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GRADED POST-LIE ALGEBRA STRUCTURES AND HOMOGENEOUS ROTA-BAXTER OPERATORS ON THE SCHRÖDINGER-VIRASORO ALGEBRA

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Abstract. In this paper, we characterize the graded post-Lie algebra structures on the Schrödinger-Virasoro Lie algebra. Furthermore, as an application, we obtain the all homogeneous Rota-Baxter operator of weight 1 on the Schrödinger-Virasoro Lie algebra.

1. **Introduction.** The Schrödinger-Virasoro algebra is an infinite-dimensional Lie algebra that was introduced (see, e.g.,[10]) in the context of non-equilibrium statistical physics. In [21], the author give a representation of the Schrödinger-Virasoro algebra by using vertex algebras, and introduced an extension of the Schrödinger-Virasoro algebra. To be precise, for $\varepsilon \in \{0, \frac{1}{2}\}$, the Schrödinger-Virasoro algebra $\mathcal{SV}(\varepsilon)$ is a Lie algebra with the $\mathbb{C}$ basis

$$\{L_i, H_j, I_i | i \in \mathbb{Z}, j \in \varepsilon + \mathbb{Z}\}$$

and Lie brackets

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n},\\
&[L_m, H_n] = (\frac{1}{2}m - n)H_{m+n},\\
&[L_m, I_n] = -nI_{m+n},\\
&[H_m, H_n] = (m-n)I_{m+n},\\
&[H_m, I_n] = [I_m, I_n] = 0.
\end{aligned}$$

The Lie algebra $\mathcal{SV}(\frac{1}{2})$ is called the original Schrödinger-Virasoro algebra, and $\mathcal{SV}(0)$ is called the twisted Schrödinger-Virasoro algebra. Recently, the theory of the structure and representations of both original and twisted Schrödinger-Virasoro algebra has been investigated in a series of studies. For instance, the Lie bialgebra structures, (bi)derivations, automorphisms, 2-cocycles, vertex algebra representations and Whittaker modules were investigated in [9, 11, 14, 15, 21].

Post-Lie algebras were introduced around 2007 by B. Vallette [22], who found the structure in a purely operadic manner as the Koszul dual of a commutative

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trialgebra. Post-Lie algebras have arose the interest of a great many authors, see [4, 5, 12, 13]. One of the most important problems in the study of post-Lie algebras is to find the post-Lie algebra structures on the (given) Lie algebras. In [13, 18, 20], the authors determined all post-Lie algebra structures on $sl(2,\mathbb{C})$ of special linear Lie algebra of order 2, the Witt algebra and the W-algebra $W(2,2)$ respectively.

In this paper, we shall study the graded post-Lie algebra structures on the Schrödinger-Virasoro algebra. We only study the twisted Schrödinger-Virasoro algebra $\mathcal{SV}(0)$, the case for the original Schrödinger-Virasoro algebra $\mathcal{SV}(\frac{1}{2})$ is similar. For convenience we denote $\mathcal{S}=\mathcal{SV}(0)$. It should be noted that the commutative post-Lie algebra structures on $\mathcal{S}$ already are given by [11], we will consider the general case.

Throughout this paper, we denote by $\mathbb{Z}$ the set of all integers. For a subset S of $\mathbb{Z}$ and a fixed integer k, denote $S^*=S\setminus\{0\}$, $S_{>k}=\{t\in S|t>k\}$, $S_{<k}=\{t\in S|t<k\}$, $S_{\geqslant k}=\{t\in S|t\geqslant k\}$ and $S_{\leqslant k}=\{t\in S|t\leqslant k\}$. We assume that the field in this paper always is the complex number field $\mathbb{C}$.

The paper is organized as follows. In Section 2, we give general results on post-Lie algebras and some lemmas which will be used to our proof. In Section 3, we completely characterize the graded post-Lie algebra structures on Schrödinger-Virasoro algebra $\mathcal{S}$. In Section 4, by using the post-Lie algebra structures we characterize the forms of the homogeneous Rota-Baxter operator on $\mathcal{S}$.

2. **Preliminaries.** We will give the essential definitions and results as follows.

Definition 2.1. A post-Lie algebra $(V,\triangleright,[,])$ is a vector space V over a field k equipped with two k-bilinear products $x\triangleright y$ and $[x,y]$ satisfying that $(V,[,])$ is a Lie algebra and

$$[x,y]\triangleright z=x\triangleright(y\triangleright z)-y\triangleright(x\triangleright z)-\langle x,y\rangle\triangleright z, \tag{1}$$

$$x\triangleright[y,z]=[x\triangleright y,z]+[y,x\triangleright z] \tag{2}$$

for all $x,y\in V$, where $\langle x,y\rangle=x\triangleright y-y\triangleright x$. We also say that $(V,\triangleright,[,])$ is a post-Lie algebra structure on the Lie algebra $(V,[,])$. If a post-Lie algebra $(V,\triangleright,[,])$ satisfies $x\triangleright y=y\triangleright x$ for all $x,y\in V$, then it is called a commutative post-Lie algebra.

Suppose that $(L,[,])$ is a Lie algebra. Two post-Lie algebras $(L,[,],\triangleright_1)$ and $(L,[,],\triangleright_2)$ on the Lie algebra L are called to the isomorphic if there is an automorphism τ of the Lie algebra $(L,[,])$ satisfies

$$\tau(x\triangleright_1 y)=\tau(x)\triangleright_2\tau(y),\forall x,y\in L.$$

Remark 1. The left multiplications of the post-Lie algebra $(V,[,],\triangleright)$ are denoted by $\mathcal{L}$, i.e., we have $\mathcal{L}(x)(y)=x\triangleright y$ for all $x,y\in V$. By (2), we see that all operator $\mathcal{L}(x)$ are Lie algebra derivations of the Lie algebra (V,[,]).

Lemma 2.2. [15] *Denote by $Der(\mathcal{S})$ and by $Inn(\mathcal{S})$ the space of derivations and the space of inner derivations of $\mathcal{S}$ respectively. Then*

$$Der(\mathcal{S})=Inn(\mathcal{S})\oplus\mathbb{C}D_1\oplus\mathbb{C}D_2\oplus\mathbb{C}D_3$$

where D_1,D_2,D_3 are outer derivations defined by

$$D_1(L_n)=0,D_1(H_n)=H_n,D_1(I_n)=2I_n,$$
$$D_2(L_n)=nI_n,D_2(H_n)=0,D_2(I_n)=0,$$
$$D_3(L_n)=I_n,D_3(H_n)=0,D_3(I_n)=0.$$

3. **The graded post-Lie algebra structures on the Schrödinger-Virasoro algebra.** Since the Schrödinger-Virasoro algebra $\mathcal{S}$ is graded, we suppose that the post-Lie algebra structure on the Schrödinger-Virasoro algebra $\mathcal{S}$ to be graded. Namely, we mainly consider the post-Lie algebra structure on Schrödinger-Virasoro algebra $\mathcal{S}$ which satisfies

$$\begin{aligned} L_m \triangleright L_n &= \phi(m,n)L_{m+n}, && (3)\\ L_m \triangleright H_n &= \varphi(m,n)H_{m+n}, && (4)\\ L_m \triangleright I_n &= \chi(m,n)I_{m+n}, && (5)\\ H_m \triangleright L_n &= \psi(m,n)H_{m+n}, && (6)\\ H_m \triangleright H_n &= \xi(m,n)I_{m+n}, && (7)\\ I_m \triangleright L_n &= \theta(m,n)I_{m+n}, && (8)\\ H_m \triangleright I_n &= I_m \triangleright H_n = I_m \triangleright I_n = 0, && (9) \end{aligned}$$

for all $m, n \in \mathbb{Z}$, where ϕ, φ, χ, ψ, ξ, θ are complex-valued functions on $\mathbb{Z} \times \mathbb{Z}$.

We start with the crucial lemma.

Lemma 3.1. *There exists a graded post-Lie algebra structure on $\mathcal{S}$ satisfying (3)-(9) if and only if there are complex-valued functions f, g, h on $\mathbb{Z}$ and complex numbers a, μ such that*

$$\phi(m,n) = (m-n)f(m), \tag{10}$$
$$\varphi(m,n) = (\frac{m}{2} - n)f(m) + \delta_{m,0}\mu, \tag{11}$$
$$\chi(m,n) = -nf(m) + 2\delta_{m,0}\mu, \tag{12}$$
$$\psi(m,n) = -(\frac{n}{2} - m)h(m), \tag{13}$$
$$\xi(m,n) = (m-n)h(m), \tag{14}$$
$$\theta(m,n) = mg(m) + \delta_{m,0}na, \tag{15}$$
$$(m-n)(f(m+n)(1+f(m)+f(n)) - f(n)f(m)) = 0, \tag{16}$$
$$(m-n)\delta_{m+n,0}\mu(1+f(m)+f(n)) = 0, \tag{17}$$
$$(\frac{m}{2} - n)(h(m+n)(1+f(m)+h(n)) - f(m)h(n)) = 0, \tag{18}$$
$$n\delta_{m+n,0}a(1+f(m)+g(n)) = 0, \tag{19}$$
$$\begin{aligned} &n(m+n)(g(m+n)(1+f(m)+g(n)) - f(m)g(n))\\ &\quad = \delta_{n,0}m^2a(f(m)-g(m)), \end{aligned} \tag{20}$$
$$(m-n)\delta_{m+n,0}a(1+h(m)+h(n)) = 0, \tag{21}$$
$$(m-n)(g(m+n)(1+h(m)+h(n)) - h(m)h(n)) = 0. \tag{22}$$

Proof. Suppose that there exists a graded post-Lie algebra structure satisfying (3)-(9) on $\mathcal{S}$. By Remark 1, $\mathcal{L}(x)$ is a derivation of $\mathcal{S}$. It follows by Lemma 2.2 that there are a linear map ψ from $\mathcal{S}$ into itself and linear functions α, β, γ on $\mathcal{S}$ such that

$$\begin{aligned} x \triangleright y &= (ad\psi(x) + \alpha(x)D_1 + \beta(x)D_2 + \gamma(x)D_3)(y)\\ &= [\psi(x), y] + \alpha(x)D_1(y) + \beta(x)D_2(y) + \gamma(x)D_3(y) \end{aligned}$$

where $D_i, i = 1, 2, 3$ are given by Lemma 2.2. This, together with (3)-(9), gives that

$$L_m \triangleright L_n = [\psi(L_m), L_n] + \beta(L_m)nI_n + \gamma(L_m)I_n = \phi(m,n)L_{m+n}, \tag{23}$$

$$\begin{aligned}
L_m \triangleright H_n &= [\psi(L_m), H_n] + \alpha(L_m)H_n = \varphi(m,n)H_{m+n}, & (24)\\
L_m \triangleright I_n &= [\psi(L_m), I_n] + \alpha(L_m)2I_n = \chi(m,n)I_{m+n}, & (25)\\
H_m \triangleright L_n &= [\psi(H_m), L_n] + \beta(H_m)nI_n + \gamma(H_m)I_n = \psi(m,n)H_{m+n}, & (26)\\
H_m \triangleright H_n &= [\psi(H_m), H_n] + \alpha(H_m)H_n = \xi(m,n)I_{m+n}, & (27)\\
H_m \triangleright I_n &= [\psi(H_m), I_n] + \alpha(H_m)2I_n = 0, & (28)\\
I_m \triangleright L_n &= [\psi(I_m), L_n] + \beta(I_m)nI_n + \gamma(I_m)I_n = \theta(m,n)I_{m+n}, & (29)\\
I_m \triangleright H_n &= [\psi(I_m), H_n] + \alpha(I_m)H_n = 0, & (30)\\
I_m \triangleright I_n &= [\psi(I_m), I_n] + \alpha(I_m)2I_n = 0. & (31)
\end{aligned}$$

Let

$$\begin{aligned}
\psi(L_m) &= \textstyle\sum_{i\in\mathbb{Z}} a_i^{(m)}L_i + \sum_{i\in\mathbb{Z}} b_i^{(m)}H_i + \sum_{i\in\mathbb{Z}} c_i^{(m)}I_i,\\
\psi(H_m) &= \textstyle\sum_{i\in\mathbb{Z}} d_i^{(m)}L_i + \sum_{i\in\mathbb{Z}} e_i^{(m)}H_i + \sum_{i\in\mathbb{Z}} f_i^{(m)}I_i,\\
\psi(I_m) &= \textstyle\sum_{i\in\mathbb{Z}} g_i^{(m)}L_i + \sum_{i\in\mathbb{Z}} h_i^{(m)}H_i + \sum_{i\in\mathbb{Z}} x_i^{(m)}I_i
\end{aligned}$$

where $a_i^{(m)}, b_i^{(m)}, c_i^{(m)}, d_i^{(m)}, e_i^{(m)}, f_i^{(m)}, g_i^{(m)}, h_i^{(m)}, x_i^{(m)} \in \mathbb{C}$ for all $i \in \mathbb{Z}$. Then by (23)-(31), similar to the proof of [18], we obtain that (10)-(22) hold.

The "if" part is a direct checking. The proof is completed. □

Lemma 3.2. *Let f, g, h be complex-valued functions on $\mathbb{Z}$ and $\mu, a \in \mathbb{C}$ satisfying (18) and (20). Then we have*

$$g(n), h(n) \in \{0, -1\} \text{ for every } n \neq 0. \tag{32}$$

Proof. By letting $m = 0$ in (18) and (20), respectively, we have $nh(n)(1+h(n)) = 0$ and $n^2g(n)(1+g(n)) = 0$. This implies (32). □

Lemma 3.3. *Let f, g, h be complex-valued functions on $\mathbb{Z}$ and μ, a be complex numbers satisfying (17)-(22). If $f(\mathbb{Z}) = 0$, then we have $\mu = a = 0$ and*

$$g(\mathbb{Z}) = h(\mathbb{Z}) = 0 \text{ or } g(\mathbb{Z}) = h(\mathbb{Z}) = -1.$$

Proof. Since $f(\mathbb{Z}) = 0$, we take $m = -n = 1$ in (17) and (19) we have $\mu = 0$ and

$$a(1 + g(-1)) = 0. \tag{33}$$

By letting $n = 0$ and $m = -1$ in (20) we deduce that $ag(-1) = 0$. This, together with (33), implies $a = 0$. As $\mu = a = 0$, so Equations (18), (20) and (22) become to

$$(\frac{m}{2} - n)(h(m+n)(1 + h(n)) = 0, \tag{34}$$

$$n(m+n)(g(m+n)(1 + g(n)) = 0, \tag{35}$$

$$(m-n)(g(m+n) - h(m)h(n) + h(m)g(m+n) + h(n)g(m+n)) = 0. \tag{36}$$

We now prove the following four claims:

Claim 1. *If $h(1) = 0$, then $h(\mathbb{Z}) = 0$.*

By (34) with $n = 1$ we see that $h(m+1) = 0$ for all $m \neq 2$. It follows that $h(\mathbb{Z}\backslash\{3\}) = 0$. Since $h(2) = 0$, by taking $n = 2, m = 1$ in (34) we have $-\frac{3}{2}h(3) = 0$, which implies $h(3) = 0$. We obtain $h(\mathbb{Z}) = 0$.

Claim 2. *If $h(1) = -1$, then $h(\mathbb{Z}) = -1$.*

By (34) with $m + n = 1$ we see that $h(n) = -1$ for all $n \in \mathbb{Z}$ with $\frac{1-3n}{2} \neq 0$. This means that $h(\mathbb{Z}) = -1$.

Claim 3. *If $g(1) = 0$, then $g(\mathbb{Z}^*) = 0$.*

By (35) with $n = 1$ we see that $g(m + 1) = 0$ for all $m \neq -1$. It follows that $g(\mathbb{Z}^*) = 0$.

Claim 4. *If $g(1) = -1$, then $g(\mathbb{Z}^*) = -1$.*

By (35) with $m + n = 1$ we see that $g(n) = -1$ for all $n \neq 0$. This means that $g(\mathbb{Z}^*) = -1$.

Now we consider the values of $h(1)$ and $g(1)$ according to (32).

Case i. If $h(1) = g(1) = 0$, then by Claims 1 and 3 we have $h(\mathbb{Z}) = 0$ and $g(\mathbb{Z}^*) = 0$. According to (36) with $n = -1$ and $m = 1$ we know $g(0) = 0$. This means that $g(\mathbb{Z}) = 0$.

Case ii. If $h(1) = g(1) = -1$, then by Claims 2 and 4 we have $h(\mathbb{Z}) = -1$ and $g(\mathbb{Z}^*) = -1$. According to (36) with $n = -1$ and $m = 1$ we see that $1 + g(0) = 0$ and so that $g(0) = -1$. This implies $g(\mathbb{Z}) = -1$.

Case iii. If $h(1) = 0$, $g(1) = -1$, then we will get a contradiction. In fact, by Claims 1 and 4, we have $h(\mathbb{Z}) = 0$ and $g(\mathbb{Z}^*) = -1$. From (36) with $m = 2, n = -1$ we see that $g(1) = 0$ which contradicts $g(1) = -1$.

Case iv. If $h(1) = -1$, $g(1) = 0$, then we will also get a contradiction. In fact, by Claims 2 and 3, we have $h(\mathbb{Z}) = -1$ and $g(\mathbb{Z}^*) = 0$. From (36) with with $m = 2$, $n = -1$ we see that $g(1) = -1$ which contradicts $g(1) = 0$. The proof is completed. □

Lemma 3.4. *Let f, g, h be complex-valued functions on $\mathbb{Z}$ and μ, a be complex numbers satisfying (17)-(22). If $f(\mathbb{Z}_{\geqslant 2}) = -1, f(\mathbb{Z}_{\leqslant 1}) = 0$, then $\mu = a = 0$ and g, h must satisfy one of the following forms:*

(i) $g(\mathbb{Z}) = h(\mathbb{Z}) = 0$;

(ii) $g(\mathbb{Z}) = h(\mathbb{Z}) = -1$;

(iii) $h(\mathbb{Z}_{\leqslant 0}) = 0$, $h(\mathbb{Z}_{\geqslant 1}) = -1$ and
$g(\mathbb{Z}_{\leqslant -1}) = 0$, $g(\mathbb{Z}_{\geqslant 1}) = -1$, $g(0) = \hat{\lambda}$ for some $\hat{\lambda} \in \mathbb{C}$.

Proof. By $f(\mathbb{Z}_{\geqslant 2}) = -1, f(\mathbb{Z}_{\leqslant 1}) = 0$, similar to the proof of Lemma 3.3, we know $\mu = a = 0$. From this, we have by (18), (20) and (22) that

$$h(m + n)(h(n) + 1) = 0 \text{ if } m \leqslant 1, \frac{m}{2} - n \neq 0, \tag{37}$$

$$g(m + n)(g(n) + 1) = 0 \text{ if } m \leqslant 1, n \neq 0, m + n \neq 0, \tag{38}$$

$$g(m + n)(1 + h(m) + h(n)) = h(m)h(n) \text{ if } m \neq n. \tag{39}$$

We first prove the following six claims:

Claim 1. *If $h(1) = 0$, then $h(\mathbb{Z}) = 0$.*

By (37) with $n = 1$ we see that $h(m + 1) = 0$ for all $\frac{m}{2} - 1 \neq 0$ with $m \leqslant 1$. Hence, we deduce that $h(\mathbb{Z}_{\leqslant 2}) = 0$. Note that $h(2) = 0$, by (37) with $n = 2$ we see that $h(m + 2) = 0$ for all $\frac{m}{2} - 2 \neq 0$ with $m \leqslant 1$. We now have $h(\mathbb{Z}_{\leqslant 3}) = 0$. If we repeat this process, we see that $h(\mathbb{Z}_{\leqslant k}) = 0$ for all $k = 1, 2, 3, \cdots$. Note that $\bigcup_{k \geqslant 1}(\mathbb{Z}_{\leqslant k}) = \mathbb{Z}$, so one has $h(\mathbb{Z}) = 0$.

Claim 2. *If $h(-1) = -1$, then $h(\mathbb{Z}) = -1$.*

By (37) with $m + n = -1$ we see that $h(n) = h(-1 - m) = -1$ for all $\frac{3m}{2} + 1 \neq 0$ with $m \leqslant 1$. This deduces that $h(\mathbb{Z}_{\geqslant -2}) = -1$. Note that $h(-2) = -1$, by (37) with $m + n = -2$ we see that $h(-m - 2) = -1$ for all $\frac{3m}{2} + 2 \neq 0$ with $m \leqslant 1$.

Thus, $h(\mathbb{Z}_{\geqslant -3}) = -1$. If we repeat this process, we see that $h(\mathbb{Z}_{\geqslant k}) = -1$ for all $k = -1, -2, -3, \cdots$. Note that $\bigcup_{k\leqslant -1}(\mathbb{Z}_{\geqslant k}) = \mathbb{Z}$, so one has $h(\mathbb{Z}) = -1$.

Claim 3. *If $h(1) = -1$, then $h(\mathbb{Z}_{\geqslant 1}) = -1$.*

By (37) with $m + n = 1$ we see that $h(n) = h(1 - m) = -1$ for all $\frac{3m}{2} - 1 \neq 0$ with $m \leqslant 1$. This implies $h(\mathbb{Z}_{\geqslant 1}) = -1$.

Claim 4. *If $h(-1) = 0$, then $h(\mathbb{Z}_{\leqslant 0}) = 0$.*

By (37) with $n = -1$ we see that $h(m - 1) = 0$ for all $m \neq -2$ with $m \leqslant 1$. It follows that $h(\mathbb{Z}_{\leqslant 0} \setminus \{-3\}) = 0$. Let $m = -1, n = -2$ in (37), from $\frac{m}{2} \neq n$ we have $h(-3) = 0$. Therefore, we get $h(\mathbb{Z}_{\leqslant 0}) = 0$.

Next, similar to Claims 1 and 3, we from (38) obtain the following claims.

Claim 5. *If $g(1) = 0$, then $g(\mathbb{Z}^*) = 0$.*

Claim 6. *If $g(1) = -1$, then $g(\mathbb{Z}_{\geqslant 1}) = -1$.*

Now we discuss the values of $h(1)$ and $h(-1)$. By (32), $h(1), h(-1) \in \{-1, 0\}$.

Case i. When $h(1) = 0$.

By Claim 1 we have $h(\mathbb{Z}) = 0$. According to (39), one has $g(m + n) = 0$ for any $m, n \in \mathbb{Z}$ with $m \neq n$. This implies $g(\mathbb{Z}) = 0$.

Case ii. When $h(-1) = -1$.

By Claim 2 we have $h(\mathbb{Z}) = -1$. According to (39), one has $g(m + n) = -1$ for any $m, n \in \mathbb{Z}$ with $m \neq n$. This implies $g(\mathbb{Z}) = -1$.

Case iii. When $h(1) = -1$ and $h(-1) = 0$.

By Claims 3 and 4 we have $h(\mathbb{Z}_{\leqslant 0}) = 0$ and $h(\mathbb{Z}_{\geqslant 1}) = -1$. This, together with (39), yields $g(m + n) = 0$ for any $m, n \in \mathbb{Z}$ with $m, n \leqslant 0$ and $m \neq n$, and $g(m + n) = -1$ for any $m, n \in \mathbb{Z}$ with $m, n \geqslant 1$ and $m \neq n$. Consequently, we obtain $g(\mathbb{Z}_{\leqslant -1}) = 0$ and $g(\mathbb{Z}_{\geqslant 3}) = -1$. By (32), $g(1) \in \{-1, 0\}$. If $g(1) = 0$, then Claim 5 tells us that $g(\mathbb{Z}^*) = 0$ which contracts $g(\mathbb{Z}_{\geqslant 3}) = -1$. Therefore, we have $g(1) = -1$. From this with Claim 6 we have $g(\mathbb{Z}_{\geqslant 1}) = -1$. Let $g(0) = \hat{\lambda}$ for some $\hat{\lambda} \in \mathbb{C}$.

It is easy to check that the values of g given in Cases i-iii above are consistent with (38). They give the conclusions (i), (ii) and (iii) respectively. The proof is completed. □

Lemma 3.5. *Let f, g, h be complex-valued functions on $\mathbb{Z}$ and μ, a be complex numbers satisfying (17)-(22). If $f(\mathbb{Z}_{>0}) = -1$, $f(\mathbb{Z}_{<0}) = 0$ and $f(0) = c$ for some $c \in \mathbb{C}$, then there are $\lambda, \hat{\tau} \in \mathbb{C}$ such that μ, a, g, h must be one of the following forms:*

(i) $a = 0$, $\mu \in \mathbb{C}$ and $g(\mathbb{Z}) = h(\mathbb{Z}) = 0$;

(ii) $a = 0$, $\mu \in \mathbb{C}$ and $g(\mathbb{Z}) = h(\mathbb{Z}) = -1$;

(iii) $\mu \in \mathbb{C}$, $h(\mathbb{Z}_{>0}) = -1$, $h(\mathbb{Z}_{<0}) = 0$, $h(0) = \lambda$ and $g(\mathbb{Z}^*_{\geqslant k}) = -1$, $g(\mathbb{Z}^*_{\leqslant k-1}) =$ 0

for some $k \in \{-2, -1, 1, 2, 3\}$, $g(0) = \hat{\tau}$ and $a = 0$ when $k \neq 1$;

(iv) $a = 0$, $\mu \in \mathbb{C}$ and $h(\mathbb{Z}_{\geqslant t}) = -1$, $h(\mathbb{Z}_{\leqslant t-1}) = 0$ for some $t \in \mathbb{Z} \setminus \{0, 1\}$ and $g(\mathbb{Z}_{\geqslant s}) = -1$, $g(\mathbb{Z}_{\leqslant s-1}) = 0$ for some $s \in \{2t - 2, 2t - 1, 2t, 2t + 1, 2t + 2\}$.

Proof. Take $m = -n \neq 0$ in (18)-(22), one has

$$h(0)(1 + f(-n) + h(n)) = f(-n)h(n), \text{ for all } n \neq 0, \tag{40}$$

$$a(1 + f(-n) + g(n)) = 0, \text{ for all } n \neq 0, \tag{41}$$

$$a(1+h(-n)+h(n))=0,\ \text{for all } n\neq 0, \tag{42}$$
$$g(0)(1+h(-n)+h(n))=h(-n)h(n),\ \text{for all } n\neq 0. \tag{43}$$

Note that $f(\mathbb{Z}_{>0})=-1, f(\mathbb{Z}_{<0})=0$ and $f(0)=c$ for some $c\in\mathbb{C}$. It is follows by (18), (20) and (22) that

$$h(n)(h(m+n)+1)=0 \text{ for all } m>0, \frac{m}{2}-n\neq 0; \tag{44}$$
$$h(m+n)(h(n)+1)=0 \text{ for all } m<0, \frac{m}{2}-n\neq 0; \tag{45}$$
$$g(n)(g(m+n)+1)=0 \text{ for all } m>0, n\neq 0, m+n\neq 0; \tag{46}$$
$$g(m+n)(g(n)+1)=0 \text{ for all } m<0, n\neq 0, m+n\neq 0; \tag{47}$$
$$g(m+n)(1+h(m)+h(n))=h(m)h(n) \text{ for all } m\neq n. \tag{48}$$

For any $t\in\mathbb{Z}^*$, we first prove some claims as follows.

Claim 1. *If $h(t)=0$, then $h(\mathbb{Z}_{\leqslant t})=0$.*

In fact, by (44) with $n=t-m$ we deduce $h(t-m)=0$ for all $m>0$ with $m\neq \frac{2}{3}t$. This implies $h(\mathbb{Z}_{\leqslant t}\setminus\{\frac{1}{3}t\})=0$. On the other hand, by (45) with $n=t$ we see that $h(m+t)=0$ for all $m<0$ with $m\neq 2t$. This gives that $h(\mathbb{Z}_{\leqslant t}\setminus\{3t\})=0$. Clearly, $3t\neq\frac{1}{3}t$ since $t\neq 0$. Thereby, we obtain $h(\mathbb{Z}_{\leqslant t})=0$.

Claim 2. *If $h(t)=-1$, then $h(\mathbb{Z}_{\geqslant t})=-1$.*

This proof is similar to Claim 1 by using (44) and (45). Also, similar to Claims 1 and 2, by (46) and (47) we can obtain the following two claims:

Claim 3. *If $g(t)=0$, then $g(\mathbb{Z}^*_{\leqslant t})=0$.*

Claim 4. *If $g(t)=-1$, then $g(\mathbb{Z}^*_{\geqslant t})=-1$.*

According to (32), by Claims 1 and 2, h must be one of the following forms:
(1) $h(\mathbb{Z}^*)=0$; (2) $h(\mathbb{Z}^*)=-1$;
(3) $h(\mathbb{Z}_{>0})=-1$, $h(\mathbb{Z}_{<0})=0$ and $h(0)=\lambda$ for some $\lambda\in\mathbb{C}$;
(4) $h(\mathbb{Z}_{\geqslant t})=-1$, $h(\mathbb{Z}_{\leqslant t-1})=0$ for some $t\in\mathbb{Z}\setminus\{0,1\}$.

In view of the above result, the next proof will be divided into the following cases.

Case i. When $h(\mathbb{Z}^*)=0$.

By taking $n=1$ in (40), one has $h(0)=0$. Hence we see that $h(\mathbb{Z})=0$. This together with (48) yields $g(\mathbb{Z})=0$. In addition, we have by (43) that $a=0$.

Case ii. When $h(\mathbb{Z}^*)=-1$.

By taking $n=-1$ in (40), one has $h(0)=-1$. Hence we see that $h(\mathbb{Z})=-1$. This together with (48) yields $g(\mathbb{Z})=-1$. In addition, by (43) we get $a=0$.

Case iii. When $h(\mathbb{Z}_{>0})=-1$, $h(\mathbb{Z}_{<0})=0$ and $h(0)=\lambda$ for some $\lambda\in\mathbb{C}$.

By (48) we see that $g(m+n)=-1$ for any $m,n\in\mathbb{Z}$ with $m,n>0$ and $m\neq n$, and $g(m+n)=0$ for any $m,n\in\mathbb{Z}$ with $m,n<0$ and $m\neq n$. Consequently, we obtain $g(\mathbb{Z}_{\leqslant -3})=0$ and $g(\mathbb{Z}_{\geqslant 3})=-1$. By (32), $g(i)\in\{-1,0\}$ for $i\in\{-2,-1,1,2\}$. In view of Claims 3 and 4, we can assume that $g(k)=-1$ and $g(k-1)=0$ for some $k\in\{-2,-1,1,2,3\}$. In all, by Claims 3 and 4 we get $g(\mathbb{Z}^*_{\geqslant k})=-1$ and $g(\mathbb{Z}^*_{\leqslant k-1})=0$. Next, if $k\in\{-1,-2\}$ then by taking $n=k$ in (41) we have $a=0$; and if $k\in\{2,3\}$ then by taking $n=k-1$ in (41) we also have $a=0$. But a can be arbitrary if $k=1$.

Case iv. When $h(\mathbb{Z}_{\geqslant t})=-1$, $h(\mathbb{Z}_{\leqslant t-1})=0$ for some $t\in\mathbb{Z}\setminus\{0,1\}$.

Note that $t \geqslant 2$ or $t \leqslant -1$, then by taking $n = 1$ in (42) we have $a = 0$. Next, by(48) we see that $g(m+n) = -1$ for any $m, n \in \mathbb{Z}$ with $m, n \geqslant t$ and $m \neq n$, and $g(m+n) = 0$ for any $m, n \in \mathbb{Z}$ with $m, n \leqslant t-1$ and $m \neq n$. Consequently, we obtain $g(\mathbb{Z}_{\leqslant 2t-3}) = 0$ and $g(\mathbb{Z}_{\geqslant 2t+1}) = -1$. By (32), $g(i) \in \{-1, 0\}$ for $i \in \{2t-2, 2t-1, 2t, 2t+1\}$. In view of Claims 3 and 4, we can assume that $g(s) = -1$ and $g(s-1) = 0$ for some $s \in \{2t-2, 2t-1, 2t, 2t+1, 2t+2\}$. Note that $0 \notin \{2t-2, 2t-1, 2t, 2t+1\}$ since $t \neq 0, 1$, by Claims 3 and 4 we get $g(\mathbb{Z}_{\geqslant s}) = -1$ and $g(\mathbb{Z}_{\leqslant s-1}) = 0$. The proof is completed. □

Lemma 3.6. *Let f, g, h be complex-valued functions on $\mathbb{Z}$ and μ, a be complex numbers. Then (17)-(22) hold if and only if f, g, h, a, μ meet one of the situations listed in Table 2.*

Proof. The proof of the "if" direction can be directly verified. We now prove the "only if" direction. In view of f satisfying (16), by Theorem 2.4 of [10] we know that f is determined by Table 1.

Cases	$f(n)$
$\mathcal{P}_1$	$f(\mathbb{Z}) = 0$
$\mathcal{P}_2$	$f(\mathbb{Z}) = -1$
$\mathcal{P}_3^c$	$f(\mathbb{Z}_{>0}) = -1, f(\mathbb{Z}_{<0}) = 0$ and $f(0) = c$
$\mathcal{P}_4^c$	$f(\mathbb{Z}_{>0}) = 0, f(\mathbb{Z}_{<0}) = -1$ and $f(0) = c$
$\mathcal{P}_5$	$f(\mathbb{Z}_{\geqslant 2}) = -1$ and $f(\mathbb{Z}_{\leqslant 1}) = 0$
$\mathcal{P}_6$	$f(\mathbb{Z}_{\geqslant 2}) = 0$ and $f(\mathbb{Z}_{\leqslant 1}) = -1$
$\mathcal{P}_7$	$f(\mathbb{Z}_{\geqslant -1}) = 0$ and $f(\mathbb{Z}_{\leqslant -2}) = -1$
$\mathcal{P}_8$	$f(\mathbb{Z}_{\geqslant -1}) = -1$ and $f(\mathbb{Z}_{\leqslant -2}) = 0$

Table 1: Values of f satisfying (16), where $c \in \mathbb{C}$

When f takes the form of Case $\mathcal{P}_1$ in Table 1, by the results of Lemma 3.3, we see that μ, a, g, h must satisfy the condition of Cases $\mathcal{W}_1^{\mathcal{P}_1}$ and $\mathcal{W}_2^{\mathcal{P}_1}$ in Table 2. From Lemma 3.3, Cases $\mathcal{W}_1^{\mathcal{P}_1}, i = 1, 2$ is easy to say. In the same way, when f takes the form of Case $\mathcal{P}_2$ in Table 1, then we obtain the forms of Cases $\mathcal{W}_1^{\mathcal{P}_2}$ and $\mathcal{W}_2^{\mathcal{P}_2}$ in Table 2.

When f takes the form of Case $\mathcal{P}_3^c$ in Table 1, by the results of Lemma 3.5, we see that μ, a, g, h must satisfy the one condition of Cases $\mathcal{W}_{i,\mu}^{\mathcal{P}_3^c}, i = 1, 2$, $\mathcal{W}_{3,\mu}^{\mathcal{P}_3^c,k}$, $\mathcal{W}_{4,a,\mu}^{\mathcal{P}_3^c,k=1}$ and $\mathcal{W}_{5,\mu}^{\mathcal{P}_3^c,s,t}$ in Table 2. From Lemma 3.5, the results of Cases $\mathcal{W}_{i,\mu}^{\mathcal{P}_3^c}, i = 1, 2$ are easily obtained; and Case $\mathcal{W}_{3,\mu}^{\mathcal{P}_3^c,k}$ satisfies $\mu \in \mathbb{C}$, $h(\mathbb{Z}_{>0}) = -1$, $h(\mathbb{Z}_{<0}) = 0$, $h(0) = \lambda$ and $g(\mathbb{Z}^*_{\geqslant k}) = -1$, $g(\mathbb{Z}^*_{\leqslant k-1}) = 0$, for some $k \in \{-2, -1, 1, 2, 3\}$, $g(0) = \hat{\tau}$ with $a = 0$ when $k \neq 1$ and a is arbitrary if $k = 1$; Case $\mathcal{W}_{4,a,\mu}^{\mathcal{P}_3^c,k=1}$ satisfies $\mu \in \mathbb{C}$, $h(\mathbb{Z}_{>0}) = -1$, $h(\mathbb{Z}_{<0}) = 0$, $h(0) = \lambda$ and $g(\mathbb{Z}_{>0}) = -1$, $g(\mathbb{Z}_{<0}) = 0$ for some $k = 1$, $g(0) = \hat{\tau}$; Case $\mathcal{W}_{5,\mu}^{\mathcal{P}_3^c,s,t}$ satisfies $a = 0$, $\mu \in \mathbb{C}$ and $h(\mathbb{Z}_{\geqslant t}) = -1$, $h(\mathbb{Z}_{\leqslant t-1}) = 0$ for some $t \in \mathbb{Z} \setminus \{0, 1\}$ and $g(\mathbb{Z}_{\geqslant s}) = -1$, $g(\mathbb{Z}_{\leqslant s-1}) = 0$ for some $s \in \{2t-2, 2t-1, 2t, 2t+1, 2t+2\}$. In the same way, when f takes the form of Case $\mathcal{P}_4^c$ in Table 1, then we obtain the results of Cases $\mathcal{W}_{i,\mu}^{\mathcal{P}_4^c}, i = 1, 2$, $\mathcal{W}_{3,\mu}^{\mathcal{P}_4^c,k}$, $\mathcal{W}_{4,a,\mu}^{\mathcal{P}_4^c,k=1}$ and $\mathcal{W}_{5,\mu}^{\mathcal{P}_4^c,s,t}$ in Table 2, respectively.

When f takes the form of Case $\mathcal{P}_5$ in Table 1, by the results of Lemma 3.4, we see that μ, a, g, h must satisfy the condition of Cases $\mathcal{W}_i^{\mathcal{P}_5}, i = 1, 2, 3$ in Table 2. From Lemma 3.4, the results of Cases $\mathcal{W}_i^{\mathcal{P}_5}, i = 1, 2$, are easily obtained; and for Case $\mathcal{W}_3^{\mathcal{P}_5}$, we get $h(\mathbb{Z}_{\leqslant 0}) = 0$, $h(\mathbb{Z}_{\geqslant 1}) = -1$ and $g(\mathbb{Z}_{\leqslant -1}) = 0$, $g(\mathbb{Z}_{\geqslant 1}) = -1$, $g(0) = \hat{\lambda}$

for some $\hat{\lambda} \in \mathbb{C}$. Similarly, when f takes the form of Case $\mathcal{P}_k, k = 6, 7, 8$ in Table 1, then we obtain the forms of Cases $\mathcal{W}_i^{\mathcal{P}_k}$, $i = 1, 2, 3$, $k = 6, 7, 8$ in Table 2. The proof is completed. □

Cases	$f(n)$ from Table 1	a, μ	$h(n), g(n)$
$\mathcal{W}_1^{\mathcal{P}_1}$	$\mathcal{P}_1$	$a = \mu = 0$	$h(\mathbb{Z}) = g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_1}$	$\mathcal{P}_1$	$a = \mu = 0$	$h(\mathbb{Z}) = g(\mathbb{Z}) = -1$
$\mathcal{W}_1^{\mathcal{P}_2}$	$\mathcal{P}_2$	$a = \mu = 0$	$h(\mathbb{Z}) = g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_2}$	$\mathcal{P}_2$	$a = \mu = 0$	$h(\mathbb{Z}) = g(\mathbb{Z}) = -1$
$\mathcal{W}_{1,\mu}^{\mathcal{P}_3^c}$	$\mathcal{P}_3^c$	$a = 0$ and $\forall \mu$	$h(\mathbb{Z}) = g(\mathbb{Z}) = 0$
$\mathcal{W}_{2,\mu}^{\mathcal{P}_3^c}$	$\mathcal{P}_3^c$	$a = 0$ and $\forall \mu$	$h(\mathbb{Z}) = g(\mathbb{Z}) = -1$
$\mathcal{W}_{3,\mu}^{\mathcal{P}_3^c,k}$	$\mathcal{P}_3^c$	$a = 0$ and $\forall \mu$	$h(\mathbb{Z}_{>0}) = -1$, $h(\mathbb{Z}_{<0}) = 0$ and $g(\mathbb{Z}^*_{\geqslant k}) = -1$, $g(\mathbb{Z}^*_{\leqslant k-1}) = 0$
$\mathcal{W}_{4,a,\mu}^{\mathcal{P}_3^c,k=1}$	$\mathcal{P}_3^c$	$\forall a$ and $\forall \mu$	$h(\mathbb{Z}_{>0}) = -1$, $h(\mathbb{Z}_{<0}) = 0$ and $g(\mathbb{Z}_{>0}) = -1$, $g(\mathbb{Z}_{<0}) = 0$
$\mathcal{W}_{5,\mu}^{\mathcal{P}_3^c,s,t}$	$\mathcal{P}_3^c$	$a = 0$ and $\forall \mu$	$h(\mathbb{Z}_{\geqslant t}) = -1$, $h(\mathbb{Z}_{\leqslant t-1}) = 0$ and $g(\mathbb{Z}_{\geqslant s}) = -1$, $g(\mathbb{Z}_{\leqslant s-1}) = 0$
$\mathcal{W}_{1,\mu}^{\mathcal{P}_4^c}$	$\mathcal{P}_4^c$	$a = 0$ and $\forall \mu$	$h(\mathbb{Z}) = g(\mathbb{Z}) = 0$
$\mathcal{W}_{2,\mu}^{\mathcal{P}_4^c}$	$\mathcal{P}_4^c$	$a = 0$ and $\forall \mu$	$h(\mathbb{Z}) = g(\mathbb{Z}) = -1$
$\mathcal{W}_{3,\mu}^{\mathcal{P}_4^c,k}$	$\mathcal{P}_4^c$	$a = 0$ and $\forall \mu$	$h(\mathbb{Z}_{>0}) = 0$, $h(\mathbb{Z}_{<0}) = -1$ and $g(\mathbb{Z}^*_{\geqslant k}) = 0$, $g(\mathbb{Z}^*_{\leqslant k-1}) = -1$
$\mathcal{W}_{4,a,\mu}^{\mathcal{P}_4^c,k=1}$	$\mathcal{P}_4^c$	$\forall a$ and $\forall \mu$	$h(\mathbb{Z}_{>0}) = 0$, $h(\mathbb{Z}_{<0}) = -1$ and $g(\mathbb{Z}_{>0}) = 0$, $g(\mathbb{Z}_{<0}) = -1$
$\mathcal{W}_{5,\mu}^{\mathcal{P}_4^c,s,t}$	$\mathcal{P}_4^c$	$a = 0$ and $\forall \mu$	$h(\mathbb{Z}_{\geqslant t}) = 0$, $h(\mathbb{Z}_{\leqslant t-1}) = -1$ and $g(\mathbb{Z}_{\geqslant s}) = 0$, $g(\mathbb{Z}_{\leqslant s-1}) = -1$
$\mathcal{W}_1^{\mathcal{P}_5}$	$\mathcal{P}_5$	$a = \mu = 0$	$h(\mathbb{Z}) = g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_5}$	$\mathcal{P}_5$	$a = \mu = 0$	$h(\mathbb{Z}) = g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_5}$	$\mathcal{P}_5$	$a = \mu = 0$	$h(\mathbb{Z}_{\leqslant 0}) = 0$, $h(\mathbb{Z}_{\geqslant 1}) = -1$ and $g(\mathbb{Z}_{\leqslant -1}) = 0$, $g(\mathbb{Z}_{\geqslant 1}) = -1$
$\mathcal{W}_1^{\mathcal{P}_6}$	$\mathcal{P}_6$	$a = \mu = 0$	$h(\mathbb{Z}) = g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_6}$	$\mathcal{P}_6$	$a = \mu = 0$	$h(\mathbb{Z}) = g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_6}$	$\mathcal{P}_6$	$a = \mu = 0$	$h(\mathbb{Z}_{\leqslant 0}) = -1$, $h(\mathbb{Z}_{\geqslant 1}) = 0$ and $g(\mathbb{Z}_{\leqslant -1}) = -1$, $g(\mathbb{Z}_{\geqslant 1}) = 0$
$\mathcal{W}_1^{\mathcal{P}_7}$	$\mathcal{P}_7$	$a = \mu = 0$	$h(\mathbb{Z}) = g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_7}$	$\mathcal{P}_7$	$a = \mu = 0$	$h(\mathbb{Z}) = g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_7}$	$\mathcal{P}_7$	$a = \mu = 0$	$h(\mathbb{Z}_{\leqslant 0}) = -1$, $h(\mathbb{Z}_{\geqslant 1}) = 0$ and $g(\mathbb{Z}_{\leqslant -1}) = -1$, $g(\mathbb{Z}_{\geqslant 1}) = 0$
$\mathcal{W}_1^{\mathcal{P}_8}$	$\mathcal{P}_8$	$a = \mu = 0$	$h(\mathbb{Z}) = g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_8}$	$\mathcal{P}_8$	$a = \mu = 0$	$h(\mathbb{Z}) = g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_8}$	$\mathcal{P}_8$	$a = \mu = 0$	$h(\mathbb{Z}_{\leqslant 0}) = 0$, $h(\mathbb{Z}_{\geqslant 1}) = -1$ and $g(\mathbb{Z}_{\leqslant -1}) = 0$, $g(\mathbb{Z}_{\geqslant 1}) = -1$

Table 2: Values of f, g, h satisfying (16)-(22), where $a, \mu \in \mathbb{C}$, $k \in \{-2, -1, 1, 2, 3\}$, $t \in \mathbb{Z} \setminus \{0, 1\}$ and $s \in \{2t-2, 2t-1, 2t, 2t+1, 2t+2\}$.

Lemma 3.7. *Let* $(\mathcal{P}(\phi_i, \varphi_i, \chi_i, \psi_i, \xi_i, \theta_i), \triangleright_i)$, $i = 1, 2$ *be two algebras with the same linear space as* $\mathcal{S}$ *and equipped with* $\mathbb{C}$*-bilinear products* $x \triangleright_i y$ *such that*

$$\begin{aligned} &L_m \triangleright_i L_n = \phi_i(m,n)L_{m+n}, \;\; L_m \triangleright_i H_n = \varphi_i(m,n)H_{m+n}, \\ &L_m \triangleright_i I_n = \chi_i(m,n)I_{m+n}, \;\; H_m \triangleright_i L_n = \psi_i(m,n)H_{m+n}, \\ &H_m \triangleright_i H_n = \xi_i(m,n)I_{m+n}, \;\; I_m \triangleright_i L_n = \theta_i(m,n)I_{m+n}, \\ &H_m \triangleright_i I_n = I_m \triangleright_i H_n = I_m \triangleright_i I_n = 0 \end{aligned}$$

for all $m, n \in \mathbb{Z}$, *where* $\phi_i, \varphi_i, \chi_i, \psi_i, \xi_i, \theta_i$, $i = 1, 2$ *are complex-valued functions on* $\mathbb{Z} \times \mathbb{Z}$. *Furthermore, let* $\tau : \mathcal{P}(\phi_1, \varphi_1, \chi_1, \psi_1, \xi_1, \theta_1) \to \mathcal{P}(\phi_2, \varphi_2, \chi_2, \psi_2, \xi_2, \theta_2)$ *be a linear map determined by*

$$\tau(L_m) = -L_{-m}, \tau(H_m) = -H_{-m}, \tau(I_m) = -I_{-m}$$

for all $m \in \mathbb{Z}$. *In addition, suppose that* $(\mathcal{P}(\phi_1, \varphi_1, \chi_1, \psi_1, \xi_1, \theta_1), [,], \triangleright_1)$ *is a post-Lie algebra. Then* $(\mathcal{P}(\phi_2, \varphi_2, \chi_2, \psi_2, \xi_2, \theta_2), [,], , \triangleright_2)$ *is a post-Lie algebra and* τ *is an isomorphism on post-Lie algebras if and only if*

$$\begin{cases} \phi_2(m,n) = -\phi_1(-m,-n); \\ \varphi_2(m,n) = -\varphi_1(-m,-n); \\ \chi_2(m,n) = -\chi_1(-m,-n); \\ \psi_2(m,n) = -\psi_1(-m,-n); \\ \xi_2(m,n) = -\xi_1(-m,-n); \\ \theta_2(m,n) = -\theta_1(-m,-n). \end{cases} \tag{49}$$

Proof. Clearly, τ is a Lie automorphism of $\mathcal{S}$. Suppose $(\mathcal{P}(\phi_2, \varphi_2, \chi_2, \psi_2, \xi_2, \theta_2), [,], \triangleright_2)$ is a post-Lie algebra and $\tau : \mathcal{P}(\phi_1, \varphi_1, \chi_1, \psi_1, \xi_1, \theta_1) \to \mathcal{P}(\phi_2, \varphi_2, \chi_2, \psi_2, \xi_2, \theta_2)$ is a post-Lie isomorphism. Then we have

$$\begin{aligned} \tau(L_m \triangleright_i L_n) &= -\phi_i(m,n)L_{-(m+n)}, \\ \tau(L_m \triangleright_i H_n) &= -\varphi_i(m,n)H_{-(m+n)}, \\ \tau(L_m \triangleright_i I_n) &= -\chi_i(m,n)I_{-(m+n)}, \\ \tau(H_m \triangleright_i L_n) &= -\psi_i(m,n)H_{-(m+n)}, \\ \tau(H_m \triangleright_i H_n) &= -\xi_i(m,n)I_{-(m+n)}, \\ \tau(I_m \triangleright_i L_n) &= -\theta_i(m,n)I_{-(m+n)} \end{aligned}$$

for $i = 1, 2$. This tell us that that (49) holds. Conversely, we first suppose that (49) hold. Then, by using Lemma 3.1 and $(\phi_1, \varphi_1, \chi_1, \psi_1, \xi_1, \theta_1, [,], \triangleright_1)$ is a post-Lie algebra, we know that there are complex-valued functions f_1, g_1, h_1 on $\mathbb{Z}$ and complex numbers a_1, μ_1 satisfying (10)-(22) with replacing $(\phi, \varphi, \chi, \psi, \xi, \theta, f, g, h, \mu, a)$ by $(\phi_1, \varphi_1, \chi_1, \psi_1, \xi_1, \theta_1, f_1, g_1, h_1, \mu_1, a_1)$. Next, let $f_2(m) = f_1(-m)$, $g_2(m) = g_1(-m)$, $h_2(m) = h_1(-m)$, $\mu_2 = -\mu_1$ and $a_2 = a_1$, then we see that (10)-(22) hold with replacing $(\phi, \varphi, \chi, \psi, \xi, \theta, f, g, h, \mu, a)$ by $(\phi_2, \varphi_2, \chi_2, \psi_2, \xi_2, \theta_2, f_1, g_1, h_1, \mu_1, a_1)$. By Lemma 3.1, $\mathcal{P}(\phi_2, \varphi_2, \chi_2, \psi_2, \xi_2, \theta_2)$ is a post-Lie algebra.

The remainder is to prove that τ is an isomorphism between post-Lie algebra. But one has

$$\begin{aligned} &\tau(L_m \triangleright_1 L_n) = -\phi_1(m,n)L_{-(m+n)} = \phi_2(-m,-n)L_{-(m+n)} = \tau(L_m) \triangleright_2 \tau(L_n), \\ &\tau(L_m \triangleright_1 H_n) = -\varphi_1(m,n)H_{-(m+n)} = \varphi_2(-m,-n)H_{-(m+n)} = \tau(L_m) \triangleright_2 \tau(H_n), \\ &\tau(L_m \triangleright_1 I_n) = -\chi_1(m,n)I_{-(m+n)} = \chi_2(-m,-n)I_{-(m+n)} = \tau(L_m) \triangleright_2 \tau(I_n), \\ &\tau(H_m \triangleright_1 L_n) = -\psi_1(m,n)H_{-(m+n)} = \psi_2(-m,-n)H_{-(m+n)} = \tau(H_m) \triangleright_2 \tau(L_n), \\ &\tau(H_m \triangleright_1 H_n) = -\varphi_1(m,n)I_{-(m+n)} = \varphi_2(-m,-n)I_{-(m+n)} = \tau(H_m) \triangleright_2 \tau(H_n), \end{aligned}$$

$$\tau(I_m \triangleright_1 L_n) = -\theta_1(m,n)I_{-(m+n)} = \phi_2(-m,-n)I_{-(m+n)} = \tau(I_m) \triangleright_2 \tau(L_n)$$

and $\tau(H_m \triangleright_1 I_n) = \tau(H_m) \triangleright_2 \tau(I_n) = 0$, $\tau(I_m \triangleright_1 H_n) = \tau(I_m) \triangleright_2 \tau(H_n)=0$, $\tau(I_m \triangleright_1 I_n) = \tau(I_m) \triangleright_2 \tau(I_n)=0$. The proof is completed. □

Theorem 3.8. *A graded post-Lie algebra structure on $\mathcal{S}$ satisfying (3)-(9) must be one of the following types, for all $m,n \in \mathbb{Z}$ (in every case $I_m \triangleright H_n = H_m \triangleright I_n = I_m \triangleright I_n = 0$),*

($\mathcal{W}_1^{\mathcal{P}_1}$): $L_m \triangleright_1^{\mathcal{P}_1} L_n = 0$, $L_m \triangleright_1^{\mathcal{P}_1} H_n = 0$, $L_m \triangleright_1^{\mathcal{P}_1} I_n = 0$, $H_m \triangleright_1^{\mathcal{P}_1} L_n = 0$, $H_m \triangleright_1^{\mathcal{P}_1} H_n = 0$, $I_m \triangleright_1^{\mathcal{P}_1} L_n = 0$;

($\mathcal{W}_2^{\mathcal{P}_1}$): $L_m \triangleright_2^{\mathcal{P}_1} L_n = 0$, $L_m \triangleright_2^{\mathcal{P}_1} H_n = 0$, $L_m \triangleright_2^{\mathcal{P}_1} I_n = 0$, $H_m \triangleright_2^{\mathcal{P}_1} L_n = (\frac{n}{2} - m)H_{m+n}$, $H_m \triangleright_2^{\mathcal{P}_1} H_n = (n-m)I_{m+n}$, $I_m \triangleright_2^{\mathcal{P}_1} L_n = -mI_{m+n}$;

($\mathcal{W}_1^{\mathcal{P}_2}$): $L_m \triangleright_1^{\mathcal{P}_2} L_n = (n-m)L_{m+n}$, $L_m \triangleright_1^{\mathcal{P}_2} H_n = (n-\frac{m}{2})H_{m+n}$, $L_m \triangleright_1^{\mathcal{P}_2} I_n = nI_{m+n}$, $H_m \triangleright_1^{\mathcal{P}_2} L_n = 0$, $H_m \triangleright_1^{\mathcal{P}_2} H_n = 0$, $I_m \triangleright_1^{\mathcal{P}_2} L_n = 0$;

($\mathcal{W}_2^{\mathcal{P}_2}$): $L_m \triangleright_2^{\mathcal{P}_2} L_n = (n-m)L_{m+n}$, $L_m \triangleright_2^{\mathcal{P}_2} H_n = (n-\frac{m}{2})H_{m+n}$, $L_m \triangleright_2^{\mathcal{P}_2} I_n = nI_{m+n}$, $H_m \triangleright_2^{\mathcal{P}_2} L_n = (\frac{n}{2}-m)H_{m+n}$, $H_m \triangleright_2^{\mathcal{P}_2} H_n = (n-m)I_{m+n}$, $I_m \triangleright_2^{\mathcal{P}_2} L_n = -mI_{m+n}$;

($\mathcal{W}_{i,a,\mu,\lambda}^{\mathcal{P}_3^c,s,k,t}$) : $i = 1,2,3,4,5$

$$L_m \triangleright_i^{\mathcal{P}_3^c} L_n = \begin{cases} (n-m)L_{m+n}, & m>0, \\ -ncL_n, & m=0, \\ 0, & m<0; \end{cases}$$

$$L_m \triangleright_i^{\mathcal{P}_3^c} H_n = \begin{cases} (n-\frac{m}{2})H_{m+n}, & m>0, \\ (-nc+\mu)H_n, & m=0, \\ 0, & m<0; \end{cases}$$

$$L_m \triangleright_i^{\mathcal{P}_3^c} I_n = \begin{cases} nI_{m+n}, & m>0, \\ (-nc+2\mu)I_n, & m=0, \\ 0, & m<0; \end{cases}$$

$$\begin{aligned} H_m \triangleright_i^{\mathcal{P}_3^c} L_n = {} & \delta_{i,2}(\tfrac{n}{2}-m)H_{m+n} \\ & +(\delta_{i,3}+\delta_{i,4}) \begin{cases} (\frac{n}{2}-m)H_{m+n}, & m>0, \\ -\frac{n}{2}\lambda H_n, & m=0, \\ 0, & m<0; \end{cases} \\ & +\delta_{i,5} \begin{cases} (\frac{n}{2}-m)H_{m+n}, & m \geqslant t, \\ 0, & m \leqslant t-1; \end{cases} \end{aligned}$$

$$\begin{aligned} H_m \triangleright_i^{\mathcal{P}_3^c} H_n = {} & \delta_{i,2}(n-m)I_{m+n} \\ & +(\delta_{i,3}+\delta_{i,4}) \begin{cases} (n-m)I_{m+n}, & m>0, \\ -n\lambda I_n, & m=0, \\ 0, & m<0; \end{cases} \\ & +\delta_{i,5} \begin{cases} (n-m)I_{m+n}, & m \geqslant t, \\ 0, & m \leqslant t-1; \end{cases} \end{aligned}$$

$$\begin{aligned} I_m \triangleright_i^{\mathcal{P}_3^c} L_n = {} & \delta_{i,2}(-m)I_{m+n} \\ & +\delta_{i,3} \begin{cases} -mI_{m+n}, & m \geqslant k, \\ 0, & m \leqslant k-1; \end{cases} \\ & +\delta_{i,4} \begin{cases} -mI_{m+n}, & m>0, \\ naI_n, & m=0, \\ 0, & m<0; \end{cases} \end{aligned}$$

$$+\delta_{i,5}\begin{cases}-mI_{m+n}, & m\geqslant s,\\ 0, & m\leqslant s-1;\end{cases}$$

$(\mathcal{W}_{i,a,\mu,\lambda}^{\mathcal{P}_4^c,s,k,t})$: $i=1,2,3,4,5$

$$L_m\triangleright_i^{\mathcal{P}_4^c}L_n=\begin{cases}(n-m)L_{m+n}, & m<0,\\ -ncL_n, & m=0,\\ 0, & m>0;\end{cases}$$

$$L_m\triangleright_i^{\mathcal{P}_4^c}H_n=\begin{cases}(n-\frac{m}{2})H_{m+n}, & m<0,\\ (-nc+\mu)H_n, & m=0,\\ 0, & m>0;\end{cases}$$

$$L_m\triangleright_i^{\mathcal{P}_4^c}I_n=\begin{cases}nI_{m+n}, & m<0,\\ (-nc+2\mu)I_n, & m=0,\\ 0, & m>0;\end{cases}$$

$$\begin{aligned}H_m\triangleright_i^{\mathcal{P}_4^c}L_n=&\ \delta_{i,2}(\tfrac{n}{2}-m)H_{n+m}\\ &+(\delta_{i,3}+\delta_{i,4})\begin{cases}0, & m>0,\\ -\frac{n}{2}\lambda H_n, & m=0,\\ (\frac{n}{2}-m)H_{m+n}, & m<0;\end{cases}\\ &+\delta_{i,5}\begin{cases}0, & m\geqslant t,\\ (\frac{n}{2}-m)H_{m+n}, & m\leqslant t-1;\end{cases}\end{aligned}$$

$$\begin{aligned}H_m\triangleright_i^{\mathcal{P}_4^c}H_n=&\ \delta_{i,2}(n-m)I_{n+m}\\ &+(\delta_{i,3}+\delta_{i,4})\begin{cases}0, & m>0,\\ -n\lambda I_n, & m=0,\\ (n-m)I_{m+n}, & m<0;\end{cases}\\ &+\delta_{i,5}\begin{cases}0, & m\geqslant t,\\ (n-m)I_{m+n}, & m\leqslant t-1;\end{cases}\end{aligned}$$

$$\begin{aligned}I_m\triangleright_i^{\mathcal{P}_4^c}L_n=&\ \delta_{i,2}(-m)I_{n+m}\\ &+\delta_{i,3}\begin{cases}0, & m\geqslant k,\\ -mI_{m+n}, & m\leqslant k-1;\end{cases}\\ &+\delta_{i,4}\begin{cases}0, & m>0,\\ naI_n, & m=0,\\ -mI_{m+n}, & m<0;\end{cases}\\ &+\delta_{i,5}\begin{cases}0, & m\geqslant s,\\ -mI_{m+n}, & m\leqslant s-1;\end{cases}\end{aligned}$$

$(\mathcal{W}_i^{\mathcal{P}_5})$: $i=1,2,3,$

$$L_m\triangleright_i^{\mathcal{P}_5}L_n=\begin{cases}(n-m)L_{m+n}, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}$$

$$L_m\triangleright_i^{\mathcal{P}_5}H_n=\begin{cases}(n-\frac{m}{2})L_{m+n}, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}$$

$$L_m\triangleright_i^{\mathcal{P}_5}I_n=\begin{cases}nI_{m+n}, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}$$

$$\begin{aligned}H_m\triangleright_i^{\mathcal{P}_5}L_n=&\ \delta_{i,2}(\tfrac{n}{2}-m)H_{m+n}\\ &+\delta_{i,3}\begin{cases}0, & m\leqslant 0,\\ (\frac{n}{2}-m)H_{m+n}, & m\geqslant 1;\end{cases}\end{aligned}$$

$$H_m \triangleright_i^{\mathcal{P}_5} H_n = \delta_{i,2}(n-m)I_{m+n} + \delta_{i,3}\begin{cases}0, & m \leqslant 0,\\ (n-m)I_{m+n}, & m \geqslant 1;\end{cases}$$

$$I_m \triangleright_i^{\mathcal{P}_5} L_n = \delta_{i,2}(-m)I_{m+n} + \delta_{i,3}\begin{cases}0, & m \leqslant 0,\\ -mI_{m+n}, & m \geqslant 1;\end{cases}$$

$(\mathcal{W}_i^{\mathcal{P}_6})$: $i = 1, 2, 3$,

$$L_m \triangleright_i^{\mathcal{P}_6} L_n = \begin{cases}(n-m)L_{m+n}, & m \leqslant 1,\\ 0, & m \geqslant 2;\end{cases}$$

$$L_m \triangleright_i^{\mathcal{P}_6} H_n = \begin{cases}(n-\frac{m}{2})H_{m+n}, & m \leqslant 1,\\ 0, & m \geqslant 2;\end{cases}$$

$$L_m \triangleright_i^{\mathcal{P}_6} I_n = \begin{cases}nI_{m+n}, & m \leqslant 1,\\ 0, & m \geqslant 2;\end{cases}$$

$$H_m \triangleright_i^{\mathcal{P}_6} L_n = \delta_{i,2}(\tfrac{n}{2}-m)H_{m+n} + \delta_{i,3}\begin{cases}(\frac{n}{2}-m)H_{m+n}, & m \leqslant 0,\\ 0, & m \geqslant 1;\end{cases}$$

$$H_m \triangleright_i^{\mathcal{P}_6} H_n = \delta_{i,2}(n-m)I_{m+n} + \delta_{i,3}\begin{cases}(n-m)I_{m+n}, & m \leqslant 0,\\ 0, & m \geqslant 1;\end{cases}$$

$$I_m \triangleright_i^{\mathcal{P}_6} L_n = \delta_{i,2}(-m)I_{m+n} + \delta_{i,3}\begin{cases}-mI_{m+n}, & m \leqslant -1,\\ 0, & m \geqslant 0;\end{cases}$$

$(\mathcal{W}_i^{\mathcal{P}_7})$: $i = 1, 2, 3$,

$$L_m \triangleright_i^{\mathcal{P}_7} L_n = \begin{cases}(n-m)L_{m+n}, & m \leqslant -2,\\ 0, & m \geqslant -1;\end{cases}$$

$$L_m \triangleright_i^{\mathcal{P}_7} H_n = \begin{cases}(n-\frac{m}{2})H_{m+n}, & m \leqslant -2,\\ 0, & m \geqslant -1;\end{cases}$$

$$L_m \triangleright_i^{\mathcal{P}_7} I_n = \begin{cases}nI_{m+n}, & m \leqslant -2,\\ 0, & m \geqslant -1;\end{cases}$$

$$H_m \triangleright_i^{\mathcal{P}_7} L_n = \delta_{i,2}(\tfrac{n}{2}-m)H_{m+n} + \delta_{i,3}\begin{cases}(\frac{n}{2}-m)H_{m+n}, & m \leqslant 0,\\ 0, & m \geqslant 1;\end{cases}$$

$$H_m \triangleright_i^{\mathcal{P}_7} H_n = \delta_{i,2}(n-m)I_{m+n} + \delta_{i,3}\begin{cases}(n-m)I_{m+n}, & m \leqslant 0,\\ 0, & m \geqslant 1;\end{cases}$$

$$I_m \triangleright_i^{\mathcal{P}_7} L_n = \delta_{i,2}(-m)I_{m+n} + \delta_{i,3}\begin{cases}-mI_{m+n}, & m \leqslant -1,\\ 0, & m \geqslant 0;\end{cases}$$

$(\mathcal{W}_i^{\mathcal{P}_8})$: $i = 1, 2, 3$,

$$L_m \triangleright_i^{\mathcal{P}_8} L_n = \begin{cases}(n-m)L_{m+n}, & m \geqslant -1,\\ 0, & m \leqslant -2;\end{cases}$$

$$L_m \triangleright_i^{\mathcal{P}_8} H_n = \begin{cases}(n-\frac{m}{2})H_{m+n}, & m \geqslant -1,\\ 0, & m \leqslant -2;\end{cases}$$

$$L_m \triangleright_i^{\mathcal{P}_8} I_n = \begin{cases} nI_{m+n}, & m \geqslant -1, \\ 0, & m \leqslant -2; \end{cases}$$
$$H_m \triangleright_i^{\mathcal{P}_8} L_n = \delta_{i,2}(\tfrac{n}{2} - m)H_{m+n} + \delta_{i,3}\begin{cases} 0, & m \leqslant 0, \\ (\tfrac{n}{2} - m)H_{m+n}, & m \geqslant 1 \end{cases}$$
$$H_m \triangleright_i^{\mathcal{P}_8} H_n = \delta_{i,2}(n - m)I_{m+n} + \delta_{i,3}\begin{cases} 0, & m \leqslant 0, \\ (n - m)I_{m+n}, & m \geqslant 1 \end{cases}$$
$$I_m \triangleright_i^{\mathcal{P}_8} L_n = \delta_{i,2}(-m)I_{m+n} + \delta_{i,3}\begin{cases} 0, & m \leqslant 0, \\ -mI_{m+n}, & m \geqslant 1 \end{cases}$$

where $c, a, \mu, \lambda \in \mathbb{C}$, $k \in \{-2, -1, 1, 2, 3\}$, $t \in \mathbb{Z}\setminus\{0, 1\}$ *and* $s \in \{2t-2, 2t-1, 2t, 2t+1, 2t+2\}$. *Conversely, the above types are all the graded post-Lie algebra structures satisfying (3)-(9) on* $\mathcal{S}$. *Furthermore, the post-Lie algebras* $\mathcal{W}_{i,a,\mu,\lambda}^{\mathcal{P}_3^c,s,k,t}$, $\mathcal{W}_j^{\mathcal{P}_5}$ *and* $\mathcal{W}_j^{\mathcal{P}_6}$ *are isomorphic to the post-Lie algebras* $\mathcal{W}_{i,a,\mu,\lambda}^{\mathcal{P}_4^c,s,k,t}$, $\mathcal{W}_j^{\mathcal{P}_7}$ *and* $\mathcal{W}_j^{\mathcal{P}_8}$, $i = 1, 2, 3, 4, 5$ *and* $j = 1, 2, 3$ *respectively, and other post-Lie algebras are not mutually isomorphic.*

Proof. Suppose that $(\mathcal{S}, [,], \triangleright)$ is a class of post-Lie algebra structures satisfying (3)-(9) on the Schrödinger-Virasoro algebra $\mathcal{S}$. By Lemma 3.3-3.5, there are complex-valued functions f, g, h on $\mathbb{Z}$ and complex numbers μ, a such that one of 26 cases in Table 2 holds. From this with Lemma 3.1, we obtain 26 classes of graded post-Lie algebra structures on $\mathcal{S}$. We claim that $h(0) = \lambda$ and $g(0) = \hat{\tau}$ in $\mathcal{W}_{i,a,\mu,\lambda}^{\mathcal{P}_j^c,s,k,t}$, $j = 3, 4$ and $i = 1, 2, 3, 4, 5$ and $g(0) = \hat{\lambda}$ in $\mathcal{W}_i^{\mathcal{P}_j}$, $j = 5, 6, 7, 8$ and $i = 1, 2, 3$. We claim that $g(0) = \hat{\lambda}$ and $g(0) = \hat{\tau}$ will not appear in every structures, when $m = 0$, for example, in Case $\mathcal{W}_i^{\mathcal{P}_5}$, $i = 1, 2, 3$, then $I_m \triangleright_3^{P_5} L_n = 0\hat{\lambda}I_{0+n} = 0$, one has $I_m \triangleright_3^{P_5} L_n = 0$ for $m \leqslant 0$, and in Case $\mathcal{W}_{i,a,\mu,\lambda}^{\mathcal{P}_3^c,s,k,t}$, $i = 1, 2, 3, 4, 5$, then $H_m \triangleright_{3,\lambda}^{P_3} L_n = -(\frac{n}{2} - 0)\lambda H_{0+n} = 0$, one has $H_m \triangleright_{3,\lambda}^{P_3} L_n = -\frac{n}{2}\lambda H_n$ for $m = 0$. Hence we can obtain 26 classes of graded post-Lie algebra structures on $\mathcal{S}$ listed in the theorem.

Conversely, every type of the 26 cases means that there are complex-valued functions f and g, h on $\mathbb{Z}$ and complex numbers a, μ such that (10)-(15) hold and, the Equations (16)-(22) are easily verified. Thus, by Lemma 3.1 we see that they are the all graded post-Lie algebra structures satisfying (3)-(9) on the Schrödinger-Virasoro algebra $\mathcal{S}$.

Finally, by Lemma 3.7 with maps $L_m \to -L_{-m}$, $H_m \to -H_{-m}$, $I_m \to -I_{-m}$ we know that the post-Lie algebras $\mathcal{W}_{i,a,\mu,\lambda}^{\mathcal{P}_3^c,s,k,t}$, $\mathcal{W}_j^{\mathcal{P}_5}$ and $\mathcal{W}_j^{\mathcal{P}_6}$ are isomorphic to the post-Lie algebras $\mathcal{W}_{i,a,\mu,\lambda}^{\mathcal{P}_4^c,s,k,t}$, $\mathcal{W}_j^{\mathcal{P}_7}$ and $\mathcal{W}_j^{\mathcal{P}_8}$, $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3$ respectively. Clearly, the other post-Lie algebras are not mutually isomorphic. The proof is completed. □

4. Application to Rota-Baxter operators.

The Rota-Baxter algebra was introduced by the mathematician Glen E. Baxter [2] in 1960 in his probability study, and was popularized mainly by the work of Rota [16, 17] and his school. Recently, the Rota-Baxter algebra relation were introduced to solve certain analytic and combinatorial problem and then applied to many fields in mathematics and mathematical physics (see [6, 7, 19, 23] and the references therein). Now let us recall the definition of Rota-Baxter operator.

Definition 4.1. Let L be a complex Lie algebra. A Rota-Baxter operator of weight $\lambda \in \mathbb{C}$ is a liner map $R : L \to L$ satisfying

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]) + \lambda R([x, y]), \ \forall \text{x,y} \in L. \tag{50}$$

Note that if R is a Rota-Baxter operator of weight $\lambda \neq 0$, then $\lambda^{-1}R$ is a Rota-Baxter operator of weight 1. Therefore, one only needs to consider Rota-Baxter operators of weight 0 and 1.

4.1. **Rota-Baxter operators of weight 1.** In this section, we mainly consider the homogeneous Rota-Baxter operator R of weight 1 on the Schrödinger-Virasoro $\mathcal{S}$ given by

$$R(L_m) = f(m)L_m, \quad R(H_m) = h(m)H_m, \quad R(I_m) = g(m)I_m \tag{51}$$

for all $m \in \mathbb{Z}$, where f, g, h are complex-valued functions on $\mathbb{Z}$.

Lemma 4.2. *(see [1]) Let $(L, [,])$ be a Lie algebra and $R : L \to L$ a Rota-Baxter operator of weight 1. Define a new operation $x \triangleright y = [R(x), y]$ on L. Then $(L, [,], \triangleright)$ is a post-Lie algebra.*

Theorem 4.3. *A homogeneous Rote-Baxrer operator R of weight 1 satisfying (51) on the Schrödinger-Virasoro $\mathcal{S}$ must be one of the following types*

$(\mathcal{R}_1^{\mathcal{P}_1})$: $R(L_m) = 0, R(H_n) = 0, R(I_n) = 0$;

$(\mathcal{R}_2^{\mathcal{P}_1})$: $R(L_m) = 0, R(H_n) = -H_n, R(I_n) = -I_n$;

$(\mathcal{R}_1^{\mathcal{P}_2})$: $R(L_m) = -L_m, R(H_n) = 0, R(I_n) = 0$;

$(\mathcal{R}_2^{\mathcal{P}_2})$: $R(L_m) = -L_m, R(H_n) = -H_n, R(I_n) = -I_n$;

$$(\mathcal{R}_1^{\mathcal{P}_3^c}):\ R(L_m) = \begin{cases} -L_m, & m > 0, \\ cL_0, & m = 0, \\ 0, & m < 0; \end{cases} \qquad R(H_n) = 0, \qquad R(I_n) = 0;$$

$$(\mathcal{R}_2^{\mathcal{P}_3^c}):\ R(L_m) = \begin{cases} -L_m, & m > 0, \\ cL_0, & m = 0, \\ 0, & m < 0; \end{cases} \qquad R(H_n) = -H_n, \qquad R(I_n) = -I_n;$$

$$(\mathcal{R}_{3,\hat{\tau},\lambda}^{\mathcal{P}_3^c,k}):\ R(L_m) = \begin{cases} -L_m, & m > 0, \\ cL_0, & m = 0, \\ 0, & m < 0; \end{cases} \qquad R(H_n) = \begin{cases} -H_n, & n > 0, \\ \lambda H_0, & n = 0, \\ 0, & n < 0; \end{cases}$$

$$R(I_n) = \begin{cases} -I_n, & n \geqslant k, \\ \hat{\tau} I_0, & n = 0, \\ 0, & n \leqslant k-1; \end{cases}$$

$$(\mathcal{R}_5^{\mathcal{P}_3^c,s,t}):\ R(L_m) = \begin{cases} -L_m, & m > 0, \\ cL_0, & m = 0, \\ 0, & m < 0; \end{cases} \qquad R(H_n) = \begin{cases} -H_n, & n \geqslant t, \\ 0, & n \leqslant t-1; \end{cases}$$

$$R(I_n) = \begin{cases} -I_n, & n \geqslant s, \\ 0, & n \leqslant s-1; \end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_4^c}):\ R(L_m) = \begin{cases} -L_m, & m < 0, \\ cL_0, & m = 0, \\ 0, & m > 0; \end{cases} \qquad R(H_n) = 0, \qquad R(I_n) = 0;$$

$$(\mathcal{R}_2^{\mathcal{P}_4^c}):\ R(L_m)=\begin{cases}-L_m, & m<0,\\ cL_0, & m=0,\\ 0, & m>0;\end{cases}\qquad R(H_n)=-H_n,\qquad R(I_n)=-I_n;$$

$$(\mathcal{R}_{3,\hat{\tau},\lambda}^{\mathcal{P}_4^c,k}):\ R(L_m)=\begin{cases}-L_m, & m<0,\\ cL_0, & m=0,\\ 0, & m>0;\end{cases}\qquad R(H_n)=\begin{cases}0, & n>0,\\ \lambda H_0, & n=0,\\ -H_n, & n<0;\end{cases}$$

$$R(I_n)=\begin{cases}0, & n\geqslant k,\\ \hat{\tau}I_0, & n=0,\\ -I_n, & n\leqslant k-1;\end{cases}$$

$$(\mathcal{R}_5^{\mathcal{P}_4^c,s,t}):\ R(L_m)=\begin{cases}-L_m, & m>0,\\ cL_0, & m=0,\\ 0, & m<0;\end{cases}\qquad R(H_n)=\begin{cases}0, & n\geqslant t,\\ -H_n, & n\leqslant t-1;\end{cases}$$

$$R(I_n)=\begin{cases}0, & n\geqslant s,\\ -I_n, & n\leqslant s-1;\end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_5}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}\qquad R(H_n)=0,\qquad R(I_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_5}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}\qquad R(H_n)=-H_n,\qquad R(I_n)=-I_n;$$

$$(\mathcal{R}_{3,\hat{\lambda}}^{\mathcal{P}_5}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}\qquad R(H_n)=\begin{cases}0, & n\leqslant 0,\\ -H_n, & n\geqslant 1;\end{cases}$$

$$R(I_n)=\begin{cases}0, & n\leqslant -1,\\ \hat{\lambda}I_0, & n=0,\\ -I_n, & n\geqslant 1;\end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_6}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}\qquad R(H_n)=0,\qquad R(I_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_6}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}\qquad R(H_n)=-H_n,\qquad R(I_n)=-I_n;$$

$$(\mathcal{R}_{3,\hat{\lambda}}^{\mathcal{P}_6}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\leqslant 0,\\ 0, & n\geqslant 1;\end{cases}$$

$$R(I_n)=\begin{cases}-I_n, & n\leqslant -1,\\ \hat{\lambda}I_0, & n=0,\\ 0, & n\geqslant 1;\end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_7}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}\qquad R(H_n)=0,\qquad R(I_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_7}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}\qquad R(H_n)=-H_n,\qquad R(I_n)=-I_n;$$

$$(\mathcal{R}_{3,\hat{\lambda}}^{\mathcal{P}_7}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}\qquad R(H_n)=\begin{cases}0, & n\geqslant 1,\\ -H_n, & n\leqslant 0;\end{cases}$$

$$R(I_n)=\begin{cases}0, & n\geqslant 1,\\ \hat{\lambda}I_0, & n=0,\\ -I_n, & n\leqslant -1;\end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_8})\text{: } R(L_m)=\begin{cases}-L_m, & m\geqslant -1,\\ 0, & m\leqslant -2;\end{cases}\qquad R(H_n)=0,\qquad R(I_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_8})\text{: } R(L_m)=\begin{cases}-L_m, & m\geqslant -1,\\ 0, & m\leqslant -2,\end{cases}\qquad R(H_n)=-H_n,\qquad R(I_n)=-I_n;$$

$$(\mathcal{R}_{3,\hat{\lambda}}^{\mathcal{P}_8})\text{: } R(L_m)=\begin{cases}-L_m, & m\geqslant -1,\\ 0, & m\leqslant -2,\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\geqslant 1,\\ 0, & n\leqslant 0;\end{cases}$$

$$R(I_n)=\begin{cases}-I_n, & n\geqslant 1,\\ \hat{\lambda}I_0, & n=0,\\ 0, & n\leqslant -1\end{cases}$$

for all $m,n\in\mathbb{Z}$, where $c,\lambda,\hat{\lambda},\hat{\tau}\in\mathbb{C}$, $k\in\{-2,-1,1,2,3\}$ with $k\neq 1$, $t\in\mathbb{Z}\setminus\{0,1\}$ and $s\in\{2t-2,2t-1,2t,2t+1,2t+2\}$.

Proof. In view of Lemma 4.2, if we define a new operation $x\triangleright y=[R(x),y]$ on $\mathcal{S}$, then $(\mathcal{S},[,],\triangleright)$ is a post-Lie algebra. By (51), we have

$$L_m\triangleright L_n=[R(L_m),L_n]=(m-n)f(m)L_{m+n}, \tag{52}$$

$$L_m\triangleright H_n=[R(L_m),H_n]=(\frac{m}{2}-n)f(m)H_{m+n}, \tag{53}$$

$$L_m\triangleright I_n=[R(L_m),I_n]=-nf(m)I_{m+n}, \tag{54}$$

$$H_m\triangleright L_n=[R(H_m),L_n]=-(\frac{n}{2}-m)h(m)H_{m+n}, \tag{55}$$

$$H_m\triangleright H_n=[R(H_m),H_n]=(m-n)h(m)I_{m+n}, \tag{56}$$

$$I_m\triangleright L_n=[R(I_m),L_n]=mg(m)I_{m+n} \tag{57}$$

and $I_m\triangleright H_n=[R(I_m),H_n]=H_m\triangleright I_n=[R(H_m),I_n]=I_m\triangleright I_n=[R(I_m),I_n]=0$ for all $m,n\in\mathbb{Z}$. This means that $(\mathcal{S},[,],\triangleright)$ is a graded post-Lie algebras structure satisfying (3)-(9) with $\phi(m,n)=(m-n)f(m)$, $\varphi(m,n)=(\frac{m}{2}-n)f(m)$, $\chi(m,n)=-nf(m)$, $\psi(m,n)=-(\frac{n}{2}-m)h(m)$, $\xi(m,n)=(m-n)h(m)$ and $\theta(m,n)=mg(m)$.

A similar discussion to Lemma 3.1 gives

$$\begin{aligned}&(m-n)(f(m+n)-f(n)f(m)+f(m)f(m+n)+f(n)f(m+n))=0,\\ &(\frac{m}{2}-n)(h(m+n)-f(m)h(n)+f(m)h(m+n)+h(n)h(m+n))=0,\\ &n(m+n)(g(m+n)(1+f(m)+g(n))-f(m)g(n))=0,\\ &(m-n)(g(m+n)-h(m)h(n)+h(m)g(m+n)+h(n)g(m+n))=0.\end{aligned}$$

From this we conclude that Equations (10)-(22) hold with $a=\mu=0$. In the same way of Lemma 3.6, we see that f,g,h must satisfy Table 2 with $a=\mu=0$. This excludes Cases $\mathcal{W}_{4,a,\mu}^{\mathcal{P}_3^c,k=1}$ and $\mathcal{W}_{4,a,\mu}^{\mathcal{P}_4^c,k=1}$. Thus, f, g, h must be of the 24 cases listed in Table 2 with $a=\mu=0$, which can yield the 24 forms of R one by one. It is easy to verify that every form of R listed in the above is a Rota-Baxter operator of weight 1 satisfying (51). The proof is completed. □

4.2. **Remark on Rota-Baxter operators of weight zero and pre-Lie algebras.** The natural question is: how we can characterize the Rota-Baxter operators of weight zero on the Schrödinger-Virasoro $\mathcal{S}$? This is related to the so called pre-Lie algebra which is a class of Lie-admissible algebras whose commutators are Lie algebras. Pre-Lie algebras appeared in many fields in mathematics and physics under different names like left-symmetric algebras, Vinberg algebras and quasi-associative algebras (see the survey article [3] and the references therein). Now we recall the definition of pre-Lie algebra as follows.

Definition 4.4. A *pre-Lie algebra* A is a vector space A with a bilinear product $\triangleright$ satisfying

$$(x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) = (y \triangleright x) \triangleright z - y \triangleright (x \triangleright z), \quad \forall x, y, z \in A. \tag{58}$$

As a parallel result of Lemma 4.2, one has the following conclusion.

Proposition 1. *(see [8]) Let $(L, [\cdot, \cdot])$ be a Lie algebra with a Rota-Baxter operator R of weight 0 on it. Define a new operation $x \triangleright y = [R(x), y]$ for any $x, y \in L$. Then $(L, \triangleright)$ is a pre-Lie algebra.*

Using a similar method on classification of Rota-Baxter operators of weight 1 in the above subsection, by Proposition 1 we can get the forms of Rota-Baxter operators of weight zero when the corresponding structure of pre-Lie algebra are known. For example, consider the homogeneous Rota-Baxter operator R of weight zero on the Schrödinger-Virasoro algebra $\mathcal{S}$ satisfying (51). According to Proposition 1, if we define a new operation $x \triangleright y = [R(x), y]$ on $\mathcal{S}$, then $(\mathcal{S}, \triangleright)$ is a pre-Lie algebra. By (51), we have Equations (52)-(57) hold. At this point we can apply the relevant results on pre-Lie algebra satisfying (52)-(57). But the classification of graded pre-Lie algebra structures on $\mathcal{S}$ is also an unsolved problem, as far as we know. In fact, we can direct characterize the Rota-Baxter operators of weight zero on the Schrödinger-Virasoro $\mathcal{S}$ satisfying (51) following the approach of [6]. Due to limited space, it will not be discussed here.

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REFERENCES

[1] C. Bai, L. Guo and X. Ni, Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and PostLie algebras, *Commun. Math. Phys.*, **297** (2010), 553–596.

[2] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.*, **10** (1960), 731–742.

[3] D. Burde, Left-symmetric algebras, or pre-Lie algebras in geometry and physics, *Cent. Eur. J. Math.*, **4** (2006), 323–357.

[4] D. Burde, K. Dekimpe and K. Vercammen, Affine actions on Lie groups and post-Lie algebra structures, *Linear Algebra Appl.*, **437** (2012), 1250–1263.

[5] D. Burde and W. A. Moens, Commutative post-Lie algebra structures on Lie algebras, *J. Algebra*, **467** (2016), 183–201.

[6] X. Gao, M. Liu, C. Bai and N. Jing, Rota-Baxter operators on Witt and Virasoro algebras, *J. Geom. Phys.*, **108** (2016), 1–20.

[7] L. Guo, *An Introduction to Rota-Baxter Algebra*, Somerville: International Press, 2012.

[8] I. Z. Golubschik and V. V. Sokolov, Generalized operator Yang-Baxter equations, integrable ODES and nonassociative algebras, *J. Nonlinear Math. Phys.*, **7** (2000), 184–197.

[9] J. Han, J. Li and Y. Su, Lie bialgebra structures on the Schrödinger-Virasoro Lie algebras, *J. Math. Phys.*, **50** (2009), 083504, 12 pp.

[10] M. Henkel, Schrödinger invariance and stringly anisotropic critical systems, *J. Stat. Phys.*, **75** (1994), 1023–1061.

[11] Y. Li and X. Tang, Biderivations and commutative post-Lie algebra structure on Schrödinger-Virasoro Lie algebras, *Bull. Iranian Math. Soc.*, **45** (2019), 1743–1754.

[12] H. Z. Munthe-Kaas and A. Lundervold, On post-Lie algebras, Lie-Butcher series and moving frames, *Found. Comput. Math.*, **13** (2013), 583–613.

[13] Y. Pan, Q. Liu, C. Bai and L. Guo, Post-Lie algebra structures on the Lie algebra $\mathrm{SL}(2,\mathbb{C})$, *Electron. J. Linear Algebra*, **23** (2012), 180–197.

[14] Y. Pei and C. Bai, Novikov algebras and Schrödinger-Virasoro Lie algebras, *J. Phys.*, **44** (2011), 045201, 18 pp.

[15] C. Roger and J. Unterberger, The Schrödinger-Virasoro Lie group and algebra: Representation theory and cohomological study, *Ann. Henri Poincaré*, **7** (2006), 1477–1529.

[16] G.-C. Rota, Baxter algebras and combinatorial identities I, *Bull. Amer. Math. Soc.*, **75** (1969), 325–329.

[17] G.-C. Rota, Baxter operators, an introduction, in "Gian-Carlo Rota on combinatorics, introductory papers and commentaries", *Joesph PS Kung, Editor, J.*, (1995), 504–512.

[18] X. Tang, Post-Lie algebra structures on the Witt algebra, *Bull. Malays. Math. Sci. Soc.*, **42** (2019), 3427–3451.

[19] X. Tang, Y. Zhang and Q. Sun, Rota-Baxter operators on 4-dimensional complex simple associative algebras, *Appl. Math. Comput.*, **229** (2014), 173–186.

[20] X. Tang and Y. Zhong, Graded post-Lie algebra structures, Rota-Baxter operators and Yang-Baxter equations on the W-algebra $W(2,2)$, *Bull. Korean Math. Soc.*, **55** (2018), 1727–1748.

[21] J. Unterberger, On vertex algrbra representations of the Schrödinger-Virasoro Lie algebra, *Nuclear Phys. B*, **823** (2009), 320–371.

[22] B. Vallette, Homology of generalized partition posets, *J. Pure. Appl. Algebra*, **208** (2007), 699–725.

[23] H. Yu, L. Guo and J.-Y. Thibon, Weak quasi-symmetric functions, Rota-Baxter algebras and Hopf algebras, *Adv. Math.*, **344** (2019), 1–34.

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