AUTOMORPHISM GROUP AND TWISTED MODULES OF THE
TWISTED HEISENBERG-VIRASORO VERTEX
OPERATOR ALGEBRA

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Abstract. We first determine the automorphism group of the twisted Heisenberg-Virasoro vertex operator algebra $V_{L}(\ell_{123}, 0)$. Then, for any integer $t > 1$, we introduce a new Lie algebra $L_t$, and show that $\sigma_t$-twisted $V_{L}(\ell_{123}, 0)(\ell_2 = 0)$-modules are in one-to-one correspondence with restricted $L_t$-modules of level $\ell_{13}$, where $\sigma_t$ is an order $t$ automorphism of $V_{L}(\ell_{123}, 0)$. At the end, we give a complete list of irreducible $\sigma_t$-twisted $V_{L}(\ell_{123}, 0)(\ell_2 = 0)$-modules.

1. Introduction. Let $L$ be the twisted Heisenberg-Virasoro algebra. It is the universal central extension of the Lie algebra of differential operators on a circle of order at most one (cf. [3]):

$$\{ f(t) \frac{d}{dt} + g(t) \mid f(t), g(t) \in \mathbb{C}[t, t^{-1}] \}.$$

$L$ contains an infinite-dimensional Heisenberg algebra and the Virasoro algebra as subalgebras (cf. [3], [4]). The induced module $V_{L}(\ell_{123}, 0) = U(L) \otimes U(L_{\leq 1}) \mathbb{C}\ell_{123}$ is a vertex operator algebra of central charge $\ell_1$ with conformal vector $\omega = L_{-2}1$ (cf. [12]). $V_{L}(\ell_{123}, 0)$ is a nonrational vertex operator algebra and is not $C_2$-cofinite. The structure theory and representation theory of the twisted Heisenberg-Virasoro vertex operator algebra $V_{L}(\ell_{123}, 0)$ are closely related to the three scalars $\ell_1, \ell_2, \ell_3$ (cf. [3], [1], [2], [4], [5], [8], [12], etc.).

Determining the automorphism group $\text{Aut}(V)$ of a vertex operator algebra $V$ is an important subject in vertex operator algebra theory. It is related to the orbifold theory which studies the fixed point subalgebras of vertex operator algebras and their modules under certain finite subgroups of the full automorphism groups. The orbifold conjecture says that under some conditions on $V$, every simple $V^G$-module is contained in some $g$-twisted $V$-module, where $G$ is a finite subgroup of $\text{Aut}(V)$, $g \in G$, $V^G$ is the fixed point subalgebra of $V$ under the group $G$.

The automorphism group of the twisted Heisenberg-Virasoro algebra $L$ has been studied in [15]. In this paper, we study the automorphism group of the twisted Heisenberg-Virasoro vertex operator algebra $V_{L}(\ell_{123}, 0)$. We know that $V_{L}(\ell_{123}, 0)$ is generated by $\omega = L_{-2}1$ and $I_{-1}1$. By definition, any homomorphism of a vertex operator algebra $(V, Y, 1, \omega)$ takes $\omega$ to $\omega$. Therefore, it suffices to determine the action on the generator $I_{-1}1$. It turns out that automorphisms of the twisted

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Heisenberg-Virasoro vertex operator algebra $V_C(\ell_{123}, 0)$ depend on the numbers $\ell_2$ and $\ell_3$, and there exists automorphism of order other than 2, which makes the study of twisted modules for $V_C(\ell_{123}, 0)$ interesting.

In [12], all irreducible modules for the vertex operator algebra $V_C(\ell_{123}, 0)$ are classified: that is, every irreducible module for $V_C(\ell_{123}, 0)$ is isomorphic to some $L_C(\ell_{123}, h_1, h_2)$, $h_1, h_2 \in \mathbb{C}$. In this paper, for any integer $t > 1$, we classify $\sigma_t$-twisted irreducible modules for $V_C(\ell_{123}, 0)$, where $\sigma_t$ is an order $t$ automorphism of $V_C(\ell_{123}, 0)$. The way we deal with this problem is similar to the one used for modules of vertex algebra (cf. [10], [11], [12], [13], etc.), but in the context of twisted modules. We first introduce another Lie algebra $L_t$. It is a Lie algebra with basis $\{L_n, I_{n+\frac{1}{t}}, k_1, k_3 \mid n \in \mathbb{Z}\}$, and Lie brackets

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0} \frac{m^3-m}{12} k_1,$$

$$[L_m, I_{n+\frac{1}{t}}] = -(n + \frac{1}{t})I_{m+n+\frac{1}{t}},$$

$$[I_{m+\frac{1}{t}}, I_{n+\frac{1}{t}}] = (m + \frac{1}{t})\delta_{m+n+\frac{2}{t},0}\delta_{t,2}k_3, \quad [L_t, k_i] = 0, \quad i = 1, 3.$$  

Note that when $t \neq 2$, $\{I_{n+\frac{1}{t}} \mid n \in \mathbb{Z}\}$ is an abelian Lie algebra. Then we construct irreducible $L_t$-modules $L_C(k_1, k_3, h)$ as quotient of the induced modules $M_C(k_1, k_3, h)$, where $k_1, k_3, h \in \mathbb{C}$. We show that $\sigma_t$-twisted $V_C(\ell_{123}, 0)$-modules ($\ell_2 = 0$) are in one-to-one correspondence with restricted $L_t$-modules of level $\ell_{13}$. Using this result, we get a complete list of irreducible $\sigma_t$-twisted $V_C(\ell_{123}, 0)$-modules, where $\ell_2 = 0$, $\ell_1$, $\ell_3 \in \mathbb{C}$. Let $V_C(\ell_{123}, 0)^{\sigma_t}$ be the fixed point subalgebra of $V_C(\ell_{123}, 0)$ under the automorphism $\sigma_t$. $V_C(\ell_{123}, 0)^{\sigma_t}$ is a vertex operator subalgebra of $V_C(\ell_{123}, 0)$. It is important and meaningful to study the representations of the fixed point subalgebra $V_C(\ell_{123}, 0)^{\sigma_t}$. We remark at the end of the paper that except for the case of order 2, the complete list of irreducible modules for $V_C(\ell_{123}, 0)^{\sigma_t}$ needs to be further investigated.

This paper is organized as follows. In Section 2, we review the notions and some results of vertex operator algebras, automorphisms and twisted modules for vertex operator algebras. In Section 3, we study the automorphism group of the twisted Heisenberg-Virasoro vertex operator algebra $V_C(\ell_{123}, 0)$. In Section 4, we first study the twisted modules of $V_C(\ell_{123}, 0)(\ell_2 = 0)$ under an order $t$ automorphism $\sigma_t$ for any integer $t > 1$. Then we give a complete list of irreducible $\sigma_t$-twisted $V_C(\ell_{123}, 0)(\ell_2 = 0)$-modules.

2. Preliminaries. For later use, we recall the following result (cf. Proposition 2.3.7 of [13]).

**Lemma 2.1.**

\[(x_1 - x_2)^m \left( \frac{\partial}{\partial x_2} \right)^n x_2^{-1} \delta \left( \frac{x_2}{x_1} \right) = 0 \quad (2.1)\]

for $m > n$, $m, n \in \mathbb{N}$, where $\delta \left( \frac{x_1}{x_2} \right) = \sum_{n \in \mathbb{Z}} x_1^n x_2^{-n}$.

Let $(V, Y, \omega, 1)$ be a vertex operator algebra (cf. Definition 3.1.22 of [13]). Then, a priori, $(V, Y, \omega, 1)$ is a vertex algebra (cf. Definition 3.1.1 of [13]). Let $D$ be the endomorphism of the vertex algebra $V$ defined by $D(v) = v_{-2} 1$ for $v \in V$. We have
Remark 2.5. Let \( u, v \) (TW7) \( \dim W \) \( Y \) (TW3) \( \left[ Y v \right] \), satisfying the following conditions:

\[k \sigma( \omega ) = \omega \; \text{and} \; \sigma(u_n v) = \sigma(u) \omega \sigma(v)\]

for any \( u, v \in V, \; n \in \mathbb{Z} \).

The group of all automorphisms of a vertex operator algebra \( V \) is denoted by \( \text{Aut}(V) \). Any automorphism of a vertex operator algebra \( V \) is grading-preserving, i.e. it preserves each homogeneous subspace \( V(n) \) of \( V, \; n \in \mathbb{Z} \). Let \( \sigma \) be an order \( t \) automorphism of a vertex operator algebra \( V \), \( t \) is a positive integer. Then \( \sigma \) acts semisimply on \( V \). Therefore

\[ V = V^0 \oplus V^1 \oplus \cdots \oplus V^{t-1}, \]

where \( V^k \) is the eigenspace of \( \sigma \) with eigenvalues \( \eta^k \), where \( \eta = \exp(\frac{2\pi i}{t}) \), \( k = 0, \ldots, t-1 \). It is easy to see that the fixed points set \( V^0 := V^\sigma = \{ v \in V \mid \sigma(v) = v \} \) is a vertex operator subalgebra of \( V \).

Now we recall some notions regarding to twisted modules from [14].

Definition 2.3. Let \((V, 1, Y)\) be a vertex algebra with an automorphism \( \sigma \) of order \( t \). A \( \sigma \)-twisted \( V \)-module is a triple \((W, d, Y_W)\) consisting of a vector space \( W \), an endomorphism \( d \) of \( W \) and a linear map

\[ Y_W(\cdot, z) : V \rightarrow (\text{End}W)[[z^{\frac{1}{t}}, z^{-\frac{1}{t}}]] \]

satisfying the following conditions:

(TW1) For any \( v \in V, w \in W, \; v_n w = 0 \) for \( n \in \frac{1}{t} \mathbb{Z} \) sufficiently large;

(TW2) \( Y_W(1, z) = Id_W \);

(TW3) \( [d, Y_W(v, z)] = Y_W(D(v), z) = \frac{d}{dz} Y_W(v, z) \) for any \( v \in V \);

(TW4) For any \( u, v \in V \), the following \( \sigma \)-twisted Jacobi identity holds:

\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_W(u, z_1) Y_W(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_W(v, z_2) Y_W(u, z_1) = z_2^{-1} \frac{1}{t} \sum_{k=0}^{t-1} \delta \left( \frac{z_1 - z_0}{z_2} \right)^k Y_W(Y(\sigma^k u, z_0) v, z_2).
\]

If \( V \) is a vertex operator algebra, a \( \sigma \)-twisted \( V \)-module for \( V \) as a vertex algebra is called a \( \sigma \)-twisted weak module for \( V \) as a vertex operator algebra.

Definition 2.4. For \( V \) a vertex operator algebra, a \( \sigma \)-twisted \( V \)-module \( W \) is a \( \sigma \)-twisted weak module for \( V \) as a vertex algebra and \( W = \prod_{h \in \mathbb{C}} W_{(h)} \) such that

(TW5) \( L(0)w = hw \) for \( h \in \mathbb{C}, \; w \in W_{(h)} \);

(TW6) For any fixed \( h, \; W_{(h+n)} = 0 \) for \( n \in \frac{1}{t} \mathbb{Z} \) sufficiently small;

(TW7) \( \dim W_{(h)} < \infty \) for any \( h \in \mathbb{C} \).

Remark 2.5. Let \( W \) be a \( \sigma \)-twisted \( V \)-module. Then

\[ [L(0), Y_W(v, z)] = z Y_W(L(-1)v, z) + Y_W(L(0)v, z) \] (2.2)
for \( v \in V \). Hence, if \( v \in V \) is homogeneous,

\[
v_n W(h) \subseteq W(h \cdot wt \; v - n - 1) \quad \text{for } n \in \mathbb{Z}, \quad h \in \mathbb{C}.
\]

It follows that a \( \sigma \)-twisted \( V \)-module \( W \) decomposes into twisted submodules corresponding to the congruence classes mod \( \frac{1}{t} \mathbb{Z} \): For \( h \in \mathbb{C}/\frac{1}{t} \mathbb{Z} \), let

\[
W[h] = \prod_{\alpha + \frac{1}{t} \mathbb{Z} = h} W(\alpha).
\]

(2.3)

Then

\[
W = \bigoplus_{h \in \mathbb{C}/\frac{1}{t} \mathbb{Z}} W[h].
\]

(2.4)

In particular, if \( W \) is irreducible, then

\[
W = W[h]
\]

(2.5)

for some \( h \).

**Remark 2.6.** Let \( W \) be a \( \sigma \)-twisted \( V \)-module. For \( u \in V^k \), \( v \in V \), there is the following twisted iterate formula (cf. (2.32) of [14])

\[
Y_W(Y(u, z_0)v, z_2) = \text{Res}_{z_1} \left( \frac{z_1 - z_0}{z_2} \right) \frac{k}{t} \cdot X,
\]

where

\[
X = z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_W(u, z_1)Y_W(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_W(v, z_2)Y_W(u, z_1).
\]

Then it can be easily deduced that for any \( n \in \mathbb{Z} \),

\[
Y_W(u, v, z) = \text{Res}_{z_1} \sum_{j=0}^{\infty} (-1)^j \left( \frac{k}{t} \right)^j \frac{z_1^{-j} z^{-\frac{k}{t} + j} \cdot Y_j}{z_1^{j}}.
\]

(2.7)

where

\[
Y_j = \left( (z_1 - z)^{n+j}Y_W(u, z_1)Y_W(v, z) - (-z + z_1)^{n+j}Y_W(v, z)Y_W(u, z_1) \right).
\]

Using the formula (2.7), similarly as Proposition 4.5.1 of [13], there is the following result.
Proposition 2.8. Let $W_1$ and $W_2$ be $\sigma$-twisted $V$-modules and let $\psi \in \text{Hom}_C(W_1, W_2)$. Suppose that
\[ Y(a, z)\psi = \psi Y(a, z) \text{ for } a \in S, \]
where $S$ is a given generating set of $V$. Then $\psi$ is a $\sigma$-twisted $V$-module homomorphism.

In the following, we review from Section 3 of [14] the local systems of twisted vertex operators.

Definition 2.9. Let $W$ be a vector space, let $t$ be a fixed positive integer. A $\mathbb{Z}_t$-twisted weak vertex operator on $W$ is a formal series $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in \langle \text{End } W \rangle[[z^\frac{1}{t}, z^{-\frac{1}{t}}]],$ such that for any $w \in W,$ $a_n w = 0$ for $n \in \frac{1}{t}\mathbb{Z}$ sufficiently large.

Definition 2.10. Two $\mathbb{Z}_t$-twisted weak vertex operators $a(z)$ and $b(z)$ are said to be mutually local if there is a positive integer $n$ such that
\[ (z_1 - z_2)^n a(z_1) b(z_2) = (z_1 - z_2)^n b(z_2) a(z_1). \]

A $\mathbb{Z}_t$-twisted weak vertex operator is called a $\mathbb{Z}_t$-twisted vertex operator if it is local with itself.

Denote by $F(W, t)$ the space of all $\mathbb{Z}_t$-twisted weak vertex operators on $W$. Let $\sigma$ be the endomorphism of $\langle \text{End } W \rangle[[z^\frac{1}{t}, z^{-\frac{1}{t}}]]$ defined by: $\sigma f(z^\frac{1}{t}) = f(\eta^{-1} z^\frac{1}{t}).$ Denote by $F(W, t)^k = \{ f(z) \in F(W, t) \mid \sigma f(z) = \eta^k f(z) \}.$ For any mutually local $\mathbb{Z}_t$-twisted vertex operators $a(z), b(z)$ on $W,$ define $a(z)_n b(z)$ as follows (cf. Definition 3.7 of [14]).

Definition 2.11. Let $W$ be a vector space and let $a(z)$ and $b(z)$ be mutually local $\mathbb{Z}_t$-twisted vertex operators on $W$ such that $a(z) \in F(W, t)^k.$ Then for any integer $n$ we define $a(z)_n b(z)$ as an element of $F(W, t)$ as follows:
\[ a(z)_n b(z) = \text{Res}_{z_1} \text{Res}_{z_0} \left( \frac{z_1 - z_0}{z} \right)^{\frac{1}{t}} z_0^{-n} \cdot X, \quad (2.10) \]
where
\[ X = z_0^{-1} \delta \left( \frac{z_1 - z}{z_0} \right) a(z_1) b(z) - z_0^{-1} \delta \left( \frac{z - z_1}{z_0} \right) b(z) a(z_1). \]

For any set $S$ of mutually local $\mathbb{Z}_t$-twisted vertex operators on $W,$ by Zorn’s Lemma, there exists a local system $A$ of $\mathbb{Z}_t$-twisted vertex operators on $W$ (cf. Section 3 of [14]). Denote by $\langle S \rangle$ the vertex algebra generated by $S$ inside $A$ via the operations $a(z)_n b(z),$ $n \in \mathbb{Z}.$

3. Automorphism group. In this section, we first recall the definition of the twisted Heisenberg-Virasoro algebra $L$ and the construction of the twisted Heisenberg-Virasoro vertex operator algebra $V_L(\ell_{123}, 0)$. Then we determine the automorphism group of the vertex operator algebra $V_L(\ell_{123}, 0)$.

Recall the definition of the twisted Heisenberg-Virasoro algebra $L$ (see [3] or [4]).

Definition 3.1. The twisted Heisenberg-Virasoro algebra $L$ is a Lie algebra with basis $\{L_n, I_n, c_1, c_2, c_3 \mid n \in \mathbb{Z} \}$, and the following Lie brackets:
\[ [L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0} \frac{m^3-m}{12} c_1, \quad (3.1) \]
\[ [L_m, I_n] = -nI_{m+n} - \delta_{m+n,0}(m^2+m)c_2, \quad (3.2) \]
[L_n, L_m] = m\delta_{m+n,0}c_3, \quad [\mathcal{L}, c_i] = 0, \quad i = 1, 2, 3. \quad (3.3)

Clearly, \text{Span}\{L_n, c_i \mid n \in \mathbb{Z}\} is a Virasoro algebra, \text{Span}\{L_n, c_3 \mid n \in \mathbb{Z}\} is an infinite-dimensional Heisenberg algebra, we denote them by \text{Vir}, \mathcal{H} respectively. Let

\begin{align*}
L(z) &= \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad I(z) = \sum_{n \in \mathbb{Z}} I_n z^{-n-1},
\end{align*}

then the defining relations of \mathcal{L} become to be

\begin{align*}
[L(z_1), L(z_2)] &= \sum_{m,n \in \mathbb{Z}} (m-n)L_{m+n} z_1^{-m-2} z_2^{-n-2} + \sum_{m \in \mathbb{Z}} \frac{m^3-m}{12} c_1 z_1^{-m-2} z_2^{-m-2} \\
&= \frac{d}{dz_2} (L(z_2)) z_1^{-1} \delta \left( \frac{z_2}{z_1} \right) + 2L(z_2) \frac{\partial}{\partial z_2} z_1^{-1} \delta \left( \frac{z_2}{z_1} \right) \\
&\quad + \frac{c_1}{12} \left( \frac{\partial}{\partial z_2} \right)^3 z_1^{-1} \delta \left( \frac{z_2}{z_1} \right), \quad (3.4)
\end{align*}

\begin{align*}
[I(z_1), I(z_2)] &= -\sum_{m,n \in \mathbb{Z}} nI_{m+n} z_1^{-m-2} z_2^{-n-1} - \sum_{m \in \mathbb{Z}} (m^2+m)c_2 z_1^{-m-2} z_2^{-m-1} \\
&= \frac{d}{dz_2} (I(z_2)) z_1^{-1} \delta \left( \frac{z_2}{z_1} \right) + I(z_2) \frac{\partial}{\partial z_2} z_1^{-1} \delta \left( \frac{z_2}{z_1} \right) \\
&\quad - \left( \frac{\partial}{\partial z_2} \right)^2 z_1^{-1} \delta \left( \frac{z_2}{z_1} \right) c_2, \quad (3.5)
\end{align*}

\begin{align*}
[I(z_1), I(z_2)] &= \sum_{m \in \mathbb{Z}} mc_3 z_1^{-m-1} z_2^{-m-1} = \frac{\partial}{\partial z_2} z_1^{-1} \delta \left( \frac{z_2}{z_1} \right) c_3. \quad (3.6)
\end{align*}

We recall the construction of the twisted Heisenberg-Virasoro vertex operator algebra \(V_\mathcal{L}(\ell_{123},0)\) from [12]. Let

\begin{align*}
\mathcal{L}_{(\leq 1)} &= \prod_{n \leq 1} \mathbb{C} L_{-n} \oplus \prod_{n \leq 0} \mathbb{C} I_{-n} \oplus \sum_{i=1}^{3} \mathbb{C} c_i, \\
\mathcal{L}_{(\geq 2)} &= \prod_{n \geq 2} \mathbb{C} L_{-n} \oplus \prod_{n \geq 1} \mathbb{C} I_{-n}.
\end{align*}

They are graded subalgebras of \(\mathcal{L}\) and

\(\mathcal{L} = \mathcal{L}_{(\leq 1)} \oplus \mathcal{L}_{(\geq 2)}\).

Let \(\ell_i, i = 1, 2, 3\), be any complex numbers. Consider \(\mathbb{C}\) as an \(\mathcal{L}_{(\leq 1)}\)-module with \(c_i\) acting as the scalar \(\ell_i\), \(i = 1, 2, 3\), and with \(\prod_{n \leq 1} \mathbb{C} L_{-n} \oplus \prod_{n \leq 0} \mathbb{C} I_{-n}\) acting trivially. Denote this \(\mathcal{L}_{(\leq 1)}\)-module by \(\mathbb{C}_{\ell_{123}}\). Form the induced module

\(V_\mathcal{L}(\ell_{123},0) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_{(\leq 1)})} \mathbb{C}_{\ell_{123}},\)

where \(U(\cdot)\) denotes the universal enveloping algebra of a Lie algebra. Set \(1 = 1 \otimes 1 \in V_\mathcal{L}(\ell_{123},0)\). \(V_\mathcal{L}(\ell_{123},0)\) is a vertex operator algebra with vacuum vector 1.
and conformal vector $\omega = L_{-2}1$. And $\{\omega = L_{-2}1, I := L_{-1}1\}$ is a generating subset of $V_L(\ell_{123}, 0)$. Recall the grading on $V_L(\ell_{123}, 0)$:

$$ V_L(\ell_{123}, 0) = \bigoplus_{n \geq 0} V_L(\ell_{123}, 0)_{(n)} , $$

where $V_L(\ell_{123}, 0)_{(0)} = \mathbb{C}$ and $V_L(\ell_{123}, 0)_{(n)}$, $n \geq 1$, has a basis consisting of the vectors

$$ I_{-k_1} \cdots I_{-k_s} L_{-m_1} \cdots L_{-m_r} 1 $$

for $r, s \geq 0$, $m_1 \geq \cdots \geq m_r \geq 2$, $k_1 \geq \cdots \geq k_s \geq 1$ with $\sum_{i=1}^{r} m_i + \sum_{j=1}^{s} k_j = n$.

Now we give our first main result. The automorphism group of $V_L(\ell_{123}, 0)$ is determined in the following theorem.

**Theorem 3.2.**  
(1) If $\ell_2 \neq 0$, then $\text{Aut}(V_L(\ell_{123}, 0)) = \{id\}$.

(2) If $\ell_2 = 0$ and $\ell_3 \neq 0$, then $\text{Aut}(V_L(\ell_{123}, 0)) \cong \mathbb{Z}_2$.

(3) If both $\ell_2$ and $\ell_3$ are zero, then $\text{Aut}(V_L(\ell_{123}, 0)) \cong \mathbb{C}^\times = \mathbb{C}\setminus\{0\}$.

**Proof.** Let

$$ \varphi : V_L(\ell_{123}, 0) \to V_L(\ell_{123}, 0) $$

be an automorphism of the vertex operator algebra $V_L(\ell_{123}, 0)$. Then $\varphi(1) = 1$ and $\varphi(\omega) = \omega$. Since $V_L(\ell_{123}, 0)$ is generated by $\omega$ and $I = L_{-1}1$, it suffices to determine $\varphi(I)$. $\varphi$ is grading-preserving, so $\varphi(I) = aI$ for some $a \in \mathbb{C}^\times$.

Then, on the one hand, we have

$$ \varphi(L_1 I) = aL_1 I = a[L_1, I_{-1}]1 = -2a\ell_21 , $$

on the other hand, we have

$$ \varphi(L_1 I) = \varphi([L_1, I_{-1}]1) = -2\ell_21 . $$

Therefore if $\ell_2 \neq 0$, we get that $a = 1$, i.e. when $\ell_2 \neq 0$, $\text{Aut}(V_L(\ell_{123}, 0)) = \{id\}$ only consists of the identity map.

Suppose now $\ell_2 = 0$. Let’s consider $\varphi(I_1 I)$. On the one hand,

$$ \varphi(I_1 I) = \varphi(I)\varphi(I) = a^2I_1 I_{-1}1 = a^2[I_1, I_{-1}]1 = a^2\ell_31 , $$

on the other hand,

$$ \varphi(I_1 I) = \varphi([I_1, I_{-1}]1) = \ell_31 . $$

So if $\ell_3 \neq 0$, we get $a^2 = 1$, i.e. $a = 1$ or $a = -1$, then $\text{Aut}(V_L(\ell_{123}, 0)) \cong \mathbb{Z}_2$.

Now let $\ell_2 = 0$ and $\ell_3 = 0$, then $a$ can be any nonzero complex number, so $\text{Aut}(V_L(\ell_{123}, 0)) \cong \mathbb{C}^\times$. \qed

4. $\sigma_t$-twisted $V_L(\ell_{123}, 0)$-modules. In this section, we always assume $\ell_2 = 0$. We study twisted modules for the vertex operator algebra $V_L(\ell_{123}, 0)$. More precisely, for any integer $t > 1$, we introduce an infinite-dimensional Lie algebra $L_t$. We show that there is a one-to-one correspondence between restricted $L_t$-modules of level $\ell_{13}$ and $\sigma_t$-twisted $V_L(\ell_{123}, 0)$-modules, where $\sigma_t$ is an order $t$ automorphism of $V_L(\ell_{123}, 0)$. And we give a complete list of irreducible $\sigma_t$-twisted $V_L(\ell_{123}, 0)$-modules.

Note that if $\ell_2 = 0$ and $\ell_3 \neq 0$, then $t$ can only be the integer 2 (Theorem 3.2). As we need, we introduce the following Lie algebra.
Definition 4.1. Let $L_t$ be a Lie algebra with basis $\{L_n, I_{n+\frac{1}{2}}, k_1, k_3 \mid n \in \mathbb{Z}\}$, and the Lie brackets are given by:

\[
[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} k_1, \tag{4.1}
\]

\[
[L_m, I_{n+\frac{1}{2}}] = -(n + \frac{1}{t})I_{m+n+\frac{1}{2}}, \tag{4.2}
\]

\[
[I_{m+\frac{1}{2}}, I_{n+\frac{1}{2}}] = (m + \frac{1}{t})\delta_{m+n+\frac{2}{2},0}\delta_{i,2}k_3, \quad [L_t, k_i] = 0, \quad i = 1, 3. \tag{4.3}
\]

Note that if $t \neq 2$, then $[I_{m+\frac{1}{2}}, I_{n+\frac{1}{2}}] = 0$ for any $m, n \in \mathbb{Z}$.

Form the generating function as

\[
L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad I_{\sigma_1}(z) = \sum_{n \in \mathbb{Z}} I_{n+\frac{1}{2}} z^{-n-\frac{1}{2}-1}.
\]

Then the defining relations (4.1), (4.2), (4.3) amount to:

\[
[L(z_1), L(z_2)]
= \frac{d}{dz_1}(L(z_2))z_1^{-1}\delta(z_2 z_1) + 2L(z_2) \frac{\partial}{\partial z_2} z_1^{-1}\delta(z_2 z_1) + \frac{1}{12} \left( \frac{\partial}{\partial z_2} \right)^3 z_1^{-1}\delta(z_2 z_1) k_1, \tag{4.4}
\]

\[
[L(z_1), I_{\sigma_1}(z_2)]
= -\sum_{m,n \in \mathbb{Z}} (n + \frac{1}{t})I_{m+n+\frac{1}{2}} z_1^{-m-2}z_2^{-n-\frac{1}{2}-1}
= \frac{d}{dz_2} (I_{\sigma_1}(z_2))z_1^{-1}\delta(z_2 z_1) + I_{\sigma_1}(z_2) \frac{\partial}{\partial z_2} z_1^{-1}\delta(z_2 z_1), \tag{4.5}
\]

\[
[I_{\sigma_1}(z_1), I_{\sigma_1}(z_2)]
= \sum_{m \in \mathbb{Z}} (m + \frac{1}{t})z_1^{-m-\frac{1}{2}}z_2^{-m+\frac{1}{2}+1}\delta_{i,2}k_3
= \frac{\partial}{\partial z_2} \left( z_1^{-1}\delta(z_2 z_1) \right) \left( z_2 z_1 \right)^{\frac{1}{2}} \delta_{i,2}k_3. \tag{4.6}
\]

Now we construct irreducible $L_t$-modules (cf. [12], [13], etc.). Let

\[
(L_t)_{\geq 0} = \left( \bigoplus_{m \geq 0} \mathbb{C} L_m \right) \oplus \left( \bigoplus_{n \geq 0} \mathbb{C} I_{n+\frac{1}{2}} \right) \oplus \mathbb{C} k_1 \oplus \mathbb{C} k_3.
\]

It is a subalgebra of $L_t$.

Let $\mathbb{C}$ be an $(L_t)_{\geq 0}$-module, where $L_m, I_{n+\frac{1}{2}}$ act trivially for all $m \geq 1, n \geq 0$, and $L_0, k_1, k_3$ act as scalar multiplications by $h, k_1, k_3$ respectively. Denote this $(L_t)_{\geq 0}$-module by $\mathbb{C}_{k_{13}, h}$. Form the induced module

\[
M_{L_t}(k_1, k_3, h) = U(L_t) \otimes_{U((L_t)_{\geq 0})} \mathbb{C}_{k_{13}, h}.
\]

Set

\[
1_{k_{13}, h} = 1 \in \mathbb{C}_{k_{13}, h} \subset M_{L_t}(k_1, k_3, h).
\]

Then $M_{L_t}(k_1, k_3, h)$ is $\mathbb{C}$-graded by $L_0$-eigenvalues:

\[
M_{L_t}(k_1, k_3, h) = \bigoplus_{n \geq 0} M_{L_t}(k_1, k_3, h)_{n+h}.
\]
where $M_{L_0}(k_1, k_3, h)(n) = \mathbb{C}_{k_1, k_3, h}$ and $M_{L_0}(k_1, k_3, h)(n+h)$ is the $L_0$-eigenspace of eigenvalue $n+h$ for $n > 0$. $M_{L_0}(k_1, k_3, h)(n+h)$ has a basis consisting of

$$I_{-k_1+\frac{1}{r}} \cdots I_{-k_s+\frac{1}{r}}I_{-m_1} \cdots I_{-m_r}1_{k_13,h}$$

for $r, s \geq 0$, $m_1 \geq \cdots \geq m_r \geq 1$, $k_1 \geq \cdots \geq k_s \geq 1$ with $\sum_{i=1}^{r} m_i + \sum_{j=1}^{s} (k_j - \frac{1}{l}) = n$, $n > 0$.

**Remark 4.2.** As a module for $L_t$, $M_{L_0}(k_1, k_3, h)$ is generated by $1_{k_13,h}$ with the relations

$$L_0 1_{k_13,h} = h 1_{k_13,h}, \quad k_i = k_i, \quad i = 1, 3,$$

and

$$L_m 1_{k_13,h} = 0, \quad I_{n+\frac{1}{l}} 1_{k_13,h} = 0 \quad \text{for} \quad m \geq 1, \quad n \geq 0.$$  

$M_{L_0}(k_1, k_3, h)$ is universal in the sense that for any $L_t$-module $W$ of level $k_{13}$ equipped with a vector $v$ such that $L_0 v = hv$, $L_m v = 0$, $I_{n+\frac{1}{l}} v = 0$ for $m \geq 1, n \geq 0$, there exists a unique $L_t$-module map $M_{L_0}(k_1, k_3, h) \to W$ sending $1_{k_13,h}$ to $v$.

In general, $M_{L_0}(k_1, k_3, h)$ as an $L_t$-module may be reducible. Since $\mathbb{C}_{k_{13},h}$ generate $M_{L_0}(k_1, k_3, h)$ as $L_t$-module, for any proper submodule $U$ of $M_{L_0}(k_1, k_3, h)$, $M_{L_0}(k_1, k_3, h)/U(h) = U \cap M_{L_0}(k_1, k_3, h)(h) = 0$. Hence there exists a maximal proper $L_t$-submodule $T_{L_0}(k_1, k_3, h)$. Set

$$L_{L_0}(k_1, k_3, h) = M_{L_0}(k_1, k_3, h)/T_{L_0}(k_1, k_3, h).$$

Then $L_{L_0}(k_1, k_3, h)$ is an irreducible $L_t$-module.

**Definition 4.3.** An $L_t$-module $W$ is said to be restricted if for any $w \in W$, $n \in \mathbb{Z}$, $L_n w = 0$ and $I_{n+\frac{1}{l}} w = 0$ for $n$ sufficiently large. We say an $L_t$-module $W$ is of level $k_{13}$ if the central element $k_i$ acts as scalar $k_i$ for $i = 1, 3$.

It is easy to see that $M_{L_0}(k_1, k_3, h)$, $L_{L_0}(k_1, k_3, h)$ are restricted $L_t$-modules of level $k_{13}$ for any $h \in \mathbb{C}$. Now we are going to relate $L_t$-modules with twisted $V_L(\ell_{123},0)$-modules. On the one hand, we have

**Theorem 4.4.** If $W$ is a restricted $L_t$-module of level $\ell_{13}$, then $W$ is a $\sigma_t$-twisted $V_L(\ell_{123},0)$-module for $V_L(\ell_{123},0)$ as a vertex algebra with

$$Y_{\sigma_t}(L_{-2} 1, z) = L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

$$Y_{\sigma_t}(I_{-1} 1, z) = I_{\sigma_t}(z) = \sum_{n \in \mathbb{Z}} I_{n+\frac{1}{l}} z^{-n-\frac{1}{l}-1}.$$  

**Proof.** Let $U_W = \{L(z), I_{\sigma_t}(z), 1_W\}$, where $1_W$ is the identity operator on $W$. Clearly, $L(z), I_{\sigma_t}(z)$ are $Z_t$-twisted weak vertex operators on $W$. From (4.4), (4.5), (4.6), using (2.1), we see that $L(z), I_{\sigma_t}(z)$ are mutually local $Z_t$-twisted vertex operators on $W$. Hence, by Corollary 3.15 of [14], $\langle U_W \rangle$ is a vertex algebra with $W$ a faithful $\sigma$-twisted module, where $\sigma$ is an order $t$ automorphism of the vertex algebra $\langle U_W \rangle$. To say that $W$ is a $\sigma_t$-twisted module for $V_L(\ell_{123},0)$, from Proposition 3.17 of [14], it suffices to show that there exists a vertex algebra homomorphism from $V_L(\ell_{123},0)$ to $\langle U_W \rangle$.

By Lemma 2.11 of [14], $Y(L(z), z_1)$ and $Y(I_{\sigma_t}(z), z_1)$ satisfy the twisted Heisenberg-Virasoro algebra relations (3.4), (3.5), (3.6). Then $\langle U_W \rangle$ is an $L$-module.
with \( L_n, I_n \) acting as \( L(z)_{n+1}, I_\sigma(z)_n \) for \( n \in \mathbb{Z} \), \( c_i \) acting as \( \ell_i \) with \( \ell_2 = 0 \), \( i = 1, 2, 3 \).

By the universal property of \( V_\mathcal{L}(\ell_{123}, 0) \) (c.f. Remark 2.7 of [12]), there exists a unique \( \mathcal{L} \)-module homomorphism

\[
\psi : V_\mathcal{L}(\ell_{123}, 0) \rightarrow \langle U_W \rangle; \quad 1 \mapsto 1_W.
\]

Then

\[
\psi(\omega_n v) = L(z)_n \psi(v) = \psi(\omega)_n \psi(v),
\]

\[
\psi(I_n v) = I_\sigma(z)_n \psi(v) = \psi(I)_n \psi(v)
\]

for all \( v \in V_\mathcal{L}(\ell_{123}, 0) \), \( n \in \mathbb{Z} \). Hence \( \psi \) is a vertex algebra homomorphism. Therefore, \( W \) is a weak \( \sigma \)-twisted \( V_\mathcal{L}(\ell_{123}, 0) \)-module with \( Y_\sigma(L_{-2}1, z) = L(z), Y_\sigma(I_{-1}1, z) = I_\sigma(z) \).

Conversely, we have

**Theorem 4.5.** If \( W \) is a \( \sigma \)-twisted \( V_\mathcal{L}(\ell_{123}, 0) (\ell_2 = 0) \)-module, then \( W \) is a restricted \( \mathcal{L} \)-module of level \( \ell_{13} \) with \( L(z) = Y_W(L_{-2}1, z) \), \( I_{\sigma}(z) = Y_W(I_{-1}1, z) \).

**Proof.** Let \( W \) be a \( \sigma \)-twisted \( V_\mathcal{L}(\ell_{123}, 0)(\ell_2 = 0) \)-module. Recall the following formula (c.f. (2.40) of [14])

\[
[Y_W(a, z_1), Y_W(b, z_2)] = \sum_{j=0}^{\infty} \frac{1}{j!} \left( \left( \frac{\partial}{\partial z_2} \right)^j z_1^{-1} \delta \left( \frac{z_2}{z_1} \right) \left( \frac{z_2}{z_1} \right)^j \right) Y_W(a^j b, z_2),
\]

where \( k \) is determined by \( a \). In our case, when \( a = L_{-2}1, k = 0 \), when \( a = I_{-1}1, k = 1 \). For \( a = b = L_{-2}1 \), we have (2.21) of [12]

\[
(L_{-2}1)_j L_{-2}1 = (j + 1) L_{j-3}1 + \delta_{j-3,0} \frac{(j - 1)^3 - (j - 1)}{12} c_1 1,
\]

so

\[
[Y_W(L_{-2}1, z_1), Y_W(L_{-2}1, z_2)] = Y_W(L_{-3}1, z_2) z_2^{-1} \delta \left( \frac{z_1}{z_2} \right)
\]

\[
+ 2 Y_W(L_{-2}1, z_2) \left( \frac{\partial}{\partial z_1} \right) z_2^{-1} \delta \left( \frac{z_1}{z_2} \right)
\]

\[
+ \frac{1}{12} \left( \frac{\partial}{\partial z_1} \right)^3 z_2^{-1} \delta \left( \frac{z_1}{z_2} \right) \ell_1 1.
\] (4.7)

For \( a = L_{-2}1, b = I_{-1}1 \), we have (2.22) of [12] with \( \ell_2 = 0 \)

\[
(L_{-2}1)_j I_{-1}1 = I_{j-2}1,
\]

so

\[
[Y_W(L_{-2}1, z_1), Y_W(I_{-1}1, z_2)] = Y_W(I_{-2}1, z_2) z_1^{-1} \delta \left( \frac{z_2}{z_1} \right) + Y_W(I_{-1}1, z_2) \left( \frac{\partial}{\partial z_2} \right) z_1^{-1} \delta \left( \frac{z_2}{z_1} \right).
\] (4.8)

For \( a = b = I_{-1}1 \), we have (2.23) of [12]

\[
(I_{-1}1)_j I_{-1}1 = j \delta_{j-1,0} c_3 1,
\]

so

\[
[Y_W(I_{-1}1, z_1), Y_W(I_{-1}1, z_2)] = \left( \frac{\partial}{\partial z_2} \right) z_1^{-1} \delta \left( \frac{z_2}{z_1} \right) \left( \frac{z_2}{z_1} \right)^j \ell_3 1.
\] (4.9)
Note that for \( t \neq 2 \), we have to require \( \ell_3 = 0 \) (Theorem 3.2). Therefore, with (4.4), (4.5) and (4.6), \( W \) is a \( \mathcal{L}_t \)-module with \( L(z) = Y_W(L_{-1}1, z) \), \( I_{\sigma_i}(z) = Y_W(L_{-1}1, z) \), \( k_i = \ell_i, i = 1, 3 \). Then \( W \) is restricted of level \( \ell_{13} \) is clear.

Let
\[
\mathcal{L}_t^{(0)} = \mathbb{C}L_0 \oplus \mathbb{C}k_1 \oplus \mathbb{C}k_3, \quad \mathcal{L}_t^{(n)} = \mathcal{C}L_{-n} \text{ for } 0 \neq n \in \mathbb{Z},
\]
\[
\mathcal{L}_t^{-\frac{1}{t} + n} = \mathcal{C}L_{-n + \frac{1}{t}}, \quad \mathcal{L}_t^{(k)} = 0 \text{ for } k \in \frac{1}{t} \mathbb{Z}, k \notin \mathbb{Z}, -\frac{1}{t} + \mathbb{Z}.
\]
Then \( \mathcal{L} = \prod_{n \in \mathbb{Z}} \mathcal{L}_t^{(\frac{1}{t})} \) is a \( \frac{1}{t} \mathbb{Z} \)-graded Lie algebra, and the grading is given by \( \text{ad}L_0 \)-eigenvalues.

By Theorem 4.4 and Theorem 4.5 we have the following result.

**Theorem 4.6.** The \( \sigma_t \)-twisted modules for \( V_L(\ell_{123}, 0) \) (\( \ell_2 = 0 \)) viewed as a vertex operator algebra (i.e. \( \mathbb{C} \)-graded by \( L_0 \)-eigenvalues and with the two grading restrictions (TW6), (TW7)) are exactly those restricted modules for the Lie algebra \( \mathcal{L}_t \) of level \( \ell_{13} \) that are \( \mathbb{C} \)-graded by \( L_0 \)-eigenvalues and with the two grading restrictions. Furthermore, for any \( \sigma_t \)-twisted \( V_L(\ell_{123}, 0) \)-module \( W \), the \( \sigma_t \)-twisted \( V_L(\ell_{123}, 0) \)-submodules of \( W \) are exactly the submodules of \( W \) for \( \mathcal{L}_t \), and these submodules are in particular graded.

Hence irreducible restricted \( \mathcal{L}_t \)-modules of level \( \ell_{13} \) corresponds to irreducible \( \sigma_t \)-twisted \( V_L(\ell_{123}, 0) \)-modules. Recall that for any \( h \in \mathbb{C} \), \( L_\mathcal{L}_t(\ell_1, \ell_3, h) \) is an irreducible restricted \( \mathcal{L}_t \)-module of level \( \ell_{13} \), so it is an irreducible \( \sigma_t \)-twisted \( V_L(\ell_{123}, 0) \)-module with \( \ell_2 = 0 \).

Now we give the complete list of irreducible \( \sigma_t \)-twisted \( V_L(\ell_{123}, 0) \)-modules.

**Theorem 4.7.** Let \( \ell_2 = 0, \ell_1, \ell_3 \in \mathbb{C} \). Then \( \{ L_\mathcal{L}_t(\ell_1, \ell_3, h) \mid h \in \mathbb{C} \} \) is a complete list of irreducible \( \sigma_t \)-twisted \( V_L(\ell_{123}, 0) \)-modules.

**Proof.** Let \( W = \prod_{r \in \mathbb{C}} W(r) \) be an irreducible \( \sigma_t \)-twisted \( V_L(\ell_{123}, 0) \)-module. By Theorem 4.6, \( W \) is an irreducible restricted \( \mathcal{L}_t \)-module of level \( \ell_{13} \). So \( k_i \) acts on \( W \) as a scalar \( \ell_i \) for \( i = 1, 3 \). From Remark 2.5, there exists \( h \in \mathbb{C} \) such that \( W(h) \neq 0 \) and \( W(h-n) = 0 \) for all \( n \in \frac{1}{t} \mathbb{Z}_{\geq 1} \). Let \( 0 \neq w \in W(h) \). Then
\[
L_0w = hw, \quad L_mw = 0, \quad I_{n+\frac{1}{t}}w = 0
\]
for \( m \geq 1, n \geq 0 \). In view of Remark 4.2, there is a unique \( \mathcal{L}_t \)-module homomorphism
\[
\varphi : M_{\mathcal{L}_t}(\ell_1, \ell_3, h) \rightarrow W
\]
such that \( \varphi(1_{k_1,h}) = w \). By Proposition 2.8, \( \varphi \) is a \( \sigma \)-twisted \( \langle U_W \rangle \)-module homomorphism (since \( \mathcal{L}_t \) generates the vertex algebra \( \langle U_W \rangle \)), where \( \sigma \) is an order \( t \) automorphism of the vertex algebra \( \langle U_W \rangle \). Recall that \( M_{\mathcal{L}_t}(\ell_1, \ell_3, h) \) is a weak \( \sigma_t \)-twisted \( V_L(\ell_{123}, 0) \)-module via the vertex algebra homomorphism \( \psi \) in Theorem 4.4. So \( \varphi \) can be viewed as a \( \sigma_t \)-twisted \( V_L(\ell_{123}, 0) \)-module homomorphism. Since \( W \) is irreducible and \( T_{\mathcal{L}_t}(\ell_1, \ell_3, h) \) is the (unique) largest proper submodule of \( M_{\mathcal{L}_t}(\ell_1, \ell_3, h) \), it follows that
\[
\varphi(M_{\mathcal{L}_t}(\ell_1, \ell_3, h)) = W
\]
and
\[
\ker \varphi = T_{\mathcal{L}_t}(\ell_1, \ell_3, h).
\]
Thus \( \varphi \) reduces to a \( \sigma_t \)-twisted \( V_L(\ell_{123}, 0) \)-module isomorphism from \( L_{\mathcal{L}_t}(\ell_1, \ell_3, h) \) to \( W \).
It is interesting and important to classify the irreducible modules for the fixed point subalgebra $V_\ell(\ell_{123},0)^\sigma := \{ v \in V_\ell(\ell_{123},0) \mid \sigma_t(v) = v \}$, $t \in \mathbb{Z}_{\geq 1}$. In the case of $\ell_2 = 0$ and $\ell_3 \neq 0$, we have $t = 2$. Then $\sigma_2$ is the order 2 automorphism of $V_\ell(\ell_{123},0)$ which is induced from its Heisenberg vertex operator subalgebra. Denote by $\sigma_2 = \sigma$. Precisely, the automorphism

$$\sigma : V_\ell(\ell_{123},0) \rightarrow V_\ell(\ell_{123},0)$$

is defined on the basis elements by

$$I_{-k_1} \cdots I_{-k_s} L_{-m_1} \cdots L_{-m_l} 1 \mapsto (-1)^s I_{-k_1} \cdots I_{-k_s} L_{-m_1} \cdots L_{-m_l} 1,$$

and extended linearly, where $r, s \geq 1$, $m_1 \geq \cdots \geq m_r \geq 2$, $k_1 \geq \cdots \geq k_s \geq 1$. Let $V_\ell(\ell_{123},0)^+ = \{ v \in V_\ell(\ell_{123},0) \mid \sigma(v) = v \}$ be the fixed point subalgebra under $\sigma$.

For $\ell_3 \neq 0$, denote by

$$c_{\bar{V}_tir} = \ell_1 - 1 + 12\frac{\ell_2^2}{\ell_3}.$$ 

Let $V_H(\ell_3,0)$ be the vertex operator algebra constructed from the Heisenberg subalgebra $H$ which is equipped with the nonstandard conformal vector $\omega_H = \frac{1}{2\ell_3} I_{-1} I_{-1} 1 + \frac{\ell_2^2}{\ell_3^3} I_{-2} 1$ (of central charge $1 - 12\frac{\ell_2^2}{\ell_3^3}$). Let $\bar{V}_tir$ be the Virasoro algebra constructed by $\bar{\omega} = \omega - \omega_H$. Let $V_{\bar{V}_tir}(c_{\bar{V}_tir},0)$ be the corresponding Virasoro vertex operator algebra. Recall that when $\ell_3 \neq 0$, we have (cf. Theorem 3.16 of [12])

$$V_\ell(\ell_{123},0) \cong V_H(\ell_3,0) \otimes V_{\bar{V}_tir}(c_{\bar{V}_tir},0)$$

as vertex operator algebras.

It is well known that there exists an order 2 isomorphism of the Heisenberg vertex operator algebra $V_H(\ell_3,0)$. Let again

$$\sigma : V_H(\ell_3,0) \rightarrow V_H(\ell_3,0)$$

be the order 2 automorphism which is defined on the basis elements by

$$I_{-k_1} \cdots I_{-k_s} 1 \mapsto (-1)^s I_{-k_1} \cdots I_{-k_s} 1$$

and extended linearly, where $s \geq 1$, $k_1 \geq \cdots \geq k_s \geq 1$. The fixed point subalgebra $V_H(\ell_3,0)^+ = \{ v \in V_H(\ell_3,0) \mid \sigma_2(v) = v \}$ has been extensively studied (cf. [6] etc.).

Then it is immediate to see that when $\ell_3 \neq 0$ and $\ell_2 = 0$, we have an isomorphism of vertex operator algebras

$$V_\ell(\ell_{123},0)^+ \cong V_H(\ell_3,0)^+ \otimes V_{\bar{V}_tir}(c_{\bar{V}_tir},0).$$

Up to isomorphism, irreducible modules for the vertex operator algebra $V_H(\ell_3,0)^+$ are (cf. [6], etc.) $V_H(\ell_3,0)^\pm$, $V_H(\ell_3,0)(\sigma)^\pm$, $V_H(\ell_3,\lambda) \cong V_H(\ell_3,-\lambda)$ for any $0 \neq \lambda \in \mathbb{C}$. Up to isomorphism, irreducible modules for the Virasoro vertex operator algebra $V_{\bar{V}_tir}(c_{\bar{V}_tir},0)$ are $L_{\bar{V}_tir}(c_{\bar{V}_tir},h)$ for all $h \in \mathbb{C}$ (cf. [9], [16], etc.). Therefore, in the case of $\ell_2 = 0$ and $\ell_3 \neq 0$, irreducible modules of $V_\ell(\ell_{123},0)^+$ are one-to-one correspond to the tensor product of irreducible modules of $V_H(\ell_3,0)^+$ and irreducible modules of $V_{\bar{V}_tir}(c_{\bar{V}_tir},0)$ (cf. Proposition 4.7.2 and Theorem 4.7.4 of [7]).

For other automorphism $\sigma_3$, the complete list of irreducible $V_\ell(\ell_{123},0)^{\sigma_3}$-modules remains to be investigated.
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