is sympathetic if and only if $I = s_{i_1} \oplus \cdots \oplus s_{i_m}$ with $\{i_1, \cdots, i_m\} \subseteq \{1, \cdots, n\}$.

5. The sympathetic decomposition of Lie superalgebras.

Lemma 5.1. *Let L be a Lie superalgebra. If I and J are two graded ideals satisfying the following conditions*

(1) I is a sympathetic graded ideal of L,

(2) there exists a vector subspace W of L such that $L = J \oplus W$ and $[I, W] \subseteq W$,

then $I \cap J$ is a sympathetic graded ideal of L.

Proof. On account of I is sympathetic, we have $I = [I, L]$, then

$$I = [I, J] \oplus [I, W].$$

Since $[[I, W], I] \subseteq W$, therefore $[I, W]$ is an ideal of I, thus $[I, J]$ is a direct factor of I. According to Lemma 3.2, it is easy to see that $[I, J]$ and $[I, W]$ are sympathetic. Thus

$$C(I/[I, J]) = \{0\}.$$

Considering the canonical surjection $\phi : I \to I/[I, J]$. Set $x \in I \cap J$, $y \in I$,

$$[\phi(x), \phi(y)] = \phi([x, y]) = 0,$$

it follows that

$$\phi(x) \in C(I/[I, J]) = \{0\},$$

we have $x \in [I, J]$. Hence, $I \cap J = [I, J]$, i.e., $I \cap J$ is a sympathetic graded ideal of L. □

Corollary 7. *Suppose that L is a Lie superalgebra, I and J are two graded ideals of L.*

(1) If I is a sympathetic graded ideal of L and J is a direct factor of L, then $I \cap J$ is a sympathetic graded ideal of L.

(2) If I and J are sympathetic graded ideals of L, then $I \cap J$ and $I + J$ are sympathetic graded ideals of L.

Proof. (1) The assertion is obvious by Lemma 5.1.

(2) According to Corollary 1, we can get J is a direct factor of L, thus $I \cap J$ is a sympathetic graded ideal of L. The following sequence is exact:

$$0 \longrightarrow I \xrightarrow{\varphi} I + J \xrightarrow{\varphi} J/I \cap J \longrightarrow 0$$

defined by $\varphi(x) = x$ for any $x \in I$ and $\varphi(x_I + x_J) = T(x_J)$ for any $x_I \in I$, $x_J \in J$ where $T : J \to J/I \cap J$ is the canonical surjection. Thus $I + J$ is an extension of sympathetic Lie superalgebras. According to Corollary 4, we obtain $I + J$ is a sympathetic graded ideal of L. □

Proposition 7. *Each Lie superalgebra contains a maximal sympathetic graded ideal.*

Proof. Using dimensions to prove. Suppose that L is a Lie superalgebra.

$$\mathcal{D} = \{\dim I \mid I \text{ is a sympathetic graded ideal of } L\}$$

is a non-empty subset of $\mathbb{N}$ bounded above by $\dim L$, then it has a maximal element m, consequently there exists a sympathetic graded ideal M of L such that $\dim M = m$. Set I be a sympathetic graded ideal of L, thus $M + I$ is a sympathetic graded ideal of L by Corollary 7. Hence $M = M + I$, thereby $I \subseteq M$, which implies that M is the maximal graded ideal of L, the proposition is certified. □

Lemma 5.2. *Suppose that L is a Lie superalgebra, set I be a graded ideal of L. If there exists a graded ideal J of L satisfies the following conditions*
(1) $L = I \oplus J$,
(2) $C(J) = \{0\}$ and $[J, J] = J$,
then I is a characteristic ideal of L.

Proof. Set $x \in I$ and $d \in \mathrm{Der}(L)$. Thus

$$d(x) = y + c$$

where $y \in I$, $c \in J$. Set $c_1 \in J$. Then

$$[c, c_1] = [d(x) - y, c_1] = [d(x), c_1] = d([x, c_1]) - (-1)^{|x||d|}[x, d(c_1)] = 0,$$

since J is a characteristic ideal by Lemma 2.4, it follows that $c \in C(J) = 0$, hence $d(x) \in I$, which implies that $d(I) \subseteq I$. □

Theorem 5.3. *Let L be a Lie superalgebra and M be a maximal sympathetic graded ideal of L. Then there exists a characteristic ideal I of L satisfying*
(1) $L = M \oplus I$,
(2) I is the greatest among the ideals of L which are direct factors of L and do not contain any non-zero sympathetic graded ideal.

Proof. (1) According to Corollary 1, M is a direct factor of L. So there exists an ideal I of L such that $L = M \oplus I$. Set $x \in I$ and $d \in \mathrm{Der}(L)$. Thus

$$d(x) = y + z$$

where $y \in I$, $z \in M$. Let $z_1 \in M$. Then

$$[z, z_1] = [d(x) - y, z_1] = [d(x), z_1] = d([x, z_1]) - (-1)^{|x||d|}[x, d(z_1)] = 0,$$

therefore $z \in C(M) = 0$, hence $d(x) \in I$, then we obtain $d(I) \subseteq I$. It follows that I is a characteristic ideal of L.

Let S be a sympathetic graded ideal of I, then S is also a sympathetic graded ideal of L, consequently $S \subseteq I \cap M = \{0\}$.

(2) Let J be a graded ideal of L which is a direct factor of L and doesn't contain any non-zero sympathetic graded ideal, then there exists a graded ideal Q of L such that $L = J \oplus Q$.

Set $M^{'}$ be the maximal sympathetic graded ideal of Q, then there exists a graded ideal $I^{'}$ of Q such that $Q = M^{'} \oplus I^{'}$ and $I^{'}$ is a characteristic ideal of Q which doesn't have any non-zero sympathetic graded ideal.

Therefore, $L = M^{'} \oplus I^{'} \oplus J$ where $M^{'}$ is a sympathetic graded ideal of L. $I^{'}$ and J are graded ideals of L which don't contain any non-zero sympathetic graded ideal.

According to Lemma 4.3, we get

$$M = M \cap M^{'} \oplus M \cap I^{'} \oplus M \oplus J.$$

Claim that $M \cap I^{'} = \{0\}$(resp. $M \cap J = \{0\}$). Otherwise, $M \cap I^{'}$(resp. $M \cap J$) is a non-zero sympathetic graded ideal of L by Corollary 7(1). Note that $[M \cap I^{'}, I^{'}] \subseteq M \cap I^{'}$(resp. $[M \cap J, J] \subseteq M \cap J$), we have $M \cap I^{'}$(resp. $M \cap J$) is a non-zero sympathetic graded ideal of $I^{'}$(resp. J), contradiction. So $M \cap I^{'} = \{0\}$(resp. $M \cap J = \{0\}$), consequently, $M = M \cap M^{'}$, which implies that $M = M^{'}$.

Let $x \in J$, then $x = y + z$ with $y \in M$ and $z \in I$. Note that $[M, J - I] = \{0\}$, we get

$$[y, M] = [x - z, M] = \{0\},$$

which implies that $y \in C(M) = 0$, consequently $x = z$, i.e., $J \subseteq I$. $\square$

Definition 5.4. Let L be a Lie superalgebra. We call the decomposition of Theorem 5.3 is a sympathetic decomposition, i.e., the decomposition $L = M \oplus I$ where M is the maximal sympathetic graded ideal of L and I is the greatest among the graded ideals of L. The I is a direct factor of L and it doesn't contain any non-zero sympathetic graded ideal.

Proposition 8. *Suppose that L is a Lie superalgebra, M is a maximal sympathetic graded ideal of L, $L = M \oplus I$ is a sympathetic decomposition. If $I = \{0\}$, then* $\mathrm{Rad}(I) = \{0\}$.

Proposition 9. *Let L be a Lie superalgebra, M be a maximal sympathetic graded ideal of L, $L = M \oplus I$ is a sympathetic decomposition. For $I = \{0\}$ if and only if L is a sympathetic Lie superalgebra.*

Proof. The necessity is obvious. Then we prove the sufficiency. By Lemma 3.2, I is a sympathetic Lie superalgebra, thus $I = \{0\}$. $\square$

Lemma 5.5. *Let the Lie superalgebra L be decomposed into the direct sum of two graded ideals. i.e., $L = I \oplus J$. Then we have the following conclusions*

(1) $C(L)$ has the decomposition $C(L) = C(I) \oplus C(J)$.
(2) If $C(L) = \{0\}$, then $\mathrm{ad}(L) = \mathrm{ad}(I) \oplus \mathrm{ad}(J)$ and $\mathrm{Der}(L) = \mathrm{Der}(I) \oplus \mathrm{Der}(J)$.
(3) If L, I and J are perfect, then L is sympathetic if and only if I and J are sympathetic.

Proof. (1) According to Proposition 4, $C(I)$ and $C(J)$ are ideals of L. $C(I) \cap C(J) = \{0\}$, since $I \cap J = \{0\}$. Set

$$a + b \in C(I) \oplus C(J)$$

with $a \in C(I)$ and $b \in C(J)$. Then $[a, I] = 0$ and $[b, J] = 0$. Set

$$c + d \in I \oplus J$$

with $c \in I$ and $d \in J$. Then

$$[a + b, c + d] = [a, c] + [b, d] = 0,$$

by virtue of $[b, c]$ and $[a, d]$ belong to $I \cap J$. Thus $a + b \in C(L)$ and $C(I) \oplus C(J) \subseteq C(L)$. Set $m = x + y \in C(L)$ with $x \in I$ and $y \in J$. Then

$$[x + y, L] = [x + y, I + J] = 0$$

and

$$[x, I] = [m - y, I] = 0,$$

by virtue of $m \in C(L)$ and $[y, I] \subseteq [J, I] = 0$. Thereby $x \in C(I)$. Similarly, $y \in C(J)$. Hence $C(L) \subseteq C(I) \oplus C(J)$.

(2) For $D \in \mathrm{Der}(I)$, extend it to a linear transformation on L by setting $D(a+b) = D(a)$, for $a \in I$ and $b \in J$. It is easy to see that $D \in \mathrm{Der}(L)$ and $\mathrm{Der}(I) \subseteq \mathrm{Der}(L)$. Similarly, $\mathrm{Der}(J) \subseteq \mathrm{Der}(L)$. Let $a \in I$, $b \in J$ and $D \in \mathrm{Der}(L)$. Then

$$[D(a), b] = D([a, b]) - (-1)^{|a||D|}[a, D(b)] = -(-1)^{|a||D|}[a, D(b)] \in I \cap J.$$

By virtue of $I \cap J = \{0\}$, thus $[D(a), b] = [a, D(b)] = 0$. Assume $D(a) = a^{'} + b^{'}$, where $a^{'} \in I$, then $b^{'} \in J$. Thus $[D(a), b] = [a^{'}, b] + [b^{'}, b] = 0$ for every $b \in J$ and $b^{'} \in C(J)$. By the hypothesis $C(L) = C(I) \oplus C(J) = \{0\}$, we have $b^{'} = 0$. Hence $D(a) = a^{'} \in I$. Consequently, $D(I) \subseteq I$. Similarly, $D(J) \subseteq J$.

Set $D \in \mathrm{Der}(L)$ and $a + b \in I \oplus J$ with $a \in I$ and $b \in J$. Define W and V by

$$W(a + b) = D(a) \text{ and } V(a + b) = D(b).$$

Thus $W \in \mathrm{Der}(I)$ and $V \in \mathrm{Der}(J)$. Hence $D = W + V \in \mathrm{Der}(I) + \mathrm{Der}(J)$. On occount of $\mathrm{Der}(I) \cap \mathrm{Der}(J) = \{0\}$, $\mathrm{Der}(L) = \mathrm{Der}(I) \oplus \mathrm{Der}(J)$ as a vector space. Set $D \in (\mathrm{Der}(I))$, $W \in (\mathrm{Der}(L))$ and $b \in J$. Then

$$[W, D](b) = WD(b) - (-1)^{|D||W|} DW(b) = 0.$$

Which implies that $\mathrm{Der}(I) \lhd \mathrm{Der}(L)$. Similarly, we can obtain $\mathrm{Der}(J) \lhd \mathrm{Der}(L)$.

(3) First prove the necessity, by virtue of L is sympathetic, $C(L) = \{0\}$. And from (1), we have $C(I) = C(J) = \{0\}$. Since $\mathrm{ad}(L) = \mathrm{Der}(L)$, $\mathrm{ad}(L) = \mathrm{ad}(L) \oplus \mathrm{ad}(J)$ and $\mathrm{Der}(L) = \mathrm{Der}(I) \oplus \mathrm{Der}(J)$ by (2). Since $\mathrm{ad}(I) \subset \mathrm{Der}(I)$ and $\mathrm{ad}(J) \subset \mathrm{Der}(J)$, then $\mathrm{ad}(I) = \mathrm{Der}(I)$ and $\mathrm{ad}(J) = \mathrm{Der}(J)$. Hence I and J are sympathetic Lie superalgebras.

On the contrary, according to $C(L) = C(I) \oplus C(J) = \{0\}$. Then $\mathrm{Der}(L) = \mathrm{Der}(I) \oplus \mathrm{Der}(J) = \mathrm{ad}(I) \oplus \mathrm{ad}(J) = \mathrm{ad}(L)$ by the (2) of this Lemma.

□

Definition 5.6. A sympathetic Lie superalgebra L is called a simply sympathetic Lie superalgebra if any non-trivial graded ideal of L is not sympathetic.

Example 2. A simple and sympathetic Lie superalgebra is a simply sympathetic Lie superalgebra.

Lemma 5.7. *(1) Any sympathetic Lie superalgebra can be decomposed into the direct sum of simply sympathetic ideals.*

(2) A sympathetic Lie superalgebra is simply sympathetic if and only if it is indecomposable.

Proof. (1) If L is simply sympathetic, the result is true. If L is not simply sympathetic, there exists a nonzero minimal sympathetic graded ideal I of L such that $L = I \oplus C_L(I)$ by Lemma 3.1. Continuing this process to $C_L(I)$, we get the decomposition of L into simply sympathetic graded ideals since a graded ideal of $C_L(I)$ is also a graded ideal of L. (2) The conclusion follows from (1). □

6. Sympathetic Lie superalgebra L/I and $\mathrm{Der}(L)$.

Theorem 6.1. *Let L be a Lie superalgebra and I be a graded ideal of L such that $C(L/I) = \{0\}$. The following conclusions hold*

(1) $C(L) \subseteq I$.

(2) If $L = s_1 \oplus \cdots \oplus s_n$ where every s_i is a graded ideal of L, then $I = I \cap s_1 \oplus \cdots I \cap s_n$.

Proof. Considering the canonical surjection $\psi : L \to L/I$.

(1) Let $x \in C(L)$ and $y \in L$,

$$[\psi(x), \psi(y)] = \psi([x, y]) = 0,$$

then $\psi(x) \in C(L/I) = \{0\}$, it follows that $x \in I$. Consequently, $C(L) \subseteq I$.

(2) Let $x \in I$ and $y \in L$,

$$x = x_1 + \cdots + x_n, \quad y = y_1 + \cdots + y_n$$

where x_i, y_i are elements of s_i for all $i \in \{1, \cdots, n\}$, then $[\psi(x_i), \psi(y)] = [\psi(x), \psi(y_i)] = 0$, conclude that $\psi(x_i) \in C(L/I) = \{0\}$, thus, $x_i \in s_i \cap I$, consequently,

$$I = I \cap s_1 \oplus \cdots I \cap s_n.$$

□

Lemma 6.2. *Let L be a Lie superalgebra. Suppose that I and J are graded ideals of L such that $L = I \oplus J$ and $I^{'}$ (resp. $J^{'}$) is a graded ideal of I (resp. J). Then the following conditions are equivalent*

(1) $L/(I^{'} \oplus J^{'})$ is a sympathetic Lie superalgebra;

(2) $I/I^{'}$ and $J/J^{'}$ are sympathetic Lie superalgebras.

Proof. By virtue of I and J are direct factors of L, $I^{'}$ and $J^{'}$ are graded ideals of L, we have $I^{'} \oplus J^{'}$ is a graded ideal of L and $L/(I^{'} \oplus J^{'})$ is isomorphic to $I/I^{'} \oplus J/J^{'}$. According to Lemma 3.2 and Proposition 3, we can obtain the conditions (1) and (2) are equivalent. □

Proposition 10. *Let L be a Lie superalgebra. Suppose that I and J are graded ideals of L such that $L = I \oplus J$ and S is a graded ideal of L. Then the following conditions are equivalent*

(1) L/S is a sympathetic Lie superalgebra;

(2) $I/I \cap S$ and $J/J \cap S$ are sympathetic Lie superalgebras.

Proof. (1) $\Rightarrow$ (2) By Theorem 6.1, $S = S \cap I \oplus S \cap J$, and by Lemma 6.2(2), we have $I/I \cap S$ and $J/J \cap S$ are sympathetic Lie superalgebras.

(2) $\Rightarrow$ (1) We claim that $S = S \cap I \oplus S \cap J$. Set $x \in S$, then $x = y + z$ with $y \in I$ and $z \in J$. Considering the canonical surjections

$$\varphi : L \to L/S \text{ and } \phi : I \to I/I \cap S.$$

Let $y^{'} \in I$, then

$$\varphi[y, y^{'}] = \varphi[x - z, y^{'}] = \varphi([x, y^{'}] - [z, y^{'}]) = 0,$$

we get that $[y, y^{'}] \in S \cap I$, therefore

$$[\phi(y), \phi(y^{'})] = \phi([y, y^{'}]) = 0.$$

Therefore

$$\phi(y) \in C(I/I \cap S) = \{0\},$$

then $y \in I \cap S$. Similarly, we have $z \in J \cap S$. It implies that $S = S \cap I \oplus S \cap J$. Consequently, L/S is a sympathetic Lie superalgebra by Lemma 6.2. □

Proposition 11. *Let L be a Lie superalgebra. Suppose that I and J are graded ideals of L such that $L = I \oplus J$ and S a graded ideal of L. Then the following conditions are equivalent*

(1) S is the smallest among the graded ideals A of L for which L/A is a sympathetic Lie superalgebra;

(2) $S \cap I$ (resp. $S \cap J$) is the smallest among the graded ideals A of I (resp. J) for which I/A (resp. J/A) is a sympathetic Lie superalgebra.

Proof. (1) $\Rightarrow$ (2) According to Proposition 10, $I/I \cap S$ and $J/J \cap S$ are sympathetic Lie superalgebras. Set $I^{'}$(resp. $J^{'}$) be a graded ideal of I(resp. J) such that $I^{'} \subseteq I \cap S$(resp. $J^{'} \subseteq J \cap S$) and $I/I^{'}$(resp. $J/J^{'}$) is a sympathetic Lie superalgebra.

According to Lemma 6.2, we get that $L/(I^{'} \oplus J^{'})$ is a sympathetic Lie superalgebra, consequently, $S = I^{'} \oplus J^{'}$. Then

$$I \cap S = I \cap I^{'} \oplus I \cap J^{'} = I \cap I^{'} = I^{'}.$$

Similarly, $J^{'} = J \cap H$.

(2) $\Rightarrow$ (1) Set $S^{'}$ be a graded ideal of L such that $S^{'} \subseteq S$ and $L/S^{'}$ is a sympathetic Lie superalgebra. According to Proposition 10, $I/I \cap S^{'}$ and $J/J \cap S^{'}$ are sympathetic Lie superalgebras. By the assumption of this proposition we have $I \cap S^{'} = I \cap S$ and $J \cap S^{'} = J \cap S$. Then we obtain $S = S \cap I \oplus S \cap J = S^{'} \cap I \oplus S^{'} \cap J = S^{'}$. □

Definition 6.3. Let L be a Lie superalgebra. An ascending central series of L is a family graded ideals $\{A_i(L)\}_{i \geq 0}$ of L satisfying

$$0 = A_0(L) \subseteq A_1(L) \subseteq \cdots \text{ and } C(L/A_n(L)) = A_{n+1}(L)/A_n(L).$$

Lemma 6.4. *Let L be a Lie superalgebra, $\{A_i(L)\}_{i \geq 0}$ the ascending central series of L and $A_\infty(L)$ the union of the ascending central series of L. Then $A_\infty(L)$ is the smallest among the graded ideals I of L such that $C(L/I) = \{0\}$.*

Proof. There exists a positive integer n such that $A_\infty(L) = A_n(L)$. Then we have

$$C(L/A_\infty(L)) = C(L/A_n(L)) = A_{n+1}(L)/A_n(L) = \{0\}.$$

Set J be a graded ideal of L such that $C(L/J) = \{0\}$. By Lemma 6.1, $C(L) \subseteq J$, i.e., $A_1(L) \subseteq J$. Suppose that $A_i(L) \subseteq J$, and show that $A_{i+1}(L) \subseteq J$.

Set $H = J/A_i(L)$ be a graded ideal of $G = L/A_i(L)$, and G/H is isomorphic to L/J, which implies that $C(G/H) = \{0\}$. By Lemma 6.1, we have that $C(G) \subseteq H$.

Considering the canonical surjection $f : L \to G$, then $J = f^{-1}(H)$ contains $A_{i+1}(L) = f^{-1}(C(G))$. It follows that $A_i(L) \subseteq J$ for all positive integers i, hence $A_\infty(L) \subseteq J$. □

Theorem 6.5. *Let L be a perfect Lie superalgebra with trivial centre and* $\mathrm{ad}(L)$ *be a characteristic ideal of* $\mathrm{Der}(L)$. *Then* $\mathrm{Der}(L)$ *is sympathetic. If, in addition, L is indecomposable, then* $\mathrm{Der}(L)$ *is simply sympathetic.*

Proof. By virtue of $C(L) = \{0\}$, we have $L \simeq \mathrm{ad}(L)$. Let $\mathfrak{f} = \mathrm{Der}(L)$, then $L \lhd \mathfrak{f}$. Set $\mathfrak{e}$ be a splitting extension by $\mathfrak{f}$, we have $\mathfrak{f} \lhd \mathfrak{e}$. Therefore we can get $\mathrm{ad}e \in \mathrm{Der}(\mathfrak{f})$ with $\forall e \in \mathfrak{e}$. On account of L is a characteristic ideal of $\mathfrak{f}$, there exists an element $f \in \mathfrak{f}$ such that $\mathrm{ad}f|_L = \mathrm{ad}e|_L$. Then $\mathrm{ad}(f-e)|_L = 0$. Thus $f - e \in C_{\mathfrak{f}}(L)$. It follows that we obtain

$$\mathfrak{e} = \mathfrak{f} + C_{\mathfrak{f}}(L).$$

But $\mathfrak{f} \cap C_{\mathfrak{e}}(L) = C_{\mathfrak{f}}(L) = 0$ and $\mathfrak{f} \lhd \mathfrak{e}$. Thereby

$$\mathfrak{e} = \mathfrak{f} \oplus C_{\mathfrak{e}}(L).$$

Immediately, we have

$$C_{\mathfrak{e}}(L) \subset C_{\mathfrak{e}}(\mathfrak{f}),$$

which implies that $\mathfrak{e} = \mathfrak{f} \oplus C_{\mathfrak{e}}(\mathfrak{f})$. Consequently, $\mathfrak{f}$ is a sympathetic Lie superalgebra by Theorem 3.6.

Assume that $\mathrm{Der}(L)$ is not simply sympathetic. Then there exists a simply sympathetic ideal I. And there exists a graded ideal J such that $\mathfrak{f} = I \oplus J$ by Lemma 3.1. For $x, y \in L$, there exist $x_1, y_1 \in I$, $x_2, y_2 \in J$ such that $x = x_l + x_2$ and $y = y_1 + y_2$. Hence

$$[x, y] = [x_1, y] + [x_2, y],$$

where $[x_1, y] \in I \cap L$ and $[x_2, y] \in J \cap L$. Thus

$$L = [L, L] = (I \cap L) \oplus (J \cap L).$$

Since L is indecomposable, then $I \cap L = \{0\}$ (or $J \cap L = \{0\}$) . Therefore $L \subset J \cap L$ and $L \subset J$. It follows that $I \subset C_{\mathfrak{f}}(L) = \{0\}$. According to Lemma 5.7, we obtain $\mathfrak{f}$ is indecomposable and $\mathrm{Der}(L)$ is simply sympathetic. The theorem holds. □

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