

## ERGODIC MEASURES OF INTERMEDIATE ENTROPY FOR AFFINE TRANSFORMATIONS OF NILMANIFOLDS

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ABSTRACT. In this paper we study ergodic measures of intermediate entropy for affine transformations of nilmanifolds. We prove that if an affine transformation  $\tau$  of nilmanifold has a periodic point, then for every  $a \in [0, h_{top}(\tau)]$  there exists an ergodic measure  $\mu_a$  of  $\tau$  such that  $h_{\mu_a}(\tau) = a$ .

**1. Introduction.** Throughout this paper, by a *topological dynamical system*  $(X, T)$  (TDS for short) we mean a compact metric space  $(X, d)$  with a homeomorphism map  $T$  from  $X$  onto itself, where  $d$  refers to the metric on  $X$ . By a *measure preserving system* (MPS for short) we mean a quadruple  $(X, \mathcal{X}, \mu, T)$ , where  $(X, \mathcal{X}, \mu)$  is a Borel probability space and  $T, T^{-1} : X \rightarrow X$  are both measurable and measure preserving, i.e.  $T^{-1}\mathcal{X} = \mathcal{X} = T\mathcal{X}$  and  $\mu(A) = \mu(T^{-1}A)$  for each  $A \in \mathcal{X}$ .

Given a TDS  $(X, T)$ , let  $\mathcal{M}(X, T)$  be the set of all  $T$ -invariant Borel probability measures of  $X$ . In weak\*-topology,  $\mathcal{M}(X, T)$  is a compact convex space. By Krylov-Bogolioubov Theorem  $\mathcal{M}(X, T) \neq \emptyset$ . For each  $\mu \in \mathcal{M}(X, T)$ ,  $(X, \mathcal{B}_X, T, \mu)$  can be viewed as a MPS, where  $\mathcal{B}_X$  is the Borel  $\sigma$ -algebra of  $X$ . Let  $\mathcal{M}^e(X, T)$  be the space of all ergodic measures of  $(X, T)$ . Then  $\mathcal{M}^e(X, T)$  is the set of extreme points of  $\mathcal{M}(X, T)$ .

Define

$$\mathcal{E}(T) = \{h_{\mu}(T) : \mu \in \mathcal{M}^e(X, T)\}$$

where  $h_{\mu}(T)$  denotes the measure-theoretic entropy of the measure preserving system  $(X, \mathcal{B}_X, T, \mu)$ . By the variational principle of entropy  $\sup \mathcal{E}(T) = h_{top}(T)$ , where  $h_{top}(T)$  is the topological entropy of  $(X, T)$ . The extreme case is that  $\mathcal{M}^e(X, T)$  consists of only one member, that is,  $(X, T)$  is *uniquely ergodic*. When  $(X, T)$  is uniquely ergodic,  $\mathcal{E}(T) = \{h_{top}(T)\}$ .

It is interesting to consider the case when  $\mathcal{E}(T)$  is big. As a direct corollary of [7, Theorem 11], Katok showed that

$$[0, h_{top}(f)] \subset \mathcal{E}(f) \tag{1.1}$$

for any  $C^{1+\alpha}$  diffeomorphism  $f$  on a two-dimensional surface, based on the fact that every ergodic measure of positive metric entropy is hyperbolic. Katok conjectured that (1.1) holds for any smooth system.

**Conjecture 1.1** (Katok). *Let  $f$  be a  $C^r$  ( $r > 1$ ) diffeomorphism on a smooth compact manifold  $M$ , then (1.1) holds, i.e. for every  $a \in [0, h_{top}(f))$ , there is  $\mu_a \in \mathcal{M}^e(M, f)$  such that  $h_{\mu_a}(f) = a$ .*

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We need to point out that Katok's conjecture implies that any positive entropy smooth system is not uniquely ergodic, though whether or not a smooth diffeomorphism of positive topological entropy can be uniquely ergodic is still in question (see [5] for Herman's example: positive entropy minimal  $C^\infty$ -smooth diffeomorphisms). In [13, 14], Quas and Soo showed that if a topological dynamical system satisfies asymptotic entropy expansiveness, almost weak specification property and small boundary property, then it is universal, which implies the conclusion of Katok's conjecture. Recently, Burguet [2], Chandgotia and Meyerovitch [4], extended the result of Quas and Soo to request only the almost weak specification property.

In this paper, we study intermediate entropy for affine transformations of nilmanifolds. Throughout this paper, by a *nilmanifold*  $G/\Gamma$  we mean that  $G$  is a connected, simply connected nilpotent Lie group, and  $\Gamma$  is a cocompact discrete subgroup of  $G$ . A homeomorphism  $\tau$  of  $G/\Gamma$  is an *affine transformation* if there exist a  $\Gamma$ -invariant automorphism  $A$  of  $G$  and a fixed element  $g_0 \in G$  such that  $\tau(g\Gamma) = g_0A(g)\Gamma$  for each  $g \in G$ . Our main result is the following.

**Theorem 1.2.** *Let  $G/\Gamma$  be a nilmanifold and  $\tau$  be an affine transformation of  $G/\Gamma$ . If  $(G/\Gamma, \tau)$  has a periodic point, then  $\mathcal{E}(\tau) = [0, h_{top}(\tau)]$ .*

Following Lind [11], we say that an affine transformation of a nilmanifold is *quasi-hyperbolic* if its associated matrix has no eigenvalue 1. As an application of Theorem 1.2, one has the following.

**Theorem 1.3.** *Let  $G/\Gamma$  be a nilmanifold and  $\tau$  be an affine transformation of  $G/\Gamma$ . If  $\tau$  is quasi-hyperbolic, then  $\mathcal{E}(\tau) = [0, h_{top}(\tau)]$ .*

The paper is organized as follows. In Section 2, we introduce some notions. In Section 3, we prove Theorem 1.2 and Theorem 1.3.

**2. Preliminary.** In this section, we recall some notions of entropy, nilmanifold and upper semicontinuity of entropy map.

**2.1. Entropy.** We summarize some basic concepts and useful properties related to topological entropy and measure-theoretic entropy here.

Let  $(X, T)$  be a TDS. A *cover* of  $X$  is a family of subsets of  $X$ , whose union is  $X$ . A *partition* of  $X$  is a cover of  $X$  whose elements are pairwise disjoint. Given two covers  $\mathcal{U}, \mathcal{V}$  of  $X$ , set  $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$  and  $T^{-i}\mathcal{U} = \{T^{-i}U : U \in \mathcal{U}\}$  for  $i \in \mathbb{Z}_+$ . Denote by  $N(\mathcal{U})$  the minimal cardinality among all cardinalities of subcovers of  $\mathcal{U}$ .

**Definition 2.1.** Let  $(X, T)$  be a TDS and  $\mathcal{U}$  be a finite open cover of  $X$ . The *topological entropy of  $\mathcal{U}$*  is defined by

$$h_{top}(T, \mathcal{U}) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right),$$

where  $\{\log N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U})\}_{n=1}^\infty$  is a sub-additive sequence and hence  $h_{top}(T, \mathcal{U})$  is well defined. The *topological entropy of  $(X, T)$*  is

$$h_{top}(T) = \sup_{\mathcal{U}} h_{top}(T, \mathcal{U}),$$

where supremum is taken over all finite open covers of  $X$ .

A subset  $E$  of  $X$  is an  $(n, \epsilon)$ -separated set with respect to  $T$  provided that for any distinct  $x, y \in E$  there is  $0 \leq j < n$  such that  $d(T^j x, T^j y) \geq \epsilon$ . Let  $K$  be a compact subset of  $X$ . Let  $s_n^{(T)}(\epsilon, K)$  be the largest cardinality of any subset  $E$  of  $K$  which is an  $(n, \epsilon)$ -separated set. Then the Bowen’s topological entropy of  $K$  with respect to  $T$  [1] is defined by

$$h_d(T, K) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\log s_n^{(T)}(\epsilon, K)}{n}.$$

Let  $Z$  be a non-empty subset of  $X$ . The Bowen’s topological entropy of  $Z$  with respect to  $T$  is defined by

$$h_d(T, Z) = \sup_{\substack{K \subset Z \\ K \text{ is compact}}} h_d(T, K).$$

And the Bowen’s topological entropy of a TDS  $(X, T)$  is defined by  $h_d(T) = h_d(T, X)$  which happens to coincide with  $h_{top}(T)$ .

Next we define measure-theoretic entropy. Let  $(X, \mathcal{X}, \mu, T)$  be a MPS and  $\mathcal{P}_X$  be the set of finite measurable partitions of  $X$ . Suppose  $\xi \in \mathcal{P}_X$ . The entropy of  $\xi$  is defined by

$$h_\mu(T, \xi) = \lim_{n \rightarrow +\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right),$$

where  $H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right) = -\sum_{A \in \mathcal{V}_{i=0}^{n-1} T^{-i}\xi} \mu(A) \log \mu(A)$  and  $\{H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right)\}_{n=1}^\infty$  is a sub-additive sequence. The entropy of  $(X, \mathcal{X}, T, \mu)$  is defined by

$$h_\mu(T) = \sup_{\xi \in \mathcal{P}_X} h_\mu(T, \xi).$$

The basic relationship between topological entropy and measure-theoretic entropy is given by the variational principle [12].

**Theorem 2.2** (The variational principle). *Let  $(X, T)$  be a TDS. Then*

$$h_{top}(T) = \sup\{h_\mu(T) : \mu \in \mathcal{M}(X, T)\} = \sup\{h_\mu(T) : \mu \in \mathcal{M}^e(X, T)\}.$$

A factor map  $\pi : (X, T) \rightarrow (Y, S)$  between the TDS  $(X, T)$  and  $(Y, S)$  is a continuous onto map with  $\pi \circ T = S \circ \pi$ ; we say that  $(Y, S)$  is a factor of  $(X, T)$  and that  $(X, T)$  is an extension of  $(Y, S)$ . The systems are said to be conjugate if  $\pi$  is bijective. In [8], Ledrappier and Walters showed that if  $\pi : (X, T) \rightarrow (Y, S)$  is a factor map and  $\nu \in \mathcal{M}(Y, S)$ , then

$$\sup_{\substack{\mu \in \mathcal{M}(X, T) \\ \pi(\mu) = \nu}} h_\mu(T) = h_\nu(S) + \int_Y h_d(T, \pi^{-1}(y)) d\nu(y) \tag{2.1}$$

where  $\pi(\mu)(B) = \mu(\pi^{-1}(B))$  for  $B \in \mathcal{B}_Y$ .

Let  $G$  be a compact metric group and  $\tau : G \rightarrow G$  be a continuous surjective map. Let  $\pi : (X, T) \rightarrow (Y, S)$  be a factor map. We say that  $\pi$  is a  $(G, \tau)$ -extension, if there exists a continuous map  $P : X \times G \rightarrow X$  (we write  $P(x, g) = xg$ ) such that:

- (1)  $\pi^{-1}(\pi(x)) = xG$  for  $x \in X$ ,
- (2) For any  $x \in X, g_1, g_2 \in G, xg_1 = xg_2$  if and only if  $g_1 = g_2$ ,
- (3)  $T(xg) = T(x)\tau(g)$  for  $x \in X$  and  $g \in G$ .

The following is from [1, Theorem 19].

**Theorem 2.3.** *Let  $\pi : (X, T) \rightarrow (Y, S)$  be a factor map. If  $\pi$  is a  $(G, \tau)$ -extension, then  $h_{top}(T) = h_{top}(S) + h_{top}(\tau)$ .*

**Remark 2.4.** (1) In the above situation, Bowen shows that

$$h_d(T, \pi^{-1}(y)) = h_{top}(\tau) \text{ for any } y \in Y, \quad (2.2)$$

where  $d$  is the metric on  $X$ . This fact is proved in the proof of [1, Theorem 19]. In fact, (2.2) holds in the more general situation of actions of amenable groups. This fact is given explicitly as Lemma 6.12 in the paper [10].

(2) If  $G$  is a Lie group,  $H$  and  $N$  are cocompact closed subgroups of  $G$  such that  $N$  is a normal subgroup of  $H$ , then  $G/N$  and  $G/H$  are compact metric spaces and  $H/N$  is a compact metric group. Given further  $g_0 \in G$  and an automorphism  $A$  of  $G$  preserving  $H$  and  $N$ , one has the affine maps  $T : G/N \rightarrow G/N$  given by  $T(gN) = g_0A(g)N$ , and  $S : G/H \rightarrow G/H$  given by  $S(gH) = g_0A(g)H$ , and the automorphism  $\tau$  of  $H/N$  given by  $\tau(hN) = A(h)N$ . Then there is a map  $\pi : G/N \rightarrow G/H$  given by  $\pi(gN) = gH$  for  $g \in G$ , and a map  $P : G/N \times H/N \rightarrow G/N$  given by  $P(gN, hN) = ghN$  for  $g, h \in G$ . These maps satisfy the conditions (1) – (3) in the definition of  $(H/N, \tau)$ -extension for the factor map  $\pi : (G/N, T) \rightarrow (G/H, S)$ . That is,  $(G/N, T)$  is an  $(H/N, \tau)$ -extension of  $(G/H, S)$ . Hence one has by (2.2) that

$$h_d(T, \pi^{-1}(y)) = h_{top}(\tau) \text{ for any } y \in G/H, \quad (2.3)$$

where  $d$  is the metric on  $G/N$ .

**2.2. Upper semicontinuity of entropy map.** Given a TDS  $(X, T)$ , the entropy map of  $(X, T)$  is the map  $\mu \mapsto h_\mu(T)$  which is defined on  $\mathcal{M}(X, T)$  and has value in  $[0, \infty]$ . For any invariant measure  $\mu$  on  $X$ , there is a unique Borel probability measure  $\rho$  on  $\mathcal{M}(X, T)$  with  $\rho(\mathcal{M}^e(X, T)) = 1$  such that

$$\int_{\mathcal{M}^e(X, T)} \int_X f(x) dm(x) d\rho(m) = \int_X f(x) d\mu(x) \text{ for all } f \in C(X).$$

We write  $\mu = \int_{\mathcal{M}^e(X, T)} md\rho(m)$  and call it the ergodic decomposition of  $\mu$ . The following is standard.

**Theorem 2.5.** *Let  $(X, T)$  be a TDS. If  $\mu \in \mathcal{M}(X, T)$  and  $\mu = \int_{\mathcal{M}^e(X, T)} md\rho(m)$  is the ergodic decomposition of  $\mu$ . Then*

$$h_\mu(T) = \int_{\mathcal{M}^e(X, T)} h_m(T) d\rho(m).$$

We say that the entropy map of  $(X, T)$  is upper semicontinuous if for  $\mu_n, \mu \in \mathcal{M}(X, T)$

$$\lim_{n \rightarrow \infty} \mu_n = \mu \text{ implies } \limsup_{n \rightarrow \infty} h_{\mu_n}(T) \leq h_\mu(T).$$

We say that a TDS  $(X, T)$  satisfies asymptotic entropy expansiveness if

$$\limsup_{\delta \rightarrow 0} \sup_{x \in X} h_d(T, \Gamma_\delta(x)) = 0.$$

Here for each  $\delta > 0$ ,

$$\Gamma_\delta(x) := \{y \in X : d(T^j x, T^j y) < \delta \text{ for all } j \geq 0\}.$$

The result of Misiurewicz [12, Corollary 4.1] gives a sufficient condition for upper semicontinuity of the entropy map.

**Theorem 2.6.** *Let  $(X, T)$  be a TDS. If  $(X, T)$  satisfies asymptotic entropy expansiveness. Then the entropy map of  $(X, T)$  is upper semicontinuous.*

The result of Buzzi [3] gives a sufficient condition for asymptotic entropy expansiveness.

**Theorem 2.7.** *Let  $f$  be a  $C^\infty$  diffeomorphism on a smooth compact manifold  $M$ , then  $(M, f)$  satisfies asymptotic entropy expansiveness. Especially, the entropy map of  $(M, f)$  is upper semicontinuous.*

**3. Proof of Theorem 1.2 and Theorem 1.3.** In this section, we prove our main results. In the first subsection, we prove that Katok’s conjecture holds for affine transformations of torus. In the second subsection, we show some properties of metrics on nilmanifolds. In the last subsection, we prove Theorem 1.2 and Theorem 1.3.

**3.1. Intermediate entropy for affine transformations of torus.** We say that a topological dynamical system  $(Y, S)$  is *universal* if for every invertible non-atomic ergodic measure preserving system  $(X, \mathcal{X}, \mu, T)$  with measure-theoretic entropy strictly less than the topological entropy of  $S$  there exists  $\nu \in \mathcal{M}^e(Y, S)$  such that  $(X, \mathcal{X}, \mu, T)$  is isomorphic to  $(Y, \mathcal{B}_Y, \nu, S)$ . In [14], Quas and Soo show that toral automorphisms are universal, which implies the conclusion of Katok’s conjecture. By using Quas and Soo’s result, we have the following.

**Theorem 3.1.** *Let  $m \in \mathbb{N}$ ,  $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$  and  $\tau$  be an affine transformation of  $\mathbb{T}^m$ . Then  $\mathcal{E}(\tau) = [0, h_{top}(\tau)]$ .*

*Proof.* We think of  $\mathbb{T}^m$  as a group. Then there exist an element  $b \in \mathbb{T}^m$  and a toral automorphism  $A$  of  $\mathbb{T}^m$  such that

$$\tau(x) = A(x) + b \text{ for each } x \in \mathbb{T}^m.$$

Let  $\mu_h$  be the Haar measure. Then  $h_{\mu_h}(\tau) = h_{top}(\tau)$ . Let  $\mu_h = \int_{\mathcal{M}^e(\mathbb{T}^m, \tau)} \nu d\rho(\nu)$  be the ergodic decomposition of  $\mu_h$ . Then by Theorem 2.5, one has

$$h_{top}(\tau) = \int_{\mathcal{M}^e(\mathbb{T}^m, \tau)} h_\nu(\tau) d\rho(\nu).$$

By variational principle, there exists  $\mu \in \mathcal{M}^e(\mathbb{T}^m, \tau)$  such that  $h_\mu(\tau) = h_{top}(\tau)$ . Now we assume that  $a \in [0, h_{top}(\tau)]$ . We have two cases.

**Case 1.**  $A$  is quasi-hyperbolic. In this case, there is  $q \in \mathbb{T}^m$  such that  $A(q) = q - b$ . We let

$$\pi(x) = x - q \text{ for each } x \in \mathbb{T}^m.$$

Then  $\pi$  is a self homeomorphism of  $\mathbb{T}^m$  and  $\pi \circ \tau = A \circ \pi$ . That is,  $(\mathbb{T}^m, \tau)$  topologically conjugates to a torus automorphism. By Quas and Soo’s result [14, Theorem 1], there exists  $\mu_a \in \mathcal{M}^e(\mathbb{T}^m, \tau)$  such that  $h_{\mu_a}(\tau) = a$ .

**Case 2.**  $A$  is not quasi-hyperbolic. In this case, we put

$$H = \{x \in \mathbb{T}^m : (A - id)^m x = 0\}.$$

Then  $H$  is a compact subgroup of  $\mathbb{T}^m$  and  $\mathbb{T}^m/H$  is a torus. We let  $Y = \mathbb{T}^m/H$  and  $\pi_Y$  be the natural projection from  $\mathbb{T}^m$  to  $Y$ . The induced map  $\tau_Y$  on  $Y$  is a quasi-hyperbolic affine transformation and the extension  $\pi_Y$  is distal. Therefore,  $h_{top}(\tau_Y) = h_{top}(\tau)$  and by Case 1 there exists  $\mu_a^Y \in \mathcal{M}^e(Y, \tau_Y)$  such that  $h_{\mu_a^Y}(\tau_Y) = a$ . There is  $\mu_a \in \mathcal{M}^e(\mathbb{T}^m, \tau)$  such that  $\pi_Y(\mu_a) = \mu_a^Y$ . Since the extension  $\pi_Y$  is distal, one has  $h_{\mu_a}(\tau) = h_{\mu_a^Y}(\tau_Y) = a$  (see [6, Theorem 4.4]).

This ends the proof of Theorem 3.1. □

**3.2. Proof of Theorem 1.2 and Theorem 1.3.** Let  $G$  be a group. For  $g, h \in G$ , we write  $[g, h] = ghg^{-1}h^{-1}$  for the commutator of  $g$  and  $h$  and we write  $[B_1, B_2]$  for the subgroup spanned by  $\{[b_1, b_2] : b_1 \in B_1, b_2 \in B_2\}$ . The commutator subgroups  $G_j, j \geq 1$ , are defined inductively by setting  $G_0 = G_1 = G$  and  $G_{j+1} = [G_j, G]$ . Let  $s \geq 1$  be an integer. We say that  $G$  is *s-step nilpotent* if  $G_{s+1}$  is the trivial subgroup. Recall that an *s-step nilmanifold* is a manifold of the form  $G/\Gamma$  where  $G$  is a connected, simply connected  $s$ -step nilpotent Lie group, and  $\Gamma$  is a cocompact discrete subgroup of  $G$ .

If  $G/\Gamma$  is an  $s$ -step nilmanifold, then for each  $j = 1, \dots, s$ ,  $G_j\Gamma$  and  $G_j$  are closed subgroups of  $G$  and  $G_j\Gamma/\Gamma$  is a closed submanifold of  $G/\Gamma$  (see Subsection 2.11 in [9]).

We fix an  $s$ -step nilmanifold of the form  $G/\Gamma$  and an affine transformation  $\tau$  of  $G/\Gamma$  such that

$$\tau(g\Gamma) = g_0A(g)\Gamma \text{ for each } g \in G$$

where  $g_0 \in G$  and  $A$  is a  $\Gamma$ -invariant automorphism of  $G$ . For each  $j \geq 1$ , we let

$$A_j : G_{j-1}\Gamma/G_j\Gamma \rightarrow G_{j-1}\Gamma/G_j\Gamma : A_j(hG_j\Gamma) = A(h)G_j\Gamma \text{ for each } h \in G_{j-1}$$

and

$$\tau_j : G/G_j\Gamma \rightarrow G/G_j\Gamma : \tau_j(hG_j\Gamma) = g_0A(h)G_j\Gamma \text{ for each } h \in G.$$

It is easy to see that  $\{A_j\}_{j \in \mathbb{N}}$  and  $\{\tau_j\}_{j \in \mathbb{N}}$  are well defined since  $A(G_j) \subset G_j$  for each  $j \geq 1$ .

For each  $j \geq 1$ , define the map  $\pi_{j+1}$  from  $G/G_{j+1}\Gamma$  to  $G/G_j\Gamma$  by

$$\pi_{j+1}(gG_{j+1}\Gamma) = gG_j\Gamma \text{ for each } g \in G. \tag{3.1}$$

It is easy to see that  $\pi_{j+1}$  is continuous and onto, and satisfies  $\pi_{j+1} \circ \tau_{j+1} = \tau_j \circ \pi_{j+1}$ . Hence, for each  $j \geq 1$ ,  $\pi_{j+1} : G/G_{j+1}\Gamma \rightarrow G/G_j\Gamma$  is a factor map. We let  $b_j = h_{top}(A_j)$  for each  $j \geq 1$ . Then we have the following.

**Lemma 3.2.** *For each  $j \geq 1$  and  $y \in G/G_j\Gamma$ ,  $h_{d_{j+1}}(\tau_{j+1}, \pi_{j+1}^{-1}(y)) = b_{j+1}$  where  $d_{j+1}$  is the metric on  $G/G_{j+1}\Gamma$ .*

*Proof.* In Remark 2.4 (2), we let  $N = G_{j+1}\Gamma$  and  $H = G_j\Gamma$ . Then both  $N$  and  $H$  are cocompact subgroup of  $G$ . Moreover,  $N$  is a normal subgroup of  $H$ . Hence  $(G/N = G/G_{j+1}\Gamma, \tau_{j+1})$  is an  $(H/N = G_j\Gamma/G_{j+1}\Gamma, A_{j+1})$ -extension of  $(G/H = G/G_j\Gamma, \tau_j)$ . By (2.3), one has

$$h_{d_{j+1}}(\tau_{j+1}, \pi_{j+1}^{-1}(y)) = h_{top}(A_{j+1}) = b_{j+1} \text{ for every } y \in G/G_j\Gamma.$$

This ends the proof of Lemma 3.2. □

The following result is immediately from Lemma 3.2, (2.1) and Theorem 2.7.

**Lemma 3.3.** *For  $j \geq 1$  and  $\nu_j \in \mathcal{M}(G/G_j\Gamma, \tau_j)$ , there exists  $\mu \in \mathcal{M}(G/G_{j+1}\Gamma, \tau_{j+1})$  such that  $h_\mu(\tau_{j+1}) = h_{\nu_j}(\tau_j) + b_{j+1}$ .*

We have the following.

**Corollary 3.4.**  *$h_{top}(\tau_j) = \sum_{i=1}^j b_i$  for  $j \geq 1$ . Especially,  $h_{top}(\tau) = \sum_{i=1}^{s+1} b_i$ .*

*Proof.* We prove the corollary by induction on  $j$ . In the case  $j = 1$ , it is obviously true. Now we assume that the corollary is valid for some  $j \in \mathbb{N}$ . Then for  $j + 1$ ,

let  $\pi_{j+1}$  be defined as in (3.1). Then by Ledrappier and Walters’s result (2.1) and variational principle Theorem 2.2, we have

$$\begin{aligned} h_{top}(\tau_{j+1}) &= \sup_{\mu \in \mathcal{M}(G/G_{j+1}\Gamma, \tau_{j+1})} h_{\mu}(\tau_{j+1}) \\ &\leq \sup_{\mu \in \mathcal{M}(G/G_j\Gamma, \tau_j)} \left( h_{\mu}(\tau_j) + \int_{G/G_j\Gamma} h_{d_{j+1}}(\tau_{j+1}, \pi_{j+1}^{-1}(y)) d\mu(y) \right) \\ &\leq h_{top}(\tau_j) + \sup_{\mu \in \mathcal{M}(G/G_j\Gamma, \tau_j)} \int_{G/G_j\Gamma} h_{d_{j+1}}(\tau_{j+1}, \pi_{j+1}^{-1}(y)) d\mu(y) \\ &= \sum_{i=1}^j b_i + b_{j+1} = \sum_{i=1}^{j+1} b_i, \end{aligned}$$

where we used Lemma 3.2. On the other hand, by Lemma 3.3 there exists  $\mu \in \mathcal{M}(G/G_{j+1}\Gamma, \tau_{j+1})$  such that  $h_{\mu}(\tau_{j+1}) = \sum_{i=1}^{j+1} b_i$ . Therefore  $h_{top}(\tau_{j+1}) = \sum_{i=1}^{j+1} b_i$ . By induction, this ends the proof of Corollary 3.4.  $\square$

**Remark 3.5.** We remark that the topological entropy of  $(G/\Gamma, \tau)$  is determined by the associated matrix of  $\tau$  [1]. That is

$$h_{top}(\tau) = h_d(\tau) = \sum_{|\lambda_i| > 1} \log |\lambda_i|$$

where  $\lambda_1, \lambda_2, \dots, \lambda_m$  are the eigenvalues of the associated matrix of  $\tau$ .

**Lemma 3.6.** For  $j \geq 1$  and  $\nu_j \in \mathcal{M}^e(G/G_j\Gamma, \tau_j)$ , there is  $\nu_{j+1} \in \mathcal{M}^e(G/G_{j+1}\Gamma, \tau_{j+1})$  such that  $h_{\nu_{j+1}}(\tau_{j+1}) = h_{\nu_j}(\tau_j) + b_{j+1}$ .

*Proof.* We fix  $\nu_j \in \mathcal{M}^e(G/G_j\Gamma, \tau_j)$ . Let  $\pi_{j+1}$  be defined as in (3.1). By Lemma 3.3, there exists  $\nu \in \mathcal{M}(G/G_{j+1}\Gamma, \tau_{j+1})$  such that

$$h_{\nu}(\tau_{j+1}) = \sup_{\substack{\mu \in \mathcal{M}(G/G_{j+1}\Gamma, \tau_{j+1}) \\ \pi_{j+1}(\mu) = \nu_j}} h_{\mu}(\tau_{j+1}) = h_{\nu_j}(\tau_j) + b_{j+1}.$$

We fix such  $\nu$  and assume that the ergodic decomposition of  $\nu$  is

$$\nu = \int_{\mathcal{M}^e(G/G_{j+1}\Gamma, \tau_{j+1})} m d\rho(m).$$

Then by property of ergodic decomposition, one has

$$\rho(\{m \in \mathcal{M}^e(G/G_{j+1}\Gamma, \tau_{j+1}) : \pi_{j+1}(m) = \nu_j\}) = 1.$$

Therefore, for  $\rho$ -a.e.  $m \in \mathcal{M}^e(G/G_{j+1}\Gamma, \tau_{j+1})$ ,

$$h_m(\tau_{j+1}) \leq h_{\nu}(\tau_{j+1}) = h_{\nu_j}(\tau_j) + b_{j+1}.$$

Hence by Theorem 2.5, one has

$$h_{\nu_j}(\tau_j) + b_{j+1} = h_{\nu}(\tau_{j+1}) = \int_{\mathcal{M}^e(G/G_{j+1}\Gamma, \tau_{j+1})} h_m(\tau_{j+1}) d\rho(m) \leq h_{\nu_j}(\tau_j) + b_{j+1}.$$

We notice that the equality holds only in the case  $h_m(\tau_{j+1}) = h_{\nu_j}(\tau_j) + b_{j+1}$  for  $\rho$ -a.e.  $m \in \mathcal{M}^e(G/G_{j+1}\Gamma, \tau_{j+1})$ . Therefore, there exists  $\nu_{j+1} \in \mathcal{M}^e(G/G_{j+1}\Gamma, \tau_{j+1})$  such that

$$h_{\nu_{j+1}}(\tau_{j+1}) = h_{\nu_j}(\tau_j) + b_{j+1} \text{ and } \pi_{j+1}(\nu_{j+1}) = \nu_j.$$

This ends the proof of Lemma 3.6.  $\square$

Now we are ready to prove our main results.

*Proof of Theorem 1.2.* Firstly we assume that  $(G/\Gamma, \tau)$  has a fixed point  $p\Gamma$ . We fix a real number  $a \in [0, h_{top}(\tau)]$ . We are going to show that there exists  $\mu_a \in \mathcal{M}^e(G/\Gamma, \tau)$  such that  $h_{\mu_a}(\tau) = a$ . By Corollary 3.4, we can find an  $i \in \{1, 2, \dots, s, s + 1\}$  such that

$$\sum_{j=i+1}^{s+1} b_j \leq a \leq \sum_{j=i}^{s+1} b_j.$$

Since  $p\Gamma$  is a fixed point of  $(G/\Gamma, \tau)$ , there exists  $\gamma \in \Gamma$  such that  $g_0A(p) = p\gamma$ . Therefore,

$$\tau_i(pG_{i-1}\Gamma/G_i\Gamma) = p\gamma G_{i-1}\Gamma/G_i\Gamma \subset p[\gamma, G_{i-1}]G_{i-1}\Gamma/G_i\Gamma \subset pG_{i-1}\Gamma/G_i\Gamma,$$

where we used the fact  $[\gamma, G_{i-1}] \subset G_{i-1}$ . That is,  $(pG_{i-1}\Gamma/G_i\Gamma, \tau_i)$  is a TDS. We let

$$\pi(phG_i\Gamma) = hG_i\Gamma \text{ for each } h \in G_{i-1}.$$

Then for each  $h \in G_{i-1}$ , one has

$$\pi \circ \tau_i(phG_i\Gamma) = p^{-1}g_0A(p)A(h)G_i\Gamma = \gamma A(h)G_i\Gamma = A(h)\gamma[\gamma, A(h)]G_i\Gamma = A(h)G_i\Gamma$$

where we used the fact  $[\gamma, A(h)] \in G_i$  since  $h \in G_{i-1}$ . Therefore  $\pi \circ \tau_i(phG_i\Gamma) = A_i \circ \tau_i(phG_i\Gamma)$  for each  $h \in G_{i-1}$ . That is  $\pi \circ \tau_i = A_i \circ \pi$ . Hence,

$$(pG_{i-1}\Gamma/G_i\Gamma, \tau_i) \text{ topologically conjugates to } (G_{i-1}\Gamma/G_i\Gamma, A_i).$$

Notice that  $(G_{i-1}\Gamma/G_i\Gamma, A_i)$  is a toral automorphism and  $h_{top}(A_i) = b_i$ . By Theorem 3.1, there exists  $\nu_i \in \mathcal{M}^e(G/G_i\Gamma, \tau_i)$  such that  $h_{\nu_i}(\tau_i) = a - \sum_{j=i+1}^{s+1} b_j$ . Combining this with Lemma 3.6, there exists an ergodic measure  $\mu_a = \nu_{s+1} \in \mathcal{M}^e(G/G_{s+1}\Gamma, \tau_{s+1}) = \mathcal{M}^e(G/\Gamma, \tau)$  such that

$$h_{\mu_a}(\tau) = h_{\nu_{s+1}}(\tau_{s+1}) = h_{\nu_i}(\tau_i) + \sum_{j=i+1}^{s+1} b_j = a.$$

Thus  $\mu_a$  is the ergodic measure as required.

Now we assume that  $(G/\Gamma, \tau)$  has a periodic point. By assumption, there exists  $m \in \mathbb{N}$  such that  $(G/\Gamma, \tau^m)$  has a fixed point. Since  $\tau^m$  is an affine transformation of  $G/\Gamma$ , by argument above, there exists  $\mu \in \mathcal{M}^e(G/\Gamma, \tau^m)$  such that  $h_{\mu}(\tau^m) = ma$ . Put  $\mu_a = \frac{1}{m} \sum_{j=0}^{m-1} \tau^j(\mu)$ . It is easy to see that  $\mu_a \in \mathcal{M}^e(G/\Gamma, \tau)$  and  $h_{\mu_a}(\tau) = \frac{h_{\mu}(\tau^m)}{m} = a$ . Thus  $\mu_a$  is the ergodic measure as required.

This ends the proof of Theorem 1.2. □

**Proposition 3.7.** *Let  $G$  be an  $s$ -step nilpotent Lie group and  $A$  be a quasi-hyperbolic automorphism of  $G$ . Then for  $g \in G$ , there exists  $p \in G$  such that  $gA(p) = p$ .*

*Proof.* We prove the proposition by induction on  $s$ . In the case  $s = 1$ , it is obviously true. Now we assume that the Proposition is valid in the case  $s = k$ . Then in the case  $s = k + 1$ , we fix  $g \in G$ . Notice that  $G/G_{k+1}$  is a  $k$ -step nilpotent Lie group. There exists  $\tilde{p} \in G$  such that  $gA(\tilde{p})G_{k+1} = \tilde{p}G_{k+1}$ . There exists  $\tilde{g} \in G_{k+1}$  such that  $gA(\tilde{p})\tilde{g} = \tilde{p}$ . There exists  $p' \in G_{k+1}$  such that  $\tilde{g}^{-1}A(p') = p'$ . In the end, we let  $p = \tilde{p}p'$ . Then

$$gA(p) = gA(\tilde{p})A(p') = \tilde{p}\tilde{g}^{-1}\tilde{g}p' = \tilde{p}p' = p.$$

By induction, we end the proof of Proposition 3.7. □

*Proof of Theorem 1.3.* This comes immediately from Proposition 3.7 and Theorem 1.2. □



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