

## LOCAL WELL-POSEDNESS OF PERTURBED NAVIER-STOKES SYSTEM AROUND LANDAU SOLUTIONS

JINGJING ZHANG AND TING ZHANG\*

School of Mathematical Sciences, Zhejiang University  
Hangzhou 310027, China

(Communicated by Shouchuan Hu)

ABSTRACT. For the incompressible Navier-Stokes system, when initial data are uniformly locally square integral, the local existence of solutions has been obtained. In this paper, we consider perturbed system and show that perturbed solutions of Landau solutions to the Navier-Stokes system exist locally under  $L^q_{\text{uloc}}$ -perturbations,  $q \geq 2$ . Furthermore, when  $q \geq 3$ , the solution is well-posed. Precisely, we give the explicit formula of the pressure term.

**1. Introduction.** The initial value problem of the Navier-Stokes system is described as follows

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f, & x \in \mathbb{R}^3, t \geq 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.1)$$

where  $u = (u_1, u_2, u_3)$  is the velocity and  $p$  is the pressure.

It is well known that Leray [16] proved the global existence of weak solutions for divergence free initial data  $u_0 \in L^2(\mathbb{R}^3)$  and  $f = 0$ . The uniqueness and regularity for the solutions still remain open, see *e.g.* [14] and references therein. For well-posedness results to the Navier-Stokes system, Kato [9] proved the local well-posedness for the general initial data in  $L^n(\mathbb{R}^n)$  and the global well-posedness for the small initial data in  $L^n(\mathbb{R}^n)$ . Giga and Miyakawa [3] and Taylor [22] gave the same result in certain Morrey spaces. In 2001, Koch and Tataru [11] proved the global well-posedness evolving from small initial data in the space  $BMO^{-1}$ , in which they need  $u \in L^2_{\text{loc}}(\mathbb{R}^n \times [0, \infty))$  in order to make sense of the system. Moreover, self-similar solutions  $u \notin L^3(\mathbb{R}^3)$ , but belong to  $L^2_{\text{uloc}}(\mathbb{R}^3)$ , see [2, 24, 26]. The definition of  $L^q_{\text{uloc}}$  will be given in (1.6) later.

For the Navier-Stokes system with  $u_0 \in L^2_{\text{uloc}}$ , there are some results on uniformly locally square integrable solutions. Basson [1] described such solutions. Lemarié-Rieusset [14, 15] gave the local existence of weak solution  $u$  for when  $u_0 \in L^2_{\text{uloc}}$ . Moreover, global weak solution exists for the decaying initial data  $u_0 \in E_2 = \{f \in L^2_{\text{uloc}} : \lim_{|x_0| \rightarrow \infty} \|f\|_{L^2(B(x_0, 1))} = 0\}$ . Kikuchi and Seregin's paper [10] extend above results which include forcing terms in the equations. Very recently, Kown and Tsai [12] generalizes the global existence with non-decaying initial data whose local oscillations decay.

For the uniformly local- $L^3$  integrable functions space  $L^3_{\text{uloc}}$ , Lemarié-Rieusset [14] gave the applications of the space  $L^3_{\text{uloc}}$  to the Navier-Stokes system. Hineman and Wang [6]

---

2020 *Mathematics Subject Classification.* Primary: 35Q30; Secondary: 49K40.

*Key words and phrases.* Local well-posedness, Navier-Stokes system, Landau solutions.

\* Corresponding author: Ting Zhang.

obtained the local well-posedness of Nematic liquid crystal flow for any initial data  $(u_0, d_0)$  with small  $L^3_{\text{uloc}}$ -norm of  $(u_0, \nabla d_0)$ .

The stationary Navier-Stokes system in  $\mathbb{R}^3$  has the form

$$\begin{cases} -\Delta v + (v \cdot \nabla)v + \nabla p = f, \\ \nabla \cdot v = 0. \end{cases} \tag{1.2}$$

When  $f = (b(c)\delta_0, 0, 0)$  with  $b(c) = \frac{8\pi c}{3(c^2-1)} \left(2 + 6c^2 - 3c(c^2 - 1) \ln\left(\frac{c+1}{c-1}\right)\right)$  and  $\delta_0$  Dirac measure, the following formulas

$$\begin{aligned} v_c^1(x) &= 2 \frac{c|x|^2 - 2x_1|x| + cx_1^2}{|x|(c|x| - x_1)^2}, \quad v_c^2(x) = 2 \frac{x_2(cx_1 - |x|)}{|x|(c|x| - x_1)^2}, \\ v_c^3(x) &= 2 \frac{x_3(cx_1 - |x|)}{|x|(c|x| - x_1)^2}, \quad p_c(x) = 4 \frac{cx_1 - |x|}{|x|(c|x| - x_1)^2}, \end{aligned} \tag{1.3}$$

with  $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$  and constant  $|c| > 1$  are the distributional solutions to system (1.2) in  $\mathbb{R}^3$ . The explicit stationary solutions (1.3) were given by Landau [13]. These solutions (1.3) are called Landau solutions. Landau solutions are in  $L^2_{\text{uloc}}$  space. Tian and Xin [23] proved that all  $(-1)$ -homogeneous, axisymmetric nonzero solutions of system (1.2) in  $C^2(\mathbb{R}^3 \setminus \{0\})$  are Landau solutions. Šverák [21] proved that Landau solutions are the only  $(-1)$ -homogeneous solutions in  $C^2(\mathbb{R}^3 \setminus \{0\})$ . More details can be seen in [13, 17, 18, 19, 20, 23].

We denote  $u(x, t)$  be the solution to the Navier-Stokes system (1.1) with the given external force  $f = (b(c)\delta_0, 0, 0)$  and initial data  $u_0 = v_c + w_0$ . By a direct calculation, functions  $w(x, t) = u(x, t) - v_c(x)$  and  $\pi(x) = p(x, t) - p_c(x)$  satisfy the following perturbed Navier-Stokes system

$$\begin{cases} w_t - \Delta w + (w \cdot \nabla)w + (w \cdot \nabla)v_c + (v_c \cdot \nabla)w + \nabla \pi = 0, \\ \nabla \cdot w = 0, \\ w(x, 0) = w_0(x). \end{cases} \tag{1.4}$$

The explicit formula of  $\pi$  is as follows

$$\pi = -\frac{1}{3}|w|^2 + \text{p.v.} \int_{\mathbb{R}^3} \partial_i \partial_j \Gamma(x - y) w_i w_j(y) dy - \frac{2}{3} v_c \cdot w + 2 \text{p.v.} \int_{\mathbb{R}^3} \partial_i \partial_j \Gamma(x - y) w_i v_{cj}(y) dy, \tag{1.5}$$

for which detailed calculation can be seen in Appendix.

Karch and Pilarczyk [7] show that perturbed solutions of Landau solutions to the Navier-Stokes system exist globally under  $L^2$ - perturbations. In 2017, Karch, Pilarczyk and Schonbek [8] generalized the work of [7]. They presented a new method to show the global existence for a large class of solutions including the Landau ones. Based on these results, we are inspired to study local well-posedness of weak solutions to the perturbed Navier-Stokes system (1.4) with initial data  $w_0 \in L^q_{\text{uloc}}(\mathbb{R}^3)$  in our work.

First, we give some notations used in this paper. Ball  $B(x, r)$  is a ball in  $\mathbb{R}^3$  centered at  $x$  with a radius  $r$ ,

$$B(x, r) = B_r(x) = \{y \in \mathbb{R}^3 : |y - x| < r\}.$$

The spaces  $L^q_{\text{uloc}}$ ,  $1 \leq q \leq \infty$ , and  $U^{s,p}(t_0, t)$  for  $1 \leq s, p \leq \infty$  and  $0 \leq t_0 < t \leq \infty$ , defined by

$$L^q_{\text{uloc}} = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^3) : \|u\|_{L^q_{\text{uloc}}} = \sup_{x_0 \in \mathbb{R}^3} \|u\|_{L^q(B_1(x_0))} < +\infty \right\} \tag{1.6}$$

and

$$U^{s,p}(t_0, t) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^3 \times (t_0, t)) : \|u\|_{U^{s,p}(t_0, t)} = \sup_{x_0 \in \mathbb{R}^3} \|u\|_{L^s(t_0, t; L^p(B_1(x_0)))} < +\infty \right\}$$

When  $t_0 = 0$ , we simply use  $U_T^{s,p} = U^{s,p}(0, T)$ . Note that  $U^{\infty,p}(t_0, t) = L^\infty(t_0, t; L^p_{\text{uloc}})$ .

Set  $L^2$  local energy space

$$\mathcal{E}_T = \{u \in L^2_{\text{loc}}([0, T] \times \mathbb{R}^3; \mathbb{R}^3) : \text{div } u = 0, \|u\|_{\mathcal{E}_T} < +\infty\}, \tag{1.7}$$

where

$$\|u\|_{\mathcal{E}_T} := \|u\|_{U_T^{\infty,2}} + \|\nabla u\|_{U_T^{2,2}}. \tag{1.8}$$

The definition of  $L^q$  local energy solution,  $q \geq 2$ , is as follows

**Definition 1.1.** ( $L^q$  local energy solution) Let  $w_0 \in L^q_{\text{uloc}}$ ,  $\text{div } w_0 = 0$ . A pair of functions  $(w, \pi)$  is a local energy solution to the perturbed Navier-Stokes system (1.4) with initial data  $w_0$  in  $\mathbb{R}^3 \times (0, T)$  for  $0 < T < \infty$ , if the functions satisfy the following conditions:

- (1)  $w \in U_T^{\infty,q}$ ,  $\nabla(|w|^{\frac{q}{2}}) \in U_T^{2,2}$  and  $\pi \in L^q_{\text{loc}}([0, T]; L^{\frac{2q}{q+1}}(\mathbb{R}^3))$ ;
- (2)  $(w, \pi)$  satisfies the perturbed Navier-Stokes system (1.4) in the sense of distributions;
- (3) the function  $t \mapsto \int_{\mathbb{R}^3} w(x, t) \cdot \varphi(x) dx$  is continuous on  $[0, T]$  for any compactly supported function  $\varphi \in C_c^\infty(\mathbb{R}^3)$ . Furthermore, for any compact set  $K \subset \mathbb{R}^3$ ,

$$\|w(\cdot, t) - w_0\|_{L^q(K)} \rightarrow 0, \quad \text{as } t \rightarrow 0^+; \tag{1.9}$$

- (4)  $(w, \pi)$  satisfies the following local energy inequality

$$\begin{aligned} & \int_{\mathbb{R}^3} |w|^2 \xi(x, t) dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla w|^2 \xi dx ds \\ & \leq \int_0^t \int_{\mathbb{R}^3} (2v_c \otimes w : \nabla w \xi + (\partial_s \xi + \Delta \xi) |w|^2 \\ & \quad + (|w|^2 + 2\pi + 2v_c \cdot w)(w \cdot \nabla) \xi + |w|^2 v_c \cdot \nabla \xi) dx ds, \end{aligned} \tag{1.10}$$

for any  $t \in (0, T)$  and for all non-negative smooth functions  $\xi \in C_c^\infty((0, T) \times \mathbb{R}^3)$ ;

- (5) For any  $x_0 \in \mathbb{R}^3$ , there exists a function  $c_{x_0}(t) \in L^q(0, T)$  such that

$$\pi(x, t) = \widehat{\pi}_{x_0}(x, t) + c_{x_0}(t), \quad \text{in } L^q_{\text{loc}}([0, T]; L^{\frac{2q}{q+1}}(B(x_0, \frac{3}{2}))), \tag{1.11}$$

where

$$\begin{aligned} \widehat{\pi}_{x_0}(x, t) &= -\frac{1}{3} |w(x, t)|^2 + \text{p.v.} \int_{B(x_0, 2)} \partial_i \partial_j \Gamma(x - y) w_i w_j(y) dy \\ &+ \text{p.v.} \int_{B(x_0, 2)^c} \partial_i \partial_j (\Gamma(x - y) - \Gamma(x_0 - y)) w_i w_j(y) dy - \frac{2}{3} v_c \cdot w(x, t) \\ &+ 2 \text{p.v.} \int_{B(x_0, 2)} \partial_i \partial_j \Gamma(x - y) w_i v_{cj}(y) dy \\ &+ 2 \text{p.v.} \int_{B(x_0, 2)^c} \partial_i \partial_j (\Gamma(x - y) - \Gamma(x_0 - y)) w_i v_{cj}(y) dy \end{aligned} \tag{1.12}$$

for  $\Gamma(x) = \frac{1}{4\pi|x|}$ .

Our main result is as follows

**Theorem 1.2.** *There exist positive universal constants  $c_3, \varepsilon_1$  and  $C$  with the following properties,*

(i) *For every  $|c| \geq c_3, w_0 \in L^q_{uloc}, q \geq 2$  with  $\operatorname{div} w_0 = 0$ , if*

$$T \leq \frac{\varepsilon_1}{1 + \|w_0\|_{L^q_{uloc}}^{2q}}, \quad (1.13)$$

*there exists a  $L^q$  local energy solution  $(w, \pi)$  on  $\mathbb{R}^3 \times (0, T)$  to the perturbed Navier-Stokes system (1.4) with initial data  $w_0$ , satisfying*

$$\|w\|_{U_T^{\infty, q}} + \|\nabla(|w|^{\frac{q}{2}})\|_{U_T^{2, 2}}^{\frac{2}{q}} \leq C \|w_0\|_{L^q_{uloc}}. \quad (1.14)$$

(ii) *Furthermore, when  $q \geq 3$ , the solution is unique.*

**Remark 1.1.** From (2.38) and (3.55), we could see a more detailed dependence of  $c_3$ .

**Scheme of the proof and organization of the paper.** In Section 2, we give some results which will be used in the proof of Theorem 1.2. In Section 3, we prove Theorem 1.2 by classical approximation theory. In Appendix, we give the details to derive the integral formula of pressure  $\pi$ , i.e. (1.5).

Let us complete this section by the notations that we shall use in this article.

**Notations.**

- We denote  $\|\cdot\|_p$  or  $\|\cdot\|_{L^p}$  the norm of the Lebesgue space  $L^p(\mathbb{R}^3)$  with  $p \in [1, \infty]$ .
- We denote  $\|\cdot\|_{L_t^p(L_x^q)}$  the norm of the Lebesgue space  $L_t^p([0, \infty); L_x^q(\mathbb{R}^3))$  with  $p, q \in [1, \infty]$ .
- We use the homogeneous Sobolev space  $\dot{H}^1(\mathbb{R}^3) = \{u \in \mathcal{S}'(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ .
- $C_0^\infty(\mathbb{R}^3)$  denotes the set of smooth and compactly supported functions.
- The  $i$ th coordinate ( $i = 1, 2, 3$ ) of a vector  $u$  will be denoted by  $u_i$ .

Set  $(\cdot, \cdot)$  be the  $L^2(\mathbb{R}^3)$  inner product. We use notation  $A \lesssim B$  to denote  $A \leq CB$ , where  $C$  is an absolute constant.

**2. Localized-mollified system.** We consider approximate solutions  $(w^\varepsilon, \pi^\varepsilon)$  to the following localized-mollified system in  $\mathbb{R}^3 \times (0, T)$

$$\begin{cases} w_t^\varepsilon - \Delta w^\varepsilon + (\mathcal{J}_\varepsilon(w^\varepsilon) \cdot \nabla)(w^\varepsilon \Phi_\varepsilon) + (\mathcal{J}_\varepsilon(w^\varepsilon) \cdot \nabla)(v_c \Phi_\varepsilon) + (\Phi_\varepsilon v_c \cdot \nabla) \mathcal{J}_\varepsilon(w^\varepsilon) + \nabla \pi^\varepsilon = 0, \\ \nabla \cdot w^\varepsilon = 0, \\ w^\varepsilon(x, 0) = w_0(x), \end{cases} \quad (2.1)$$

where  $\mathcal{J}_\varepsilon(v) = v * \eta_\varepsilon, \varepsilon > 0$ , the mollifier  $\eta_\varepsilon(x) = \varepsilon^{-3} \eta(\frac{x}{\varepsilon})$  with positive  $\eta \in C_c^\infty(B(0, 1)), \int \eta dx = 1$ . Localization factor  $\Phi_\varepsilon(x) = \Phi(\varepsilon x), \varepsilon > 0$  with  $\Phi \in C_c^\infty(\mathbb{R}^3)$ ,

$$\Phi = \begin{cases} 1 & \text{in } B(0, 1), \\ 0 & \text{in } B(0, \frac{3}{2})^c. \end{cases} \quad (2.2)$$

We will construct approximate solution  $w^\varepsilon$  directly in  $\mathcal{E}_T$  since  $w_0 \in L^2_{uloc}$  has no decay. First, we give a property of Landau solution  $v_c$  which can be obtained by direct calculation.

**Lemma 2.1.** *The explicit formula of  $v_c$  is (1.3), we have*

$$\| |x| v_c \|_{L^\infty} \leq \frac{2\sqrt{2}}{|c| - 1} := K_c. \quad (2.3)$$

Then, we give a fundamental inequality with the singular weight in Sobolev spaces: the so-called Hardy inequality which go back to the pioneering work by G.H. Hardy [4, 5].

**Lemma 2.2.** *For any  $f$  in  $\dot{H}^1(\mathbb{R}^3)$ , there holds*

$$\left( \int_{\mathbb{R}^3} \frac{|f(x)|^2}{|x|^2} dx \right)^{\frac{1}{2}} \leq 2 \|\nabla f\|_{L^2}. \tag{2.4}$$

By the Duhamel principle, we can write the solution to system (2.1) into the following integral formulation

$$w^\epsilon(t) = e^{t\Delta} w_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (\mathcal{J}_\epsilon(w^\epsilon) \otimes w^\epsilon \Phi_\epsilon + \mathcal{J}_\epsilon(w^\epsilon) \otimes v_c \Phi_\epsilon + \Phi_\epsilon v_c \otimes \mathcal{J}_\epsilon(w^\epsilon))(s) ds. \tag{2.5}$$

The following lemma give the construction of mild solution (see Chap. 5 in [25]) to system (2.1) in the space  $\mathcal{E}_T$ .

**Lemma 2.3.** *For each  $0 < \epsilon < 1$ ,  $\|w_0\|_{L^2_{uloc}} \leq B$  and  $\|v_c\|_{L^2_{uloc}} \leq 2C_0 B$ . If  $0 < T < \min(1, C\epsilon^3 B^{-2})$ , we can find a unique solution  $w = w^\epsilon$  to the integral form of (2.1) such that*

$$w(t) = e^{t\Delta} w_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (\mathcal{J}_\epsilon(w) \otimes w \Phi_\epsilon + \mathcal{J}_\epsilon(w) \otimes v_c \Phi_\epsilon + \Phi_\epsilon v_c \otimes \mathcal{J}_\epsilon(w))(s) ds \tag{2.6}$$

satisfying

$$\|w\|_{\mathcal{E}_T} \leq 2C_0 B \tag{2.7}$$

where  $C > 0$  and  $C_0 > 1$  are absolute constants.

*Proof.* Set the map

$$\Psi(w) = e^{t\Delta} w_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (\mathcal{J}_\epsilon(w) \otimes w \Phi_\epsilon + \mathcal{J}_\epsilon(w) \otimes v_c \Phi_\epsilon + \Phi_\epsilon v_c \otimes \mathcal{J}_\epsilon(w))(s) ds. \tag{2.8}$$

We will do contraction mapping in the local energy space  $\mathcal{E}_T$  which is defined in (1.7). According to Lemma 2.4 in [12], for any  $T > 0$ , we have

$$\|e^{t\Delta} f\|_{\mathcal{E}_T} \lesssim \left(1 + T^{\frac{1}{2}}\right) \|f\|_{L^2_{uloc}}, \tag{2.9}$$

for  $f \in L^2_{uloc}$ , and

$$\left\| \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot F(s) ds \right\|_{\mathcal{E}_T} \lesssim (1 + T) \|F\|_{U_T^{2,2}}, \tag{2.10}$$

for  $F \in U_T^{2,2}$ . Hence, by (2.8)-(2.10) and  $T \leq 1$ , we obtain

$$\|\Psi(w)\|_{\mathcal{E}_T} \lesssim \|w_0\|_{L^2_{uloc}} + \|\mathcal{J}_\epsilon(w) \otimes w \Phi_\epsilon\|_{U_T^{2,2}} + 2 \|\mathcal{J}_\epsilon(w) \otimes v_c \Phi_\epsilon\|_{U_T^{2,2}}. \tag{2.11}$$

Note that

$$\begin{aligned} \|\mathcal{J}_\epsilon(w) \otimes w \Phi_\epsilon\|_{U_T^{2,2}} &\lesssim \|w * \eta_\epsilon\|_{L^2(0,T;L^\infty(\mathbb{R}^3))} \|w\|_{U_T^{\infty,2}} \\ &\lesssim \|w\|_{U_T^{2,2}} \|\eta_\epsilon\|_{L^2} \|w\|_{U_T^{\infty,2}} \\ &\lesssim \epsilon^{-\frac{3}{2}} \|w\|_{U_T^{2,2}} \|w\|_{U_T^{\infty,2}} \\ &\lesssim \epsilon^{-\frac{3}{2}} \sqrt{T} \|w\|_{U_T^{\infty,2}}^2, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \|\mathcal{J}_\epsilon(w) \otimes v_c \Phi_\epsilon\|_{U_T^{2,2}} &\lesssim \|w * \eta_\epsilon\|_{L^2(0,T;L^\infty(\mathbb{R}^3))} \|v_c\|_{L^2_{uloc}} \\ &\lesssim \epsilon^{-\frac{3}{2}} \sqrt{T} \|w\|_{U_T^{\infty,2}} \|v_c\|_{L^2_{uloc}}. \end{aligned} \tag{2.13}$$

We have

$$\begin{aligned} \|\Psi(w)\|_{\mathcal{E}_T} &\lesssim \|w_0\|_{L^2_{\text{uloc}}} + \epsilon^{-\frac{3}{2}}\sqrt{T}\|w\|_{U_T^{\infty,2}}^2 + \epsilon^{-\frac{3}{2}}\sqrt{T}\|w\|_{U_T^{\infty,2}}\|v_c\|_{L^2_{\text{uloc}}} \\ &\lesssim \|w_0\|_{L^2_{\text{uloc}}} + \epsilon^{-\frac{3}{2}}\sqrt{T}\left(\|w\|_{U_T^{\infty,2}} + 2\|v_c\|_{L^2_{\text{uloc}}}\right)\|w\|_{U_T^{\infty,2}} \\ &\leq C_0\|w_0\|_{L^2_{\text{uloc}}} + C_1\epsilon^{-\frac{3}{2}}\sqrt{T}\left(\|w\|_{\mathcal{E}_T} + 2\|v_c\|_{L^2_{\text{uloc}}}\right)\|w\|_{\mathcal{E}_T}, \end{aligned} \quad (2.14)$$

for some constants  $C_0, C_1$ . Hence, for any  $w, z \in \mathcal{E}_T$ , there holds

$$\|\Psi(w) - \Psi(z)\|_{\mathcal{E}_T} \leq C_1\epsilon^{-\frac{3}{2}}\sqrt{T}\left(\|w\|_{\mathcal{E}_T} + \|z\|_{\mathcal{E}_T} + 2\|v_c\|_{L^2_{\text{uloc}}}\right)\|w - z\|_{\mathcal{E}_T}. \quad (2.15)$$

By Picard contraction theorem and  $\|v_c\|_{L^2_{\text{uloc}}} \leq 2C_0B$ , if  $T$  satisfies

$$T < \frac{\epsilon^3}{256(C_0C_1B)^2} = C\epsilon^3B^{-2}, \quad (2.16)$$

there exists a fixed point  $w \in \mathcal{E}_T$  of  $w = \Psi(w)$  satisfying

$$\|w\|_{\mathcal{E}_T} \leq 2C_0B. \quad (2.17)$$

□

We will give a uniform bound of  $(w^\epsilon, \pi^\epsilon)$  on a uniform time  $[0, T]$  in the following lemma

**Lemma 2.4.** *For each  $0 < \epsilon < 1$ , let  $(w^\epsilon, \pi^\epsilon)$  be the solution to system (2.1) on  $\mathbb{R}^3 \times [0, T_\epsilon]$ . If  $|c| \geq c_3$  and  $w_0 \in L^2_{\text{uloc}}$  with  $\text{div } w_0 = 0$ , there exists a small constant positive  $\varepsilon_1$  independent of  $\epsilon$  and  $\|w_0\|_{L^2_{\text{uloc}}}$  such that, if  $T_\epsilon \leq \tilde{T} = \varepsilon_1 \left(1 + \|w_0\|_{L^2_{\text{uloc}}}^4\right)^{-1}$ , then*

$$\|w^\epsilon\|_{\mathcal{E}_{T_\epsilon}} \leq C\|w_0\|_{L^2_{\text{uloc}}}, \quad (2.18)$$

where the constant  $C$  is independent of  $\epsilon$  and  $T_\epsilon$ .

*Proof.* Note that we can derive an integral formula of pressure  $\pi^\epsilon$  similar to  $\pi$  for which the detailed proof can be seen in Appendix

$$\begin{aligned} \pi^\epsilon(x, t) & \\ &= -\frac{1}{3}\mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon\Phi_\epsilon(x, t) + \text{p.v.} \int \partial_i\partial_j\Gamma(x-y)\mathcal{J}_\epsilon(w_i^\epsilon)w_j^\epsilon(y, t)\Phi_\epsilon(y)dy \\ &\quad -\frac{2}{3}v_c \cdot \mathcal{J}_\epsilon(w^\epsilon)\Phi_\epsilon(x, t) + 2\text{p.v.} \int \partial_i\partial_j\Gamma(x-y)\mathcal{J}_\epsilon(w_i^\epsilon)v_{cj}(y, t)\Phi_\epsilon(y)dy, \end{aligned} \quad (2.19)$$

for  $\Gamma(x) = \frac{1}{4\pi|x|}$ . For any fixed point  $x_0$ , we define  $\widehat{\pi}_{x_0}^\epsilon(x, t)$  on  $B(x_0, \frac{3}{2}) \times [0, T]$  by

$$\begin{aligned} &\widehat{\pi}_{x_0}^\epsilon(x, t) \\ &= -\frac{1}{3}\mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon\Phi_\epsilon(x, t) + \text{p.v.} \int_{B(x_0, 2)} \partial_i\partial_j\Gamma(x-y)\mathcal{J}_\epsilon(w_i^\epsilon)w_j^\epsilon(y, t)\Phi_\epsilon(y)dy \\ &\quad + \text{p.v.} \int_{B(x_0, 2)^c} \partial_i\partial_j(\Gamma(x-y) - \Gamma(x_0-y))\mathcal{J}_\epsilon(w_i^\epsilon)w_j^\epsilon(y, t)\Phi_\epsilon(y)dy \\ &\quad -\frac{2}{3}v_c \cdot \mathcal{J}_\epsilon(w^\epsilon)\Phi_\epsilon(x, t) + 2\text{p.v.} \int_{B(x_0, 2)} \partial_i\partial_j\Gamma(x-y)\mathcal{J}_\epsilon(w_i^\epsilon)v_{cj}(y, t)\Phi_\epsilon(y)dy \\ &\quad + 2\text{p.v.} \int_{B(x_0, 2)^c} \partial_i\partial_j(\Gamma(x-y) - \Gamma(x_0-y))\mathcal{J}_\epsilon(w_i^\epsilon)v_{cj}(y, t)\Phi_\epsilon(y)dy \\ &:= \pi_1 + \dots + \pi_6. \end{aligned} \quad (2.20)$$

Therefore,  $\pi^\epsilon - \widehat{\pi}_{x_0}^\epsilon$  depends only on  $x_0$  and  $t$ . Hence,  $\nabla \pi^\epsilon = \nabla \widehat{\pi}_{x_0}^\epsilon$  on  $B(x_0, \frac{3}{2}) \times [0, T]$ . Hence,  $(w^\epsilon, \widehat{\pi}_{x_0}^\epsilon)$  is another solution to system (2.1). We will replace  $\pi^\epsilon$  by  $\widehat{\pi}_{x_0}^\epsilon$  in the following procedure.

Take  $\psi(x, s) = \phi^2(x)\theta(s)$  with  $\text{supp}\phi(x) \in B(x_0, \frac{3}{2})$ ,  $\theta(s) \in C_c^\infty$  on  $[0, T_\epsilon]$  and  $\theta(s) = 1$  on  $[0, t]$ . Using  $2w^\epsilon\psi$  as a test function in (2.1)<sub>1</sub>, we have

$$\begin{aligned} & \int |w^\epsilon|^2 \psi(x, t) dx + 2 \int_0^t \int |\nabla w^\epsilon|^2 \psi dx ds \\ = & \int |w_0|^2 \psi(\cdot, 0) dx + \int_0^t \int |w^\epsilon|^2 (\partial_s \psi + \Delta \psi) dx ds + \int_0^t \int |w^\epsilon|^2 \Phi_\epsilon (\mathcal{J}_\epsilon(w^\epsilon) \cdot \nabla) \psi dx ds \\ & + \int_0^t \int |w^\epsilon|^2 \psi (\mathcal{J}_\epsilon(w^\epsilon) \cdot \nabla) \Phi_\epsilon dx ds + 2 \int_0^t \int \widehat{\pi}_{x_0}^\epsilon w^\epsilon \cdot \nabla \psi dx ds \\ & + 2 \int_0^t \int \psi v_c \cdot w^\epsilon (\mathcal{J}_\epsilon(w^\epsilon) \cdot \nabla) \Phi_\epsilon dx ds + 2 \int_0^t \int (\mathcal{J}_\epsilon(w^\epsilon) \cdot \nabla) w^\epsilon \cdot v_c \Phi_\epsilon \psi dx ds \\ & + 2 \int_0^t \int \Phi_\epsilon v_c \cdot w^\epsilon (\mathcal{J}_\epsilon(w^\epsilon) \cdot \nabla) \psi dx ds + 2 \int_0^t \int (\mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon) \psi (v_c \cdot \nabla) \Phi_\epsilon dx ds \\ & + 2 \int_0^t \int (\Phi_\epsilon v_c \cdot \nabla) w^\epsilon \cdot \mathcal{J}_\epsilon(w^\epsilon) \psi dx ds + 2 \int_0^t \int (\Phi_\epsilon v_c \cdot \nabla) \psi \mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon dx ds, \end{aligned} \tag{2.21}$$

for any  $0 < t < T_\epsilon$ . Then we have

$$\begin{aligned} & \|w^\epsilon(\cdot, t)\phi(\cdot)\|_{L^2}^2 + 2\|\nabla w^\epsilon(\cdot, t)\phi(\cdot)\|_{L^2([0,t] \times \mathbb{R}^3)}^2 \\ \lesssim & \|w_0\|_{L^2_{\text{uloc}}} + \left| \int_0^t \int |w^\epsilon|^2 |\Delta \phi^2| dx ds \right| + \left| \int_0^t \int |w^\epsilon|^2 \Phi_\epsilon (\mathcal{J}_\epsilon(w^\epsilon) \cdot \nabla) \phi^2 dx ds \right| \\ & + \left| \int_0^t \int |w^\epsilon|^2 \phi^2 (\mathcal{J}_\epsilon(w^\epsilon) \cdot \nabla) \Phi_\epsilon dx ds \right| + \left| \int_0^t \int 2\widehat{\pi}_{x_0}^\epsilon (w^\epsilon \cdot \nabla) \phi^2 dx ds \right| \\ & + \left| 2 \int_0^t \int \phi^2 v_c \cdot w^\epsilon ((\mathcal{J}_\epsilon(w^\epsilon) \cdot \nabla) \Phi_\epsilon) dx ds \right| + \left| 2 \int_0^t \int \mathcal{J}_\epsilon(w^\epsilon) \cdot \nabla w^\epsilon \cdot v_c \Phi_\epsilon \phi^2 dx ds \right| \\ & + \left| 2 \int_0^t \int \Phi_\epsilon v_c \cdot w^\epsilon (\mathcal{J}_\epsilon(w^\epsilon \cdot \nabla)) \phi^2 dx ds \right| + \left| 2 \int_0^t \int (\mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon) \phi^2 (v_c \cdot \nabla) \Phi_\epsilon dx ds \right| \\ & + \left| 2 \int_0^t \int (\Phi_\epsilon v_c \cdot \nabla) w^\epsilon \cdot \mathcal{J}_\epsilon(w^\epsilon) \phi^2 dx ds \right| + \left| 2 \int_0^t \int (\Phi_\epsilon v_c \cdot \nabla) \phi^2 \mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon dx ds \right|, \\ := & \|w_0\|_{L^2_{\text{uloc}}} + J_1 + \dots + J_{10}. \end{aligned} \tag{2.22}$$

By Hölder's inequality and  $|\nabla \Phi_\epsilon| \lesssim \epsilon \leq 1$ , we obtain

$$J_1 \leq C \|w^\epsilon\|_{U_t^{2,2}}^2 \tag{2.23}$$

and

$$J_2, J_3 \leq C \|w^\epsilon\|_{U_t^{3,3}}^3. \tag{2.24}$$

By Hölder's inequality, we have

$$J_4 \leq C \|\widehat{\pi}_{x_0}^\epsilon w^\epsilon\|_{U_t^{1,1}}. \tag{2.25}$$

According to (2.20), we have

$$\|\pi_1\|_{L^{\frac{3}{2}}([0,t] \times B(x_0, \frac{3}{2}))} \leq C \|w^\epsilon\|_{U_t^{3,3}}^2. \tag{2.26}$$

Moreover, by Calderon-Zygmund theorem, there holds

$$\begin{aligned} \|\pi_2\|_{L^{\frac{3}{2}}([0,t] \times B(x_0, \frac{3}{2}))} &\leq C \|\mathcal{J}_\epsilon(w_i^\epsilon) w_j^\epsilon \Phi_\epsilon\|_{L^{\frac{3}{2}}([0,t] \times B(x_0, 2))} \\ &\leq C \|w^\epsilon\|_{U_t^{3,3}}^2. \end{aligned} \quad (2.27)$$

Since  $x \in B(x_0, \frac{3}{2})$  and  $y \in B(x_0, 2)^c$ , we have

$$|\partial_i \partial_j \Gamma(x-y) - \partial_i \partial_j \Gamma(x_0-y)| \leq \frac{|x-x_0|}{|x_0-y|^4} \leq \frac{2}{3} \frac{1}{|x_0-y|^4}. \quad (2.28)$$

Hence,

$$\begin{aligned} &\|\pi_3\|_{L^{\frac{3}{2}}([0,t] \times B(x_0, \frac{3}{2}))} \\ &\leq C \left\| \int_{B(x_0, 2)^c} \frac{1}{|x_0-y|^4} \mathcal{J}_\epsilon(w_i^\epsilon) w_j^\epsilon(y, s) \Phi_\epsilon(y) dy \right\|_{L^{\frac{3}{2}}(0,t)} \\ &\leq C \left\| \sum_{k=1}^{\infty} \frac{1}{2^{4k}} \int_{B(x_0, 2^{k+1})} |\mathcal{J}_\epsilon(w_i^\epsilon) w_j^\epsilon|(y, s) dy \right\|_{L^{\frac{3}{2}}(0,t)} \\ &\leq C \sum_{k=1}^{\infty} \frac{1}{2^{4k}} \left\| \sum_{j=1}^{a_k} \int_{B(x_j^k, 1)} |\mathcal{J}_\epsilon(w_i^\epsilon) w_j^\epsilon|(y, s) dy \right\|_{L^{\frac{3}{2}}(0,t)} \\ &\leq C \sum_{k=1}^{\infty} \frac{a_k}{2^{4k}} \|\mathcal{J}_\epsilon(w_i^\epsilon) w_j^\epsilon\|_{U_t^{\frac{3}{2}, \frac{3}{2}}} \\ &\leq C \|w^\epsilon\|_{U_t^{3,3}}^2, \end{aligned} \quad (2.29)$$

where we take  $B(x_0, 2^{k+1}) \subset \cup_{j=1}^{a_k} B(x_j^k, 1)$  with  $a_k \leq 2^{3k}$ . Therefore

$$\|\pi_i w^\epsilon\|_{U_t^{1,1}} \leq C \|\pi_i\|_{L^{\frac{3}{2}}([0,t] \times B(x_0, \frac{3}{2}))} \|w^\epsilon\|_{U_t^{3,3}} \leq C \|w^\epsilon\|_{U_t^{3,3}}^3, \quad (2.30)$$

for  $i = 1, 2, 3$ . By interpolation and Young's inequality, we have

$$\begin{aligned} \|\pi_4 w^\epsilon\|_{U_t^{1,1}} &\leq C \|\pi_4\|_{U_t^{2, \frac{4}{3}}} \|w^\epsilon\|_{U_t^{2,4}} \\ &\leq C \|v_c\|_{L_{\text{uloc}}^2} \|w^\epsilon\|_{U_t^{2,4}}^2 \\ &\leq C \|v_c\|_{L_{\text{uloc}}^2} (\|w^\epsilon\|_{U_t^{2,2}}^2 + \|\nabla w^\epsilon\|_{U_t^{2,2}}^2). \end{aligned} \quad (2.31)$$

By Calderon-Zygmund theorem, there holds

$$\begin{aligned} \|\pi_5 w^\epsilon\|_{U_t^{1,1}} &\leq C \|\pi_5\|_{U_t^{2, \frac{4}{3}}} \|w^\epsilon\|_{U_t^{2,4}} \\ &\leq C \|v_c\|_{L_{\text{uloc}}^2} \|w^\epsilon\|_{U_t^{2,4}}^2 \\ &\leq C \|v_c\|_{L_{\text{uloc}}^2} (\|w^\epsilon\|_{U_t^{2,2}}^2 + \|\nabla w^\epsilon\|_{U_t^{2,2}}^2). \end{aligned} \quad (2.32)$$

Similar to (2.29), we have

$$\begin{aligned} \|\pi_6 w^\epsilon\|_{U_t^{1,1}} &\leq C \|\pi_6\|_{U_t^{2, \frac{4}{3}}} \|w^\epsilon\|_{U_t^{2,4}} \\ &\leq C \sum_{k=1}^{\infty} \frac{a_k}{2^{4k}} \|v_c\|_{L_{\text{uloc}}^2} \|w^\epsilon\|_{U_t^{2,4}}^2 \\ &\leq C \|v_c\|_{L_{\text{uloc}}^2} \|w^\epsilon\|_{U_t^{2,4}}^2 \end{aligned}$$



$$\leq C\|v_c\|_{L^2_{\text{uloc}}} (\|w^\epsilon\|_{U^{2,2}_t}^2 + \|\nabla w^\epsilon\|_{U^{2,2}_t}^2). \tag{2.33}$$

Combining with (2.25) and (2.30)-(2.33), we obtain

$$J_4 \leq C\|w^\epsilon\|_{U^{3,3}_t}^3 + C\|v_c\|_{L^2_{\text{uloc}}} (\|w^\epsilon\|_{U^{2,2}_t}^2 + \|\nabla w^\epsilon\|_{U^{2,2}_t}^2). \tag{2.34}$$

Similar to (2.31), we have

$$J_5, J_7, J_8, J_9, J_{10} \leq C\|v_c\|_{L^2_{\text{uloc}}} (\|w^\epsilon\|_{U^{2,2}_t}^2 + \|\nabla w^\epsilon\|_{U^{2,2}_t}^2). \tag{2.35}$$

For  $J_6 = 2 \int_0^t \int w^\epsilon \cdot \nabla w^\epsilon \cdot v_c \Phi_\epsilon \phi^2 dx ds$ , we have

$$\begin{aligned} J_6 &= 2 \int_0^t \int \frac{\phi w^\epsilon}{|x|} \cdot \nabla w^\epsilon \phi \cdot |x| v_c \cdot \Phi_\epsilon dx ds \\ &\leq 2 \int_0^t \|\nabla(\phi w^\epsilon)\|_{L^2} \|\nabla w^\epsilon \phi\|_{L^2} \| |x| v_c \|_{L^\infty} ds \\ &\leq CK_c (\|w^\epsilon\|_{U^{2,2}_t}^2 + \|\nabla w^\epsilon\|_{U^{2,2}_t}^2), \end{aligned} \tag{2.36}$$

where the first inequality holds because of Hardy inequality and Hölder’s inequality.

Therefore, we obtain

$$\begin{aligned} &\|w^\epsilon(\cdot, t)\|_{L^2_{\text{uloc}}}^2 + 2\|\nabla w^\epsilon\|_{U^{2,2}_t}^2 \\ &\leq C\|w_0\|_{L^2_{\text{uloc}}}^2 + C\|w^\epsilon\|_{U^{3,3}_t}^3 + C_2(\|v_c\|_{L^2_{\text{uloc}}} + K_c)(\|w^\epsilon\|_{U^{2,2}_t}^2 + \|\nabla w^\epsilon\|_{U^{2,2}_t}^2) \\ &\leq C\|w_0\|_{L^2_{\text{uloc}}}^2 + C\|w^\epsilon\|_{U^{3,3}_t}^3 + \frac{1}{4}\|w^\epsilon\|_{U^{2,2}_t}^2 + \frac{1}{4}\|\nabla w^\epsilon\|_{U^{2,2}_t}^2, \end{aligned} \tag{2.37}$$

where the last inequality holds because of the assumption that

$$C_2(\|v_c\|_{L^2_{\text{uloc}}} + K_c) \leq \frac{1}{4}. \tag{2.38}$$

Using the interpolation inequality and Young’s inequality,

$$\begin{aligned} C\|w^\epsilon\|_{U^{3,3}_t}^3 &\leq C\|w^\epsilon\|_{U^{6,2}_t}^{3/2} \|w^\epsilon\|_{U^{2,6}_t}^{3/2} \\ &\leq C\|w^\epsilon\|_{L^6([0,t];L^2_{\text{uloc}})}^6 + \frac{1}{4}\|w^\epsilon\|_{U^{2,2}_t}^2 + \frac{1}{4}\|\nabla w^\epsilon\|_{U^{2,2}_t}^2. \end{aligned} \tag{2.39}$$

Combining with (2.37), we have

$$\|w^\epsilon\|_{L^2_{\text{uloc}}}^2 + \frac{3}{2}\|\nabla w^\epsilon\|_{U^{2,2}_t}^2 \leq C\|w_0\|_{L^2_{\text{uloc}}}^2 + C \int_0^t (\|w^\epsilon(\cdot, s)\|_{L^2_{\text{uloc}}}^2 + \|w^\epsilon(\cdot, s)\|_{L^2_{\text{uloc}}}^6) ds. \tag{2.40}$$

Hence, there exists a small constant  $\varepsilon_1 > 0$  such that, if  $w^\epsilon$  exists on  $[0, T_\epsilon]$  for  $T_\epsilon \leq \tilde{T} = \varepsilon_1 \left(1 + \|w_0\|_{L^2_{\text{uloc}}}^4\right)^{-1}$ , then we have

$$\sup_{0 < t < T} \|w^\epsilon(\cdot, t)\|_{L^2_{\text{uloc}}} \leq C\|w_0\|_{L^2_{\text{uloc}}}. \tag{2.41}$$

Combining with (2.40), we have (2.18). □

Then, we can obtain the following lemma easily. We omit the details.

**Lemma 2.5.** *The distribution solutions  $\{(w^\epsilon, \pi^\epsilon)\}_{0 < \epsilon < 1}$  of (2.1) can be extended to the uniform time interval  $[0, \tilde{T}]$ , where  $\tilde{T}$  is as in Lemma 2.4.*

**3. Proof of Theorem 1.2.** First, when  $q = 2$ , we give the following existence result

**Proposition 3.1.** *Let  $|c| \geq c_3$  and  $w_0 \in L^2_{uloc}$  with  $\operatorname{div} w_0 = 0$ . If*

$$T \leq \frac{\varepsilon_1}{1 + \|w_0\|^4_{L^2_{uloc}}}, \tag{3.1}$$

for some small positive constant  $\varepsilon_1$  independent of  $\epsilon$  and  $\|w_0\|_{L^2_{uloc}}$ , there exists a  $L^2$  local energy solution  $(w, \pi)$  on  $\mathbb{R}^3 \times (0, T)$  to the perturbed Navier-Stokes system (1.4) with initial data  $w_0$ , satisfying

$$\|w\|_{\mathcal{E}_T} \leq C \|v_0\|_{L^2_{uloc}}. \tag{3.2}$$

*Proof of Proposition 3.1.* Our method is inspired by Theorem 3.2 in [12]. We will prove our result in the following four steps.

**Step 1. Construct  $\{(w^\epsilon, \pi^\epsilon)\}$  on  $[0, T^\epsilon]$ .**

Let  $(w^\epsilon, \pi^\epsilon)$  be the solution to the localized-mollified system (2.1). According to Lemmas 2.3 and 2.4, we construct  $w^\epsilon \in \mathcal{E}_{T^\epsilon}$  on  $\mathbb{R}^3 \times [0, T^\epsilon]$ , where  $T^\epsilon \leq T = \varepsilon_1 \left(1 + \|w_0\|^4_{L^2_{uloc}}\right)^{-1}$  with constant  $\varepsilon_1$  independent of  $\epsilon$  and  $\|w_0\|_{L^2_{uloc}}$ . By Lemma 2.5, time interval can be extended to  $[0, \tilde{T}]$ . We construct pressure  $\pi^\epsilon$  as follows

$$\begin{aligned} \pi^\epsilon(x, t) = & -\frac{1}{3} \mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon \Phi_\epsilon(x, t) + \text{p.v.} \int_{B_2} \partial_i \partial_j \Gamma(x - y) \mathcal{J}_\epsilon(w_i^\epsilon) w_j^\epsilon(y, t) \Phi_\epsilon(y) dy \\ & + \text{p.v.} \int_{B_2^c} \partial_i \partial_j (\Gamma(x - y) - \Gamma(-y)) \mathcal{J}_\epsilon(w_i^\epsilon) w_j^\epsilon(y, t) \Phi_\epsilon(y) dy \\ & - \frac{2}{3} v_c \cdot \mathcal{J}_\epsilon(w^\epsilon) \Phi_\epsilon(x, t) + 2 \text{p.v.} \int_{B_2} \partial_i \partial_j \Gamma(x - y) \mathcal{J}_\epsilon(w_i^\epsilon) v_{cj}(y, t) \Phi_\epsilon(y) dy \\ & + 2 \text{p.v.} \int_{B_2^c} \partial_i \partial_j (\Gamma(x - y) - \Gamma(-y)) \mathcal{J}_\epsilon(w_i^\epsilon) v_{cj}(y, t) \Phi_\epsilon(y) dy, \end{aligned} \tag{3.3}$$

for  $\Gamma(x) = \frac{1}{4\pi|x|}$ . It is easy to check  $\pi^\epsilon \in L^2_{loc}([0, T]; L^{\frac{4}{3}}_{loc}(\mathbb{R}^3))$ .

**Step 2. Prove that  $\|w^\epsilon\|_{\mathcal{E}_T}$  and  $\|\pi^\epsilon\|_{L^2(0, T; L^{\frac{4}{3}}_{uloc})}$  is uniformly bounded.**

According to Lemma 2.4, we have

$$\|w^\epsilon\|_{\mathcal{E}_T} \leq C \|w_0\|_{L^2_{uloc}}, \tag{3.4}$$

where the constant  $C$  is independent of  $\epsilon$  and  $T$ . We consider  $\pi^\epsilon$  in  $B_{2^n}$  for each  $n \in \mathbb{N}$ . We rewrite (3.3) as follows

$$\begin{aligned} & \pi^\epsilon(x, t) \\ = & -\frac{1}{3} \mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon \Phi_\epsilon(x, t) + \text{p.v.} \int_{B_2} \partial_i \partial_j \Gamma(x - y) \mathcal{J}_\epsilon(w_i^\epsilon) w_j^\epsilon(y, t) \Phi_\epsilon(y) dy \\ & + \left( \text{p.v.} \int_{B_{2^{n+1}} \setminus B_2} + \text{p.v.} \int_{B_{2^{n+1}}^c} \right) \partial_i \partial_j (\Gamma(x - y) - \Gamma(-y)) \mathcal{J}_\epsilon(w_i^\epsilon) w_j^\epsilon(y, t) \Phi_\epsilon(y) dy \\ & - \frac{2}{3} v_c \cdot \mathcal{J}_\epsilon(w^\epsilon) \Phi_\epsilon(x, t) + 2 \text{p.v.} \int_{B_2} \partial_i \partial_j \Gamma(x - y) \mathcal{J}_\epsilon(w_i^\epsilon) v_{cj}(y, t) \Phi_\epsilon(y) dy \\ & + 2 \left( \text{p.v.} \int_{B_{2^{n+1}} \setminus B_2} + \text{p.v.} \int_{B_{2^{n+1}}^c} \right) \partial_i \partial_j (\Gamma(x - y) - \Gamma(-y)) \mathcal{J}_\epsilon(w_i^\epsilon) v_{cj}(y, t) \Phi_\epsilon(y) dy \\ = & \pi_1^\epsilon + \dots + \pi_8^\epsilon. \end{aligned} \tag{3.5}$$

For  $\pi_1^\epsilon$ , we have

$$\begin{aligned} \|\pi_1^\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} &\leq \|\mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon \Phi_\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} \\ &\leq \|\mathcal{J}_\epsilon(w^\epsilon)\|_{L^\infty(0,T;L^2(B_{2^n}))} \|w^\epsilon\|_{L^2(0,T;L^4(B_{2^n}))} \\ &\leq C(n) \|w^\epsilon\|_{\mathcal{E}_T} \\ &\leq C(n, \|w_0\|_{L^2_{\text{uloc}}}). \end{aligned} \tag{3.6}$$

For  $\pi_2^\epsilon$ , by Calderon-Zygmund theorem, there holds

$$\begin{aligned} \|\pi_2^\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} &\leq \|\mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon \Phi_\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_2))} \\ &\leq C(\|w_0\|_{L^2_{\text{uloc}}}). \end{aligned} \tag{3.7}$$

For the third term, we have

$$\begin{aligned} \pi_3^\epsilon &= \text{p.v.} \int_{B_{2^{n+1}} \setminus B_2} \partial_i \partial_j \Gamma(x-y) \mathcal{J}_\epsilon(w_i^\epsilon) w_j^\epsilon(y,t) \Phi_\epsilon(y) dy \\ &\quad - \text{p.v.} \int_{B_{2^{n+1}} \setminus B_2} \partial_i \partial_j \Gamma(-y) \mathcal{J}_\epsilon(w_i^\epsilon) w_j^\epsilon(y,t) \Phi_\epsilon(y) dy \\ &:= \pi_{31}^\epsilon + \pi_{32}^\epsilon. \end{aligned} \tag{3.8}$$

Using Calderon-Zygmund theorem, we have

$$\begin{aligned} \|\pi_{31}^\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} &\leq \|\mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon \Phi_\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^{n+1}}))} \\ &\leq C(n, \|w_0\|_{L^2_{\text{uloc}}}). \end{aligned} \tag{3.9}$$

On the other hand

$$\begin{aligned} \|\pi_{32}^\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} &\lesssim 2^{2n} \|\mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon \Phi_\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^{n+1}}))} \left\| \frac{1}{|y|^3} \right\|_{L^4(B_{2^{n+1}} \setminus B_2)} \\ &\lesssim C(n, \|w_0\|_{L^2_{\text{uloc}}}). \end{aligned} \tag{3.10}$$

For  $\pi_4^\epsilon$ , since  $x \in B_{2^n}$  and  $y \in B_{2^{n+1}}^c$ , we have

$$|\partial_i \partial_j \Gamma(x-y) - \partial_i \partial_j \Gamma(-y)| \lesssim \frac{|x|}{|y|^4} \lesssim \frac{2^n}{|y|^4}. \tag{3.11}$$

Similar to (2.29)-(2.30), we obtain

$$\begin{aligned} \|\pi_4^\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} &\lesssim 2^{3n} \sum_{k=n+1}^\infty \frac{1}{2^{4k}} \|\mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon \Phi_\epsilon\|_{L^2(0,T;L^1(B_{2^{k+1}}))} \\ &\lesssim C(n) \|\mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon \Phi_\epsilon\|_{U_T^{2,\frac{4}{3}}} \\ &\leq C(n, \|w_0\|_{L^2_{\text{uloc}}}). \end{aligned} \tag{3.12}$$

Similar to (2.31), we have

$$\begin{aligned} \|\pi_5^\epsilon, \pi_6^\epsilon, \pi_{71}^\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} &\lesssim 2^{2n} \|\mathcal{J}_\epsilon(w^\epsilon) \cdot v_c \Phi_\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^{n+1}}))} \\ &\lesssim C(n) \|w^\epsilon\|_{U_T^{2,4}} \|v_c\|_{L^2_{\text{uloc}}} \\ &\lesssim C(n, \|w_0\|_{L^2_{\text{uloc}}}). \end{aligned} \tag{3.13}$$

Similar to (3.10), there holds

$$\begin{aligned} \|\pi_{72}^\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} &\lesssim 2^{2n} \|\mathcal{J}_\epsilon(w^\epsilon) \cdot v_c \Phi_\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^{n+1}}))} \left\| \frac{1}{|y|^3} \right\|_{L^4(B_{2^{n+1}} \setminus B_2)} \\ &\lesssim C(n, \|w_0\|_{L^2_{\text{uloc}}}). \end{aligned} \quad (3.14)$$

For the last term  $\pi_8^\epsilon$ , since  $x \in B_{2^n}$  and  $y \in B_{2^{n+1}}^c$ , we have

$$|\partial_i \partial_j \Gamma(x-y) - \partial_i \partial_j \Gamma(-y)| \lesssim \frac{|x|}{|y|^4} \lesssim \frac{2^n}{|y|^4}. \quad (3.15)$$

Therefore, we deduce

$$\begin{aligned} \|\pi_8^\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} &\lesssim 2^{3n} \sum_{k=n+1}^{\infty} \frac{1}{2^{4k}} \|\mathcal{J}_\epsilon(w^\epsilon) \cdot w^\epsilon \Phi_\epsilon\|_{L^2(0,T;L^1(B_{2^{k+1}}))} \\ &\leq C(n, \|w_0\|_{L^2_{\text{uloc}}}). \end{aligned} \quad (3.16)$$

Combining with above estimates, we conclude

$$\|\pi^\epsilon\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} \leq C(n, \|w_0\|_{L^2_{\text{uloc}}}). \quad (3.17)$$

**Step 3. Find subsequence of  $(w^\epsilon, \pi^\epsilon)$ , then show the subsequence converge to  $(w, \pi)$ .** Similar method has been used in [10, 12]. For each  $n \in \mathbb{N}$ , we find a limit solution of  $(w^\epsilon, \pi^\epsilon)$  up to subsequence on each  $[0, T] \times B_{2^n}$ . First, we construct  $w$  on the compact set  $[0, T] \times B_2$ . By uniform bounds on  $w^\epsilon$  and the compactness argument, we can find sequences  $w^{1,k}$  form  $w^\epsilon$  such that

$$\begin{aligned} w^{1,k} &\overset{*}{\rightharpoonup} w^1 \quad \text{in } L^\infty(0, T; L^2(B_2)), \\ w^{1,k} &\rightharpoonup w^1 \quad \text{in } L^2(0, T; H^1(B_2)), \\ w^{1,k} &\rightarrow w^1 \quad \text{in } L^2(0, T; L^4(B_2)), \\ \mathcal{J}_{1,k}(w^{1,k}) &\rightarrow w^1 \quad \text{in } L^2(0, T; L^4(B_{\delta_1})), \end{aligned} \quad (3.18)$$

for any  $\delta_1 < 2$ , as  $k \rightarrow \infty$ . Let  $w = w^1$  on  $[0, T] \times B_2$ .

Then, we extend  $w$  to  $[0, T] \times B_4$ . By the same arguments as above, we can find sequences  $w^{2,k}$  form  $w^{1,k}$  such that

$$\begin{aligned} w^{2,k} &\overset{*}{\rightharpoonup} w^2 \quad \text{in } L^\infty(0, T; L^2(B_4)), \\ w^{2,k} &\rightharpoonup w^2 \quad \text{in } L^2(0, T; H^1(B_4)), \\ w^{2,k} &\rightarrow w^2 \quad \text{in } L^2(0, T; L^4(B_4)), \\ \mathcal{J}_{2,k}(w^{2,k}) &\rightarrow w^2 \quad \text{in } L^2(0, T; L^4(B_{\delta_2})), \end{aligned} \quad (3.19)$$

for any  $\delta_2 < 4$ , as  $k \rightarrow \infty$ .

Continuing this process, we can construct sequence  $w^{n,k}$  and its limit  $w$ . By diagonal argument, we have

$$w^{(k)} = \begin{cases} w^{k,k} & [0, T] \times B_{2^k}, \\ 0 & \text{otherwise,} \end{cases} \quad \forall k \in \mathbb{N}, \quad (3.20)$$

satisfying

$$\begin{aligned}
 w^{(k)} &\xrightarrow{*} w && \text{in } L^\infty(0, T; L^2(B_{2^n})), \\
 w^{(k)} &\rightharpoonup w && \text{in } L^2(0, T; H^1(B_{2^n})), \\
 w^{(k)} &\rightarrow w && \text{in } L^2(0, T; L^4(B_{2^n})), \\
 \mathcal{J}_{2,k}(w^{(k)}) &\rightarrow w && \text{in } L^2(0, T; L^4(B_{\delta_n})),
 \end{aligned}
 \tag{3.21}$$

for any  $\delta_n < 2^n$ , as  $k \rightarrow \infty$ . Furthermore,

$$\|w\|_{\mathcal{E}_T} + \|w\|_{L^2(0,T;L^4(\mathbb{R}^3))} \leq C \|w_0\|_{L^2_{\text{loc}}}.
 \tag{3.22}$$

Next, we will prove

$$\pi^{(k)} \rightarrow \pi \quad \text{in } L^2\left(0, T; L^{\frac{4}{3}}(B_{2^n})\right),
 \tag{3.23}$$

for each  $n \in \mathbb{N}$ . According to formula (3.3) of  $\pi^\epsilon$ , we define  $\pi^{(k)}$  as follows

$$\begin{aligned}
 \pi^{(k)}(x, t) &= -\frac{1}{3} \mathcal{J}_{(k)}(w^{(k)}) \cdot w^{(k)} \Phi_{(k)}(x, t) \\
 &+ \text{p.v.} \int_{B_2} \partial_i \partial_j \Gamma(x-y) \mathcal{J}_{(k)}(w_i^{(k)}) w_j^{(k)}(y, t) \Phi_{(k)}(y) dy \\
 &+ \text{p.v.} \int_{B_2^c} \partial_i \partial_j (\Gamma(x-y) - \Gamma(-y)) \mathcal{J}_{(k)}(w_i^{(k)}) w_j^{(k)}(y, t) \Phi_{(k)}(y) dy \\
 &- \frac{2}{3} v_c^{(k)} \cdot \mathcal{J}_{(k)}(w^{(k)}) \Phi_{(k)}(x, t) \\
 &+ 2 \text{p.v.} \int_{B_2} \partial_i \partial_j \Gamma(x-y) \mathcal{J}_{(k)}(w_i^{(k)}) v_{cj}^{(k)}(y, t) \Phi_{(k)}(y) dy \\
 &+ 2 \text{p.v.} \int_{B_2^c} \partial_i \partial_j (\Gamma(x-y) - \Gamma(-y)) \mathcal{J}_{(k)}(w_i^{(k)}) v_{cj}^{(k)}(y, t) \Phi_{(k)}(y) dy.
 \end{aligned}$$

And pressure

$$\begin{aligned}
 \pi(x, t) &= -\frac{1}{3} \mathcal{J}(w) \cdot w \Phi(x, t) + \text{p.v.} \int_{B_2} \partial_i \partial_j \Gamma(x-y) \mathcal{J}(w_i) w_j(y, t) \Phi(y) dy \\
 &+ \text{p.v.} \int_{B_2^c} \partial_i \partial_j (\Gamma(x-y) - \Gamma(-y)) \mathcal{J}(w_i) w_j(y, t) \Phi(y) dy - \frac{2}{3} v_c \cdot \mathcal{J}(w) \Phi(x, t) \\
 &+ 2 \text{p.v.} \int_{B_2} \partial_i \partial_j \Gamma(x-y) \mathcal{J}(w_i) v_{cj}(y, t) \Phi(y) dy \\
 &+ 2 \text{p.v.} \int_{B_2^c} \partial_i \partial_j (\Gamma(x-y) - \Gamma(-y)) \mathcal{J}(w_i) v_{cj}(y, t) \Phi(y) dy.
 \end{aligned}
 \tag{3.24}$$

Hence, for any  $m > n$

$$\begin{aligned}
 \pi^{(k)} - \pi &= p_1 + p_2 + \left( \text{p.v.} \int_{B_{2^{n+1}} \setminus B_2} + \text{p.v.} \int_{B_{2^m} \setminus B_{2^{n+1}}} + \text{p.v.} \int_{B_{2^m}^c} \right) \cdots \\
 &+ p_6 + p_7 + \left( \text{p.v.} \int_{B_{2^{n+1}} \setminus B_2} + \text{p.v.} \int_{B_{2^m} \setminus B_{2^{n+1}}} + \text{p.v.} \int_{B_{2^m}^c} \right) \cdots \\
 &= p_1 + \cdots + p_{10}.
 \end{aligned}$$

Set

$$N_{ij}^{(k)} = \mathcal{J}_{(k)}(w_i^{(k)}) w_j^{(k)} \Phi_{(k)},
 \tag{3.25}$$

and

$$N_{ij} = w_i w_j. \tag{3.26}$$

Note that for fixed  $R > 0$ , we have

$$\begin{aligned} & \left\| N_{ij}^{(k)} - N_{ij} \right\|_{L^2(0,T;L^{\frac{4}{3}}(B_R))} \\ & \leq \left\| \left( \mathcal{J}_{(k)}(w_i^{(k)}) - w_i \right) w_j^{(k)} \Phi_{(k)} \right\|_{L^2(0,T;L^{\frac{4}{3}}(B_R))} \\ & \quad + \left\| w_i \left( w_j^{(k)} - w_j \right) \Phi_{(k)} \right\|_{L^2(0,T;L^{\frac{4}{3}}(B_R))} + \left\| w_i w_j (1 - \Phi_{(k)}) \right\|_{L^2(0,T;L^{\frac{4}{3}}(B_R))} \\ & \lesssim \left\| \mathcal{J}_{(k)} \left( w^{(k)} \right) - w \right\|_{L^2(0,T;L^4(B_R))} \left\| w^{(k)} \right\|_{L^\infty(0,T;L^2(B_R))} \\ & \quad + \left\| w^{(k)} - v \right\|_{L^2(0,T;L^4(B_R))} \left\| w \right\|_{L^\infty(0,T;L^2(B_R))} + \left\| |w|^2 (1 - \Phi_{(k)}) \right\|_{L^2(0,T;L^{\frac{4}{3}}(B_R))}. \end{aligned} \tag{3.27}$$

By (3.21) and Lebesgue dominated convergence theorem, we have

$$\left\| N_{ij}^{(k)} - N_{ij} \right\|_{L^2(0,T;L^{\frac{4}{3}}(B_R))} \rightarrow 0, \tag{3.28}$$

as  $k \rightarrow \infty$ . Similar to estimates in Step 3, we have

$$\|p_1, p_2, p_3\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} \lesssim_n \|N_{ij}^{(k)} - N_{ij}\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))}, \tag{3.29}$$

and

$$\|p_4\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} \lesssim \|N_{ij}^{(k)} - N_{ij}\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^m}))}. \tag{3.30}$$

Combined with (3.28), these four terms become very small for sufficiently large  $k$ .

Note that

$$\|w_i^{(k)} v_{cj}^{(k)} \Phi_{(k)} - w_i v_{cj}\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} \rightarrow 0, \tag{3.31}$$

as  $k \rightarrow \infty$ .

$$\|p_6, p_7, p_8\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} \lesssim_n \|w_i^{(k)} v_{cj}^{(k)} \Phi_{(k)} - w_i v_{cj}\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))}, \tag{3.32}$$

and

$$\|p_9\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} \lesssim_n \|w_i^{(k)} v_{cj}^{(k)} \Phi_{(k)} - w_i v_{cj}\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^m}))}. \tag{3.33}$$

Combing with (3.31), we have  $p_6, p_7, p_8, p_9 \rightarrow 0$  as  $k \rightarrow \infty$ .

For  $p_5$ , there holds

$$\begin{aligned} & \|p_5\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} \\ & \lesssim \frac{2^{3n}}{2^m} \left( \|w_i w_j\|_{U_T^{2,\frac{4}{3}}} + \|\mathcal{J}_{(k)} w^{(k)}\|_{L^\infty(0,T;L^2(B_{2^n}))} \|w^{(k)}\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} \right) \\ & \lesssim C(n, \|w_0\|_{L^2_{\text{uloc}}}, T) \frac{1}{2^m}. \end{aligned} \tag{3.34}$$

Also

$$\begin{aligned} \|p_{10}\|_{L^2(0,T;L^{\frac{4}{3}}(B_{2^n}))} & \lesssim \frac{2^{3n}}{2^m} \left( \|w_i^{(k)} v_{cj}^{(k)}\|_{U_T^{2,1}} + \|w_i v_{cj}\|_{U_T^{2,1}} \right) \\ & \lesssim \frac{2^{3n}}{2^m} \left( \|w_i^{(k)} v_{cj}^{(k)}\|_{U_T^{2,\frac{4}{3}}} + \|w_i v_{cj}\|_{U_T^{2,\frac{4}{3}}} \right) \\ & \lesssim C(n, \|w_0\|_{L^2_{\text{uloc}}}, T) \frac{1}{2^m}. \end{aligned} \tag{3.35}$$

Take  $m$  large enough, we can make  $p_5$  and  $p_{10}$  very small in the space  $L^2\left(0, T; L^{\frac{4}{3}}(B_{2^n})\right)$ . These give the convergence (3.23).

**Step 4. Check  $(w, \pi)$  is a local energy solution.** Proof in this step is very similar to the proof of Theorem 3.2 in [12]. For simplicity, we omit the details.  $\square$

For  $q > 2$ ,  $w_0 \in L^q_{\text{uloc}}$  implies that  $w_0 \in L^2_{\text{uloc}}$ . By the existence results for  $q = 2$  in Proposition 3.1, we have  $w^\epsilon \in \mathcal{E}_T$  with initial data  $w_0 \in L^q_{\text{uloc}}$ . Then, we will prove

$$\|w^\epsilon\|_{U_T^{\infty, q}} + \|\nabla(|w^\epsilon|^{\frac{q}{2}})\|_{U_T^{\frac{2}{q}, 2}} \leq \|w_0\|_{L^q_{\text{uloc}}}.$$

For simplicity, we only give crucial *a-priori* estimates.

Similar to (2.30), we have

$$\|\pi_i |w^\epsilon|^{q-2} w^\epsilon\|_{U_t^{1,1}} \leq \|\pi_i\|_{L^{\frac{q+1}{q}}([0,t] \times B(x_0, \frac{3}{2}))} \|w^\epsilon\|_{U_t^{q+1, q+1}} \leq \|w^\epsilon\|_{U_t^{q+1, q+1}}^{q+1}, \tag{3.36}$$

for  $i = 1, 2, 3$ . By interpolation and Young’s inequality, similar to (2.31), there holds

$$\begin{aligned} \|\pi_4 |w^\epsilon|^{q-2} w^\epsilon\|_{U_t^{1,1}} &\leq \|\pi_4\|_{U_t^{q, \frac{2q}{q+1}}} \| |w^\epsilon|^{q-2} w^\epsilon \|_{U_t^{\frac{q}{q-1}, \frac{2q}{q-1}}} \\ &\leq \|v_c\|_{L^2_{\text{uloc}}} \|w^\epsilon\|_{U_t^{q, 2q}}^q \\ &\leq C \|v_c\|_{L^2_{\text{uloc}}} (\|w^\epsilon\|_{U_t^{q, q}}^q + \|\nabla(|w^\epsilon|^{\frac{q}{2}})\|_{U_t^{2,2}}^2). \end{aligned} \tag{3.37}$$

By Calderon-Zygmund theorem, there holds

$$\begin{aligned} \|\pi_5 |w^\epsilon|^{q-2} w^\epsilon\|_{U_t^{1,1}} &\leq \|\pi_5\|_{U_t^{q, \frac{2q}{q+1}}} \| |w^\epsilon|^{q-2} w^\epsilon \|_{U_t^{\frac{q}{q-1}, \frac{2q}{q-1}}} \\ &\leq \|v_c\|_{L^2_{\text{uloc}}} \|w^\epsilon\|_{U_t^{q, 2q}}^q \\ &\leq C \|v_c\|_{L^2_{\text{uloc}}} (\|w^\epsilon\|_{U_t^{q, q}}^q + \|\nabla(|w^\epsilon|^{\frac{q}{2}})\|_{U_t^{2,2}}^2). \end{aligned} \tag{3.38}$$

Similar to (2.33), we have

$$\begin{aligned} \|\pi_6 |w^\epsilon|^{q-2} w^\epsilon\|_{U_t^{1,1}} &\leq \|\pi_6\|_{U_t^{q, \frac{2q}{q+1}}} \| |w^\epsilon|^{q-2} w^\epsilon \|_{U_t^{\frac{q}{q-1}, \frac{2q}{q-1}}} \\ &\leq \sum_{k=1}^{\infty} \frac{a_k}{2^{4k}} \|v_c\|_{L^2_{\text{uloc}}} \|w^\epsilon\|_{U_t^{q, 2q}}^q \\ &\leq \|v_c\|_{L^2_{\text{uloc}}} \|w^\epsilon\|_{U_t^{q, 2q}}^q \\ &\leq C \|v_c\|_{L^2_{\text{uloc}}} (\|w^\epsilon\|_{U_t^{q, q}}^q + \|\nabla(|w^\epsilon|^{\frac{q}{2}})\|_{U_t^{2,2}}^2). \end{aligned} \tag{3.39}$$

Combining with (2.25) and (3.36)-(3.39), we obtain

$$J_4 \leq \|w^\epsilon\|_{U_t^{3,3}}^3 + C \|v_c\|_{L^2_{\text{uloc}}} (\|w^\epsilon\|_{U_t^{2,2}}^2 + \|\nabla w^\epsilon\|_{U_t^{2,2}}^2). \tag{3.40}$$

Similar to (3.37), we have

$$J_5, J_7, J_8, J_9, J_{10} \leq C \|v_c\|_{L^2_{\text{uloc}}} (\|w^\epsilon\|_{U_t^{2,2}}^2 + \|\nabla w^\epsilon\|_{U_t^{2,2}}^2). \tag{3.41}$$

For  $J_6 = 2 \int_0^t \int w^\epsilon \cdot \nabla w^\epsilon \cdot v_c \Phi_\epsilon \phi^2 dx ds$ , we have

$$\begin{aligned} J_6 &= 2 \int_0^t \int \frac{\phi w^\epsilon}{|x|} \cdot \nabla w^\epsilon \phi \cdot |x| v_c \cdot \Phi_\epsilon dx ds \\ &\leq 2 \int_0^t \|\nabla(\phi w^\epsilon)\|_{L^2} \|\nabla w^\epsilon \phi\|_{L^2} \| |x| v_c \|_{L^\infty} ds \\ &\leq CK_c (\|w^\epsilon\|_{U_t^{2,2}}^2 + \|\nabla w^\epsilon\|_{U_t^{2,2}}^2), \end{aligned} \tag{3.42}$$

where the first inequality holds because of Hardy inequality and Hölder’s inequality.

By interpolation inequality and Young’s inequality, we have

$$\begin{aligned} \|w^\epsilon\|_{U_t^{q+1,q+1}}^{q+1} &\lesssim \|w^\epsilon\|_{U_t^{3q,q}}^{3/2} \|w^\epsilon\|_{U_t^{q,3q}}^{\frac{2q-1}{2}} \\ &\lesssim \|w^\epsilon\|_{L^{3q}([0,t];L^2_{\text{uloc}})}^{3q} + \|\nabla(|w^\epsilon|^{\frac{q}{2}})\|_{U_t^{2,2}}^2. \end{aligned} \tag{3.43}$$

Therefore, we have

$$\|w^\epsilon\|_{L^q_{\text{uloc}}}^q + \|\nabla(|w^\epsilon|^{\frac{q}{2}})\|_{U_t^{2,2}}^2 \lesssim \|w_0\|_{L^q_{\text{uloc}}}^q + \int_0^t \left( \|w^\epsilon(\cdot, s)\|_{L^q_{\text{uloc}}}^q + \|w^\epsilon(\cdot, s)\|_{L^q_{\text{uloc}}}^{3q} \right) ds. \tag{3.44}$$

Hence, there exists a small constant  $\epsilon_1 > 0$  such that, if  $w^\epsilon$  exists on  $[0, T_\epsilon]$  for  $T_\epsilon \leq \tilde{T} = \epsilon_1 \left(1 + \|w_0\|_{L^q_{\text{uloc}}}^{2q}\right)^{-1}$ , then we have

$$\sup_{0 < t < T} \|w^\epsilon(\cdot, t)\|_{L^q_{\text{uloc}}} \leq C \|w_0\|_{L^q_{\text{uloc}}}. \tag{3.45}$$

Following the procedure in the proof of Lemma 3.1, we have the existence results when  $q \geq 2$ .

Then, we will prove the uniqueness when  $q \geq 3$ . Let  $u, v$  be two solutions to the perturbed Navier-Stokes system (1.4) on  $\mathbb{R}^3 \times (0, T)$  with the same initial data  $w_0$ . The uniqueness can be proved by the method in the proof of Theorem 4.4 in Tsai [25]. We sketch it here.

From (1.14), using interpolation theory, we have

$$\|u\|_{U_T^{p,s}} \leq C \|u\|_{U_T^{\infty,q}} + C \|\nabla(|u|^{\frac{q}{2}})\|_{U_T^{2,2}}^{\frac{2}{q}} \leq C \|w_0\|_{L^q_{\text{uloc}}}, \tag{3.46}$$

with  $\frac{2}{p} + \frac{3}{s} = \frac{3}{q}$ . Then, there exists  $t \in (0, 1)$  sufficient small such that

$$C_4 \|u\|_{U_t^{4,6}} \leq \frac{1}{4}, \tag{3.47}$$

where  $C_4$  is given in (3.54). Set  $g = v - u$ , we have

$$\begin{cases} \partial_t g - \Delta g + \nabla \pi = -(u + g) \cdot \nabla g - g \cdot \nabla u - g \cdot \nabla v_c - v_c \cdot \nabla g, \\ \nabla \cdot g = 0, \\ g(x, 0) = 0. \end{cases} \tag{3.48}$$

Using  $2g\psi$  with  $\psi \in C_c^\infty((0, T) \times \mathbb{R}^3)$  as a test function, multiplying the equation (3.48)<sub>1</sub> by  $2g\psi$ , then integrating it, we have

$$\begin{aligned} &\int |g|^2 \psi(x, t) dx + 2 \int_0^t \int |\nabla g|^2 \psi dx ds \\ &= \int_0^t \int |g|^2 (\partial_s \psi + \Delta \psi) dx ds + \int_0^t \int |g|^2 g \cdot \nabla \psi dx ds + 2 \int_0^t \int \pi g \cdot \nabla \psi dx ds \\ &\quad + 2 \int_0^t \int g \cdot \nabla g \cdot (v_c + u) \psi dx ds + 2 \int_0^t \int (v_c + u) \cdot g (g \cdot \nabla) \psi dx ds \\ &\quad + 2 \int_0^t \int (v_c + u) \cdot \nabla g \cdot g \psi dx ds + 2 \int_0^t \int (v_c + u) \cdot \nabla \psi g \cdot g dx ds. \end{aligned}$$

Crucial part is to estimate

$$\int_0^t \int (u + v_c) \cdot (g \cdot \nabla) g \psi dx dt. \tag{3.49}$$



Denote  $E(t) = \text{ess sup}_{s < t} \|g(s)\|_{L^2_{\text{uloc}}}^2 + \int_0^t \|\nabla g\|_{L^2_{\text{uloc}}}^2 d\tau$ . Since

$$|\iint uv \nabla w dx dt| \leq C \|u\|_{L^4_t L^6_x} \|v\|_{L^\infty_t L^2_x} \|v\|_{L^2_t L^6_x} \|\nabla w\|_{L^2_{t,x}}, \tag{3.50}$$

we have

$$\left| \int_0^t \int u \cdot (g \cdot \nabla) g \psi dx d\tau \right| \leq C \|u\|_{U^{4,6}_t} E(t). \tag{3.51}$$

By Hölder inequality, Hardy inequality and Lemma 2.1, we have

$$\begin{aligned} \left| \int_0^t \int v_c \cdot (g \cdot \nabla) g \psi dx d\tau \right| &\leq \int_0^t \| |x| v_c \|_{L^\infty} \left\| \frac{g}{|x|} \right\|_{L^2_{\text{uloc}}} \|\nabla g\|_{L^2_{\text{uloc}}} d\tau \\ &\leq C \int_0^t \| |x| v_c \|_{L^\infty} (\|\nabla g\|_{L^2_{\text{uloc}}} + \|g\|_{L^2_{\text{uloc}}}) \|\nabla g\|_{L^2_{\text{uloc}}} d\tau \\ &\leq CK_c (\|\nabla g\|_{U^{2,2}_t}^2 + \|g\|_{U^{2,2}_t}^2) \\ &\leq CK_c E(t). \end{aligned} \tag{3.52}$$

For term  $\int_0^t \int \pi g \cdot \nabla \psi dx ds$ , we use the similar decomposition as (2.20) and obtain

$$\begin{aligned} \int_0^t \int \pi g \cdot \nabla \psi dx ds &\leq \|\pi g\|_{U^{1,1}_t} \\ &\leq \|g\|_{U^{3,3}_t}^3 + C \|v_c\|_{L^2_{\text{uloc}}} (\|g\|_{U^{2,2}_t}^2 + \|\nabla g\|_{U^{2,2}_t}^2). \end{aligned} \tag{3.53}$$

There holds

$$E(t) \leq C_4 \|u\|_{U^{4,6}_t} E(t) + C_4 (K_c + \|v_c\|_{L^2_{\text{uloc}}}) E(t). \tag{3.54}$$

Combining with (3.47) and

$$C_4 (K_c + \|v_c\|_{L^2_{\text{uloc}}}) < \frac{1}{2}, \tag{3.55}$$

we have  $E(t) \leq 0$ , and finish the proof of Theorem 1.2. □

**4. Appendix. Integral formula of the pressure  $\pi$ .** Our goal is to derive the integral formula of the pressure  $\pi$ , i.e. (1.5). Our method is inspired by [25] and [27]. According to the perturbed Navier-Stokes system (1.4), we have

$$-\Delta \pi = \partial_i \partial_j (w_i w_j + v_{ci} w_j + w_i v_{cj}). \tag{4.1}$$

Fix  $x$ , take a smooth compact supported function  $\xi \in C_c^\infty(B(x, 2R))$  such that

$$\xi = \begin{cases} 1 & \text{in } B(x, R) \\ 0 & \text{in } B(x, 2R)^c. \end{cases} \tag{4.2}$$

Therefore, we have  $|\nabla \xi| \leq \frac{C}{R}$  and  $|\nabla^2 \xi| \leq \frac{C}{R^2}$ . Since

$$\Delta(\xi \pi) = \Delta \pi \xi + 2 \nabla \xi \cdot \nabla \pi + \Delta \xi \pi, \tag{4.3}$$

we obtain

$$-\Delta(\xi \pi) = \partial_i \partial_j (w_i w_j + v_{ci} w_j + w_i v_{cj}) \xi - 2 \nabla \xi \cdot \nabla \pi - \Delta \xi \pi. \tag{4.4}$$

Therefore,

$$\begin{aligned}
 \xi\pi &= \int_{\mathbb{R}^3} \Gamma(x-y)\partial_i\partial_j(w_iw_j + v_{ci}w_j + w_iv_{cj})(y,t)\xi(y)dy \\
 &\quad - 2 \int_{\mathbb{R}^3} \Gamma(x-y)\nabla_y\xi(y) : \nabla_y\pi(y,t)dy - \int_{\mathbb{R}^3} \Gamma(x-y)\Delta_y\xi(y)\pi(y,t)dy \\
 &= \int_{\mathbb{R}^3} \Gamma(x-y)\partial_i\partial_j(w_iw_j + v_{ci}w_j + w_iv_{cj})(y,t)\xi(y)dy + 2 \int_{\mathbb{R}^3} \Gamma(x-y)\Delta_y\xi(y)\pi(y,t)dy \\
 &\quad + 2 \int_{\mathbb{R}^3} \nabla_y\Gamma(x-y) \cdot \nabla_y\xi(y)\pi(y,t)dy - \int_{\mathbb{R}^3} \Gamma(x-y)\Delta_y\xi(y)\pi(y,t)dy \\
 &= \int_{\mathbb{R}^3} \Gamma(x-y)\partial_i\partial_j(w_iw_j + v_{ci}w_j + w_iv_{cj})(y,t)\xi(y)dy \\
 &\quad + \int_{\mathbb{R}^3} \Gamma(x-y)\Delta_y\xi(y)\pi(y,t)dy + 2 \int_{\mathbb{R}^3} \nabla_y\Gamma(x-y) \cdot \nabla_y\xi(y)\pi(y,t)dy.
 \end{aligned}
 \tag{4.5}$$

Note that

$$\int_{\mathbb{R}^3} \Gamma(x-y)\partial_i\partial_j(u_iv_j)(y,t)\xi(y)dy \tag{4.6}$$

$$= \lim_{\varepsilon \rightarrow 0} \int_{B_{2R}/B_\varepsilon} \Gamma(x-y)\partial_i\partial_j(u_iv_j)(y,t)\xi(y)dy \tag{4.7}$$

$$= - \lim_{\varepsilon \rightarrow 0} \int_{B_{2R}/B_\varepsilon} \partial_i\Gamma(x-y)\partial_j(u_iv_j)(y,t)\xi(y)dy \tag{4.8}$$

$$- \lim_{\varepsilon \rightarrow 0} \int_{B_{2R}/B_\varepsilon} \Gamma(x-y)\partial_j(u_iv_j)(y,t)\partial_i\xi(y)dy \tag{4.9}$$

$$+ \int_{\partial B_{2R}} \Gamma(x-y)\partial_j(u_iv_j)n_i\xi ds - \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \Gamma(x-y)\partial_j(u_iv_j)n_i\xi ds \tag{4.10}$$

$$:= I + II + III + IV, \tag{4.11}$$

where  $\nabla \cdot u = 0$  and  $n_i = \frac{y_i - x_i}{|x - y|}$  denotes the  $i$ th component of the outer normal vector of Ball  $B_{2R}$ . Since  $\xi = 0$  on  $\partial B_{2R}$ , we have

$$III = 0. \tag{4.12}$$

For term  $IV$ , we have the following estimate

$$\begin{aligned}
 IV &\lesssim \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \frac{1}{|x-y|} ds \max_{y \in \partial B_\varepsilon} |\nabla u|^2 \\
 &\lesssim \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \cdot 4\pi\varepsilon^2 \cdot \max_{y \in \partial B_\varepsilon} |\nabla u|^2 = 0.
 \end{aligned}
 \tag{4.13}$$

For term  $I$ , integration by parts yields

$$\begin{aligned}
 I &= \lim_{\varepsilon \rightarrow 0} \int_{B_{2R}/B_\varepsilon} \partial_i\partial_j\Gamma(x-y)(u_iv_j)(y,t)\xi(y)dy \\
 &\quad + \lim_{\varepsilon \rightarrow 0} \int_{B_{2R}/B_\varepsilon} \partial_i\Gamma(x-y)(u_iv_j)(y,t)\partial_j\xi(y)dy \\
 &\quad - \int_{\partial B_{2R}} \partial_i\Gamma(x-y)(u_iv_j)n_j\xi ds + \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \partial_i\Gamma(x-y)(u_iv_j)n_j\xi ds \\
 &= \text{p.v.} \int \partial_i\partial_j\Gamma(x-y)(u_iv_j)(y,t)\xi(y)dy
 \end{aligned}$$

$$+ \int \partial_i \Gamma(x - y) (u_i v_j) (y, t) \partial_j \xi(y) dy - \frac{1}{3} u \cdot v, \tag{4.14}$$

where

$$\lim_{\varepsilon \rightarrow 0} \int_{B_{2R}/B_\varepsilon} \partial_i \partial_j \Gamma(x - y) (u_i v_j) (y, t) \xi(y) dy = \text{p.v.} \int_{B_{2R}} \partial_i \partial_j \Gamma(x - y) (u_i v_j) (y, t) \xi(y) dy,$$

and

$$\int_{\partial B_{2R}} \partial_i \Gamma(x - y) (u_i v_j) n_j \xi ds = 0,$$

for  $\xi = 0$  on  $\partial B_{2R}$ . The last term  $\lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \partial_i \Gamma(x - y) (u_i v_j) n_j \xi ds$  can be dealt as follows

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \partial_i \Gamma(x - y) (u_i v_j) n_j \xi ds \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \frac{x_i - y_i}{4\pi|x - y|^3} \frac{y_j - x_j}{|x - y|} (u_i v_j) (y) ds \\ &= - \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \frac{1}{4\pi\varepsilon^4} (x_i - y_i)(x_j - y_j) (u_i v_j) (y) ds \\ &= - \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \frac{1}{4\pi\varepsilon^4} (x_i - y_i)(x_j - y_j) [(u_i v_j)(y) - (u_i v_j)(x)] ds \\ &\quad - \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \frac{u_i v_j(x)}{4\pi\varepsilon^4} (x_i - y_i)(x_j - y_j) ds \\ &:= J_1 + J_2. \end{aligned} \tag{4.15}$$

By the mean value inequality, we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \frac{1}{4\pi\varepsilon^4} (x_i - y_i)(x_j - y_j) [(u_i v_j)(y) - (u_i v_j)(x)] ds \\ & \leq \lim_{\varepsilon \rightarrow 0} \frac{1}{4\pi\varepsilon^4} \cdot \varepsilon^3 \cdot 4\pi\varepsilon^2 \max_{y \in \partial B_\varepsilon} \nabla(u \cdot v) = 0. \end{aligned} \tag{4.16}$$

Hence, we obtain  $J_1 = 0$ . For  $J_2$ , when  $i = j$ ,

$$\begin{aligned} & - \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \frac{u_i v_i(x)}{4\pi\varepsilon^4} (x_i - y_i)(x_i - y_i) ds \\ &= - \lim_{\varepsilon \rightarrow 0} \frac{u_i v_i(x)}{4\pi\varepsilon^4} \int_{\partial B_\varepsilon} (x_i - y_i)(x_i - y_i) ds \\ &= - \lim_{\varepsilon \rightarrow 0} \frac{u \cdot v}{4\pi\varepsilon^4} \frac{1}{3} \int_{\partial B_\varepsilon} |x - y|^2 ds \\ &= - \lim_{\varepsilon \rightarrow 0} \frac{u \cdot v}{4\pi\varepsilon^4} \frac{1}{3} \varepsilon^2 4\pi\varepsilon^2 \\ &= -\frac{1}{3} u \cdot v. \end{aligned} \tag{4.17}$$

When  $i \neq j$ , according to the symmetry

$$\begin{aligned} & \int_{\partial B_\varepsilon} (x_i - y_i)(x_j - y_j) ds \\ &= \int_{y \in \partial B_\varepsilon, x_i - y_i > 0} (x_i - y_i)(x_j - y_j) ds + \int_{y \in \partial B_\varepsilon, x_i - y_i < 0} (x_i - y_i)(x_j - y_j) ds \\ &= 0. \end{aligned} \tag{4.18}$$

Combining with (4.15)-(4.18), we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \partial_i \Gamma(x-y) (u_i v_j) n_j \xi ds = -\frac{1}{3} u \cdot v. \quad (4.19)$$

Therefore, (4.14) holds. Combining with (4.6)-(4.13), we have

$$\begin{aligned} & \int_{\mathbb{R}^3} \Gamma(x-y) \partial_i \partial_j (u_i v_j)(y, t) \xi(y) dy \\ = & \text{p.v.} \int \partial_i \partial_j \Gamma(x-y) (u_i v_j)(y, t) \xi(y) dy - \int_{B_{2R}} \Gamma(x-y) \partial_j (u_i v_j)(y, t) \partial_i \xi(y) dy \\ & + \int \partial_i \Gamma(x-y) (u_i v_j)(y, t) \partial_j \xi(y) dy - \frac{1}{3} u \cdot v. \end{aligned} \quad (4.20)$$

Take  $u_i v_j = w_i w_j, v_{ci} w_j, w_i v_{cj}$ , separately, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^3} \Gamma(x-y) \partial_i \partial_j (w_i w_j + v_{ci} w_j + w_i v_{cj})(y, t) \xi(y) dy \\ = & -\frac{1}{3} |w|^2 + \text{p.v.} \int_{\mathbb{R}^3} \partial_i \partial_j \Gamma(x-y) w_i w_j(y) dy \\ & - \frac{2}{3} v_c \cdot w + 2 \text{p.v.} \int_{\mathbb{R}^3} \partial_i \partial_j \Gamma(x-y) w_i v_{cj}(y) dy \\ & + \int \partial_i \Gamma(x-y) (w_i w_j)(y, t) \partial_j \xi(y) dy - \int_{B_{2R}} \Gamma(x-y) \partial_j (w_i w_j)(y, t) \partial_i \xi(y) dy \\ & + \int \partial_i \Gamma(x-y) (v_{ci} w_j)(y, t) \partial_j \xi(y) dy - \int_{B_{2R}} \Gamma(x-y) \partial_j (v_{ci} w_j)(y, t) \partial_i \xi(y) dy \\ & + \int \partial_i \Gamma(x-y) (w_i v_{cj})(y, t) \partial_j \xi(y) dy - \int_{B_{2R}} \Gamma(x-y) \partial_j (w_i v_{cj})(y, t) \partial_i \xi(y) dy. \end{aligned} \quad (4.21)$$

Setting  $R \rightarrow \infty$ , combining with (4.5), we obtain (1.5).  $\square$

**Acknowledgments.** The author is grateful to Prof. Yanyan Li for bringing to our attention the question studied in this paper and much useful advice. This work is partially supported by the National Natural Science Foundation of China 11771389, 11931010 and 11621101. We sincerely thank the anonymous reviewers for their constructive revision suggestions.

#### REFERENCES

- [1] A. Basson, *Solutions Spatialement Homogènes Adaptées des équations de Navier-Stokes*, Thesis. University of Evry., 2006.
- [2] Z. Bradshaw and T.-P. Tsai, Self-similar solutions to the Navier-Stokes equations: A survey of recent results, *Nonlinear Analysis in Geometry and Applied Mathematics*, **2** (2018), 159–181.
- [3] Y. Giga and T. Miyakawa, [Navier-Stokes flow in  \$\mathbf{R}^3\$  with measures as initial vorticity and Morrey spaces](#), *Comm. Partial Differential Equations*, **14** (1989), 577–618.
- [4] G. H. Hardy, [Note on a theorem of Hilbert](#), *Math. Z.*, **6** (1920), 314–317.
- [5] G. H. Hardy, An inequality between integrals, *Messenger of Mathematics*, **54** (1925), 150–156.
- [6] J. L. Hineman and C. Wang, [Well-posedness of nematic liquid crystal flow in  \$L^3\_{\text{loc}}\(\mathbf{R}^3\)\$](#) , *Arch. Ration. Mech. Anal.*, **210** (2013), 177–218.
- [7] G. Karch and D. Pilarczyk, [Asymptotic stability of Landau solutions to Navier-Stokes system](#), *Arch. Ration. Mech. Anal.*, **202** (2011), 115–131.
- [8] G. Karch, D. Pilarczyk and M. E. Schonbek,  [\$L^2\$ -asymptotic stability of singular solutions to the Navier-Stokes system of equations in  \$\mathbf{R}^3\$](#) , *J. Math. Pures Appl.*, **108** (2017), 14–40.
- [9] T. Kato, [Strong  \$L^p\$ -solutions of the Navier-Stokes equation in  \$\mathbf{R}^m\$ , with applications to weak solutions](#), *Math. Z.*, **187** (1984), 471–480.

- [10] N. Kikuchi and G. Seregin, [Weak solutions to the Cauchy problem for the Navier-Stokes equations satisfying the local energy inequality](#), *Nonlinear Equations and Spectral Theory*, Amer. Math. Soc. Transl. Ser. 2, 220, Amer. Math. Soc., Providence, RI, **220** (2007), 141–164.
- [11] H. Koch and D. Tataru, [Well posedness for the Navier-Stokes equations](#), *Adv. Math.*, **157** (2001), 22–35.
- [12] H. Kwon and T.-P. Tsai, [Global Navier-Stokes flows for non-decaying initial data with slowly decaying oscillation](#), *Comm. Math. Phys.*, **375** (2020), 1665–1715.
- [13] L. Landau, [A new exact solution of Navier-Stokes equations](#), *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **43** (1944), 286–288.
- [14] P. G. Lemarié-Rieusset, [Recent Developments in the Navier-Stokes Problem](#), Chapman & Hall/CRC Research Notes in Mathematics, vol. 431, 2002.
- [15] P. G. Lemarié-Rieusset, [The Navier-Stokes Problem in the 21st Century](#), CRC Press, Boca Raton, FL, 2016.
- [16] J. Leray, [Sur le mouvement d'un liquide visqueux emplissant l'espace](#), *Acta Math.*, **63** (1934), 193–248.
- [17] L. Li, Y. Y. Li and X. Yan, [Homogeneous solutions of stationary Navier-Stokes equations with isolated singularities on the unit sphere. I. One singularity](#), *Arch. Ration. Mech. Anal.*, **227** (2018), 1091–1163.
- [18] L. Li, Y. Y. Li and X. Yan, [Homogeneous solutions of stationary Navier-Stokes equations with isolated singularities on the unit sphere. II. Classification of axisymmetric no-swirl solutions](#), *J. Differential Equations*, **264** (2018), 6082–6108.
- [19] L. Li, Y. Y. Li and X. Yan, [Vanishing viscosity limit for homogeneous axisymmetric no-swirl solutions of stationary Navier-Stokes equations](#), *J. Funct. Anal.*, **277** (2019), 3599–3652.
- [20] L. Li, Y. Y. Li and X. Yan, [Homogeneous solutions of stationary Navier-Stokes equations with isolated singularities on the unit sphere. III. Two singularities](#), *Discrete Contin. Dyn. Syst.*, **39** (2019), 7163–7211.
- [21] V. Šverák, [On Landau's solutions of the Navier-Stokes equations](#), *Problems in mathematical analysis*, No. 61, *J. Math. Sci. (N.Y.)*, **179** (2011), 208–228.
- [22] M. E. Taylor, [Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations](#), *Comm. Partial Differential Equations*, **17** (1992), 1407–1456.
- [23] G. Tian and Z. Xin, [One-point singular solutions to the Navier-Stokes equations](#), *Topol. Methods Nonlinear Anal.*, **11** (1998), 135–145.
- [24] T.-P. Tsai, [Forward discretely self-similar solutions of the Navier-Stokes equations](#), *Comm. Math. Phys.*, **328** (2014), 29–44.
- [25] T.-P. Tsai, [Lectures on Navier-Stokes Equations](#), American Mathematical Society, Providence, RI, vol. 192, 2018.
- [26] J. Zhang and T. Zhang, [Global existence of discretely self-similar solutions to the generalized MHD system in Besov space](#), *J. Math. Phys.*, **60** (2019), 081515, 18 pp.
- [27] N. Zhao, [A Liouville theorem for axially symmetric  \$D\$ -solutions to steady Navier-Stokes equations](#), *Nonlinear Anal.*, **187** (2019), 247–258.

Received for publication December 2020.

*E-mail address:* [zhangjingjing94@126.com](mailto:zhangjingjing94@126.com)

*E-mail address:* [zhangting79@zju.edu.cn](mailto:zhangting79@zju.edu.cn)