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ON A GENERAL HOMOGENEOUS THREE-DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS

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Abstract. In this work, we study the behavior of the solutions of following three-dimensional system of difference equations

$$x_{n+1} = f(y_n, y_{n-1}),\ y_{n+1} = g(z_n, z_{n-1}),\ z_{n+1} = h(x_n, x_{n-1})$$

where $n \in \mathbb{N}_0$, the initial values x_{-1}, x_0, y_{-1}, y_0 z_{-1}, z_0 are positive real numbers, the functions f, g, h : $(0, +\infty)^2 \to (0, +\infty)$ are continuous and homogeneous of degree zero. By proving some general convergence theorems, we have established conditions for the global stability of the corresponding unique equilibrium point. We give necessary and sufficient conditions on existence of prime period two solutions of the above mentioned system. Also, we prove a result on oscillatory solutions. As applications of the obtained results, some particular systems of difference equations defined by homogeneous functions of degree zero are investigated. Our results generalize some existing ones in the literature.

1. **Introduction and preliminaries.** In this paper, we study the behavior of the solutions of the following three-dimensional system of difference equations of second order

$$x_{n+1} = f(y_n, y_{n-1}),\ y_{n+1} = g(z_n, z_{n-1}),\ z_{n+1} = h(x_n, x_{n-1}) \tag{1}$$

where $n \in \mathbb{N}_0$, the initial values x_{-i}, y_{-i} and z_{-i}, $i = 0, 1$, are positive real numbers, the functions f, g, h : $(0, +\infty)^2 \to (0, +\infty)$ are continuous and homogeneous of degree zero. We establish results on local and global stability of the unique positive equilibrium point. To do this we prove some general convergence theorems, that can be applied to generalize a lot of existing systems and to study new ones. Some results on existence of periodic and oscillatory solutions are also proved.

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Now, we explain our motivation for doing this work. Clearly if we take $z_{-i} = x_{-i}$, $i = 0, 1$, and $h \equiv g$, then the system (1), will be

$$x_{n+1} = f(y_n, y_{n-1}),\ y_{n+1} = g(x_n, x_{n-1}). \tag{2}$$

Noting also that if we choose $z_{-i} = y_{-i} = x_{-i}$, $i = 0, 1$, and $h \equiv g \equiv f$, then System (1) will be

$$x_{n+1} = f(x_n, x_{n-1}). \tag{3}$$

In [23], the behavior of the solutions of System (2) has been investigated. System (2) is a generalization of Equation (3), studied in [17]. The present System (1) is the three-dimensional generalization of System (2).

In the literature there are many studies on difference equations defined by homogeneous functions, see for instance [1, 2, 5, 7, 11, 16]. Noting that also a lot of studies are devoted to various models of difference equations and systems, not necessary defined by homogeneous functions, see for example [4, 9, 10, 12, 14, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29].

Before we state our results, we recall the following definitions and results. For more details we refer to the following references [3, 6, 8, 13].

Let $F : (0, +\infty)^6 \to (0, +\infty)^6$ be a continuous function and consider the system of difference equations

$$Y_{n+1} = F(Y_n),\ n \in \mathbb{N}_0, \tag{4}$$

where the initial value $Y_0 \in (0, +\infty)^6$. A point $\overline{Y} \in (0, +\infty)^6$ is an equilibrium point of (4), if it is a solution of $\overline{Y} = F(\overline{Y})$.

Definition 1.1. Let $\overline{Y}$ be an equilibrium point of System (4), and let $\|.\|$ any convenient vector norm.

1. We say that the equilibrium point $\overline{Y}$ is stable (or locally stable) if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every initial condition Y_0: $\|Y_0 - \overline{Y}\| < \delta$ implies $\|Y_n - \overline{Y}\| < \epsilon$. Otherwise, the equilibrium point $\overline{Y}$ is unstable.
2. We say that the equilibrium point $\overline{Y}$ is asymptotically stable (or locally asymptotically stable) if it is stable and there exists $\gamma > 0$ such that $\|Y_0 - \overline{Y}\| < \gamma$ implies
$$\lim_{n\to\infty} Y_n = \overline{Y}.$$
3. We say that the equilibrium point $\overline{Y}$ is a global attractor if for every Y_0,
$$\lim_{n\to\infty} Y_n = \overline{Y}.$$
4. We say that the equilibrium point $\overline{Y}$ is globally (asymptotically) stable if it is stable and a global attractor.

Assume that F is C^1 on $(0, +\infty)^6$. To System (4), we associate a linear system, about the equilibrium point $\overline{Y}$, given by

$$Z_{n+1} = F_J(\overline{Y})Z_n, \quad n \in \mathbb{N}_0,\ Z_n = Y_n - \overline{Y},$$

where F_J is the Jacobian matrix of the function F evaluated at the equilibrium point $\overline{Y}$.

To study the stability of the equilibrium point $\overline{Y}$, we need the following theorem.

Theorem 1.2. *Let $\overline{Y}$ be an equilibrium point of System (4). Then, the following statements are true:*

(i) If all the eigenvalues of the Jacobian matrix F_J lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{Y}$ is asymptotically stable.

(ii) If at least one eigenvalue of F_J has absolute value greater than one, then the equilibrium $\overline{Y}$ is unstable.

Definition 1.3. A solution $(x_n, y_n, z_n)_{n\geq -1}$ of System (1) is said to be periodic of period $p \in \mathbb{N}$ if

$$x_{n+p} = x_n,\ y_{n+p} = y_n,\ z_{n+p} = z_n,\ n \geq -1. \tag{5}$$

The solution $(x_n, y_n, z_n)_{n\geq -1}$ is said to be periodic with prime period $p \in \mathbb{N}$, if it is periodic with period p and p is the least positive integer for which (1.3) holds.

Definition 1.4. Let $(x_n, y_n, z_n)_{n\geq -1}$ be a solution of System (1). We say that the sequence $(x_n)_{n\geq -1}$ (resp. $(y_n)_{n\geq -1}$, $(z_n)_{n\geq -1}$) oscillates about $\overline{x}$ (resp. $\overline{y}$, $\overline{z}$) with a semi-cycle of length one if: $(x_n - \overline{x})(x_{n+1} - \overline{x}) < 0, n \geq -1$ (resp. $(y_n - \overline{y})(y_{n+1} - \overline{y}) < 0, n \geq -1$, $(z_n - \overline{z})(z_{n+1} - \overline{z}) < 0, n \geq -1$).

Remark 1. For every term x_{n_0} of the sequence $(x_n)_{n\geq -1}$, the notation "+" means $x_{n_0} - \overline{x} > 0$ and the notation "−" means $x_{n_0} - \overline{x} < 0$. The same notations will be used for the terms of the sequences $(y_n)_{n\geq -1}$ and $(z_n)_{n\geq -1}$.

Definition 1.5. A function $\Phi : (0, +\infty)^2 \to (0, +\infty)$ is said to be homogeneous of degree $m \in \mathbb{R}$ if we have

$$\Phi(\lambda u, \lambda v) = \lambda^m \Phi(u, v)$$

for all $(u, v) \in (0, +\infty)^2$ and for all $\lambda > 0$.

Theorem 1.6. *Let $\Phi : (0, +\infty)^2 \to (0, +\infty)$ be a C^1 function on $(0, +\infty)^2$.*

1. *Then, Φ is homogeneous of degree m if and only if*

$$u\frac{\partial \Phi}{\partial u}(u, v) + v\frac{\partial \Phi}{\partial v}(u, v) = m\Phi(u, v),\ (u, v) \in (0, +\infty)^2 .$$

*(This statement is usually called **Euler's Theorem**).*

2. *If Φ is homogeneous of degree m on $(0, +\infty)^2$, then $\frac{\partial \Phi}{\partial u}$ and $\frac{\partial \Phi}{\partial v}$ are homogenous of degree $m - 1$ on $(0, +\infty)^2$.*

2. **Stability of the equilibrium points.** A point $(\overline{x}, \overline{y}, \overline{z}) \in (0, +\infty)^3$ is an equilibrium point of System (1) if it is a solution of the following system

$$\overline{x} = f(\overline{y}, \overline{y}),\ \overline{y} = g(\overline{z}, \overline{z}),\ \overline{z} = g(\overline{x}, \overline{x}).$$

Using the fact that f, g and h are homogeneous of degree zero, we get that

$$(\overline{x}, \overline{y}, \overline{z}) = (f(1, 1), g(1, 1), h(1, 1))$$

is the unique equilibrium point of System (1).

Let $F : (0, +\infty)^6 \to (0, +\infty)^6$ be the function defined by

$$F(W) = (f_1(W), f_2(W), g_1(W), g_2(W), h_1(W), h_2(W)) ,\ W = (u, v, w, t, r, s)$$

with

$$f_1(W) = f(w, t),\ f_2(W) = u,\ g_1(W) = g(r, s),\ g_2(W) = w,\ h_1(W) = h(u, v),\ g_2(W) = r.$$

Then, System (1) can be written as follows

$$W_{n+1} = F(W_n),\ W_n = (x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1})^t ,\ n \in \mathbb{N}_0.$$

So, $(\overline{x}, \overline{y}, \overline{z}) = (f(1, 1), g(1, 1), h(1, 1))$ is an equilibrium point of system (1) if and only if

$$\overline{W} = (\overline{x}, \overline{x}, \overline{y}, \overline{y}, \overline{z}, \overline{z}) = (f(1, 1), f(1, 1), g(1, 1), g(1, 1), h(1, 1), h(1, 1))$$

is an equilibrium point of $W_{n+1} = F(W_n)$.

Assume that the functions f, g and h are C^1 on $(0,+\infty)^2$. To System (1), we associate about the equilibrium point $\overline{W}$ the following linear system

$$X_{n+1} = J_F X_n,\ n \in \mathbb{N}_0$$

where J_F is the Jacobian matrix associated to the function F evaluated at

$$\overline{W} = (f(1,1), f(1,1), g(1,1), g(1,1), h(1,1), h(1,1)).$$

We have

$$J_F = \begin{pmatrix} 0 & 0 & \frac{\partial f}{\partial w}(\overline{y},\overline{y}) & \frac{\partial f}{\partial t}(\overline{y},\overline{y}) & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial g}{\partial r}(\overline{z},\overline{z}) & \frac{\partial g}{\partial s}(\overline{z},\overline{z}) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\partial h}{\partial u}(\overline{x},\overline{x}) & \frac{\partial h}{\partial v}(\overline{x},\overline{x}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

As f, g and h are homogeneous of degree 0, then using Part 1. of Theorem 1.6, we get

$$\overline{y}\frac{\partial f}{\partial w}(\overline{y},\overline{y}) + \overline{y}\frac{\partial f}{\partial t}(\overline{y},\overline{y}) = 0$$

which implies

$$\frac{\partial f}{\partial t}(\overline{y},\overline{y}) = -\frac{\partial f}{\partial w}(\overline{y},\overline{y}).$$

Similarly we get

$$\frac{\partial g}{\partial s}(\overline{z},\overline{z}) = -\frac{\partial g}{\partial r}(\overline{z},\overline{z}),\ \frac{\partial h}{\partial v}(\overline{x},\overline{x}) = -\frac{\partial h}{\partial u}(\overline{x},\overline{x}).$$

It follows that J_F takes the form:

$$J_F = \begin{pmatrix} 0 & 0 & \frac{\partial f}{\partial w}(\overline{y},\overline{y}) & -\frac{\partial f}{\partial w}(\overline{y},\overline{y}) & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial g}{\partial r}(\overline{z},\overline{z}) & -\frac{\partial g}{\partial r}(\overline{z},\overline{z}) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\partial h}{\partial u}(\overline{x},\overline{x}) & -\frac{\partial h}{\partial u}(\overline{x},\overline{x}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of the matrix J_F is given by

$$\begin{aligned} P(\lambda) &= \lambda^6 - \frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\lambda^3 + 3\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\lambda^2 \\ &\quad -3\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\lambda + \frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y}). \end{aligned}$$

Now assume that

$$\left|\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\right| < \frac{1}{8}$$

and consider the two functions

$$\Phi(\lambda) = \lambda^6,$$

$$\begin{aligned} \Psi(\lambda) &= -\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\lambda^3 + 3\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\lambda^2 \\ &\quad -3\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\lambda + \frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y}). \end{aligned}$$

We have

$$|\Psi(\lambda)| \le 8\left|\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\right| < 1 = |\Phi(\lambda)|\,,\ \forall\,\lambda \in \mathbb{C}:\ |\lambda| = 1.$$

So, by Rouché's Theorem it follows that all roots of $P(\lambda)$ lie inside the unit disk.

Hence, by Theorem 1.2, we deduce from the above consideration that the equilibrium point $(\overline{x},\overline{y},\overline{z}) = (f(1,1), g(1,1), h(1,1))$ is locally asymptotically stable.

Using Part 2. of Theorem 1.6 and the fact that the function f, g, h are homogeneous of degree zero, we get that $\frac{\partial f}{\partial w}$, $\frac{\partial g}{\partial r}$ and $\frac{\partial h}{\partial u}$ are homogeneous of degree -1. So, it follows that

$$\frac{\partial f}{\partial w}(\overline{y},\overline{y}) = \frac{\frac{\partial f}{\partial w}(1,1)}{\overline{y}},\ \frac{\partial g}{\partial r}(\overline{z},\overline{z}) = \frac{\frac{\partial g}{\partial r}(1,1)}{\overline{z}},\ \frac{\partial h}{\partial u}(\overline{x},\overline{x}) = \frac{\frac{\partial h}{\partial u}(1,1)}{\overline{x}}.$$

In summary, we have proved the following result.

Theorem 2.1. *Assume that $f(u,v)$, $g(u,v)$ and $h(u,v)$ are C^1 on $(0,+\infty)^2$. The equilibrium point*

$$(\overline{x},\overline{y},\overline{z}) = (f(1,1), g(1,1), h(1,1))$$

of System (1) is locally asymptotically stable if

$$\left|\frac{\partial f}{\partial u}(1,1)\frac{\partial g}{\partial u}(1,1)\frac{\partial h}{\partial u}(1,1)\right| < \frac{f(1,1)g(1,1)h(1,1)}{8}.$$

Now, we will prove some general convergence results. The obtained results allow us to deal with the global attractivity of the equilibrium point $(\overline{x},\overline{y},\overline{z}) = (f(1,1), g(1,1), h(1,1))$ and so the global stability.

Theorem 2.2. *Consider System (1). Assume that the following statements are true:*

1. *H_1: There exist a, b, α, β, λ, $\gamma \in (0,+\infty)$ such that*
   $$a \le f(u,v) \le b,\ \alpha \le g(u,v) \le \beta,\ \lambda \le h(u,v) \le \gamma,\ \forall (u,v) \in (0,+\infty)^2.$$
2. *H_2: $f(u,v)$, $g(u,v)$, $h(u,v)$ are increasing in u for all v and decreasing in v for all u.*
3. *H_3: If $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b]^2 \times [\alpha,\beta]^2 \times [\lambda,\gamma]^2$ is a solution of the system*
   $$\begin{aligned} m_1 &= f(m_2, M_2),\ M_1 = f(M_2, m_2),\\ m_2 &= g(m_3, M_3),\ M_2 = g(M_3, m_3),\\ m_3 &= h(m_1, M_1),\ M_3 = h(M_1, m_1) \end{aligned}$$
   then
   $$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

   Then every solution of System (1) converges to the unique equilibrium point
   $$(\overline{x},\overline{y},\overline{z}) = (f(1,1), g(1,1), h(1,1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, ...,$

$$m_1^{i+1} := f(m_2^i, M_2^i),\ M_1^{i+1} := f(M_2^i, m_2^i),$$
$$m_2^{i+1} := g(m_3^i, M_3^i),\ M_2^{i+1} := g(M_3^i, m_3^i),$$

$$m_3^{i+1} := h(m_1^i, M_1^i),\ M_3^{i+1} := h(M_1^i, m_1^i).$$

We have

$$a \le f(\alpha,\beta) \le f(\beta,\alpha) \le b,$$
$$\alpha \le g(\lambda,\gamma) \le g(\gamma,\lambda) \le \beta,$$
$$\lambda \le h(a,b) \le h(b,a) \le \gamma,$$

and so,

$$m_1^0 = a \le f(m_2^0, M_2^0) \le f(M_2^0, m_2^0) \le b = M_1^0,$$
$$m_2^0 = \alpha \le g(m_3^0, M_3^0) \le g(M_3^0, m_3^0) \le \beta = M_2^0,$$

and

$$m_3^0 = \lambda \le h(m_1^0, M_1^0) \le g(M_1^0, m_1^0) \le \gamma = M_3^0.$$

Hence,

$$m_1^0 \le m_1^1 \le M_1^1 \le M_1^0,$$
$$m_2^0 \le m_2^1 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le M_3^1 \le M_3^0.$$

Now, we have

$$m_1^1 = f(m_2^0, M_2^0) \le f(m_2^1, M_2^1) = m_1^2 \le f(M_2^1, m_2^1) = M_1^2 \le f(M_2^0, m_2^0) = M_1^1,$$
$$m_2^1 = g(m_3^0, M_3^0) \le g(m_3^1, M_3^1) = m_2^2 \le g(M_3^1, m_3^1) = M_2^2 \le g(M_3^0, m_3^0) = M_2^1,$$
$$m_3^1 = h(m_1^0, M_1^0) \le h(m_1^1, M_1^1) = m_3^2 \le h(M_1^1, m_1^1) = M_3^2 \le h(M_1^0, m_1^0) = M_3^1,$$

and it follows that

$$m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0,$$
$$m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0.$$

By induction, we get for $i = 0, 1, ...$, that

$$a = m_1^0 \le m_1^1 \le ... \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le ... \le M_1^1 \le M_1^0 = b,$$
$$\alpha = m_2^0 \le m_2^1 \le ... \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le ... \le M_2^1 \le M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \le m_3^1 \le ... \le m_3^{i-1} \le m_3^i \le M_3^i \le M_3^{i-1} \le ... \le M_3^1 \le M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$, $(m_3^i)_{i\in\mathbb{N}_0}$ (resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ M_1 = \lim_{i\to+\infty} M_1^i,$$
$$m_2 = \lim_{i\to+\infty} m_2^i,\ M_2 = \lim_{i\to+\infty} M_2^i.$$
$$m_3 = \lim_{i\to+\infty} m_3^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \le m_1 \le M_1 \le b,\ \alpha \le m_2 \le M_2 \le \beta,\ \lambda \le m_3 \le M_3 \le \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(m_2^i, M_2^i),\ M_1^{i+1} = f(M_2^i, m_2^i),$$
$$m_2^{i+1} = g(m_3^i, M_3^i),\ M_2^{i+1} = g(M_3^i, m_3^i),$$
$$m_3^{i+1} = h(m_1^i, M_1^i),\ M_3^{i+1} = h(M_1^i, m_1^i),$$

and using the continuity of f, g and h we obtain

$$\begin{aligned} m_1 &= f(m_2, M_2),\ M_1 = f(M_2, m_2), \\ m_2 &= g(m_3, M_3),\ M_2 = g(M_3, m_3), \\ m_3 &= h(m_1, M_1),\ M_3 = h(M_1, m_1) \end{aligned}$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, for $n = 1, 2, \cdots$, we get

$$m_1^0 = a \le x_n \le b = M_1^0,\ m_2^0 = \alpha \le y_n \le \beta = M_2^0,\ m_3^0 = \lambda \le z_n \le \gamma = M_3^0.$$

For $n = 2, 3, ...$, we have

$$m_1^1 = f(m_2^0, M_2^0) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^0, m_2^0) = M_1^1,$$
$$m_2^1 = g(m_3^0, M_3^0) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^0, m_3^0) = M_2^1,$$

and

$$m_3^1 = h(m_1^0, M_1^0) \le z_{n+1} = h(x_n, x_{n-1}) \le h(M_1^0, m_1^0) = M_3^1,$$

that is

$$m_1^1 \le x_n \le M_1^1,\ m_2^1 \le y_n \le M_2^1,\ m_3^1 \le z_n \le M_3^1,\ n = 3, 4, \cdots.$$

Now, for $n = 4, 5, ...$, we have

$$m_1^2 = f(m_2^1, M_2^1) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^1, m_2^1) = M_1^2,$$

and

$$m_2^2 = g(m_3^1, M_3^1) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^1, m_3^1) = M_2^2,$$
$$m_3^2 = h(m_1^1, M_1^1) \le z_{n+1} = h(x_n, x_{n-1}) \le h(M_1^1, m_1^1) = M_3^2,$$

that is

$$m_1^2 \le x_n \le M_1^2,\ m_2^2 \le y_n \le M_2^2,\ m_3^2 \le z_n \le M_3^2,\ n = 5, 6, \cdots.$$

Similarly, for $n = 6, 7, ...$, we have

$$m_1^3 = f(m_2^2, M_2^2) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^2, m_2^2) = M_1^3,$$
$$m_2^3 = g(m_3^2, M_3^2) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^2, m_3^2) = M_2^3,$$

and

$$m_3^3 = h(m_1^2, M_1^2) \le z_{n+1} = h(x_n, x_{n-1}) \le h(M_1^2, m_1^2) = M_3^3,$$

that is

$$m_1^3 \le x_n \le M_1^3,\ m_2^3 \le y_n \le M_2^3,\ m_3^3 \le z_n \le M_3^3,\ n = 7, 8, \cdots.$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \le x_n \le M_1^i,\ m_2^i \le y_n \le M_2^i,\ m_3^i \le z_n \le M_3^i,\ n \ge 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n \to +\infty} x_n = M_1,\ \lim_{n \to +\infty} y_n = M_2,\ \lim_{n \to +\infty} z_n = M_3.$$

From (1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1, 1),\ M_2 = g(M_3, M_3) = g(1, 1),\ M_3 = h(M_1, M_1) = h(1, 1).$$

□

Theorem 2.3. *Consider System (1). Assume that the following statements are true:*

1. H_1*: There exist* $a,\, b,\, \alpha,\, \beta, \lambda,\, \gamma \in (0,+\infty)$ *such that*
$$a \leq f(u,v) \leq b,\, \alpha \leq g(u,v) \leq \beta,\, \lambda \leq h(u,v) \leq \gamma,\, \forall (u,v) \in (0,+\infty)^2 .$$
2. H_2*:* $f(u,v)$, $g(u,v)$ *are increasing in* u *for all* v *and decreasing in* v *for all* u *and* $h(u,v)$ *is decreasing in* u *for all* v *and increasing in* v *for all* u.
3. H_3*: If* $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b]^2 \times [\alpha,\beta]^2 \times [\lambda,\gamma]^2$ *is a solution of the system*
$$\begin{aligned} m_1 &= f(m_2,M_2),\, M_1 = f(M_2,m_2),\\ m_2 &= g(m_3,M_3),\, M_2 = g(M_3,m_3),\\ m_3 &= h(M_1,m_1),\, M_3 = h(m_1,M_1) \end{aligned}$$
then
$$m_1 = M_1,\, m_2 = M_2,\, m_3 = M_3.$$

Then every solution of System (1) converges to the unique equilibrium point
$$(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1)).$$

Proof. Let
$$m_1^0 := a,\, M_1^0 := b,\, m_2^0 := \alpha,\, M_2^0 := \beta,\, m_3^0 := \lambda,\, M_3^0 := \gamma$$
and for each $i = 0, 1, ...,$
$$\begin{aligned} m_1^{i+1} &:= f(m_2^i, M_2^i),\, M_1^{i+1} := f(M_2^i, m_2^i),\\ m_2^{i+1} &:= g(m_3^i, M_3^i),\, M_2^{i+1} := g(M_3^i, m_3^i),\\ m_3^{i+1} &:= h(M_1^i, m_1^i),\, M_3^{i+1} := h(m_1^i, M_1^i). \end{aligned}$$
We have
$$\begin{aligned} a \leq f(\alpha,\beta) &\leq f(\beta,\alpha) \leq b,\\ \alpha \leq g(\lambda,\gamma) &\leq g(\gamma,\lambda) \leq \beta,\\ \lambda \leq h(b,a) &\leq h(a,b) \leq \gamma, \end{aligned}$$
and so,
$$\begin{aligned} m_1^0 = a \leq f(m_2^0, M_2^0) \leq f(M_2^0, m_2^0) \leq b = M_1^0,\\ m_2^0 = \alpha \leq g(m_3^0, M_3^0) \leq g(M_3^0, m_3^0) \leq \beta = M_2^0, \end{aligned}$$
and
$$m_3^0 = \lambda \leq h(M_1^0, m_1^0) \leq h(m_1^0, M_1^0) \leq \gamma = M_3^0.$$
Hence,
$$\begin{aligned} m_1^0 \leq m_1^1 \leq M_1^1 \leq M_1^0,\\ m_2^0 \leq m_2^1 \leq M_2^1 \leq M_2^0, \end{aligned}$$
and
$$m_3^0 \leq m_3^1 \leq M_3^1 \leq M_3^0.$$
Now, we have
$$\begin{aligned} m_1^1 = f(m_2^0, M_2^0) \leq f(m_2^1, M_2^1) = m_1^2 \leq f(M_2^1, m_2^1) = M_1^2 \leq f(M_2^0, m_2^0) = M_1^1,\\ m_2^1 = g(m_3^0, M_3^0) \leq g(m_3^1, M_3^1) = m_2^2 \leq g(M_3^1, m_3^1) = M_2^2 \leq g(M_3^0, m_3^0) = M_2^1,\\ m_3^1 = h(M_1^0, m_1^0) \leq h(M_1^1, m_1^1) = m_3^2 \leq h(m_1^1, M_1^1) = M_3^2 \leq h(m_1^0, M_1^0) = M_3^1, \end{aligned}$$
and it follows that
$$m_1^0 \leq m_1^1 \leq m_1^2 \leq M_1^2 \leq M_1^1 \leq M_1^0,$$

$$m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0.$$

By induction, we get for $i = 0, 1, ...$, that

$$a = m_1^0 \le m_1^1 \le ... \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le ... \le M_1^1 \le M_1^0 = b,$$
$$\alpha = m_2^0 \le m_2^1 \le ... \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le ... \le M_2^1 \le M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \le m_3^1 \le ... \le m_3^{i-1} \le m_3^i \le M_3^i \le M_3^{i-1} \le ... \le M_3^1 \le M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$, $(m_3^i)_{i\in\mathbb{N}_0}$ (resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ M_1 = \lim_{i\to+\infty} M_1^i,$$
$$m_2 = \lim_{i\to+\infty} m_2^i,\ M_2 = \lim_{i\to+\infty} M_2^i.$$
$$m_3 = \lim_{i\to+\infty} m_3^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \le m_1 \le M_1 \le b,\ \alpha \le m_2 \le M_2 \le \beta,\ \lambda \le m_3 \le M_3 \le \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(m_2^i, M_2^i),\ M_1^{i+1} = f(M_2^i, m_2^i),$$
$$m_2^{i+1} = g(m_3^i, M_3^i),\ M_2^{i+1} = g(M_3^i, m_3^i),$$
$$m_3^{i+1} = h(M_1^i, m_1^i),\ M_3^{i+1} = h(m_1^i, M_1^i),$$

and using the continuity of f, g and h we obtain

$$\begin{aligned} m_1 &= f(m_2, M_2),\ m_2 = f(M_2, m_2), \\ m_2 &= g(m_3, M_3),\ M_2 = g(M_3, m_3), \\ m_3 &= h(M_1, m_1),\ M_3 = h(m_1, M_1) \end{aligned}$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, for $n = 1, 2, \cdots$, we get

$$m_1^0 = a \le x_n \le b = M_1^0,\ m_2^0 = \alpha \le y_n \le \beta = M_2^0,\ m_3^0 = \lambda \le z_n \le \gamma = M_3^0\,.$$

For $n = 2, 3, ...$, we have

$$m_1^1 = f(m_2^0, M_2^0) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^0, m_2^0) = M_1^1,$$
$$m_2^1 = g(m_3^0, M_3^0) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^0, m_3^0) = M_2^1,$$

and

$$m_3^1 = h(M_1^0, m_1^0) \le z_{n+1} = h(x_n, x_{n-1}) \le h(m_1^0, M_1^0) = M_3^1,$$

that is

$$m_1^1 \le x_n \le M_1^1,\ m_2^1 \le y_n \le M_2^1,\ m_3^1 \le z_n \le M_3^1,\ n = 3, 4, \cdots.$$

Now, for $n = 4, 5, ...$, we have

$$m_1^2 = f(m_2^1, M_2^1) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^1, m_2^1) = M_1^2,$$

and

$$m_2^2 = g(m_3^1, M_3^1) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^1, m_3^1) = M_2^2,$$

$$m_3^2 = h(M_1^1, m_1^1) \le z_{n+1} = h(x_n, x_{n-1}) \le h(m_1^1, M_1^1) = M_3^2,$$

that is

$$m_1^2 \le x_n \le M_1^2,\ m_2^2 \le y_n \le M_2^2,\ m_3^2 \le z_n \le M_3^2,\ n = 5, 6, \cdots .$$

Similarly, for $n = 6, 7, ...$, we have

$$m_1^3 = f(m_2^2, M_2^2) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^2, m_2^2) = M_1^3,$$

$$m_2^3 = g(m_3^2, M_3^2) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^2, m_3^2) = M_2^3,$$

and

$$m_3^3 = h(M_1^2, m_1^2) \le z_{n+1} = h(x_n, x_{n-1}) \le h(m_1^2, M_1^2) = M_3^3,$$

that is

$$m_1^3 \le x_n \le M_1^3,\ m_2^3 \le y_n \le M_2^3,\ m_3^3 \le z_n \le M_3^3,\ n = 7, 8, \cdots .$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \le x_n \le M_1^i,\ m_2^i \le y_n \le M_2^i,\ m_3^i \le z_n \le M_3^i,\ n \ge 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ \lim_{n\to+\infty} z_n = M_3.$$

From (1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1, 1),\ M_2 = g(M_3, M_3) = g(1, 1),\ M_3 = h(M_1, M_1) = h(1, 1).$$

□

Theorem 2.4. *Consider System (1). Assume that the following statements are true:*

1. H_1*: There exist* a, b, α, β, λ, $\gamma \in (0, +\infty)$ *such that*

$$a \le f(u, v) \le b,\ \alpha \le g(u, v) \le \beta,\ \lambda \le h(u, v) \le \gamma,\ \forall(u, v) \in (0, +\infty)^2 .$$

2. H_2*:* $f(u, v)$ *is increasing in* u *for all* v *and decreasing in* v *for all* u *and* $g(u, v)$, $h(u, v)$ *are decreasing in* u *for all* v *and increasing in* v *for all* u*.*
3. H_3*: If* $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a, b]^2 \times [\alpha, \beta]^2 \times [\lambda, \gamma]^2$ *is a solution of the system*

$$\begin{aligned} m_1 &= f(m_2, M_2),\ M_1 = f(M_2, m_2), \\ m_2 &= g(M_3, m_3),\ M_2 = g(m_3, M_3), \\ m_3 &= h(M_1, m_1),\ M_3 = h(m_1, M_1) \end{aligned}$$

then

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

Then every solution of System (1) converges to the unique equilibrium point

$$(\overline{x}, \overline{y}, \overline{z}) = (f(1, 1), g(1, 1), h(1, 1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, ...,$

$$m_1^{i+1} := f(m_2^i, M_2^i),\ M_1^{i+1} := f(M_2^i, m_2^i),$$
$$m_2^{i+1} := g(M_3^i, m_3^i),\ M_2^{i+1} := g(m_3^i, M_3^i),$$
$$m_3^{i+1} := h(M_1^i, m_1^i),\ M_3^{i+1} := h(m_1^i, M_1^i).$$

We have

$$a \le f(\alpha, \beta) \le f(\beta, \alpha) \le b,$$
$$\alpha \le g(\gamma, \lambda) \le g(\lambda, \gamma) \le \beta,$$
$$\lambda \le h(b, a) \le h(a, b) \le \gamma,$$

and so,

$$m_1^0 = a \le f(m_2^0, M_2^0) \le f(M_2^0, m_2^0) \le b = M_1^0,$$
$$m_2^0 = \alpha \le g(M_3^0, m_3^0) \le g(m_3^0, M_3^0) \le \beta = M_2^0,$$

and

$$m_3^0 = \lambda \le h(M_1^0, m_1^0) \le h(m_1^0, M_1^0) \le \gamma = M_3^0.$$

Hence,

$$m_1^0 \le m_1^1 \le M_1^1 \le M_1^0,$$
$$m_2^0 \le m_2^1 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le M_3^1 \le M_3^0.$$

Now, we have

$$m_1^1 = f(m_2^0, M_2^0) \le f(m_2^1, M_2^1) = m_1^2 \le f(M_2^1, m_2^1) = M_1^2 \le f(M_2^0, m_2^0) = M_1^1,$$

$$m_2^1 = g(M_3^0, m_3^0) \le g(M_3^1, m_3^1) = m_2^2 \le g(m_3^1, M_3^1) = M_2^2 \le g(m_3^0, M_3^0) = M_2^1,$$
$$m_3^1 = h(M_1^0, m_1^0) \le h(M_1^1, m_1^1) = m_3^2 \le h(m_1^1, M_1^1) = M_3^2 \le h(m_1^0, M_1^0) = M_3^1,$$

and it follows that

$$m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0,$$
$$m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0.$$

By induction, we get for $i = 0, 1, ...,$ that

$$a = m_1^0 \le m_1^1 \le ... \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le ... \le M_1^1 \le M_1^0 = b,$$
$$\alpha = m_2^0 \le m_2^1 \le ... \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le ... \le M_2^1 \le M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \le m_3^1 \le ... \le m_3^{i-1} \le m_3^i \le M_3^i \le M_3^{i-1} \le ... \le M_3^1 \le M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$, $(m_3^i)_{i\in\mathbb{N}_0}$ (resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ M_1 = \lim_{i\to+\infty} M_1^i,$$
$$m_2 = \lim_{i\to+\infty} m_2^i,\ M_2 = \lim_{i\to+\infty} M_2^i.$$

$$m_3 = \lim_{i\to+\infty} m_3^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \le m_1 \le M_1 \le b,\ \alpha \le m_2 \le M_2 \le \beta,\ \lambda \le m_3 \le M_3 \le \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(m_2^i, M_2^i),\ M_1^{i+1} = f(M_2^i, m_2^i),$$
$$m_2^{i+1} = g(M_3^i, m_3^i),\ M_2^{i+1} = g(m_3^i, M_3^i),$$
$$m_3^{i+1} = h(M_1^i, m_1^i),\ M_3^{i+1} = h(m_1^i, M_1^i),$$

and using the continuity of f, g and h we obtain

$$\begin{aligned} m_1 &= f(m_2, M_2),\ M_1 = f(M_2, m_2),\\ m_2 &= g(M_3, m_3),\ M_2 = g(m_3, M_3),\\ m_3 &= h(M_1, m_1),\ M_3 = h(m_1, M_1) \end{aligned}$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, for $n = 1, 2, \cdots$, we get

$$m_1^0 = a \le x_n \le b = M_1^0,\ m_2^0 = \alpha \le y_n \le \beta = M_2^0,\ m_3^0 = \lambda \le z_n \le \gamma = M_3^0.$$

For $n = 2, 3, ...$, we have

$$m_1^1 = f(m_2^0, M_2^0) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^0, m_2^0) = M_1^1,$$
$$m_2^1 = g(M_3^0, m_3^0) \le y_{n+1} = g(z_n, z_{n-1}) \le g(m_3^0, M_3^0) = M_3^1,$$

and

$$m_3^1 = h(M_1^0, m_1^0) \le z_{n+1} = h(x_n, x_{n-1}) \le h(m_1^0, M_1^0) = M_3^1,$$

that is

$$m_1^1 \le x_n \le M_1^1,\ m_2^1 \le y_n \le M_2^1,\ m_3^1 \le z_n \le M_3^1,\ n = 3, 4, \cdots.$$

Now, for $n = 4, 5, ...$, we have

$$m_1^2 = f(m_2^1, M_2^1) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^1, m_2^1) = M_1^2,$$

and

$$m_2^2 = g(M_3^1, m_3^1) \le y_{n+1} = g(z_n, z_{n-1}) \le g(m_3^1, M_3^1) = M_2^2,$$
$$m_3^2 = h(M_1^1, m_1^1) \le z_{n+1} = h(x_n, x_{n-1}) \le h(m_1^1, M_1^1) = M_3^2,$$

that is

$$m_1^2 \le x_n \le M_1^2,\ m_2^2 \le y_n \le M_2^2,\ m_3^2 \le z_n \le M_3^2,\ n = 5, 6, \cdots.$$

Similarly, for $n = 6, 7, ...$, we have

$$m_1^3 = f(m_2^2, M_2^2) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^2, m_2^2) = M_1^3,$$
$$m_2^3 = g(M_3^2, m_3^2) \le y_{n+1} = g(z_n, z_{n-1}) \le g(m_3^2, M_3^2) = M_2^3,$$

and

$$m_3^3 = h(M_1^2, m_1^2) \le z_{n+1} = h(x_n, x_{n-1}) \le h(m_1^2, M_1^2) = M_3^3,$$

that is

$$m_1^3 \le x_n \le M_1^3,\ m_2^3 \le y_n \le M_2^3,\ m_3^3 \le z_n \le M_3^3,\ n = 7, 8, \cdots.$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \le x_n \le M_1^i,\ m_2^i \le y_n \le M_2^i,\ m_3^i \le z_n \le M_3^i,\ n \ge 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ \lim_{n\to+\infty} z_n = M_3.$$

From (1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1,1),\ M_2 = g(M_3, M_3) = g(1,1),\ M_3 = h(M_1, M_1) = h(1,1).$$

□

Theorem 2.5. *Consider System (1). Assume that the following statements are true:*

1. *H_1: There exist a, b, α, β, λ, $\gamma \in (0,+\infty)$ such that*
$$a \le f(u,v) \le b,\ \alpha \le g(u,v) \le \beta,\ \lambda \le h(u,v) \le \gamma,\ \forall (u,v) \in (0,+\infty)^2.$$
2. *H_2: $f(u,v)$, $h(u,v)$ are increasing in u for all v and decreasing in v for all u and $g(u,v)$ is decreasing in u for all v and increasing in v for all u.*
3. *H_3: If $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b]^2 \times [\alpha,\beta]^2 \times [\lambda,\gamma]^2$ is a solution of the system*
$$\begin{aligned} m_1 &= f(m_2, M_2),\ M_1 = f(M_2, m_2),\\ m_2 &= g(M_3, m_3),\ M_2 = g(m_3, M_3),\\ m_3 &= h(m_1, M_1),\ M_3 = h(M_1, m_1)\end{aligned}$$
then
$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

Then every solution of System (1) converges to the unique equilibrium point
$$(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1)).$$

Proof. Let
$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$
and for each $i = 0, 1, ...,$
$$\begin{aligned} m_1^{i+1} &:= f(m_2^i, M_2^i),\ M_1^{i+1} := f(M_2^i, m_2^i),\\ m_2^{i+1} &:= g(M_3^i, m_3^i),\ M_2^{i+1} := g(m_3^i, M_3^i),\\ m_3^{i+1} &:= h(m_1^i, M_1^i),\ M_3^{i+1} := h(M_1^i, m_1^i).\end{aligned}$$
We have
$$\begin{aligned} a \le f(\alpha,\beta) \le f(\beta,\alpha) \le b,\\ \alpha \le g(\gamma,\lambda) \le g(\lambda,\gamma) \le \beta,\\ \lambda \le h(a,b) \le h(b,a) \le \gamma,\end{aligned}$$
and so,
$$\begin{aligned} m_1^0 = a \le f(m_2^0, M_2^0) \le f(M_2^0, m_2^0) \le b = M_1^0,\\ m_2^0 = \alpha \le g(M_3^0, m_3^0) \le g(m_3^0, M_3^0) \le \beta = M_2^0,\end{aligned}$$
and
$$m_3^0 = \lambda \le h(m_1^0, M_1^0) \le h(M_1^0, m_1^0) \le \gamma = M_3^0.$$
Hence,
$$\begin{aligned} m_1^0 \le m_1^1 \le M_1^1 \le M_1^0,\\ m_2^0 \le m_2^1 \le M_2^1 \le M_2^0,\end{aligned}$$

and

$$m_3^0 \le m_3^1 \le M_3^1 \le M_3^0.$$

Now, we have

$$m_1^1 = f(m_2^0, M_2^0) \le f(m_2^1, M_2^1) = m_1^2 \le f(M_2^1, m_2^1) = M_1^2 \le f(M_2^0, m_2^0) = M_1^1,$$
$$m_2^1 = g(M_3^0, m_3^0) \le g(M_3^1, m_3^1) = m_2^2 \le g(m_3^1, M_3^1) = M_2^2 \le g(m_3^0, M_3^0) = M_2^1,$$
$$m_3^1 = h(m_1^0, M_1^0) \le h(m_1^1, M_1^1) = m_3^2 \le h(M_1^1, m_1^1) = M_3^2 \le h(M_1^0, m_1^0) = M_3^1,$$

and it follows that

$$m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0,$$
$$m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0.$$

By induction, we get for $i = 0, 1, ...$, that

$$a = m_1^0 \le m_1^1 \le ... \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le ... \le M_1^1 \le M_1^0 = b,$$
$$\alpha = m_2^0 \le m_2^1 \le ... \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le ... \le M_2^1 \le M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \le m_3^1 \le ... \le m_3^{i-1} \le m_3^i \le M_3^i \le M_3^{i-1} \le ... \le M_3^1 \le M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$, $(m_3^i)_{i\in\mathbb{N}_0}$ (resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ M_1 = \lim_{i\to+\infty} M_1^i,$$
$$m_2 = \lim_{i\to+\infty} m_2^i,\ M_2 = \lim_{i\to+\infty} M_2^i.$$
$$m_3 = \lim_{i\to+\infty} m_3^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \le m_1 \le M_1 \le b,\ \alpha \le m_2 \le M_2 \le \beta,\ \lambda \le m_3 \le M_3 \le \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(m_2^i, M_2^i),\ M_1^{i+1} = f(M_2^i, m_2^i),$$
$$m_2^{i+1} = g(M_3^i, m_3^i),\ M_2^{i+1} = g(m_3^i, M_3^i),$$
$$m_3^{i+1} = h(m_1^i, M_1^i),\ M_3^{i+1} = h(M_1^i, m_1^i),$$

and using the continuity of f, g and h we obtain

$$\begin{aligned} m_1 &= f(m_2, M_2),\ M_1 = f(M_2, m_2), \\ m_2 &= g(M_3, m_3),\ M_2 = g(m_3, M_3), \\ m_3 &= h(m_1, M_1),\ M_3 = h(M_1, m_1) \end{aligned}$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, for $n = 1, 2, \cdots$, we get

$$m_1^0 = a \le x_n \le b = M_1^0,\ m_2^0 = \alpha \le y_n \le \beta = M_2^0,\ m_3^0 = \lambda \le z_n \le \gamma = M_3^0\,.$$

For $n = 2, 3, ...$, we have

$$m_1^1 = f(m_2^0, M_2^0) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^0, m_2^0) = M_1^1,$$
$$m_2^1 = g(M_3^0, m_3^0) \le y_{n+1} = g(z_n, z_{n-1}) \le g(m_3^0, M_3^0) = M_3^1,$$

and

$$m_3^1 = h(m_1^0, M_1^0) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(M_1^0, m_1^0) = M_3^1,$$

that is

$$m_1^1 \leq x_n \leq M_1^1,\ m_2^1 \leq y_n \leq M_2^1,\ m_3^1 \leq z_n \leq M_3^1,\ n = 3, 4, \cdots.$$

Now, for $n = 4, 5, ...$, we have

$$m_1^2 = f(m_2^1, M_2^1) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(M_2^1, m_2^1) = M_1^2,$$

and

$$m_2^2 = g(M_3^1, m_3^1) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(m_3^1, M_3^1) = M_2^2,$$

$$m_3^2 = h(m_1^1, M_1^1) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(M_1^1, m_1^1) = M_3^2,$$

that is

$$m_1^2 \leq x_n \leq M_1^2,\ m_2^2 \leq y_n \leq M_2^2,\ m_3^2 \leq z_n \leq M_3^2,\ n = 5, 6, \cdots.$$

Similarly, for $n = 6, 7, ...$, we have

$$m_1^3 = f(m_2^2, M_2^2) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(M_2^2, m_2^2) = M_1^3,$$

$$m_2^3 = g(M_3^2, m_3^2) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(m_3^2, M_3^2) = M_2^3,$$

and

$$m_3^3 = h(m_1^2, M_1^2) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(M_1^2, m_1^2) = M_3^3,$$

that is

$$m_1^3 \leq x_n \leq M_1^3,\ m_2^3 \leq y_n \leq M_2^3,\ m_3^3 \leq z_n \leq M_3^3,\ n = 7, 8, \cdots.$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \leq x_n \leq M_1^i,\ m_2^i \leq y_n \leq M_2^i,\ m_3^i \leq z_n \leq M_3^i,\ n \geq 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ \lim_{n\to+\infty} z_n = M_3.$$

From (1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1, 1),\ M_2 = g(M_3, M_3) = g(1, 1),\ M_3 = h(M_1, M_1) = h(1, 1).$$

□

Theorem 2.6. *Consider System (1). Assume that the following statements are true:*

1. H_1*: There exist* a, b, α, β, λ, $\gamma \in (0, +\infty)$ *such that*

$$a \leq f(u, v) \leq b,\ \alpha \leq g(u, v) \leq \beta,\ \lambda \leq h(u, v) \leq \gamma,\ \forall (u, v) \in (0, +\infty)^2.$$

2. H_2*:* $f(u, v)$, $g(u, v)$, $h(u, v)$ *are decreasing in* u *for all* v *and increasing in* v *for all* u*.*

3. *H_3: If $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b]^2 \times [\alpha,\beta]^2 \times [\lambda,\gamma]^2$ is a solution of the system*

$$\begin{aligned} m_1 &= f(M_2, m_2),\ M_1 = f(m_2, M_2), \\ m_2 &= g(M_3, m_3),\ M_2 = g(m_3, M_3), \\ m_3 &= h(M_1, m_1),\ M_3 = h(m_1, M_1) \end{aligned}$$

then

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

Then every solution of System (1) converges to the unique equilibrium point

$$(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, ...,$

$$\begin{aligned} m_1^{i+1} &:= f(M_2^i, m_2^i),\ M_1^{i+1} := f(m_2^i, M_2^i), \\ m_2^{i+1} &:= g(M_3^i, m_3^i),\ M_2^{i+1} := g(m_3^i, M_3^i), \\ m_3^{i+1} &:= h(M_1^i, m_1^i),\ M_3^{i+1} := h(m_1^i, M_1^i). \end{aligned}$$

We have

$$\begin{aligned} a \le f(\beta, \alpha) \le f(\alpha, \beta) \le b, \\ \alpha \le g(\gamma, \lambda) \le g(\lambda, \gamma) \le \beta, \\ \lambda \le h(b, a) \le h(a, b) \le \gamma \end{aligned}$$

and so,

$$\begin{aligned} m_1^0 = a \le f(M_2^0, m_2^0) \le f(m_2^0, M_2^0) \le b = M_1^0, \\ m_2^0 = \alpha \le g(M_3^0, m_3^0) \le g(m_3^0, M_3^0) \le \beta = M_2^0, \\ m_3^0 = \lambda \le h(M_1^0, m_1^0) \le h(m_1^0, M_1^0) \le \gamma = M_3^0. \end{aligned}$$

Hence,

$$\begin{aligned} m_1^0 \le m_1^1 \le M_1^1 \le M_1^0, \\ m_2^0 \le m_2^1 \le M_2^1 \le M_2^0, \end{aligned}$$

and

$$m_3^0 \le m_3^1 \le M_3^1 \le M_3^0.$$

Now, we have

$$\begin{aligned} m_1^1 = f(M_2^0, m_2^0) \le f(M_2^1, m_2^1) = m_1^2 \le f(m_2^1, M_2^1) = M_1^2 \le f(m_2^0, M_2^0) = M_1^1, \\ m_2^1 = g(M_3^0, m_3^0) \le g(M_3^1, m_3^1) = m_2^2 \le g(m_3^1, M_3^1) = M_2^2 \le g(m_3^0, M_3^0) = M_2^1, \\ m_3^1 = h(M_1^0, m_1^0) \le h(M_1^1, m_1^1) = m_3^2 \le h(m_1^1, M_1^1) = M_3^2 \le h(m_1^0, M_1^0) = M_3^1 \end{aligned}$$

and it follows that

$$\begin{aligned} m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0, \\ m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0, \\ m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0. \end{aligned}$$

By induction, we get for $i = 0, 1, ...,$ that

$$\begin{aligned} a = m_1^0 \le m_1^1 \le ... \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le ... \le M_1^1 \le M_1^0 = b, \\ \alpha = m_2^0 \le m_2^1 \le ... \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le ... \le M_2^1 \le M_2^0 = \beta, \end{aligned}$$

and

$$\lambda = m_3^0 \le m_3^1 \le ... \le m_3^{i-1} \le m_3^i \le M_3^i \le M_3^{i-1} \le ... \le M_3^1 \le M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$ $(m_3^i)_{i\in\mathbb{N}_0}$(resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ M_1 = \lim_{i\to+\infty} M_1^i,$$

$$m_2 = \lim_{i\to+\infty} m_2^i,\ M_2 = \lim_{i\to+\infty} M_2^i.$$

$$m_3 = \lim_{i\to+\infty} m_3^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \le m_1 \le M_1 \le b,\ \alpha \le m_2 \le M_2 \le \beta,\ \lambda \le m_3 \le M_3 \le \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(M_2^i, m_2^i),\ M_1^{i+1} = f(m_2^i, M_2^i),$$
$$m_2^{i+1} = g(M_3^i, m_3^i),\ M_2^{i+1} = g(m_3^i, M_3^i),$$
$$m_3^{i+1} = h(M_1^i, m_1^i),\ M_3^{i+1} = h(m_1^i, M_1^i)$$

and using the continuity of f, g, and h we obtain

$$\begin{aligned} m_1 &= f(M_2, m_2),\ M_1 = f(m_2, M_2), \\ m_2 &= g(M_3, m_3),\ M_2 = g(m_3, M_3), \\ m_3 &= h(M_1, m_1),\ M_3 = h(m_1, M_1) \end{aligned}$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, for $n = 1, 2, \cdots$, we get

$$m_1^0 = a \le x_n \le b = M_1^0,\ m_2^0 = \alpha \le y_n \le \beta = M_2^0,\ m_3^0 = \lambda \le z_n \le \gamma = M_3^0\,.$$

For $n = 2, 3, ...$, we have

$$m_1^1 = f(M_2^0, m_2^0) \le x_{n+1} = f(y_n, y_{n-1}) \le f(m_2^0, M_2^0) = M_1^1,$$
$$m_2^1 = g(M_3^0, m_3^0) \le y_{n+1} = g(z_n, z_{n-1}) \le g(m_3^0, M_3^0) = M_2^1,$$
$$m_3^1 = h(M_1^0, m_1^0) \le z_{n+1} = h(x_n, x_{n-1}) \le h(m_1^0, M_1^0) = M_3^1,$$

that is

$$m_1^1 \le x_n \le M_1^1,\ m_2^1 \le y_n \le M_2^1,\ m_3^1 \le z_n \le M_3^1,\ n = 3, 4, \cdots.$$

Now, for $n = 4, 5, ...$, we have

$$m_1^2 = f(M_2^1, m_2^1) \le x_{n+1} = f(y_n, y_{n-1}) \le f(m_2^1, M_2^1) = M_1^2,$$
$$m_2^2 = g(M_3^1, m_1^3) \le y_{n+1} = g(z_n, z_{n-1}) \le g(m_3^1, M_1^3) = M_2^2,$$
$$m_3^2 = h(M_1^1, m_1^1) \le z_{n+1} = h(x_n, x_{n-1}) \le h(m_1^1, M_1^1) = M_3^2,$$

that is

$$m_1^2 \le x_n \le M_1^2,\ m_2^2 \le y_n \le M_2^2,\ m_3^2 \le z_n \le M_3^2,\ n = 5, 6, \cdots.$$

Similarly, for $n = 6, 7, ...$, we have

$$m_1^3 = f(M_2^2, m_2^2) \le x_{n+1} = f(y_n, y_{n-1}) \le f(m_2^2, M_2^2) = M_1^3,$$
$$m_2^3 = g(M_3^2, m_3^2) \le y_{n+1} = g(z_n, z_{n-1}) \le g(m_3^2, M_3^2) = M_2^3,$$
$$m_3^3 = h(M_1^2, m_1^2) \le z_{n+1} = h(x_n, x_{n-1}) \le h(m_1^2, M_1^2) = M_3^3,$$

that is

$$m_1^3 \le x_n \le M_1^3,\ m_2^3 \le y_n \le M_2^3,\ m_3^3 \le z_n \le M_3^3,\ n = 7, 8, \cdots.$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \leq x_n \leq M_1^i,\ m_2^i \leq y_n \leq M_2^i,\ m_3^i \leq z_n \leq M_3^i,\ n \geq 2i+1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ \lim_{n\to+\infty} z_n = M_3.$$

From (1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1,1),\ M_2 = g(M_3, M_3) = g(1,1),\ M_3 = h(M_1, M_1) = h(1,1).$$

□

Theorem 2.7. *Consider System (1). Assume that the following statements are true:*

1. *H_1: There exist a, b, α, β, λ, $\gamma \in (0, +\infty)$ such that*
$$a \leq f(u,v) \leq b,\ \alpha \leq g(u,v) \leq \beta,\ \lambda \leq h(u,v) \leq \gamma,\ \forall (u,v) \in (0,+\infty)^2.$$
2. *H_2: $f(u,v)$, $g(u,v)$ are decreasing in u for all v and increasing in v for all u, however $h(u,v)$ is increasing in u for all v and decreasing in v for all u.*
3. *H_3: If $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b]^2 \times [\alpha,\beta]^2 \times [\lambda,\gamma]^2$ is a solution of the system*
$$\begin{aligned} m_1 &= f(M_2, m_2),\ M_1 = f(m_2, M_2), \\ m_2 &= g(M_3, m_3),\ M_2 = g(m_3, M_3), \\ m_3 &= h(m_1, M_1),\ M_3 = h(M_1, m_1) \end{aligned}$$
then
$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

Then every solution of System (1) converges to the unique equilibrium point

$$(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, ...,$

$$\begin{aligned} m_1^{i+1} &:= f(M_2^i, m_2^i),\ M_1^{i+1} := f(m_2^i, M_2^i), \\ m_2^{i+1} &:= g(M_3^i, m_3^i),\ M_2^{i+1} := g(m_3^i, M_3^i), \\ m_3^{i+1} &:= h(m_1^i, M_1^i),\ M_3^{i+1} := h(M_1^i, m_1^i). \end{aligned}$$

We have

$$\begin{aligned} a \leq f(\beta, \alpha) \leq f(\alpha, \beta) \leq b, \\ \alpha \leq g(\gamma, \lambda) \leq g(\lambda, \gamma) \leq \beta, \\ \lambda \leq h(a,b) \leq h(b,a) \leq \gamma, \end{aligned}$$

and so,

$$\begin{aligned} m_1^0 = a \leq f(M_2^0, m_2^0) \leq f(m_2^0, M_2^0) \leq b = M_1^0, \\ m_2^0 = \alpha \leq g(M_3^0, m_3^0) \leq g(m_3^0, M_3^0) \leq \beta = M_2^0, \end{aligned}$$

and

$$m_3^0 = \lambda \leq h(m_1^0, M_1^0) \leq h(M_1^0, m_1^0) \leq \gamma = M_3^0.$$

Hence,

$$m_1^0 \le m_1^1 \le M_1^1 \le M_1^0,$$
$$m_2^0 \le m_2^1 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le M_3^1 \le M_3^0.$$

Now, we have

$$m_1^1 = f(M_2^0, m_2^0) \le f(M_2^1, m_2^1) = m_1^2 \le f(m_2^1, M_2^1) = M_1^2 \le f(m_2^0, M_2^0) = M_1^1,$$
$$m_2^1 = g(M_3^0, m_3^0) \le g(M_3^1, m_3^1) = m_2^2 \le g(m_3^1, M_3^1) = M_2^2 \le g(m_3^0, M_3^0) = M_2^1,$$
$$m_3^1 = h(m_1^0, M_1^0) \le h(m_1^1, M_1^1) = m_3^2 \le h(M_1^1, m_1^1) = M_3^2 \le h(M_1^0, m_1^0) = M_3^1,$$

and it follows that

$$m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0,$$
$$m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0.$$

By induction, we get for $i = 0, 1, ...,$ that

$$a = m_1^0 \le m_1^1 \le ... \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le ... \le M_1^1 \le M_1^0 = b,$$
$$\alpha = m_2^0 \le m_2^1 \le ... \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le ... \le M_2^1 \le M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \le m_3^1 \le ... \le m_3^{i-1} \le m_3^i \le M_3^i \le M_3^{i-1} \le ... \le M_3^1 \le M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$ $(m_3^i)_{i\in\mathbb{N}_0}$(resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ M_1 = \lim_{i\to+\infty} M_1^i,$$
$$m_2 = \lim_{i\to+\infty} m_2^i,\ M_2 = \lim_{i\to+\infty} M_2^i.$$
$$m_3 = \lim_{i\to+\infty} m_3^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \le m_1 \le M_1 \le b,\ \alpha \le m_2 \le M_2 \le \beta,\ \lambda \le m_3 \le M_3 \le \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(M_2^i, m_2^i),\ M_1^{i+1} = f(m_2^i, M_2^i),$$
$$m_2^{i+1} = g(M_3^i, m_3^i),\ M_2^{i+1} = g(m_3^i, M_3^i),$$
$$m_3^{i+1} = h(m_1^i, M_1^i),\ M_3^{i+1} = h(M_1^i, m_1^i),$$

and using the continuity of f, g, and h we obtain

$$\begin{aligned} m_1 &= f(M_2, m_2),\ M_1 = f(m_2, M_2), \\ m_2 &= g(M_3, m_3),\ M_2 = g(m_3, M_3), \\ m_3 &= h(m_1, M_1),\ M_3 = h(M_1, m_1) \end{aligned}$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, for $, n = 1, 2, \ldots,$ we get

$$m_1^0 = a \le x_n \le b = M_1^0,\ m_2^0 = \alpha \le y_n \le \beta = M_2^0,\ m_3^0 = \lambda \le z_n \le \gamma = M_3^0.$$

For $n = 2, 3, ...,$ we have

$$m_1^1 = f(M_2^0, m_2^0) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(m_2^0, M_2^0) = M_1^1,$$

$$m_2^1 = g(M_3^0, m_3^0) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(m_3^0, M_3^0) = M_2^1,$$

and

$$m_3^1 = h(m_1^0, M_1^0) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(M_1^0, m_1^0) = M_3^1,$$

that is

$$m_1^1 \leq x_n \leq M_1^1,\ m_2^1 \leq y_n \leq M_2^1,\ m_3^1 \leq z_n \leq M_3^1,\ n = 3, 4, \cdots .$$

Now, for $n = 4, 5, ...,$ we have

$$m_1^2 = f(M_2^1, m_2^1) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(m_2^1, M_2^1) = M_1^2,$$

$$m_2^2 = g(M_3^1, m_1^3) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(m_3^1, M_1^3) = M_2^2,$$

and

$$m_3^2 = h(m_1^1, M_1^1) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(M_1^1, m_1^1) = M_3^2,$$

that is

$$m_1^2 \leq x_n \leq M_1^2,\ m_2^2 \leq y_n \leq M_2^2,\ m_3^2 \leq z_n \leq M_3^2,\ n = 5, 6, \cdots .$$

Similarly, for $n = 6, 7, ...,$ we have

$$m_1^3 = f(M_2^2, m_2^2) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(m_2^2, M_2^2) = M_1^3,$$

$$m_2^3 = g(M_3^2, m_3^2) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(m_3^2, M_3^2) = M_2^3,$$

and

$$m_3^3 = h(m_1^2, M_1^2) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(M_1^2, m_1^2) = M_3^3,$$

that is

$$m_1^3 \leq x_n \leq M_1^3,\ m_2^3 \leq y_n \leq M_2^3,\ m_3^3 \leq z_n \leq M_3^3,\ n = 7, 8, \cdots .$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \leq x_n \leq M_1^i,\ m_2^i \leq y_n \leq M_2^i,\ m_3^i \leq z_n \leq M_3^i,\ n \geq 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ \lim_{n\to+\infty} z_n = M_3.$$

From (1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1, 1),\ M_2 = g(M_3, M_3) = g(1, 1),\ M_3 = h(M_1, M_1) = h(1, 1).$$

□

Theorem 2.8. *Consider System (1). Assume that the following statements are true:*

1. H_1: *There exist* a, b, α, β, λ, $\gamma \in (0, +\infty)$ *such that*

$$a \leq f(u, v) \leq b,\ \alpha \leq g(u, v) \leq \beta,\ \lambda \leq h(u, v) \leq \gamma,\ \forall (u, v) \in (0, +\infty)^2 .$$

2. H_2: $f(u, v)$ *is decreasing in* u *for all* v *and increasing in* v *for all* u*, however* $g(u, v)$, $h(u, v)$ *are increasing in* u *for all* v *and decreasing in* v *for all* u *.*

3. H_3: If $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b]^2 \times [\alpha,\beta]^2 \times [\lambda,\gamma]^2$ is a solution of the system

$$\begin{aligned} m_1 &= f(M_2, m_2),\ M_1 = f(m_2, M_2), \\ m_2 &= g(m_3, M_3),\ M_2 = g(M_3, m_3), \\ m_3 &= h(m_1, M_1),\ M_3 = h(M_1, m_1) \end{aligned}$$

then

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

Then every solution of System (1) converges to the unique equilibrium point

$$(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, ...,$

$$\begin{aligned} m_1^{i+1} &:= f(M_2^i, m_2^i),\ M_1^{i+1} := f(m_2^i, M_2^i), \\ m_2^{i+1} &:= g(m_3^i, M_3^i),\ M_2^{i+1} := g(M_3^i, m_3^i), \\ m_3^{i+1} &:= h(m_1^i, M_1^i),\ M_3^{i+1} := h(M_1^i, m_1^i). \end{aligned}$$

We have

$$\begin{aligned} a \le f(\beta, \alpha) \le f(\alpha, \beta) \le b, \\ \alpha \le g(\lambda, \gamma) \le g(\gamma, \lambda) \le \beta, \\ \lambda \le h(a, b) \le h(b, a) \le \gamma, \end{aligned}$$

and so,

$$\begin{aligned} m_1^0 = a \le f(M_2^0, m_2^0) \le f(m_2^0, M_2^0) \le b = M_1^0, \\ m_2^0 = \alpha \le g(m_3^0, M_3^0) \le g(M_3^0, m_3^0) \le \beta = M_2^0, \end{aligned}$$

and

$$m_3^0 = \lambda \le h(m_1^0, M_1^0) \le g(M_1^0, m_1^0) \le \gamma = M_3^0.$$

Hence,

$$\begin{aligned} m_1^0 \le m_1^1 \le M_1^1 \le M_1^0, \\ m_2^0 \le m_2^1 \le M_2^1 \le M_2^0, \end{aligned}$$

and

$$m_3^0 \le m_3^1 \le M_3^1 \le M_3^0.$$

Now, we have

$$\begin{aligned} m_1^1 = f(M_2^0, m_2^0) \le f(M_2^1, m_2^1) = m_1^2 \le f(m_2^1, M_2^1) = M_1^2 \le f(m_2^0, M_2^0) = M_1^1, \\ m_2^1 = g(m_3^0, M_3^0) \le g(m_3^1, M_3^1) = m_2^2 \le g(M_3^1, m_3^1) = M_2^2 \le g(M_3^0, m_3^0) = M_2^1, \\ m_3^1 = h(m_1^0, M_1^0) \le h(m_1^1, M_1^1) = m_3^2 \le h(M_1^1, m_1^1) = M_3^2 \le h(M_1^0, m_1^0) = M_3^1, \end{aligned}$$

and it follows that

$$\begin{aligned} m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0, \\ m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0, \end{aligned}$$

and

$$m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0.$$

By induction, we get for $i = 0, 1, ...,$ that

$$\begin{aligned} a = m_1^0 \le m_1^1 \le ... \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le ... \le M_1^1 \le M_1^0 = b, \\ \alpha = m_2^0 \le m_2^1 \le ... \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le ... \le M_2^1 \le M_2^0 = \beta, \end{aligned}$$

and

$$\lambda = m_3^0 \le m_3^1 \le ... \le m_3^{i-1} \le m_3^i \le M_3^i \le M_3^{i-1} \le ... \le M_3^1 \le M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$, $(m_3^i)_{i\in\mathbb{N}_0}$ (resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ M_1 = \lim_{i\to+\infty} M_1^i,$$
$$m_2 = \lim_{i\to+\infty} m_2^i,\ M_2 = \lim_{i\to+\infty} M_2^i.$$
$$m_3 = \lim_{i\to+\infty} m_3^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \le m_1 \le M_1 \le b,\ \alpha \le m_2 \le M_2 \le \beta,\ \lambda \le m_3 \le M_3 \le \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(M_2^i, m_2^i),\ M_1^{i+1} = f(m_2^i, M_2^i),$$
$$m_2^{i+1} = g(m_3^i, M_3^i),\ M_2^{i+1} = g(M_3^i, m_3^i),$$
$$m_3^{i+1} = h(m_1^i, M_1^i),\ M_3^{i+1} = h(M_1^i, m_1^i),$$

and using the continuity of f, g and h we obtain

$$\begin{aligned} m_1 &= f(M_2, m_2),\ M_1 = f(m_2, M_2), \\ m_2 &= g(m_3, M_3),\ M_2 = g(M_3, m_3), \\ m_3 &= h(m_1, M_1),\ M_3 = h(M_1, m_1) \end{aligned}$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, for $n = 1, 2, \ldots$, we get

$$m_1^0 = a \le x_n \le b = M_1^0,\ m_2^0 = \alpha \le y_n \le \beta = M_2^0,\ m_3^0 = \lambda \le z_n \le \gamma = M_3^0\,.$$

For $n = 2, 3, ...$, we have

$$m_1^1 = f(M_2^0, m_2^0) \le x_{n+1} = f(y_n, y_{n-1}) \le f(m_2^0, M_2^0) = M_1^1,$$
$$m_2^1 = g(m_3^0, M_3^0) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^0, m_3^0) = M_3^1,$$

and

$$m_3^1 = h(m_1^0, M_1^0) \le z_{n+1} = h(x_n, x_{n-1}) \le h(M_1^0, m_1^0) = M_3^1,$$

that is

$$m_1^1 \le x_n \le M_1^1,\ m_2^1 \le y_n \le M_2^1,\ m_3^1 \le z_n \le M_3^1,\ n = 3, 4, \cdots.$$

Now, for $n = 4, 5, ...$, we have

$$m_1^2 = f(M_2^1, m_2^1) \le x_{n+1} = f(y_n, y_{n-1}) \le f(m_2^1, M_2^1) = M_1^2,$$
$$m_2^2 = g(m_3^1, M_3^1) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^1, m_3^1) = M_2^2,$$

and

$$m_3^2 = h(m_1^1, M_1^1) \le z_{n+1} = h(x_n, x_{n-1}) \le h(M_1^1, m_1^1) = M_3^2,$$

that is

$$m_1^2 \le x_n \le M_1^2,\ m_2^2 \le y_n \le M_2^2,\ m_3^2 \le z_n \le M_3^2,\ n = 5, 6, \cdots.$$

Similarly, for $n = 6, 7, ...$, we have

$$m_1^3 = f(M_2^2, m_2^2) \le x_{n+1} = f(y_n, y_{n-1}) \le f(m_2^2, M_2^2) = M_1^3,$$
$$m_2^3 = g(m_3^2, M_3^2) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^2, m_3^2) = M_2^3,$$

and

$$m_3^3 = h(m_1^2, M_1^2) \le z_{n+1} = h(x_n, x_{n-1}) \le h(M_1^2, m_1^2) = M_3^3,$$

that is

$$m_1^3 \le x_n \le M_1^3,\ m_2^3 \le y_n \le M_2^3,\ m_3^3 \le z_n \le M_3^3,\ n = 7, 8, \cdots .$$

It follows by induction that for $i = 0, 1, \ldots$ we get

$$m_1^i \le x_n \le M_1^i,\ m_2^i \le y_n \le M_2^i,\ m_3^i \le z_n \le M_3^i,\ n \ge 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ \lim_{n\to+\infty} z_n = M_3.$$

From (1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1,1),\ M_2 = g(M_3, M_3) = g(1,1),\ M_3 = h(M_1, M_1) = h(1,1).$$

□

Theorem 2.9. *Consider System (1). Assume that the following statements are true:*

1. *H_1: There exist a, b, α, β, λ, $\gamma \in (0, +\infty)$ such that*

   $$a \le f(u,v) \le b,\ \alpha \le g(u,v) \le \beta,\ \lambda \le h(u,v) \le \gamma,\ \forall (u,v) \in (0,+\infty)^2.$$

2. *H_2: $f(u,v)$, $h(u,v)$ are decreasing in u for all v and increasing in v for all u, however $g(u,v)$ is increasing in u for all v and decreasing in v for all u .*
3. *H_3: If $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b]^2 \times [\alpha,\beta]^2 \times [\lambda,\gamma]^2$ is a solution of the system*

   $$\begin{aligned} m_1 &= f(M_2, m_2),\ M_1 = f(m_2, M_2),\\ m_2 &= g(m_3, M_3),\ M_2 = g(M_3, m_3),\\ m_3 &= h(M_1, m_1),\ M_3 = h(m_1, M_1) \end{aligned}$$

   then

   $$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

   Then every solution of System (1) converges to the unique equilibrium point

   $$(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, \ldots,$

$$\begin{aligned} m_1^{i+1} &:= f(M_2^i, m_2^i),\ M_1^{i+1} := f(m_2^i, M_2^i),\\ m_2^{i+1} &:= g(m_3^i, M_3^i),\ M_2^{i+1} := g(M_3^i, m_3^i),\\ m_3^{i+1} &:= h(M_1^i, m_1^i),\ M_3^{i+1} := h(m_1^i, M_1^i). \end{aligned}$$

We have

$$\begin{aligned} a &\le f(\beta,\alpha) \le f(\alpha,\beta) \le b,\\ \alpha &\le g(\lambda,\gamma) \le g(\gamma,\lambda) \le \beta,\\ \lambda &\le h(b,a) \le h(a,b) \le \gamma, \end{aligned}$$

and so,

$$m_1^0 = a \le f(M_2^0, m_2^0) \le f(m_2^0, M_2^0) \le b = M_1^0,$$

$$m_2^0 = \alpha \le g(m_3^0, M_3^0) \le g(M_3^0, m_3^0) \le \beta = M_2^0,$$

and

$$m_3^0 = \lambda \le h(M_1^0, m_1^0) \le h(m_1^0, M_1^0) \le \gamma = M_3^0.$$

Hence,

$$m_1^0 \le m_1^1 \le M_1^1 \le M_1^0,$$
$$m_2^0 \le m_2^1 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le M_3^1 \le M_3^0.$$

Now, we have

$$m_1^1 = f(M_2^0, m_2^0) \le f(M_2^1, m_2^1) = m_1^2 \le f(m_2^1, M_2^1) = M_1^2 \le f(m_2^0, M_2^0) = M_1^1,$$
$$m_2^1 = g(m_3^0, M_3^0) \le g(m_3^1, M_3^1) = m_2^2 \le g(M_3^1, m_3^1) = M_2^2 \le g(M_3^0, m_3^0) = M_2^1,$$
$$m_3^1 = h(M_1^0, m_1^0) \le h(M_1^1, m_1^1) = m_3^2 \le h(m_1^1, M_1^1) = M_3^2 \le h(m_1^0, M_1^0) = M_3^1,$$

and it follows that

$$m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0,$$
$$m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0.$$

By induction, we get for $i = 0, 1, ...,$ that

$$a = m_1^0 \le m_1^1 \le ... \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le ... \le M_1^1 \le M_1^0 = b,$$
$$\alpha = m_2^0 \le m_2^1 \le ... \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le ... \le M_2^1 \le M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \le m_3^1 \le ... \le m_3^{i-1} \le m_3^i \le M_3^i \le M_3^{i-1} \le ... \le M_3^1 \le M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$, $(m_3^i)_{i\in\mathbb{N}_0}$ (resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ M_1 = \lim_{i\to+\infty} M_1^i,$$
$$m_2 = \lim_{i\to+\infty} m_2^i,\ M_2 = \lim_{i\to+\infty} M_2^i.$$
$$m_3 = \lim_{i\to+\infty} m_3^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \le m_1 \le M_1 \le b,\ \alpha \le m_2 \le M_2 \le \beta,\ \lambda \le m_3 \le M_3 \le \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(M_2^i, m_2^i),\ M_1^{i+1} = f(m_2^i, M_2^i),$$
$$m_2^{i+1} = g(m_3^i, M_3^i),\ M_2^{i+1} = g(M_3^i, m_3^i),$$
$$m_3^{i+1} = h(M_1^i, m_1^i),\ M_3^{i+1} = h(m_1^i, M_1^i),$$

and using the continuity of f, g and h we obtain

$$m_1 = f(M_2, m_2),\ M_1 = f(m_2, M_2),$$
$$m_2 = g(m_3, M_3),\ M_2 = g(M_3, m_3),$$
$$m_3 = h(M_1, m_1),\ M_3 = h(m_1, M_1)$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, for $n = 1, 2, \ldots$, we get

$$m_1^0 = a \le x_n \le b = M_1^0,\ m_2^0 = \alpha \le y_n \le \beta = M_2^0,\ m_3^0 = \lambda \le z_n \le \gamma = M_3^0\,.$$

For $n = 2, 3, \ldots$, we have

$$m_1^1 = f(M_2^0, m_2^0) \le x_{n+1} = f(y_n, y_{n-1}) \le f(m_2^0, M_2^0) = M_1^1,$$

$$m_2^1 = g(m_3^0, M_3^0) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^0, m_3^0) = M_3^1,$$

and

$$m_3^1 = h(M_1^0, m_1^0) \le z_{n+1} = h(x_n, x_{n-1}) \le h(m_1^0, M_1^0) = M_3^1,$$

that is

$$m_1^1 \le x_n \le M_1^1,\ m_2^1 \le y_n \le M_2^1,\ m_3^1 \le z_n \le M_3^1,\ n = 3, 4, \cdots .$$

Now, for $n = 4, 5, \ldots$, we have

$$m_1^2 = f(M_2^1, m_2^1) \le x_{n+1} = f(y_n, y_{n-1}) \le f(m_2^1, M_2^1) = M_1^2,$$

$$m_2^2 = g(m_3^1, M_3^1) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^1, m_3^1) = M_2^2,$$

and

$$m_3^2 = h(M_1^1, m_1^1) \le z_{n+1} = h(x_n, x_{n-1}) \le h(m_1^1, M_1^1) = M_3^2,$$

that is

$$m_1^2 \le x_n \le M_1^2,\ m_2^2 \le y_n \le M_2^2,\ m_3^2 \le z_n \le M_3^2,\ n = 5, 6, \cdots .$$

Similarly, for $n = 6, 7, \ldots$, we have

$$m_1^3 = f(M_2^2, m_2^2) \le x_{n+1} = f(y_n, y_{n-1}) \le f(m_2^2, M_2^2) = M_1^3,$$

$$m_2^3 = g(m_3^2, M_3^2) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^2, m_3^2) = M_2^3,$$

and

$$m_3^3 = h(M_1^2, m_1^2) \le z_{n+1} = h(x_n, x_{n-1}) \le h(m_1^2, M_1^2) = M_3^3,$$

that is

$$m_1^3 \le x_n \le M_1^3,\ m_2^3 \le y_n \le M_2^3,\ m_3^3 \le z_n \le M_3^3,\ n = 7, 8, \cdots .$$

It follows by induction that for $i = 0, 1, \ldots$ we get

$$m_1^i \le x_n \le M_1^i,\ m_2^i \le y_n \le M_2^i,\ m_3^i \le z_n \le M_3^i,\ n \ge 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ \lim_{n\to+\infty} z_n = M_3.$$

From (1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1, 1),\ M_2 = g(M_3, M_3) = g(1, 1),\ M_3 = h(M_1, M_1) = h(1, 1).$$

□

The following theorem is devoted to global stability of the equilibrium point $(\overline{x}, \overline{y}, \overline{z}) = (f(1, 1), g(1, 1), h(1, 1))$.

Theorem 2.10. *Under the hypotheses of Theorem 2.1 and one of Theorems 2.2–2.9, the equilibrium point* $(\overline{x}, \overline{y}, \overline{z}) = (f(1, 1), g(1, 1), h(1, 1))$ *is globally stable.*

Now as an application of the previous results, we give an example.

Example 1. Consider the following system of difference equations

$$x_{n+1} = f(y_n, y_{n-1}),\ y_{n+1} = g(z_n, z_{n-1}),\ z_{n+1} = h(x_n, x_{n-1}),\ n \in \mathbb{N}_0, \tag{6}$$

where x_{-1}, x_0, y_{-1}, y_0, z_{-1}, $z_0 \in (0, +\infty)$ and

$$f(u,v) = \frac{a_1u + b_1v}{\alpha_1u + \beta_1v},\ g(u,v) = \frac{a_2u + b_2v}{\alpha_2u + \beta_2v},\ h(u,v) = \frac{a_3u + b_3v}{\alpha_3u + \beta_3v}.$$

Assume that $r_i := a_i\beta_i - \alpha_i b_i$, $i = 1, 2, 3$ are positive. For all $(u, v) \in (0, +\infty)$, we have

$$\frac{\partial f}{\partial u}(u,v) = \frac{r_1v}{(\alpha_1u + \beta_1v)^2},\ \frac{\partial f}{\partial v}(u,v) = -\frac{r_1u}{(\alpha_1u + \beta_1v)^2},\ a := \frac{b_1}{\beta_1} \le f(u,v) \le b := \frac{a_1}{\alpha_1}$$

$$\frac{\partial g}{\partial u}(u,v) = \frac{r_2v}{(\alpha_2u + \beta_2v)^2},\ \frac{\partial g}{\partial v}(u,v) = -\frac{r_2u}{(\alpha_2u + \beta_2v)^2},\ \alpha := \frac{b_2}{\beta_2} \le g(u,v) \le \beta := \frac{a_2}{\alpha_2}$$

$$\frac{\partial h}{\partial u}(u,v) = \frac{r_3v}{(\alpha_3u + \beta_3v)^2},\ \frac{\partial h}{\partial v}(u,v) = -\frac{r_3u}{(\alpha_3u + \beta_3v)^2},\ \lambda := \frac{b_3}{\beta_3} \le h(u,v) \le \gamma := \frac{a_3}{\alpha_3}$$

It follows from Theorem 2.1 that the unique equilibrium point

$$(f(1,1), g(1,1), h(1,1)) = \left(\frac{a_1 + b_1}{\alpha_1 + \beta_1}, \frac{a_2 + b_2}{\alpha_2 + \beta_2}, \frac{a_3 + b_3}{\alpha_3 + \beta_3}\right)$$

of System (6) will be locally stable if

$$\frac{\partial f}{\partial u}(1,1)\frac{\partial g}{\partial u}(1,1)\frac{\partial h}{\partial u}(1,1) < \frac{f(1,1)g(1,1)h(1,1)}{8}$$

which is equivalent to

$$\frac{r_1r_2r_3}{(\alpha_1 + \beta_1)^2(\alpha_2 + \beta_2)^2(\alpha_3 + \beta_3)^2} < \frac{(a_1 + b_1)(a_2 + b_2)(a_3 + b_3)}{8(\alpha_1 + \beta_1)(\alpha_2 + \beta_2)(\alpha_3 + \beta_3)}.$$

Also, we have conditions H_1 and H_2 of Theorem 2.2 are satisfied. So, to prove the global stability of the equilibrium point $(f(1,1), g(1,1), h(1,1))$ it suffices to check condition H_3 of Theorem 2.2.

For this purpose, let $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b] \times [\alpha, \beta] \times [\gamma, \lambda]$ such that

$$m_1 = f(m_2, M_2) = \frac{a_1m_2 + b_1M_2}{\alpha_1m_2 + \beta_1M_2},\ M_1 = f(M_2, m_2) = \frac{a_1M_2 + b_1m_2}{\alpha_1M_2 + \beta_1m_2}, \tag{7}$$

$$m_2 = g(m_3, M_3) = \frac{a_2m_3 + b_2M_3}{\alpha_2m_3 + \beta_2M_3},\ M_2 = g(M_3, m_3) = \frac{a_2M_3 + b_2m_3}{\alpha_2M_3 + \beta_2m_3}, \tag{8}$$

$$m_3 = h(m_1, M_1) = \frac{a_3m_1 + b_3M_1}{\alpha_3m_1 + \beta_3M_1},\ M_3 = h(M_1, m_1) = \frac{a_3M_1 + b_3m_1}{\alpha_3M_1 + \beta_3m_1}. \tag{9}$$

From (7)-(9), we get

$$m_1 - M_1 = \frac{r_1(m_2 - M_2)(m_2 + M_2)}{(\alpha_1m_2 + \beta_1M_2)(\alpha_1M_2 + \beta_1m_2)}, \tag{10}$$

$$m_2 - M_2 = \frac{r_2(m_3 - M_3)(m_3 + M_3)}{(\alpha_2m_3 + \beta_2M_3)(\alpha_2M_3 + \beta_2m_3)}, \tag{11}$$

$$m_3 - M_3 = \frac{r_3(m_1 - M_1)(m_1 + M_1)}{(\alpha_3m_1 + \beta_3M_1)(\alpha_3M_1 + \beta_3m_1)}. \tag{12}$$

Now, from (10)-(12), we get

$$(m_1 - M_1)(m_2 - M_2)(m_3 - M_3) = (m_1 - M_1)(m_2 - M_2)(m_3 - M_3)\Theta(m_1, M_1, m_2, M_2, m_3, M_3), \tag{13}$$

where $\Theta(m_1, M_1, m_2, M_2, m_3, M_3)$ is equal to

$$\frac{r_1 r_2 r_3 (m_1 + M_1)(m_2 + M_2)(m_3 + M_3)}{(\alpha_3 m_1 + \beta_3 M_1)(\alpha_3 M_1 + \beta_3 m_1)(\alpha_1 m_2 + \beta_1 M_2)(\alpha_1 M_2 + \beta_1 m_2)(\alpha_2 m_3 + \beta_2 M_3)(\alpha_2 M_3 + \beta_2 m_3)}.$$

It is not hard to see that

$$8\prod_{i=1}^{3} \frac{\alpha_i^2 b_i r_i}{a_i^2 \beta_i (\alpha_i + \beta_i)^2} \leq \Theta(m_1, M_1, m_2, M_2, m_3, M_3) \leq 8\prod_{i=1}^{3} \frac{a_i \beta_i^2 r_i}{\alpha_i b_i^2 (\alpha_i + \beta_i)^2}.$$

Using the fact

$$r_i \leq a_i \beta_i,\ (\alpha_i + \beta_i)^2 \geq 2\alpha_i \beta_i,\ \alpha_i b_i < a_i \beta_i,\ i = 1, 2, 3,$$

we get

$$8\prod_{i=1}^{3} \frac{\alpha_i^2 b_i r_i}{a_i^2 \beta_i (\alpha_i + \beta_i)^2} \leq \prod_{i=1}^{3} \frac{\alpha_i b_i}{a_i \beta_i} < 1.$$

If we choose the parameters a_i, b_i, α_i, β_i, $i = 1, 2, 3$ such that

$$\prod_{i=1}^{3} \frac{a_i \beta_i^2 r_i}{\alpha_i b_i^2 (\alpha_i + \beta_i)^2} < \frac{1}{8},$$

then we get that

$$\Theta(m_1, M_1, m_2, M_2, m_3, M_3) \neq 1$$

and so it follows from (13) that

$$(m_1 - M_1)(m_2 - M_2)(m_3 - M_3) = 0.$$

Using this and (10)-(12), we obtain $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$. That is the condition H_3 is satisfied.

In summary we have the following result.

Theorem 2.11. *Assume that that parameters a_i, b_i, α_i, β_i, $i = 1, 2, 3$ are such that*

- *C_1: $a_i \beta_i > \alpha_i b_i$, $i = 1, 2, 3$.*
- *C_2:*
$$\prod_{i=1}^{3} \frac{a_i \beta_i - \alpha_i b_i}{(\alpha_i + \beta_i)^2} < \frac{1}{8} \prod_{i=1}^{3} \frac{a_i + b_i}{\alpha_i + \beta_i}.$$
- *C_3:*
$$\prod_{i=1}^{3} \frac{a_i \beta_i^2 r_i}{\alpha_i b_i^2 (\alpha_i + \beta_i)^2} < \frac{1}{8}.$$

Then the equilibrium point $(f(1,1), g(1,1), h(1,1)) = \left(\frac{a_1+b_1}{\alpha_1+\beta_1}, \frac{a_2+b_2}{\alpha_2+\beta_2}, \frac{a_3+b_3}{\alpha_3+\beta_3}\right)$ is globally stable.

We visualize the solutions of System (6) in Figures 1-3. In Figure 1 and Figure 2, we give the solution and corresponding global attractor of System (6) for $x_{-1} = 2.13, x_0 = 3.1, y_{-1} = 4.03, y_0 = 2.21, z_{-1} = 2.76, z_0 = 3.12$ and $a_1 = 3, \alpha_1 = 3.2, b_1 = 4, \beta_1 = 5, a_2 = 1.2, \alpha_2 = 2, b_2 = 3.2, \beta_2 = 6, a_3 = 2.4, \alpha_3 = 2.3, b_3 = 1.3, \beta_3 = 1.3$, respectively. Note that the conditions C_1, C_2, C_3 is satisfied for these values.

However Figure 3 shows the unstable solution corresponding to the values $x_{-1} = 2.13, x_0 = 3.1, y_{-1} = 4.03, y_0 = 2.21, z_{-1} = 2.76, z_0 = 3.12$ and $a_1 = 0.3, \alpha_1 = 4, b_1 = 3, \beta_1 = 1.1, a_2 = 1.2, \alpha_2 = 4, b_2 = 3.7, \beta_2 = 1, a_3 = 2.7, \alpha_3 = 0.2, b_3 = 1.3, \beta_3 = 3$, which do not satisfy the conditions C_1, C_2, C_3.

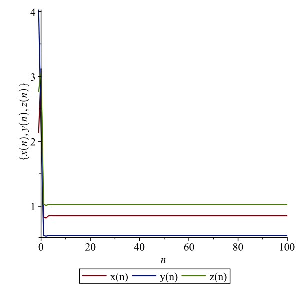

FIGURE 1.

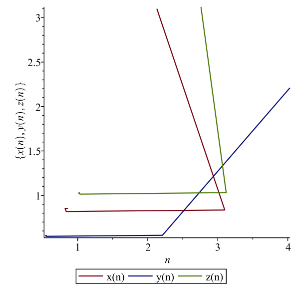

FIGURE 2.

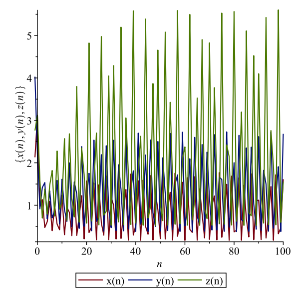

FIGURE 3.

3. **Existence of periodic solutions.** Here, we are interested in existence of periodic solutions for System (1). In the following result we will established a necessary and sufficient condition for which there exist prime period two solutions of System (1).

Theorem 3.1. *Assume that α, β and γ are positive real numbers such that $(\alpha-1)(\beta-1)(\gamma-1)\neq 0$. Then, System (1) has a prime period two solution in the form of*

$$..., (\alpha p, \beta q, \gamma r),\ (p, q, r), (\alpha p, \beta q, \gamma r),\ (p, q, r), ...$$

if and only if

$$f(1, \beta) = \alpha f(\beta, 1),\ g(1, \gamma) = \beta g(\gamma, 1),\ h(1, \alpha) = \gamma h(\alpha, 1).$$

Proof. Let α, β, γ be positive real numbers such that $(\alpha-1)(\beta-1)(\gamma-1)\neq 0$ and assume that

$$..., (\alpha p, \beta q, \gamma r),\ (p, q, r), (\alpha p, \beta q, \gamma r),\ (p, q, r), ...$$

is a solution for System (1). Then, we have

$$\alpha p = f(q, \beta q) = f(1, \beta) \tag{14}$$

$$p = f(\beta q, q) = f(\beta, 1) \tag{15}$$

$$\beta q = g(r, \gamma r) = g(1, \gamma) \tag{16}$$

$$q = g(\gamma r, r) = g(\gamma, 1) \tag{17}$$

$$\gamma r = h(p, \alpha p) = h(1, \alpha) \tag{18}$$

$$r = h(\alpha p, p) = h(\alpha, 1). \tag{19}$$

From (14)-(19), it follows that

$$f(1, \beta) = \alpha f(\beta, 1),\ g(1, \gamma) = \beta g(\gamma, 1),\ h(1, \alpha) = \gamma h(\alpha, 1).$$

Now, assume that

$$f(1,\beta)=\alpha f(\beta,1),\ g(1,\gamma)=\beta g(\gamma,1),\ h(1,\alpha)=\gamma h(\alpha,1).$$

and let

$$x_0=f(\beta,1),\ x_{-1}=f(1,\beta),\ y_0=g(\gamma,1),\ y_{-1}=g(1,\gamma),\ z_0=h(\alpha,1),\ z_{-1}=h(1,\alpha).$$

We have

$$x_1=f(y_0,y_{-1})=f(g(\gamma,1),g(1,\gamma))=f(g(\gamma,1),\beta g(\gamma,1))=f(1,\beta)=x_{-1},$$

$$y_1=g(z_0,z_{-1})=g(h(\alpha,1),h(1,\alpha))=g(h(\alpha,1),\gamma h(\alpha,1))=g(1,\gamma)=y_{-1},$$

$$z_1=h(x_0,x_{-1})=h(f(\beta,1),f(1,\beta))=h(f(\beta,1),\alpha f(\beta,1))=h(1,\alpha)=z_{-1},$$

$$x_2=f(y_1,y_0)=f(g(1,\gamma),g(\gamma,1))=f(\beta g(\gamma,1),g(\gamma,1))=f(\beta,1)=x_0,$$

$$y_2=g(z_1,z_0)=g(h(1,\alpha),h(\alpha,1))=g(\gamma h(\alpha,1),h(\alpha,1))=g(1,\gamma)=y_0,$$

$$z_1=h(x_1,x_0)=h(f(1,\beta),f(\beta,1))=h(\alpha f(\beta,1),f(\beta,1))=h(1,\alpha)=z_0.$$

By induction we get

$$x_{2n-1}=x_{-1},\ x_{2n}=x_0,\ y_{2n-1}=y_{-1},\ y_{2n}=y_0,\ z_{2n-1}=z_{-1},\ z_{2n}=z_0,\ n\in\mathbb{N}_0.$$

□

In the following, we apply our result in finding prime period two solutions of two special Systems.

Consider the three dimensional system of difference equations

$$\begin{cases} x_{n+1}=a_1+b_1\frac{y_n}{y_{n-1}}+c_1\frac{y_{n-1}}{y_n},\\ y_{n+1}=a_2+b_2\frac{z_n}{z_{n-1}}+c_2\frac{z_{n-1}}{z_n},\\ z_{n+1}=a_3+b_3\frac{x_n}{x_{n-1}}+c_3\frac{x_{n-1}}{x_n}\end{cases} \tag{20}$$

where $n\in\mathbb{N}_0$, the initial values $x_{-i},\ y_{-i},\ z_{-i},\ i=0,1$ and the $a_i,b_i,\ c_i,\ i=1,2,3$ are positive real numbers.

System (20) can be seen as a generalization of the system

$$x_{n+1}=a_1+b_1\frac{y_n}{y_{n-1}}+c_1\frac{y_{n-1}}{y_n},\ y_{n+1}=a_2+b_2\frac{x_n}{x_{n-1}}+c_2\frac{x_{n-1}}{x_n},$$

studied in [23]. This last one is also a generalization of the equation

$$x_{n+1}=a_1+b_1\frac{x_n}{x_{n-1}}+c_1\frac{x_{n-1}}{x_n}$$

studied in [7] and [15].

Corollary 1. *Assume that* $(\alpha-1)(\beta-1)(\gamma-1)\neq 0$*, then System* (20) *has prime period two solution of the form*

$$...,(\alpha f(\beta,1),\beta g(\gamma,1),\gamma h(\alpha,1)),\ (f(\beta,1),g(\gamma,1),h(\alpha,1)),...$$

if and only if

$$\begin{cases}(b_1\alpha-c_1)\beta^2+a_1\beta(\alpha-1)+c_1\alpha-b_1=0,\\ (b_2\beta-c_2)\gamma^2+a_2\gamma(\beta-1)+c_2\beta-b_2=0,\\ (b_3\gamma-c_3)\alpha^2+a_3\alpha(\gamma-1)+c_3\gamma-b_3=0.\end{cases} \tag{21}$$

Proof. System (20) can be written as

$$x_{n+1} = f(y_n, y_{n-1}),\ y_{n+1} = g(z_n, z_{n-1}),\ z_{n+1} = h(x_n, x_{n-1}),$$

where

$$f(u,v) = a_1 + b_1\frac{u}{v} + c_1\frac{v}{u},\ g(u,v) = a_2 + b_2\frac{u}{v} + c_2\frac{v}{u},\ h(u,v) = a_3 + b_3\frac{u}{v} + c_3\frac{v}{u}.$$

So, from Theorem 3.1,

$$\ldots, (\alpha f(\beta,1), \beta g(\gamma,1), \gamma h(\alpha,1)),\ (f(\beta,1), g(\gamma,1), h(\alpha,1)), \ldots$$

will be a period prime two solution of System (20) if and only if

$$f(1,\beta) = \alpha f(\beta,1),\ g(1,\gamma) = \beta g(\gamma,1),\ h(1,\alpha) = \gamma h(\alpha,1).$$

Clearly this condition is equivalent to (21). □

Example 2. If we choose $\alpha = 2$, $\beta = 3$, $\gamma = \frac{1}{2}$ then, condition (21) will be

$$3a_1 + 17b_1 - 7c_1 = 0,\ 4a_2 - b_2 + 11c_2 = 0,\ -2a_3 + 2b_3 - 7c_3 = 0.$$

The last condition is satisfied for the choice

$$a_1 = \frac{4}{3},\ b_1 = 1,\ c_1 = 3,\ a_2 = \frac{1}{4},\ b_2 = 2,\ c_2 = \frac{1}{11},\ a_3 = \frac{1}{2},\ b_3 = 4,\ c_3 = 1$$

of the parameters. The corresponding prime period two solution will be

$$x_{2n-1} = x_{-1} = \alpha f(\beta,1) = \frac{32}{3},$$

$$y_{2n-1} = y_{-1} = \beta g(\gamma,1) = \frac{189}{44},$$

$$z_{2n-1} = z_{-1} = \gamma h(\alpha,1) = \frac{9}{2},$$

and

$$x_{2n} = x_0 = f(\beta,1) = \frac{16}{3},$$

$$y_{2n} = y_0 = g(\gamma,1) = \frac{63}{44},$$

$$z_{2n} = z_0 = h(\alpha,1) = 9,$$

that is

$$\left\{\left(\frac{32}{3}, \frac{189}{44}, \frac{9}{2}\right), \left(\frac{16}{3}, \frac{63}{44}, 9\right), \left(\frac{32}{3}, \frac{189}{44}, \frac{9}{2}\right), \left(\frac{16}{3}, \frac{63}{44}, 9\right), \cdots\right\}.$$

Now, consider the following system of difference equations

$$\begin{cases} x_{n+1} = a_1 + b_1\frac{y_n}{y_{n-1}} + c_1\left(\frac{y_{n-1}}{y_n}\right)^2, \\ y_{n+1} = a_2 + b_2\frac{z_{n-1}}{z_n} + c_2\left(\frac{z_{n-1}}{z_n}\right)^2, \quad n \in \mathbb{N}_0 \\ z_{n+1} = a_3 + b_3\frac{x_{n-1}}{x_n} + 3_2\left(\frac{x_{n-1}}{x_n}\right)^2, \end{cases} \tag{22}$$

where the initial values x_{-i}, y_{-i}, z_{-i}, $i = 0, 1$ and a_i, b_i, c_i, $i = 1, 2, 3$ are positive real numbers.

Corollary 2. *Assume that* $(\alpha-1)(\beta-1)(\gamma-1)\neq 0$, *then System* (22) *has prime period two solution of the form*

$$..., (\alpha f(\beta,1), \beta g(\gamma,1), \gamma h(\alpha,1)),\ (f(\beta,1), g(\gamma,1), h(\alpha,1)), ...$$

if and only if

$$\begin{cases} a_1\beta^2(\alpha-1)+b_1\beta(\alpha\beta^2-1)+c_1(\alpha-\beta^4)=0, \\ a_2\gamma^2(\beta-1)+b_2\gamma(\beta\gamma^2-1)+c_2(\beta-\gamma^4)=0, \\ a_3\alpha^2(\gamma-1)+b_3\alpha(\gamma\alpha^2-1)+c_3(\gamma-\alpha^4)=0. \end{cases} \quad (23)$$

Proof. System (22) can be written as

$$x_{n+1}=f(y_n,y_{n-1}),\ y_{n+1}=g(z_n,z_{n-1}),\ z_{n+1}=h(x_n,x_{n-1}),$$

where

$$\begin{cases} f(u,v)=a_1+b_1\frac{u}{v}+c_1\left(\frac{v}{u}\right)^2, \\ g(u,v)=a_2+b_2\frac{v}{u}+c_2\left(\frac{v}{u}\right)^2, \\ h(u,v)=a_3+b_3\frac{v}{u}+c_3\left(\frac{v}{u}\right)^2. \end{cases}$$

So, from Theorem 3.1,

$$..., (\alpha f(\beta,1), \beta g(\gamma,1), \gamma h(\alpha,1)),\ (f(\beta,1), g(\gamma,1), h(\alpha,1)), ...$$

will be a period prime two solution of System (20) if and only if

$$f(1,\beta)=\alpha f(\beta,1),\ g(1,\gamma)=\beta g(\gamma,1),\ h(1,\alpha)=\gamma h(\alpha,1).$$

Clearly this condition is equivalent to

$$\begin{cases} a_1\beta^2(\alpha-1)+b_1\beta(\alpha\beta^2-1)+c_1(\alpha-\beta^4)=0, \\ a_2\gamma^2(\beta-1)+b_2\gamma(\beta\gamma^2-1)+c_2(\beta-\gamma^4)=0, \\ a_3\alpha^2(\gamma-1)+b_3\alpha(\gamma\alpha^2-1)+c_3(\gamma-\alpha^4)=0. \end{cases}$$

□

Example 3. For $\alpha=3$, $\beta=2$, $\gamma=\frac{1}{3}$ then, condition (23) will be

$$8a_1+22b_1-13c_1=0,\ 9a_2-21b_2+161c_2=0,\ 18a_3-18b_3+242c_3=0.$$

The last condition is satisfied for the choice

$$a_1=1,\ b_1=1,\ c_1=\frac{30}{13},\ a_2=\frac{19}{9},\ b_2=2,\ c_2=\frac{1}{7},\ a_3=\frac{1}{6},\ b_3=\frac{7}{9},\ c_3=\frac{1}{22}$$

of the parameters. The corresponding prime period two solution will be

$$x_{2n-1}=x_{-1}=3f(2,1)=\frac{297}{26},$$
$$y_{2n-1}=y_{-1}=2g(\frac{1}{3},1)=\frac{512}{63},$$
$$z_{2n-1}=z_{-1}=\frac{1}{3}h(3,1)=\frac{248}{297},$$

and

$$x_{2n}=x_0=f(2,1)=\frac{93}{26},$$
$$y_{2n}=y_0=g(\frac{1}{3},1)=\frac{256}{63},$$
$$z_{2n}=z_0=h(3,1)=\frac{744}{297},$$

that is

$$\left\{\left(\frac{297}{26},\frac{512}{63},\frac{248}{297}\right),\left(\frac{93}{26},\frac{256}{63},\frac{744}{297}\right),\left(\frac{297}{26},\frac{512}{63},\frac{248}{297}\right),\cdots\right\}.$$

4. **Existence of oscillatory solutions.** Here, we are interested in the oscillation of the solutions of System (1) about the equilibrium point $(\overline{x},\overline{y},\overline{z}) = (f(1,1), g(1,1), h(1,1))$.

Theorem 4.1. *Let $(x_n, y_n, z_n)_{n\geq -1}$ be a solution of System (1) and assume that $f(x,y)$, $g(x,y)$ $h(x,y)$ are decreasing in x for all y and are increasing in y for all x.*

1. *If*
$$x_0 < \overline{x},\ x_{-1} > \overline{x},\ y_0 < \overline{y},\ y_{-1} > \overline{y},\ z_0 < \overline{z},\ z_{-1} > \overline{z},$$
then we get
$$x_{2n} < \overline{x},\ x_{2n-1} > \overline{x},\ y_{2n} < \overline{y},\ y_{2n-1} > \overline{y},\ z_{2n} < \overline{z},\ z_{2n-1} > \overline{z},\ n \in \mathbb{N}_0.$$
That is for the sequences $(x_n)_{n\geq -1}$, $(y_n)_{n\geq -1}$ and $(z_n)_{n\geq -1}$ we have semi-cycles of length one of the form
$$+-+-+-\cdots.$$
2. *If*
$$x_0 > \overline{x},\ x_{-1} < \overline{x},\ y_0 > \overline{y},\ y_{-1} < \overline{y},\ z_0 > \overline{z},\ z_{-1} < \overline{z},$$
then we get
$$x_{2n} > \overline{x},\ x_{2n-1} < \overline{x},\ y_{2n} > \overline{y},\ y_{2n-1} < \overline{y},\ z_{2n} > \overline{z},\ z_{2n-1} < \overline{z},\ n \in \mathbb{N}_0.$$
That is for the sequences $(x_n)_{n\geq -1}$, $(y_n)_{n\geq -1}$ and $(z_n)_{n\geq -1}$ we have semi-cycles of length one of the form
$$-+-+-+\cdots.$$

Proof. 1. Assume that
$$x_0 < \overline{x},\ x_{-1} > \overline{x},\ y_0 < \overline{y},\ y_{-1} > \overline{y},\ z_0 < \overline{z},\ z_{-1} > \overline{z}.$$
We have
$$\begin{aligned}
x_1 &= f(y_0, y_{-1}) > f(\overline{y}, y_{-1}) > f(\overline{y},\overline{y}) = f(1,1) = \overline{x},\\
y_1 &= g(z_0, z_{-1}) > g(\overline{z}, z_{-1}) > g(\overline{z},\overline{z}) = g(1,1) = \overline{y},\\
z_1 &= h(x_0, x_{-1}) > h(\overline{x}, x_{-1}) > h(\overline{x},\overline{x}) = h(1,1) = \overline{z},\\
x_2 &= f(y_1, y_0) < f(\overline{y}, y_0) < f(\overline{y},\overline{y}) = f(1,1) = \overline{x},\\
y_2 &= g(z_1, z_0) < g(\overline{z}, z_0) < g(\overline{z},\overline{z}) = g(1,1) = \overline{y},\\
z_2 &= h(x_1, x_0) < h(\overline{x}, x_0) < h(\overline{x},\overline{x}) = h(1,1) = \overline{z}.
\end{aligned}$$
By induction, we get
$$x_{2n} < \overline{x},\ x_{2n-1} > \overline{x},\ y_{2n} < \overline{y},\ y_{2n-1} > \overline{y},\ z_{2n} < \overline{z},\ z_{2n-1} > \overline{z},\ n \in \mathbb{N}_0.$$

2. Assume that
$$x_0 > \overline{x},\ x_{-1} < \overline{x},\ y_0 > \overline{y},\ y_{-1} < \overline{y},\ z_0 > \overline{z},\ z_{-1} < \overline{z}.$$
We have
$$\begin{aligned}
x_1 &= f(y_0, y_{-1}) < f(\overline{y}, y_{-1}) < f(\overline{y},\overline{y}) = f(1,1) = \overline{x},\\
y_1 &= g(z_0, z_{-1}) < g(\overline{z}, z_{-1}) < g(\overline{z},\overline{z}) = g(1,1) = \overline{y},\\
z_1 &= h(x_0, x_{-1}) < h(\overline{x}, x_{-1}) < h(\overline{x},\overline{x}) = h(1,1) = \overline{z},\\
x_2 &= f(y_1, y_0) > f(\overline{y}, y_0) > f(\overline{y},\overline{y}) = f(1,1) = \overline{x},\\
y_2 &= g(z_1, z_0) > g(\overline{z}, z_0) > g(\overline{z},\overline{z}) = g(1,1) = \overline{y},\\
x_2 &= h(x_1, x_0) > h(\overline{x}, x_0) > h(\overline{x},\overline{x}) = h(1,1) = \overline{y}.
\end{aligned}$$

By induction, we get

$$x_{2n} > \overline{x},\ x_{2n-1} < \overline{x},\ y_{2n} > \overline{y},\ y_{2n-1} < \overline{y},\ z_{2n} > \overline{z},\ z_{2n-1} < \overline{z},\ n \in \mathbb{N}_0.$$

So, the proof is completed. □

Now, we will apply the results of this section on the following particular system.

Example 4. Consider the system of difference equations

$$x_{n+1} = a_1 + b_1 \left(\frac{y_{n-1}}{y_n}\right)^p,\ y_{n+1} = a_2 + b_2 \left(\frac{z_{n-1}}{z_n}\right)^q,\ z_{n+1} = a_3 + b_3 \left(\frac{x_{n-1}}{x_n}\right)^r,\ n \in \mathbb{N}_0, \tag{24}$$

where $p,\ q,\ r \in \mathbb{N}$, the initial values $x_{-i},\ y_{-i},\ z_{-i},\ i = 0, 1$ and the parameters $a_i,\ b_i,\ i = 1, 2, 3$ are positive real numbers.

Let f, g and h be the functions defined by

$$f(u,v) = a_1 + b_1 \left(\frac{v}{u}\right)^p,\ g(u,v) = a_2 + b_2 \left(\frac{v}{u}\right)^q,\ h(u,v) = a_3 + b_3 \left(\frac{v}{u}\right)^r,\ u, v \in (0, +\infty).$$

It is not hard to see that

$$\frac{\partial f}{\partial u}(u,v) < 0,\ \frac{\partial f}{\partial v}(u,v) > 0,\ \frac{\partial g}{\partial u}(u,v) < 0,\ \frac{\partial g}{\partial v}(u,v) > 0\ \frac{\partial h}{\partial u}(u,v) < 0,\ \frac{\partial h}{\partial v}(u,v) > 0.$$

System (24) has the unique equilibrium point $(\overline{x}, \overline{y}, \overline{z}) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

Corollary 3. *Let $(x_n, y_n, z_n)_{n \geq -1}$ be a solution of System (24). The following statements holds true:*

1. *Let*

$$x_0 < \overline{x},\ x_{-1} > \overline{x},\ y_0 < \overline{y},\ y_{-1} > \overline{y},\ z_0 < \overline{z},\ z_{-1} > \overline{z}.$$

Then the sequences $(x_n)_n$ (resp. $(y_n)_n$, $(z_n)_n$) oscillates about $\overline{x}$ (resp. about $\overline{y}$, $\overline{z}$) with semi-cycle of length one and every semi-cycle is in the form

$$+ - + - + - \cdots.$$

2. *Let*

$$x_0 > \overline{x},\ x_{-1} < \overline{x},\ y_0 > \overline{y},\ y_{-1} < \overline{y},\ z_0 > \overline{z},\ z_{-1} < \overline{z}.$$

Then the sequences $(x_n)_n$ (resp. $(y_n)_n$, $(z_n)_n$) oscillates about $\overline{x}$ (resp. about $\overline{y}$, $\overline{z}$) with semi-cycle of length one and every semi-cycle is in the form

$$- + - + - + \cdots.$$

Proof. 1. Let

$$x_0 < \overline{x},\ x_{-1} > \overline{x},\ y_0 < \overline{y},\ y_{-1} > \overline{y},\ z_0 < \overline{z},\ z_{-1} > \overline{z}.$$

We have

$$\frac{y_{-1}}{y_0} > \frac{\overline{y}}{\overline{y}} = 1,$$

which implies that

$$x_1 = a_1 + b_1 \left(\frac{y_{-1}}{y_0}\right)^p > a_1 + b_1 = \overline{x}.$$

Using the fact that

$$\frac{z_{-1}}{z_0} > \frac{\overline{z}}{\overline{z}} = 1,$$

we get

$$y_1 = a_2 + b_2 \left(\frac{z_{-1}}{z_0}\right)^q > a_2 + b_2 = \overline{y}.$$

Also as,

$$\frac{x_{-1}}{x_0} > \frac{\overline{x}}{\overline{x}} = 1,$$

we get

$$z_1 = a_3 + b_3 \left(\frac{x_{-1}}{x_0}\right)^r > a_3 + b_3 = \overline{z}.$$

Now, as

$$\frac{y_0}{y_1} < \frac{\overline{y}}{\overline{y}} = 1,$$

we obtain

$$x_2 = a_1 + b_1 \left(\frac{y_0}{y_1}\right)^p < a_1 + b_1 = \overline{x}.$$

Similarly,

$$\frac{z_0}{z_1} < \frac{\overline{z}}{\overline{z}} = 1 \Rightarrow y_2 = a_2 + b_2 \left(\frac{z_0}{z_1}\right)^q < a_2 + b_2 = \overline{y},$$

and

$$\frac{x_0}{x_1} < \frac{\overline{x}}{\overline{x}} = 1 \Rightarrow z_2 = a_3 + b_3 \left(\frac{x_0}{x_1}\right)^r < a_3 + b_3 = \overline{z},$$

and by induction we get that

$$x_{2n} - \overline{x} < 0,\ y_{2n} - \overline{y} < 0,\ z_{2n} - \overline{z} < 0,$$

$$x_{2n-1} - \overline{x} > 0,\ y_{2n-1} - \overline{y} > 0,\ z_{2n-1} - \overline{z} > 0,$$

for $n \in \mathbb{N}_0$. That is, the sequences $(x_n)_n$ (resp. $(y_n)_n$, $(z_n)_n$) oscillates about $\overline{x}$ (resp. about $\overline{y}$, $\overline{z}$) with semi-cycle of length one and every semi-cycle is in the form

$$+ - + - + - \cdots$$

and this is the statement of Part 1. of Theorem 4.1.

2. Let

$$x_0 > \overline{x},\ x_{-1} < \overline{x},\ y_0 > \overline{y},\ y_{-1} < \overline{y},\ z_0 > \overline{z},\ z_{-1} < \overline{z}.$$

We have

$$\frac{y_{-1}}{y_0} < \frac{\overline{y}}{\overline{y}} = 1,$$

which implies that

$$x_1 = a_1 + b_1 \left(\frac{y_{-1}}{y_0}\right)^p < a_1 + b_1 = \overline{x}.$$

Using the fact that

$$\frac{z_{-1}}{z_0} < \frac{\overline{z}}{\overline{z}} = 1,$$

we get

$$y_1 = a_2 + b_2 \left(\frac{z_{-1}}{z_0}\right)^q < a_2 + b_2 = \overline{y}.$$

Also as,

$$\frac{x_{-1}}{x_0} < \frac{\overline{x}}{\overline{x}} = 1,$$

we get

$$z_1 = a_3 + b_3 \left(\frac{x_{-1}}{x_0}\right)^r < a_3 + b_3 = \overline{z}.$$

Now, as

$$\frac{y_0}{y_1} > \frac{\overline{y}}{\overline{y}} = 1,$$

we obtain

$$x_2 = a_1 + b_1 \left(\frac{y_0}{y_1}\right)^p > a_1 + b_1 = \overline{x}.$$

Similarly,

$$\frac{z_0}{z_1} > \frac{\overline{z}}{\overline{z}} = 1 \Rightarrow y_2 = a_2 + b_2 \left(\frac{z_0}{z_1}\right)^q > a_2 + b_2 = \overline{y},$$

and

$$\frac{x_0}{x_1} > \frac{\overline{x}}{\overline{x}} = 1 \Rightarrow z_2 = a_3 + b_3 \left(\frac{x_0}{x_1}\right)^r > a_3 + b_3 = \overline{z}.$$

Thus, by induction we get that

$$x_{2n} - \overline{x} > 0,\ y_{2n} - \overline{y} > 0,\ z_{2n} - \overline{z} > 0,$$

$$x_{2n-1} - \overline{x} < 0,\ y_{2n-1} - \overline{y} < 0,\ z_{2n-1} - \overline{z} < 0$$

for $n \in \mathbb{N}_0$. That is the sequences $(x_n)_n$ (resp. $(y_n)_n$, $(z_n)_n$) oscillates about $\overline{x}$ (resp. about $\overline{y}$, $\overline{z}$) with semi-cycle of length one and every semi-cycle is in the form

$$- + - + - + \cdots$$

and this is the statement of Part 2. of Theorem 4.1.

□

5. **Conclusion.** In this study, the global stability of the unique positive equilibrium point of a three-dimensional general system of difference equations defined by positive and homogeneous functions of degree zero was studied. For this, general convergence theorems were given considering all possible monotonicity cases in arguments of functions f, g and h. In addition, the periodic nature and oscillation of the general system considered was also discussed and successful results were obtained. It is noteworthy that the results obtained on our general three-dimensional system have high applicability.

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