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A LIOUVILLE THEOREM FOR ANCIENT SOLUTIONS TO A SEMILINEAR HEAT EQUATION AND ITS ELLIPTIC COUNTERPART

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Abstract. We establish the nonexistence of nontrivial ancient solutions to the nonlinear heat equation $u_t = \Delta u + |u|^{p-1}u$ which are smaller in absolute value than the self-similar radial singular steady state, provided that the exponent p is strictly between Serrin's exponent and that of Joseph and Lundgren. This result was previously established by Fila and Yanagida [Tohoku Math. J. (2011)] by using forward self-similar solutions as barriers. In contrast, we apply a sweeping argument with a family of time independent weak supersolutions. Our approach naturally lends itself to yield an analogous Liouville type result for the steady state problem in higher dimensions. In fact, in the case of the critical Sobolev exponent we show the validity of our results for solutions that are smaller in absolute value than a 'Delaunay'-type singular solution.

1. **Introduction.** We consider classical solutions to the semilinear equation

$$u_t = \Delta u + |u|^{p-1}u, \quad x \in \mathbb{R}^N, \ t \le 0, \tag{1}$$

with $p > 1$. For obvious reasons, such solutions are frequently called ancient. Our interest will be in conditions which imply $u \equiv 0$, a Liouville type theorem that is.

In the past few decades there have been intensive studies of Liouville type theorems for the equation in (1), either when $t \le 0$, $t \in \mathbb{R}$ (entire solutions) or $t \ge 0$ (global solutions). At the same time, these have emerged as a fundamental tool in deriving various qualitative properties of the solutions to the corresponding Cauchy problem in a general domain or for a nonlinearity that behaves like a power as $u \to \infty$. The best general reference here is the monograph [19]. For a recent account of the theory and some further developments, we refer to [13, 18].

The following three exponents play an important role in the study of the equation in (1):

$$\text{Serrin's exponent} \ \ p_{sg} = \frac{N}{N-2} \ \text{ if } N \ge 3, \ \ p_{sg} = \infty \text{ if } N = 1, 2;$$

$$\text{the critical Sobolev exponent} \ \ p_S = \frac{N+2}{N-2} \ \text{ if } N \ge 3, \ \ p_S = \infty \text{ if } N = 1, 2;$$

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the Joseph-Lundgren exponent $p_{JL} = \begin{cases} \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N > 10, \\ \infty & \text{if } N \leq 10. \end{cases}$

We note that $p_{sg} < p_S < p_{JL}$ if $N \geq 3$. These exponents arise naturally in the study of the ordinary differential equation that is satisfied by the positive radial steady states [9, 19]. In this regard, let us list some well known properties which we will need in the sequel. First, for $p > p_{sg}$ there exists an explicit radial singular steady state

$$\varphi_\infty(x) = L|x|^{-2/(p-1)} \quad \text{with} \quad L = \left(\frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right)\right)^{1/(p-1)}. \tag{2}$$

For any $p > 1$, the radial ODE for the steady states admits a unique solution Φ such that $\Phi(0) = 1$, $\Phi_r(0) = 0$. This solution is defined in a maximal interval of the form $[0, R_{max})$ with $0 < R_{max} \leq \infty$ and is decreasing as long as it stays positive. The following qualitative properties of Φ will be useful for our purposes.

- If $p \in (1, p_S)$, then Φ has a first root $\rho > 0$. Actually, there are no positive steady states in this regime (see [7]). In fact, under the further restriction that $p \in (p_{sg}, p_S)$, it intersects twice with φ_∞ in $(0, \rho)$ (see in particular [9, Fig. 2]). We point out that these intersections are transverse thanks to the uniqueness of solutions to the corresponding IVP;
- If $p = p_S$, then $R_{max} = \infty$, $\Phi > 0$, $\Phi(\infty) = 0$ and Φ has exactly two intersections with φ_∞. We point out that this Φ has a simple explicit formula and decays to zero faster than φ_∞ as $r \to \infty$ (see also the discussion following (9) below);
- If $p \in (p_S, p_{JL})$, then $R_{max} = \infty$, $\Phi > 0$, $\Phi(\infty) = 0$ and Φ has infinitely many intersections with φ_∞. Moreover, $\Phi/\varphi_\infty \to 1$ as $r \to \infty$;
- If $p \geq p_{JL}$, then $R_{max} = \infty$, $\Phi > 0$, $\Phi(\infty) = 0$ and $\Phi < \varphi_\infty$. Moreover, $\Phi/\varphi_\infty \to 1$ as $r \to \infty$ (detailed information on this asymptotic behaviour can be found in [8]).

We emphasize that the rescaling

$$\varphi_\lambda(x) = \lambda\Phi\left(\lambda^{(p-1)/2}|x|\right), \quad \lambda \in (0, \infty), \tag{3}$$

furnishes a family of radial steady states such that $\varphi_\lambda(0) = \lambda$. Actually, this family includes all the positive radial steady states. We point out that φ_∞ is invariant under the above scaling. Therefore, in light of the above, if $p > p_S$ we see that $\varphi_\lambda \to \varphi_\infty$ pointwise in $\mathbb{R}^N \setminus \{0\}$ as $\lambda \to \infty$. For completeness, let us note that for $0 < \lambda < \mu < \infty$ the following hold: φ_λ has a unique radial intersection with φ_μ if $p = p_S$; φ_λ has infinitely many radial intersections with φ_μ if $p \in (p_S, p_{JL})$; $\varphi_\lambda < \varphi_\mu$ if $p \geq p_{JL}$.

We are now in position to state our main result.

Theorem 1.1. *If u satisfies (1) with $p \in (p_{sg}, p_{JL})$ and*

$$|u(x,t)| \leq \varphi_\infty(x), \; x \in \mathbb{R}^N \setminus \{0\}, \; t \leq 0, \tag{4}$$

then $u \equiv 0$.

For $p \in [p_S, p_{JL}) \subset (p_{sg}, p_{JL})$ the above theorem was proven previously by Fila and Yanagida in [6] by a different approach. Roughly, they 'squeezed' u between two forward self-similar solutions. We note that forward self-similar solutions exist also for subcritical p, and the required properties of theirs that are needed to show the Liouville property are well known (see [12, Thm. 1.1 and Cor. 1.2]). So, the proof in [6] applies in the above subcritical range as well. In fact, the approach in the aforementioned reference even yields the nonexistence in the class of functions satisfying the weaker condition

$$|u(x,t)| \leq (1+\varepsilon)\varphi_\infty(x), \ x \in \mathbb{R}^N, \ t \leq 0, \tag{5}$$

where $\varepsilon > 0$ is given by the properties of the forward self-similar solutions. Loosely speaking, time can be considered as the 'squeezing parameter' in their proof.

In contrast, our proof does not make use of (time dependent) similarity variables. Instead of using time dependent solutions as barriers, we will plainly use φ_∞ after appropriately 'covering' its singularity with a piece of φ_λ from (3) (a 'surgery' type of argument in some sense). The result is a weak supersolution of (1). Our 'squeezing parameter' will plainly be $\lambda > 0$ through the use of Serrin's sweeping principle (see [11, Thm. 9] for the elliptic case) in the spirit of the sliding method [1]. However, to be able to start such a continuity argument, we need that u is bounded. Thankfully, as it turns out, this can be assumed without loss of generality in light of the scaling and doubling arguments of [15]. On the other hand, most likely, our approach cannot be used to prove the nonexistence in the case of the weaker condition (5).

Remarkably, it was shown in [6] that the equation in (1) admits positive, entire solutions of homoclinic and heteroclinic type for $p \in (p_S, p_L)$ and $p \in [p_S, p_{JL})$, respectively, where $p_L > p_{JL}$ stands for Lepin's exponent.

Our approach, being elliptic in nature, carries over with only minor modifications to establish the following elliptic counterpart of Theorem 1.1 (as we will point out, the solutions in the latter can be extended for $t \in \mathbb{R}$). In contrast, the approach of [6] is intrinsically parabolic and seems to be inapplicable for this purpose.

Theorem 1.2. *If u satisfies*

$$\Delta u + |u|^{p-1}u = 0, \ z = (x,y) \in \mathbb{R}^{N+M}, \ \textit{with } p \in (p_{sg}(N), p_{JL}(N)), \ N \geq 3, \ M \geq 0, \tag{6}$$

and

$$|u(x,y)| \leq \varphi_\infty(x), \ x \in \mathbb{R}^N \setminus \{0\}, \ y \in \mathbb{R}^M, \tag{7}$$

then $u \equiv 0$.

If $M = 0$, the assumption (7) implies that u is a stable solution of (6) in $\mathbb{R}^N \setminus \{0\}$[1] (see [4, Ch. 1] for the definition). Indeed, as in [4, Prop. 1.3.2], it is easy to check that the difference $\varphi_\infty - |u|$ is a positive weak supersolution of the linearized operator $-\Delta - p|u|^{p-1}$ in $\mathbb{R}^N \setminus \{0\}$. Consequently, by the obvious weak version of [4, Prop. 1.2.1], we infer that u is stable in $\mathbb{R}^N \setminus \{0\}$. Therefore, in the special case $M = 0$, our result follows from [5, Thm. 2] which asserts that in that case (6) cannot have a nontrivial solution that is stable outside a compact set. On the other hand, we note that this viewpoint cannot be applied for general $M > 0$ because the exponent $p_{JL}(K)$ is decreasing with respect to K.

An analogous Liouville type result to Theorem 1.1 for $p \geq p_{JL}$, which takes into account that $\varphi_\lambda < \varphi_\infty$, $\lambda \in (0, \infty)$, can be found in our recent paper [21]. In the aforementioned work we have extended, again with a sweeping argument, the Liouville type result of Polacik and Yanagida from [16] who relied on (time dependent) similarity variables and invariant manifold ideas. A version of Theorem 1.2 for $p \geq p_{JL}(N)$ is contained in an extended remark in the same paper of ours.

[1] We were informed of this property by L. Dupaigne after the first version of the paper, we borrow his argument.

In the case of the critical Sobolev exponent $p = p_S$, a famous result of Caffarelli, Gidas and Spruck [2] asserts that all positive solutions of the steady state problem in $\mathbb{R}^N \setminus \{0\}$ are radial (whether they have a removable singularity at the origin or not). Using this information, Schoen [20] observed that all such solutions with a singularity at the origin can be completely classified by standard ODE phase-plane analysis. They are of the form

$$u(x) = |x|^{-\frac{N-2}{2}} v\left(\ln |x|\right), \tag{8}$$

where v is a positive periodic solution of

$$-v'' + \frac{(N-2)^2}{4} v - v^{\frac{N+2}{N-2}} = 0 \text{ in } \mathbb{R}. \tag{9}$$

Besides of the constant solution $\left(\frac{N-2}{2}\right)^{\frac{N-2}{2}}$, which gives rise to the self-similar singular solution φ_∞, there is a family of periodic solutions that can be uniquely parametrized, up to translations, by their minimal value which spans the interval $\left(0, \left(\frac{N-2}{2}\right)^{\frac{N-2}{2}}\right)$. These periodic solutions have a unique local maximum and minimum per period. In fact, they are symmetric with respect to their local extrema. The singular solutions of (1) that they produce via (8) are frequently called of Delaunay-type in comparison with Delaunay surfaces which are singly periodic, rotationally symmetric surfaces with constant mean curvature. We point out that each Delaunay-type singular solution has infinitely many radial intersections with φ_∞. Actually, the radial regular steady state Φ of (1) is given by (8) with v an appropriate translation of the positive, even homoclinic solution of (9). Remarkably, the latter solution can be computed explicitly and is equal to $\left(N(N-2)\right)^{\frac{N-2}{4}} \left(2\cosh(\cdot)\right)^{-\frac{N-2}{2}}$. Let us note in passing that the translation invariance of (9) echoes the scaling invariance (3) of (1). It is worth mentioning that an analogous transformation to (8) also applies for $p \neq p_S$. However, the corresponding second order autonomous ODE for v is dissipative and thus has no nonconstant periodic or homoclinic solutions (it has, however, heteroclinic solutions for $p \in (p_{sg}, p_S)$ that give rise to fast decaying singular solutions, see Remark 2 and the references therein).

Armed with the above information and by suitably adapting our approach, we can complement our main results with the following.

Theorem 1.3. *If $p = p_S(N)$, the assertions of Theorems 1.1 and 1.2 hold with the righthand side of (4) and (7), respectively, being an arbitrary Delaunay-type singular solution.*

To illustrate the delicacy of our result, at least in the parabolic case, we remark that the previously mentioned heteroclinic solutions of [6] connect φ_λ, $\lambda \in (0, \infty)$, as $t \to -\infty$ to the trivial solution as $t \to +\infty$ and are decreasing in time.

The rest of the paper is essentially devoted to the proofs of our main results in the next section. In Subsection 2.1, we will prove Theorem 1.1. After its proof, in Remark 1, we will hint at a perhaps unexpected connection between our supersolution and a well known argument from the theory of minimal surfaces. As we have already mentioned, the proof of Theorem 1.2 requires only minor modifications and will therefore be omitted. In Subsection 2.2, we will prove Theorem 1.3. Subsequently, in Remark 2, we will give a partial analog of this theorem for subcritical exponents. Lastly, for the reader's convenience, in Appendix A we will state a reduced version of the doubling lemma from [14] that is needed for our results.

2. **Proofs of the main results.** In this section we will prove Theorems 1.1 and 1.3. In order to avoid confusion, we mention again that the proof of Theorem 1.2 will be omitted as it requires only minor adaptations.

2.1. **Proof of Theorem 1.1.**

Proof. The main idea of the proof is to construct a family of weak supersolutions of (1) by appropriately modifying the singular solution φ_∞ around the origin. Our construction will hinge on the fact that, as we have already mentioned, the radial regular steady state Φ intersects at least once with φ_∞ since $p \in (p_{sg}, p_{JL})$. We denote by $r_1 > 0$ the smallest radius at which such an intersection takes place, and define a function $Z : \mathbb{R}^N \to \mathbb{R}$ with radial profile given by

$$Z(r) = \begin{cases} \Phi(r), & 0 \leq r \leq r_1, \\ \\ \varphi_\infty(r), & r > r_1. \end{cases} \tag{10}$$

Clearly, Z is continuous by our choice of r_1. The point is that it is a weak supersolution of (1) (see for instance [10, Ch. 5] for the definition) because

$$\Phi'(r_1) > \varphi'_\infty(r_1) \tag{11}$$

holds. Next, according to (3), we let

$$z_\lambda(x) = \lambda Z\left(\lambda^{(p-1)/2}|x|\right) = \begin{cases} \varphi_\lambda(r), & 0 \leq r \leq s_\lambda, \\ \\ \varphi_\infty(r), & r > s_\lambda, \end{cases} \qquad r = |x|,\ \lambda > 0,$$

where we have denoted

$$s_\lambda = r_1 \lambda^{-(p-1)/2}.$$

We emphasize that we have used that φ_∞ is invariant under the above rescaling. We point out that $z_\lambda \to \infty$ uniformly on $|x| \leq s_\lambda$ as $\lambda \to \infty$. On the other hand, $z_\lambda \to 0$ as $\lambda \to 0$, uniformly in $\mathbb{R}^N$. Clearly, z_λ is still a weak supersolution to (1). Actually, we will not use any weak form of the maximum principle in the sequel. Nevertheless, the fact that z_λ is a weak supersolution of (1) will serve as an important guideline.

By making partial use of our supersolution, we will first show that u can be extended as a solution of the equation in (1) for $t \in \mathbb{R}$. To this end, the standard existence and uniqueness theory for the corresponding Cauchy problem (it is well-posed in $L^\infty(\mathbb{R}^N)$, see [19, Prop. 51.40]) guarantees that u can be extended in a maximal time interval of the form $(-\infty, T)$ for some $T \in (0, \infty]$. Moreover, by the strong maximum principle for linear parabolic equations [10, Ch. II], we assert from (4) that

$$|u| < \varphi_\infty,\ x \in \mathbb{R}^N \setminus \{0\},\ t \in (-\infty, T). \tag{12}$$

Since $u(\cdot, 0)$ is a bounded function, there exists a $\lambda^* \gg 1$ such that

$$u(x, 0) < \varphi_{\lambda^*}(x), \quad |x| \leq s_{\lambda^*}.$$

Let $\varepsilon \in (0, T)$ be arbitrary. By virtue of the above two relations, since u and φ_{λ^*} are bounded on $\{|x| \leq s_{\lambda^*},\ t \in [0, T - \varepsilon]\}$, the parabolic maximum principle [10, Lem. 2.3] (applied to the linear equation for the difference of these two solutions of (1)) yields

$$u \leq \varphi_{\lambda^*} \quad \text{for } |x| \leq s_{\lambda^*},\ t \in [0, T - \varepsilon].$$

Since $\varepsilon > 0$ is arbitrary, we obtain $u \leq \varphi_{\lambda^*}$ for $|x| \leq s_{\lambda^*}$, $t \in [0, T)$. Applying the same argument with $-u$ in place of u, and keeping in mind (12), we conclude that

u remains bounded as $t \to T^-$. This means that $T = \infty$ as desired (if not, then u could be continued further as a solution in contradiction to the maximality of T).

Having disposed of this preliminary step, we can now turn our attention to the Liouville property. By nowadays standard doubling and scaling arguments [15], we can assume that u is bounded. In fact, we can do better and assume that

$$|u| \le 1 \text{ in } \mathbb{R}^N \times \mathbb{R}. \tag{13}$$

Indeed, let us suppose that $|u(x_0, t_0)| > 1$ for some $(x_0, t_0) \in \mathbb{R}^N \times \mathbb{R}$. Motivated from [17], we will apply Lemma A.1 from Appendix A with $X = \mathbb{R}^N \times \mathbb{R}$, equipped with the parabolic distance

$$d\left((x,t),(\tilde{x},\tilde{t})\right) = |x - \tilde{x}| + \sqrt{|t - \tilde{t}|},$$

and

$$M(x,t) = |u|^{(p-1)/2}(x,t).$$

For $\mathtt{y} = (x_0, t_0)$ and any $k \in \mathbb{N}$, the aforementioned lemma provides (x_k, t_k) such that

$$M_k := |u|^{(p-1)/2}(x_k, t_k) \ge |u|^{(p-1)/2}(x_0, t_0)$$

and

$$|u|^{(p-1)/2}(x,t) \le 2M_k \text{ whenever } |x - x_k| + \sqrt{|t - t_k|} \le \frac{k}{M_k}.$$

We note that (4) and the definition of M_k force

$$M_k |x_k| \le L^{(p-1)/2}.$$

Hence, passing to a subsequence if necessary, we may assume that

$$M_k x_k \to y_\infty \text{ for some } y_\infty \in \mathbb{R}^N. \tag{14}$$

The rescaled functions

$$v_k(y,s) = \rho_k^{2/(p-1)} u(x_k + \rho_k y, t_k + \rho_k^2 s), \text{ where } \rho_k = \frac{1}{2M_k},$$

are entire solutions of (1) and satisfy $|v_k(0,0)| = 2^{-2/(p-1)}$, $|v_k(y,s)| \le 1$ for $|y| + \sqrt{|s|} \le 2k$. The parabolic regularity theory [10, Chs. IV, VII] guarantees that the sequence $\{v_k\}$ is relatively compact in $C^{2+\theta,1+\theta/2}_{loc}$ for some $\theta \in (0,1)$. Hence, using the usual diagonal argument, passing to a further subsequence if needed, we may assume that

$$v_k \to V \text{ in } C^{2,1}_{loc}(\mathbb{R}^N \times \mathbb{R}),$$

where V is an entire solution to (1) such that $|V| \le 1$ and $V(0,0) \ne 0$. Furthermore, on account of (4), we have

$$|v_k(y,s)| \le \frac{L\rho_k^{2/(p-1)}}{|x_k + \rho_k y|^{2/(p-1)}} = \frac{L}{|x_k/\rho_k + y|^{2/(p-1)}} = \frac{L}{|2M_k x_k + y|^{2/(p-1)}},\ y \ne -\frac{x_k}{\rho_k}.$$

Thus, by letting $k \to \infty$ and using (14), we obtain

$$|V(y,s)| \le \frac{L}{|2y_\infty + y|^{2/(p-1)}},\ y \ne -2y_\infty.$$

Now, the spatially shifted solution

$$W(y,s) = V(y - 2y_\infty, s)$$

satisfies $|W| \le 1$, $W(2y_\infty, 0) \ne 0$ and (4). Consequently, it is sufficient to prove the theorem for entire solutions that satisfy (4) with $t \in \mathbb{R}$ and (13). This task will take up the rest of the proof.

The main tool in the proof is Serrin's sweeping principle (see [11, Thm. 9] for the elliptic case) using the family of supersolutions $\{z_\lambda\}$. Since u is bounded and satisfies (4), there exists a $\bar{\lambda} \gg 1$ such that

$$u \le z_\mu,\ x \in \mathbb{R}^N,\ t \in \mathbb{R},\ \text{for any } \mu \ge \bar{\lambda}.$$

Starting from $\bar{\lambda}$, we proceed to decrease λ while keeping the above ordering. There are only two possibilities. Either we can continue all the way until we reach $\lambda = 0$ or we will get 'stuck' at some first $\lambda_0 > 0$ and cannot continue further. Our goal is to show that the latter scenario (to be described in more detail below) cannot happen. This will imply that $u \le 0$. Then, the assertion of the theorem follows readily by carrying out the same procedure with $-u$ in place of u.

Let us suppose, to the contrary, that there exists some $\lambda_0 \in (0, \bar{\lambda}]$ where we get stuck in the sense that the set

$$\Lambda = \big\{\lambda \ge 0\ :\ z_\mu \ge u \text{ in } \mathbb{R}^N \times \mathbb{R} \text{ for every } \mu \ge \lambda\big\}$$

coincides with $[\lambda_0, \infty)$ (by its definition Λ is a semi infinite interval, while it is closed thanks to the continuity of z_μ with respect to μ). Clearly, we have

$$u \le z_{\lambda_0},\ x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{15}$$

Keeping in mind that z_λ depends nontrivially on λ only in the space-time cylinder $\{|x| < s_\lambda,\ t \in \mathbb{R}\}$ (where it is equal to φ_λ), and (12) with $T = \infty$, we get $\lambda_k \in (0, \lambda_0)$ such that $\lambda_k \to \lambda_0$ as $k \to \infty$, $x_k \in \mathbb{R}^N$ with $|x_k| < s_{\lambda_k}$, and $t_k \in \mathbb{R}$ such that

$$u(x_k, t_k) > \varphi_{\lambda_k}(x_k), \tag{16}$$

(the reader should not be confused with the repeated use of notation in different contexts within the proof). The whole argument is actually reminiscent to the famous sliding method [1] for elliptic problems, when translating a compactly supported subsolution. We also note that z_λ and z_μ with $\lambda < \mu$ may intersect each other in $|x| < s_\mu$ as is the case in the aforementioned procedure. Passing to a subsequence if necessary, we may assume that

$$x_k \to x_\infty \text{ for some } x_\infty \in \mathbb{R}^N \text{ such that } |x_\infty| \le s_{\lambda_0}. \tag{17}$$

If the sequence $\{t_k\}$ is bounded, passing to a further subsequence if needed, we may assume that $t_k \to t_\infty$ for some $t_\infty \in \mathbb{R}$. From (15) and (16), it follows that $u(x_\infty, t_\infty) = \varphi_{\lambda_0}(x_\infty)$. In fact, thanks to (12) with $T = \infty$, we see that $|x_\infty| \ne s_{\lambda_0}$. Thus, by virtue of (15) and the parabolic strong maximum principle [10] (applied in the linear equation for the difference $u - \varphi_{\lambda_0}$ sufficiently close to (x_∞, t_∞)), we deduce that u coincides with φ_{λ_0} in some neighborhood of (x_∞, t_∞). In turn, by repeated applications of the strong maximum principle, we obtain $u \equiv \varphi_{\lambda_0}$ which is clearly absurd on account of (4).

It remains to deal with the case where, up to a subsequence, $t_k \to -\infty$ (the case where $t_k \to +\infty$ can be handled similarly). To this end, we consider the time translated solutions

$$u_k(x, t) = u(x, t + t_k),\ x \in \mathbb{R}^N,\ t \in \mathbb{R}.$$

From (15) and (16) it follows that

$$z_{\lambda_0} \ge u_k \text{ in } \mathbb{R}^N \times \mathbb{R} \text{ and } u_k(x_k, 0) > \varphi_{\lambda_k}(x_k),$$

respectively. Since u is bounded, as before, by the usual diagonal-compactness argument, possibly up to a further subsequence, we have

$$u_k \to U \text{ in } C^{2,1}_{loc}(\mathbb{R}^N \times \mathbb{R}),$$

where U is an entire solution to (1) such that

$$U \leq z_{\lambda_0} \text{ in } \mathbb{R}^N \times \mathbb{R} \text{ and } U(x_\infty, 0) \geq \varphi_{\lambda_0}(x_\infty).$$

In particular, we get $U(x_\infty, 0) = \varphi_{\lambda_0}(x_\infty)$. Intuitively, keeping in mind (11), it is clear that we have been led to a contradiction. The rigorous justification is easy. Indeed, by virtue of (11), the above imply that

$$U \leq \varphi_{\lambda_0}, \ |x| < s_{\lambda_0} + \delta, \ |t| < 1,$$

for some sufficiently small $\delta > 0$. Then, as before, we deduce by the strong maximum principle that $U \equiv \varphi_{\lambda_0}$ which is impossible. □

Remark 1. In [21] we highlighted a heuristic connection of (1) to ancient solutions of the mean curvature flow. In that context, our time independent supersolution in (10) relates to the competitor that is used in order to show that the symmetric minimal cones are not area minimizers in low dimensions.

2.2. **Proof of Theorem 1.3.**

Proof. The proof is similar to that of Theorems 1.1 and 1.2 apart from some technical modifications. We will give a sketch of the proof only for the parabolic problem (the elliptic case is analogous) and point out the main differences.

Let us denote by

$$\psi(r) = h(\ln r) r^{-\frac{N-2}{2}}, \ r = |x| > 0,$$

with $h > 0$ a T-periodic solution of (9), a Delaunay-type singular solution that bounds the absolute value of u. For each $\lambda \in (0, \infty)$, the homoclinic solution of (9) that gives φ_λ via (8) intersects at least twice with h (this can be seen easily from the phase plane portrait). Hence, there exists a first radius $\tau_\lambda > 0$ at which φ_λ and ψ intersect. Clearly, we have $\tau_\lambda \to 0$ as $\lambda \to \infty$ and $\tau_\lambda \to \infty$ as $\lambda \to 0$. Moreover, since such an intersection is transverse (by the uniqueness of the IVP for the radial ODE), the implicit function theorem guarantees that τ_λ varies smoothly with respect to $\lambda > 0$. Keeping in mind that ψ is not invariant under the scaling in (3) (unless $h \equiv L$ of course), we now define our supersolution z_λ directly as

$$z_\lambda(x) = \begin{cases} \varphi_\lambda(r), & 0 \leq r \leq \tau_\lambda, \\ \psi(r), & r > \tau_\lambda, \end{cases} \quad r = |x|.$$

As before, we can use z_λ as a barrier in order to show that u cannot blow up in finite time. Therefore, we may again assume that u is an entire solution to (1) that satisfies

$$|u(x,t)| \leq h(\ln|x|)|x|^{-\frac{N-2}{2}}, \quad x \in \mathbb{R}^N \setminus \{0\}, \ t \in \mathbb{R}. \tag{18}$$

As in the proof of Theorem 1.1, by applying Serrin's sweeping principle, we can conclude that $u \equiv 0$ under the additional assumption that it is bounded.

It remains to verify that, by the doubling lemma as in the proof of Theorem 1.1, we can assume without loss of generality that (13) holds. To this end, assuming that this was not the case, we define M, (x_k, t_k), M_k, ρ_k and $v_k(y,s)$ analogously to the aforementioned proof. We quickly come across a minor difference which is that now we have

$$M_k |x_k| \leq \|h\|_{L^\infty(\mathbb{R})}^{\frac{2}{N-2}}.$$

Nevertheless, up to a subsequence, we still have $M_k x_k \to y_\infty$ for some $y_\infty \in \mathbb{R}^N$. Moreover, we still have the local convergence of v_k to some bounded, nontrivial limiting solution V. However, the main differences arise when passing to the limit in the rescaled form of (18). More precisely, the latter gives

$$|v_k(y,s)| \le \frac{\rho_k^{\frac{N-2}{2}} h\left(\ln|x_k+\rho_k y|\right)}{|x_k+\rho_k y|^{\frac{N-2}{2}}} = \frac{h\left(\ln|x_k+\rho_k y|\right)}{|x_k/\rho_k+y|^{\frac{N-2}{2}}}$$

for $y \neq -x_k/\rho_k$, $s \in \mathbb{R}$. Based on the identity

$$\ln|x_k+\rho_k y| = \ln\rho_k + \ln|\frac{x_k}{\rho_k}+y|,$$

we decompose $\ln\rho_k$ as

$$\ln\rho_k = m_k T + d_k,$$

with $m_k \in \mathbb{Z}$ and $|d_k| \le T$. By passing to a further subsequence if necessary, we may assume that $d_k \to d_\infty$ for some $d_\infty \in \mathbb{R}$. Since h is T-periodic, we obtain

$$h\left(\ln|x_k+\rho_k y|\right) = h\left(m_k T + d_k + \ln|\frac{x_k}{\rho_k}+y|\right) = h\left(d_k + \ln|\frac{x_k}{\rho_k}+y|\right).$$

Consequently, recalling the definition of M_k, we get

$$|v_k(y,s)| \le \frac{h\left(d_k+\ln|2M_k x_k+y|\right)}{|2M_k x_k+y|^{\frac{N-2}{2}}}, \quad y \neq -2M_k x_k.$$

Letting $k\to\infty$, we deduce that

$$|V(y,s)| \le \frac{h\left(d_\infty+\ln|2y_\infty+y|\right)}{|2y_\infty+y|^{\frac{N-2}{2}}} = \frac{h\left(\ln\left(e^{d_\infty}|2y_\infty+y|\right)\right)}{|2y_\infty+y|^{\frac{N-2}{2}}}, \quad y \neq -2y_\infty.$$

We remark that the righthand side of the above relation is plainly a rescaling (according to (3)) and a translation of the Delaunay-type singular solution ψ. In other words, after a translation, V satisfies (18) with h replaced by a positive $(e^{-d_\infty}T)$-periodic solution of (9). Hence, V is a bounded solution that satisfies the assumptions of the theorem, which is what we wanted. □

Remark 2. If $p \in (p_{sg}, p_S)$, for any $a>0$, there exists a positive, radial singular solution ϕ to the steady state problem such that

$$\lim_{r\to 0} r^{\frac{2}{p-1}}\phi(r) = L \text{ and } \lim_{r\to\infty} r^{N-2}\phi(r) = a$$

(see [3, Prop. 2.2]). We note that these singular solutions decay faster than the self-similar one as $|x|\to\infty$. We observe that φ_λ with $\lambda\in(0,\infty)$ must intersect at least twice with each such fast decaying singular solution. Indeed, if not then by the discussion following Theorem 1.2 we would have that φ_λ is a stable solution of the steady state problem in its support which is absurd (see for instance [4, Ex. 1.2.3]). In light of this property, by arguing as in the proof of Theorem 1.3 we can show that the Liouville property holds for bounded, ancient solutions to (1) that are smaller in absolute value than such a fast decaying singular steady state (one can also prove a corresponding elliptic result in the spirit of Theorem 1.2).

Appendix A. **A doubling lemma from** [14]**.** In this small appendix, we will state for the reader's convenience the following reduced version of [14, Lem. 5.1] that we referred to in the proof of Theorem 1.1.

Lemma A.1. *Let (X, d) be a complete metric space and $M : X \to [0, \infty)$ be bounded on compact subsets of X. Fix a $\boldsymbol{y} \in X$ such that $M(\boldsymbol{y}) > 0$ and a real $k > 0$. Then, there exists $\boldsymbol{x} \in X$ such that*

$$M(\boldsymbol{x}) \geq M(\boldsymbol{y})$$

and

$$M(\boldsymbol{z}) \leq 2M(\boldsymbol{x}) \quad \text{whenever} \quad d(\boldsymbol{z}, \boldsymbol{x}) \leq \frac{k}{M(\boldsymbol{x})}.$$

Remark 3. Our formulation of the doubling lemma is restricted to the whole metric space X. We also note that we assume M to be nonnegative instead of strictly positive, as was the case in the aforementioned reference. However, if $M(\mathrm{y}) > 0$ then throughout the proof of [14, Lem. 5.1] we observed that M is evaluated only at points where $M \geq M(\mathrm{y})$. Thus, there is no loss of generality.

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