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AUTOMORPHISM GROUP AND TWISTED MODULES OF THE TWISTED HEISENBERG-VIRASORO VERTEX OPERATOR ALGEBRA

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Abstract. We first determine the automorphism group of the twisted Heisenberg-Virasoro vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$. Then, for any integer $t > 1$, we introduce a new Lie algebra $\mathcal{L}_t$, and show that σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)(\ell_2 = 0)$-modules are in one-to-one correspondence with restricted $\mathcal{L}_t$-modules of level ℓ_{13}, where σ_t is an order t automorphism of $V_{\mathcal{L}}(\ell_{123}, 0)$. At the end, we give a complete list of irreducible σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)(\ell_2 = 0)$-modules.

1. **Introduction.** Let $\mathcal{L}$ be the twisted Heisenberg-Virasoro algebra. It is the universal central extension of the Lie algebra of differential operators on a circle of order at most one (cf. [3]):

$$\{f(t)\frac{d}{dt} + g(t) \mid f(t), g(t) \in \mathbb{C}[t, t^{-1}]\}.$$

$\mathcal{L}$ contains an infinite-dimensional Heisenberg algebra and the Virasoro algebra as subalgebras (cf. [3], [4]). The induced module $V_{\mathcal{L}}(\ell_{123}, 0) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_{(\leq 1)})} \mathbb{C}_{\ell_{123}}$ is a vertex operator algebra of central charge ℓ_1 with conformal vector $\omega = L_{-2}\mathbf{1}$ (cf. [12]). $V_{\mathcal{L}}(\ell_{123}, 0)$ is a nonrational vertex operator algebra and is not C_2-cofinite. The structure theory and representation theory of the twisted Heisenberg-Virasoro vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$ are closely related to the three scalars ℓ_1, ℓ_2, ℓ_3 (cf. [3], [1], [2], [4], [5], [8], [12], etc.).

Determining the automorphism group $\mathrm{Aut}(V)$ of a vertex operator algebra V is an important subject in vertex operator algebra theory. It is related to the orbifold theory which studies the fixed point subalgebras of vertex operator algebras and their modules under certain finite subgroups of the full automorphism groups. The orbifold conjecture says that under some conditions on V, every simple V^G-module is contained in some g-twisted V-module, where G is a finite subgroup of $\mathrm{Aut}(V)$, $g \in G$, V^G is the fixed point subalgebra of V under the group G.

The automorphism group of the twisted Heisenberg-Virasoro algebra $\mathcal{L}$ has been studied in [15]. In this paper, we study the automorphism group of the twisted Heisenberg-Virasoro vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$. We know that $V_{\mathcal{L}}(\ell_{123}, 0)$ is generated by $\omega = L_{-2}\mathbf{1}$ and $I_{-1}\mathbf{1}$. By definition, any homomorphism of a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ takes ω to ω. Therefore, it suffices to determine the action on the generator $I_{-1}\mathbf{1}$. It turns out that automorphisms of the twisted

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Heisenberg-Virasoro vertex operator algebra $V_{\mathcal{L}}(\ell_{123},0)$ depend on the numbers ℓ_2 and ℓ_3, and there exists automorphism of order other than 2, which makes the study of twisted modules for $V_{\mathcal{L}}(\ell_{123},0)$ interesting.

In [12], all irreducible modules for the vertex operator algebra $V_{\mathcal{L}}(\ell_{123},0)$ are classified: that is, every irreducible module for $V_{\mathcal{L}}(\ell_{123},0)$ is isomorphic to some $L_{\mathcal{L}}(\ell_{123},h_1,h_2)$, $h_1,h_2\in\mathbb{C}$. In this paper, for any integer $t>1$, we classify σ_t-twisted irreducible modules for $V_{\mathcal{L}}(\ell_{123},0)$, where σ_t is an order t automorphism of $V_{\mathcal{L}}(\ell_{123},0)$. The way we deal with this problem is similar to the one used for modules of vertex algebra (cf. [10], [11], [12], [13], etc.), but in the context of twisted modules. We first introduce another Lie algebra $\mathcal{L}_t$. It is a Lie algebra with basis $\{L_n, I_{n+\frac{1}{t}}, k_1, k_3 \mid n\in\mathbb{Z}\}$, and Lie brackets

$$[L_m,L_n]=(m-n)L_{m+n}+\delta_{m+n,0}\frac{m^3-m}{12}k_1,$$

$$[L_m,I_{n+\frac{1}{t}}]=-(n+\frac{1}{t})I_{m+n+\frac{1}{t}},$$

$$[I_{m+\frac{1}{t}},I_{n+\frac{1}{t}}]=(m+\frac{1}{t})\delta_{m+n+\frac{2}{t},0}\delta_{t,2}k_3,\quad [\mathcal{L}_t,k_i]=0,\ i=1,3.$$

Note that when $t\neq 2$, $\{I_{n+\frac{1}{t}}\mid n\in\mathbb{Z}\}$ is an abelian Lie algebra. Then we construct irreducible $\mathcal{L}_t$-modules $L_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)$ as quotient of the induced modules $M_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)$, where $\Bbbk_1,\Bbbk_3,h\in\mathbb{C}$. We show that σ_t-twisted $V_{\mathcal{L}}(\ell_{123},0)$-modules ($\ell_2=0$) are in one-to-one correspondence with restricted $\mathcal{L}_t$-modules of level ℓ_{13}. Using this result, we get a complete list of irreducible σ_t-twisted $V_{\mathcal{L}}(\ell_{123},0)$-modules, where $\ell_2=0$, ℓ_1, $\ell_3\in\mathbb{C}$. Let $V_{\mathcal{L}}(\ell_{123},0)^{\sigma_t}$ be the fixed point subalgebra of $V_{\mathcal{L}}(\ell_{123},0)$ under the automorphism σ_t. $V_{\mathcal{L}}(\ell_{123},0)^{\sigma_t}$ is a vertex operator subalgebra of $V_{\mathcal{L}}(\ell_{123},0)$. It is important and meaningful to study the representations of the fixed point subalgebra $V_{\mathcal{L}}(\ell_{123},0)^{\sigma_t}$. We remark at the end of the paper that except for the case of order 2, the complete list of irreducible modules for $V_{\mathcal{L}}(\ell_{123},0)^{\sigma_t}$ needs to be further investigated.

This paper is organized as follows. In Section 2, we review the notions and some results of vertex operator algebras, automorphisms and twisted modules for vertex operator algebras. In Section 3, we study the automorphism group of the twisted Heisenberg-Virasoro vertex operator algebra $V_{\mathcal{L}}(\ell_{123},0)$. In Section 4, we first study the twisted modules of $V_{\mathcal{L}}(\ell_{123},0)(\ell_2=0)$ under an order t automorphism σ_t for any integer $t>1$. Then we give a complete list of irreducible σ_t-twisted $V_{\mathcal{L}}(\ell_{123},0)(\ell_2=0)$-modules.

2. **Preliminaries.** For later use, we recall the following result (cf. Proposition 2.3.7 of [13]).

Lemma 2.1.

$$(x_1-x_2)^m\Big(\frac{\partial}{\partial x_2}\Big)^n x_1^{-1}\delta\Big(\frac{x_2}{x_1}\Big)=0 \tag{2.1}$$

for $m>n$, $m,n\in\mathbb{N}$, where $\delta\Big(\frac{x_1}{x_2}\Big)=\sum_{n\in\mathbb{Z}}x_1^n x_2^{-n}$.

Let $(V,Y,\omega,\mathbf{1})$ be a vertex operator algebra (cf. Definition 3.1.22 of [13]). Then, a priori, $(V,Y,\mathbf{1})$ is a vertex algebra (cf. Definition 3.1.1 of [13]). Let D be the endomorphism of the vertex algebra V defined by $D(v)=v_{-2}\mathbf{1}$ for $v\in V$. We have

$Y(Dv,x)=\dfrac{d}{dx}Y(v,x)$. Denote by $L(n)=\omega_{n+1}$ for any $n\in\mathbb{Z}$. It can be proved that $\mathbf{1}\in V_{(0)}$ and $D=L(-1)$.

Definition 2.2. An *automorphism* of a vertex operator algebra $(V,Y,\omega,\mathbf{1})$ is a linear isomorphism σ of V such that

$$\sigma(\mathbf{1})=\mathbf{1},\ \ \sigma(\omega)=\omega\ \text{ and }\ \sigma(u_nv)=\sigma(u)_n\sigma(v)$$

for any $u,v\in V$, $n\in\mathbb{Z}$.

The group of all automorphisms of a vertex operator algebra V is denoted by $\mathrm{Aut}(V)$. Any automorphism of a vertex operator algebra V is *grading-preserving*, i.e. it preserves each homogeneous subspace $V_{(n)}$ of V, $n\in\mathbb{Z}$. Let σ be an order t automorphism of a vertex operator algebra V, t is a positive integer. Then σ acts semisimply on V. Therefore

$$V=V^0\oplus V^1\oplus\cdots\oplus V^{t-1},$$

where V^k is the eigenspace of V for σ with eigenvalues η^k, where $\eta=\exp(\frac{2\pi\sqrt{-1}}{t})$, $k=0,\dots,t-1$. It is easy to see that the fixed points set $V^0:=V^\sigma=\{v\in V\mid\sigma(v)=v\}$ is a vertex operator subalgebra of V

Now we recall some notions regarding to twisted modules from [14].

Definition 2.3. Let $(V,\mathbf{1},Y)$ be a vertex algebra with an automorphism σ of order t. A *σ-twisted V-module* is a triple (W,d,Y_W) consisting of a vector space W, an endomorphism d of W and a linear map

$$Y_W(\cdot,z):V\longrightarrow(\mathrm{End}W)[[z^{\frac{1}{t}},z^{-\frac{1}{t}}]]$$

satisfying the following conditions:

(TW1) For any $v\in V,w\in W$, $v_nw=0$ for $n\in\frac{1}{t}\mathbb{Z}$ sufficiently large;
(TW2) $Y_W(\mathbf{1},z)=Id_W$;
(TW3) $[d,Y_W(v,z)]=Y_W(D(v),z)=\dfrac{d}{dz}Y_W(v,z)$ for any $v\in V$;
(TW4) For any $u,v\in V$, the following *σ-twisted Jacobi identity* holds:

$$\begin{aligned}z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)&Y_W(u,z_1)Y_W(v,z_2)-z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_W(v,z_2)Y_W(u,z_1)\\&=z_2^{-1}\frac{1}{t}\sum_{k=0}^{t-1}\delta\left(\left(\frac{z_1-z_0}{z_2}\right)^{\frac{1}{t}}\right)Y_W(Y(\sigma^ku,z_0)v,z_2).\end{aligned}$$

If V is a vertex operator algebra, a σ-twisted V-module for V as a vertex algebra is called a *σ-twisted weak module for V* as a vertex operator algebra.

Definition 2.4. For V a vertex operator algebra, a *σ-twisted V-module* W is a σ-twisted weak module for V as a vertex algebra and $W=\coprod_{h\in\mathbb{C}}W_{(h)}$ such that

(TW5) $L(0)w=hw$ for $h\in\mathbb{C},w\in W_{(h)}$;
(TW6) For any fixed $h,W_{(h+n)}=0$ for $n\in\frac{1}{t}\mathbb{Z}$ sufficiently small;
(TW7) $\dim W_{(h)}<\infty$ for any $h\in\mathbb{C}$.

Remark 2.5. Let W be a σ-twisted V-module. Then

$$[L(0),Y_W(v,z)]=zY_W(L(-1)v,z)+Y_W(L(0)v,z)\tag{2.2}$$

for $v \in V$. Hence, if $v \in V$ is homogeneous,

$$v_n W_{(h)} \subseteq W_{(h+wt\ v-n-1)} \text{ for } n \in \frac{1}{t}\mathbb{Z}, h \in \mathbb{C}.$$

It follows that a σ-twisted V-module W decomposes into twisted submodules corresponding to the congruence classes mod $\frac{1}{t}\mathbb{Z}$: For $h \in \mathbb{C}/\frac{1}{t}\mathbb{Z}$, let

$$W_{[h]} = \coprod_{\alpha+\frac{1}{t}\mathbb{Z}=h} W_{(\alpha)}. \tag{2.3}$$

Then

$$W = \coprod_{h\in\mathbb{C}/\frac{1}{t}\mathbb{Z}} W_{[h]}. \tag{2.4}$$

In particular, if W is irreducible, then

$$W = W_{[h]} \tag{2.5}$$

for some h.

Remark 2.6. Let W be a σ-twisted V-module. For $u \in V^k$, $v \in V$, there is the following *twisted iterate formula* (cf. (2.32) of [14])

$$Y_W(Y(u,z_0)v,z_2) = \mathrm{Res}_{z_1}\left(\frac{z_1-z_0}{z_2}\right)^{\frac{k}{t}} \cdot X, \tag{2.6}$$

where

$$X = z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_W(u,z_1)Y_W(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_W(v,z_2)Y_W(u,z_1).$$

Then it can be easily deduced that for any $n \in \mathbb{Z}$,

$$Y_W(u_n v,z) = \mathrm{Res}_{z_1}\sum_{j=0}^{\infty}(-1)^j\binom{\frac{k}{t}}{j}z_1^{\frac{k}{t}-j}z^{-\frac{k}{t}} \cdot Y_j, \tag{2.7}$$

where

$$Y_j = \Big((z_1-z)^{n+j}Y_W(u,z_1)Y_W(v,z) - (-z+z_1)^{n+j}Y_W(v,z)Y_W(u,z_1)\Big),$$

$\binom{\frac{k}{t}}{j} = \dfrac{\frac{k}{t}(\frac{k}{t}-1)\cdots(\frac{k}{t}-j+1)}{j!}$. Note that when $k=0$, $\dfrac{k}{t}=0$, we have $j=0$, and then (2.7) is the usual formula for (untwisted) V-modules (cf. (3.8.16) of [13]).

Definition 2.7. A *homomorphism* between two σ-twisted weak V-modules M and W is a linear map $f: M \longrightarrow W$ such that for any $v \in V$,

$$fY_M(v,z) = Y_W(v,z)f. \tag{2.8}$$

If V is a vertex operator algebra, then a V-module homomorphism f is compatible with the gradings:

$$f(M_{(h)}) \subseteq W_{(h)} \text{ for } h \in \mathbb{C}. \tag{2.9}$$

Using the formula (2.7), similarly as Proposition 4.5.1 of [13], there is the following result.

Proposition 2.8. *Let W_1 and W_2 be σ-twisted V-modules and let $\psi \in Hom_{\mathbb{C}}(W_1, W_2)$. Suppose that*

$$Y(a,z)\psi = \psi Y(a,z) \ \textit{ for } a \in S,$$

where S is a given generating set of V. Then ψ is a σ-twisted V-module homomorphism.

In the following, we review from Section 3 of [14] the local systems of twisted vertex operators.

Definition 2.9. Let W be a vector space, let t be a fixed positive integer. A $\mathbb{Z}_t$*-twisted weak vertex operator on* W is a formal series $a(z) = \sum_{n\in\frac{1}{t}\mathbb{Z}} a_n z^{-n-1} \in (\mathrm{End}\, W)[[z^{\frac{1}{t}}, z^{-\frac{1}{t}}]]$ such that for any $w \in W$, $a_n w = 0$ for $n \in \frac{1}{t}\mathbb{Z}$ sufficiently large.

Definition 2.10. Two $\mathbb{Z}_t$-twisted weak vertex operators $a(z)$ and $b(z)$ are said to be *mutually local* if there is a positive integer n such that

$$(z_1 - z_2)^n a(z_1)b(z_2) = (z_1 - z_2)^n b(z_2)a(z_1).$$

A $\mathbb{Z}_t$-twisted weak vertex operator is called a $\mathbb{Z}_t$*-twisted vertex operator* if it is local with itself.

Denote by $F(W,t)$ the space of all $\mathbb{Z}_t$-twisted weak vertex operators on W. Let σ be the endomorphism of $(\mathrm{End}\, W)[[z^{\frac{1}{t}}, z^{-\frac{1}{t}}]]$ defined by: $\sigma f(z^{\frac{1}{t}}) = f(\eta^{-1} z^{\frac{1}{t}})$. Denote by $F(W,t)^k = \{f(z) \in F(W,t) \mid \sigma f(z) = \eta^k f(z)\}$ for $0 \le k \le t-1$. For any mutually local $\mathbb{Z}_t$-twisted vertex operators $a(z), b(z)$ on W, define $a(z)_n b(z)$ as follows (cf. Definition 3.7 of [14]).

Definition 2.11. Let W be a vector space and let $a(z)$ and $b(z)$ be mutually local $\mathbb{Z}_t$-twisted vertex operators on W such that $a(z) \in F(W,t)^k$. Then for any integer n we define $a(z)_n b(z)$ as an element of $F(W,t)$ as follows:

$$a(z)_n b(z) = Res_{z_1} Res_{z_0} \left(\frac{z_1 - z_0}{z}\right)^{\frac{k}{t}} z_0^n \cdot X, \tag{2.10}$$

where

$$X = z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right) a(z_1)b(z) - z_0^{-1}\delta\left(\frac{z - z_1}{-z_0}\right) b(z)a(z_1).$$

For any set S of mutually local $\mathbb{Z}_t$-twisted vertex operators on W, by Zorn's Lemma, there exists a local system A of $\mathbb{Z}_t$-twisted vertex operators on W (cf. Section 3 of [14]). Denote by $\langle S\rangle$ the vertex algebra generated by S inside A via the operations $a(z)_n b(z)$, $n \in \mathbb{Z}$.

3. **Automorphism group.**

In this section, we first recall the definition of the twisted Heisenberg-Virasoro algebra $\mathcal{L}$ and the construction of the twisted Heisenberg-Virasoro vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$. Then we determine the automorphism group of the vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$.

Recall the definition of the twisted Heisenberg-Virasoro algebra $\mathcal{L}$ (see [3] or [4]).

Definition 3.1. The twisted Heisenberg-Virasoro algebra $\mathcal{L}$ is a Lie algebra with basis $\{L_n, I_n, c_1, c_2, c_3 \mid n \in \mathbb{Z}\}$, and the following Lie brackets:

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3 - m}{12}c_1, \tag{3.1}$$

$$[L_m, I_n] = -nI_{m+n} - \delta_{m+n,0}(m^2 + m)c_2, \tag{3.2}$$

$$[I_m, I_n] = m\delta_{m+n,0}c_3, \quad [\mathcal{L}, c_i] = 0,\ i = 1, 2, 3. \tag{3.3}$$

Clearly, $\mathrm{Span}\{L_n,\ c_1 \mid n \in \mathbb{Z}\}$ is a Virasoro algebra, $\mathrm{Span}\{I_n,\ c_3 \mid n \in \mathbb{Z}\backslash\{0\}\}$ is an infinite-dimensional Heisenberg algebra, we denote them by Vir, $\mathcal{H}$ respectively.

Let

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad I(z) = \sum_{n\in\mathbb{Z}} I_n z^{-n-1},$$

then the defining relations of $\mathcal{L}$ become to be

$$\begin{aligned}
&[L(z_1), L(z_2)]\\
&= \sum_{m,n\in\mathbb{Z}} (m-n)L_{m+n} z_1^{-m-2} z_2^{-n-2} + \sum_{m\in\mathbb{Z}} \frac{m^3-m}{12} c_1 z_1^{-m-2} z_2^{m-2}\\
&= \frac{d}{dz_2}\Big(L(z_2)\Big) z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) + 2L(z_2)\frac{\partial}{\partial z_2} z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\\
&\quad + \frac{c_1}{12}\left(\frac{\partial}{\partial z_2}\right)^3 z_1^{-1}\delta\left(\frac{z_2}{z_1}\right),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&[L(z_1), I(z_2)]\\
&= -\sum_{m,n\in\mathbb{Z}} nI_{m+n} z_1^{-m-2} z_2^{-n-1} - \sum_{m\in\mathbb{Z}} (m^2+m)c_2 z_1^{-m-2} z_2^{m-1}\\
&= \frac{d}{dz_2}\Big(I(z_2)\Big) z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) + I(z_2)\frac{\partial}{\partial z_2} z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\\
&\quad - \left(\frac{\partial}{\partial z_2}\right)^2 z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) c_2,
\end{aligned} \tag{3.5}$$

$$[I(z_1), I(z_2)] = \sum_{m\in\mathbb{Z}} mc_3 z_1^{-m-1} z_2^{m-1} = \frac{\partial}{\partial z_2} z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) c_3. \tag{3.6}$$

We recall the construction of the twisted Heisenberg-Virasoro vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$ from [12]. Let

$$\mathcal{L}_{(\le 1)} = \coprod_{n\le 1} \mathbb{C}L_{-n} \oplus \coprod_{n\le 0} \mathbb{C}I_{-n} \oplus \sum_{i=1}^{3} \mathbb{C}c_i,$$

$$\mathcal{L}_{(\ge 2)} = \coprod_{n\ge 2} \mathbb{C}L_{-n} \oplus \coprod_{n\ge 1} \mathbb{C}I_{-n}.$$

They are graded subalgebras of $\mathcal{L}$ and

$$\mathcal{L} = \mathcal{L}_{(\le 1)} \oplus \mathcal{L}_{(\ge 2)}.$$

Let $\ell_i, i = 1, 2, 3$, be any complex numbers. Consider $\mathbb{C}$ as an $\mathcal{L}_{(\le 1)}$-module with c_i acting as the scalar $\ell_i, i = 1, 2, 3$, and with $\coprod_{n\le 1} \mathbb{C}L_{-n} \oplus \coprod_{n\le 0} \mathbb{C}I_{-n}$ acting trivially. Denote this $\mathcal{L}_{(\le 1)}$-module by $\mathbb{C}_{\ell_{123}}$. Form the induced module

$$V_{\mathcal{L}}(\ell_{123}, 0) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_{(\le 1)})} \mathbb{C}_{\ell_{123}}, \tag{3.7}$$

where $U(\cdot)$ denotes the universal enveloping algebra of a Lie algebra. Set $\mathbf{1} = 1 \otimes 1 \in V_{\mathcal{L}}(\ell_{123}, 0)$. $V_{\mathcal{L}}(\ell_{123}, 0)$ is a vertex operator algebra with vacuum vector $\mathbf{1}$

and conformal vector $\omega = L_{-2}\mathbf{1}$. And $\{\omega = L_{-2}\mathbf{1}, I := I_{-1}\mathbf{1}\}$ is a generating subset of $V_{\mathcal{L}}(\ell_{123}, 0)$. Recall the grading on $V_{\mathcal{L}}(\ell_{123}, 0)$:

$$V_{\mathcal{L}}(\ell_{123}, 0) = \coprod_{n\geq 0} V_{\mathcal{L}}(\ell_{123}, 0)_{(n)},$$

where $V_{\mathcal{L}}(\ell_{123}, 0)_{(0)} = \mathbb{C}$ and $V_{\mathcal{L}}(\ell_{123}, 0)_{(n)}$, $n \geq 1$, has a basis consisting of the vectors

$$I_{-k_1} \cdots I_{-k_s} L_{-m_1} \cdots L_{-m_r}\mathbf{1}$$

for $r, s \geq 0$, $m_1 \geq \cdots \geq m_r \geq 2$, $k_1 \geq \cdots \geq k_s \geq 1$ with $\sum\limits_{i=1}^{r} m_i + \sum\limits_{j=1}^{s} k_j = n$.

Now we give our first main result. The automorphism group of $V_{\mathcal{L}}(\ell_{123}, 0)$ is determined in the following theorem.

Theorem 3.2. *(1) If $\ell_2 \neq 0$, then $Aut(V_{\mathcal{L}}(\ell_{123}, 0)) = \{id\}$.*

(2) If $\ell_2 = 0$ and $\ell_3 \neq 0$, then $Aut(V_{\mathcal{L}}(\ell_{123}, 0)) \cong \mathbb{Z}_2$.

(3) If both ℓ_2 and ℓ_3 are 0, then $Aut(V_{\mathcal{L}}(\ell_{123}, 0)) \cong \mathbb{C}^{\times} = \mathbb{C}\backslash\{0\}$.

Proof. Let

$$\varphi : V_{\mathcal{L}}(\ell_{123}, 0) \longrightarrow V_{\mathcal{L}}(\ell_{123}, 0)$$

be an automorphism of the vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$. Then $\varphi(\mathbf{1}) = \mathbf{1}$ and $\varphi(\omega) = \omega$. Since $V_{\mathcal{L}}(\ell_{123}, 0)$ is generated by ω and $I = I_{-1}\mathbf{1}$, it suffices to determine $\varphi(I)$. φ is grading-preserving, so $\varphi(I) = aI$ for some $a \in \mathbb{C}^{\times}$.

Then, on the one hand, we have

$$\varphi(L_1 I) = aL_1 I = a[L_1, I_{-1}]\mathbf{1} = -2a\ell_2\mathbf{1},$$

on the other hand, we have

$$\varphi(L_1 I) = \varphi([L_1, I_{-1}]\mathbf{1}) = -2\ell_2\mathbf{1}.$$

Therefore if $\ell_2 \neq 0$, we get that $a = 1$, i.e. when $\ell_2 \neq 0$, $\mathrm{Aut}(V_{\mathcal{L}}(\ell_{123}, 0)) = \{id\}$ only consists of the identity map.

Suppose now $\ell_2 = 0$. Let's consider $\varphi(I_1 I)$. On the one hand,

$$\varphi(I_1 I) = \varphi(I)_1\varphi(I) = a^2 I_1 I_{-1}\mathbf{1} = a^2[I_1, I_{-1}]\mathbf{1} = a^2\ell_3\mathbf{1},$$

on the other hand,

$$\varphi(I_1 I) = \varphi([I_1, I_{-1}]\mathbf{1}) = \ell_3\mathbf{1}.$$

So if $\ell_3 \neq 0$, we get $a^2 = 1$, i.e. $a = 1$ or $a = -1$, then $\mathrm{Aut}(V_{\mathcal{L}}(\ell_{123}, 0)) \cong \mathbb{Z}_2$.

Now let $\ell_2 = 0$ and $\ell_3 = 0$, then a can be any nonzero complex number, so $\mathrm{Aut}(V_{\mathcal{L}}(\ell_{123}, 0)) \cong \mathbb{C}^{\times}$. □

4. **σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-modules.** In this section, we always assume $\ell_2 = 0$. We study twisted modules for the vertex operator algebra $V_{\mathcal{L}}(\ell_{123}, 0)$. More precisely, for any integer $t > 1$, we introduce an infinite-dimensional Lie algebra $\mathcal{L}_t$. We show that there is a one-to-one correspondence between restricted $\mathcal{L}_t$-modules of level ℓ_{13} and σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-modules, where σ_t is an order t automorphism of $V_{\mathcal{L}}(\ell_{123}, 0)$. And we give a complete list of irreducible σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-modules.

Note that if $\ell_2 = 0$ and $\ell_3 \neq 0$, then t can only be the integer 2 (Theorem 3.2). As we need, we introduce the following Lie algebra.

Definition 4.1. Let $\mathcal{L}_t$ be a Lie algebra with basis $\{L_n, I_{n+\frac{1}{t}}, k_1, k_3 \mid n \in \mathbb{Z}\}$, and the Lie brackets are given by:

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}k_1, \tag{4.1}$$

$$[L_m, I_{n+\frac{1}{t}}] = -(n+\frac{1}{t})I_{m+n+\frac{1}{t}}, \tag{4.2}$$

$$[I_{m+\frac{1}{t}}, I_{n+\frac{1}{t}}] = (m+\frac{1}{t})\delta_{m+n+\frac{2}{t},0}\delta_{t,2}k_3, \quad [\mathcal{L}_t, k_i] = 0,\ i = 1, 3. \tag{4.3}$$

Note that if $t \neq 2$, then $[I_{m+\frac{1}{t}}, I_{n+\frac{1}{t}}] = 0$ for any $m, n \in \mathbb{Z}$.

Form the generating function as

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad I_{\sigma_t}(z) = \sum_{n\in\mathbb{Z}} I_{n+\frac{1}{t}} z^{-n-\frac{1}{t}-1}.$$

Then the defining relations (4.1), (4.2), (4.3) amount to:

$$\begin{aligned} &[L(z_1), L(z_2)] \\ &= \frac{d}{dz_2}(L(z_2))z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) + 2L(z_2)\frac{\partial}{\partial z_2}z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) \\ &\quad + \frac{1}{12}\left(\frac{\partial}{\partial z_2}\right)^3 z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)k_1, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &[L(z_1), I_{\sigma_t}(z_2)] \\ &= -\sum_{m,n\in\mathbb{Z}}(n+\frac{1}{t})I_{m+n+\frac{1}{t}}z_1^{-m-2}z_2^{-n-\frac{1}{t}-1} \\ &= \frac{d}{dz_2}\Big(I_{\sigma_t}(z_2)\Big)z_1^{-1}\delta\left(\frac{z_2}{z_1}\right) + I_{\sigma_t}(z_2)\frac{\partial}{\partial z_2}z_1^{-1}\delta\left(\frac{z_2}{z_1}\right), \end{aligned} \tag{4.5}$$

$$\begin{aligned} [I_{\sigma_t}(z_1), I_{\sigma_t}(z_2)] &= \sum_{m\in\mathbb{Z}}(m+\frac{1}{t})z_1^{-m-\frac{1}{t}-1}z_2^{m+\frac{1}{t}-1}\delta_{t,2}k_3 \\ &= \frac{\partial}{\partial z_2}\left(z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\left(\frac{z_2}{z_1}\right)^{\frac{1}{t}}\right)\delta_{t,2}k_3. \end{aligned} \tag{4.6}$$

Now we construct irreducible $\mathcal{L}_t$-modules (cf. [12], [13], etc.). Let

$$(\mathcal{L}_t)_{\geq 0} = (\coprod_{m\geq 0}\mathbb{C}L_m) \oplus (\coprod_{n\geq 0}\mathbb{C}I_{n+\frac{1}{t}}) \oplus \mathbb{C}k_1 \oplus \mathbb{C}k_3.$$

It is a subalgebra of $\mathcal{L}_t$.

Let $\mathbb{C}$ be an $(\mathcal{L}_t)_{\geq 0}$-module, where L_m, $I_{n+\frac{1}{t}}$ act trivially for all $m \geq 1$, $n \geq 0$, and L_0, k_1, k_3 act as scalar multiplications by $h, \Bbbk_1, \Bbbk_3$ respectively. Denote this $(\mathcal{L}_t)_{\geq 0}$-module by $\mathbb{C}_{\Bbbk_{13},h}$. Form the induced module

$$M_{\mathcal{L}_t}(\Bbbk_1, \Bbbk_3, h) = U(\mathcal{L}_t) \otimes_{U((\mathcal{L}_t)_{\geq 0})} \mathbb{C}_{\Bbbk_{13},h}.$$

Set

$$\mathbf{1}_{\Bbbk_{13},h} = 1 \in \mathbb{C}_{\Bbbk_{13},h} \subset M_{\mathcal{L}_t}(\Bbbk_1, \Bbbk_3, h).$$

Then $M_{\mathcal{L}_t}(\Bbbk_1, \Bbbk_3, h)$ is $\mathbb{C}$-graded by L_0-eigenvalues:

$$M_{\mathcal{L}_t}(\Bbbk_1, \Bbbk_3, h) = \coprod_{n\geq 0} M_{\mathcal{L}_t}(\Bbbk_1, \Bbbk_3, h)_{n+h},$$

where $M_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)_{(h)}=\mathbb{C}_{\Bbbk_{13},h}$ and $M_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)_{(n+h)}$ is the L_0-eigenspace of eigenvalue $n+h$ for $n>0$. $M_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)_{(n+h)}$ has a basis consisting of

$$I_{-k_1+\frac{1}{t}}\cdots I_{-k_s+\frac{1}{t}}L_{-m_1}\cdots L_{-m_r}\mathbf{1}_{\Bbbk_{13},h}$$

for $r,s\geq 0$, $m_1\geq\cdots\geq m_r\geq 1$, $k_1\geq\cdots\geq k_s\geq 1$ with $\sum_{i=1}^{r}m_i+\sum_{j=1}^{s}(k_j-\frac{1}{t})=n$, $n>0$.

Remark 4.2. As a module for $\mathcal{L}_t$, $M_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)$ is generated by $\mathbf{1}_{\Bbbk_{13},h}$ with the relations

$$L_0\mathbf{1}_{\Bbbk_{13},h}=h\mathbf{1}_{\Bbbk_{13},h},\ k_i=\Bbbk_i,\ i=1,3,$$

and

$$L_m\mathbf{1}_{\Bbbk_{13},h}=0, I_{n+\frac{1}{t}}\mathbf{1}_{\Bbbk_{13},h}=0 \text{ for } m\geq 1, n\geq 0.$$

$M_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)$ is *universal* in the sense that for any $\mathcal{L}_t$-module W of level $\Bbbk_{13}$ equipped with a vector v such that $L_0v=hv,\ L_mv=0, I_{n+\frac{1}{t}}v=0$ for $m\geq 1, n\geq 0$, there exists a unique $\mathcal{L}_t$-module map $M_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)\longrightarrow W$ sending $\mathbf{1}_{\Bbbk_{13},h}$ to v.

In general, $M_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)$ as an $\mathcal{L}_t$-module may be reducible. Since $\mathbb{C}_{\Bbbk_{13},h}$ generate $M_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)$ as $\mathcal{L}_t$-module, for any proper submodule U of $M_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)$, $U_{(h)}=U\bigcap M_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)_{(h)}=0$. Hence there exists a maximal proper $\mathcal{L}_t$-submodule $T_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)$. Set

$$L_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)=M_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)/T_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h).$$

Then $L_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)$ is an irreducible $\mathcal{L}_t$-module.

Definition 4.3. An $\mathcal{L}_t$-module W is said to be *restricted* if for any $w\in W, n\in\mathbb{Z}$, $L_nw=0$ and $I_{n+\frac{1}{t}}w=0$ for n sufficiently large. We say an $\mathcal{L}_t$-module W is of *level* $\Bbbk_{13}$ if the central element k_i acts as scalar $\Bbbk_i$ for $i=1,3$.

It is easy to see that $M_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)$, $L_{\mathcal{L}_t}(\Bbbk_1,\Bbbk_3,h)$ are restricted $\mathcal{L}_t$-modules of level $\Bbbk_{13}$ for any $h\in\mathbb{C}$. Now we are going to relate $\mathcal{L}_t$-modules with twisted $V_{\mathcal{L}}(\ell_{123},0)$-modules. On the one hand, we have

Theorem 4.4. *If W is a restricted $\mathcal{L}_t$-module of level ℓ_{13}, then W is a σ_t-twisted $V_{\mathcal{L}}(\ell_{123},0)$-module for $V_{\mathcal{L}}(\ell_{123},0)$ as a vertex algebra with*

$$Y_{\sigma_t}(L_{-2}\mathbf{1},z)=L(z)=\sum_{n\in\mathbb{Z}}L_nz^{-n-2},$$

$$Y_{\sigma_t}(I_{-1}\mathbf{1},z)=I_{\sigma_t}(z)=\sum_{n\in\mathbb{Z}}I_{n+\frac{1}{t}}z^{-n-\frac{1}{t}-1}.$$

Proof. Let $U_W=\{L(z),I_{\sigma_t}(z),\mathbf{1}_W\}$, where $\mathbf{1}_W$ is the identity operator on W. Clearly, $L(z),I_{\sigma_t}(z)$ are $\mathbb{Z}_t$-twisted weak vertex operators on W. From (4.4), (4.5), (4.6), using (2.1), we see that $L(z),I_{\sigma_t}(z)$ are mutually local $\mathbb{Z}_t$-twisted vertex operators on W. Hence, by Corollary 3.15 of [14], $\langle U_W\rangle$ is a vertex algebra with W a faithful σ-twisted module, where σ is an order t automorphism of the vertex algebra $\langle U_W\rangle$. To say that W is a σ_t-twisted module for $V_{\mathcal{L}}(\ell_{123},0)$, from Proposition 3.17 of [14], it suffices to show that there exists a vertex algebra homomorphism from $V_{\mathcal{L}}(\ell_{123},0)$ to $\langle U_W\rangle$.

By Lemma 2.11 of [14], $Y(L(z),z_1)$ and $Y(I_{\sigma_t}(z),z_1)$ satisfy the twisted Heisenberg-Virasoro algebra relations (3.4), (3.5), (3.6). Then $\langle U_W\rangle$ is an $\mathcal{L}$-module with L_n, I_n acting as $L(z)_{n+1}$, $I_{\sigma_t}(z)_n$ for $n \in \mathbb{Z}$, c_i acting as ℓ_i with $\ell_2 = 0$, $i = 1, 2, 3$.

By the universal property of $V_{\mathcal{L}}(\ell_{123}, 0)$ (c.f. Remark 2.7 of [12]), there exists a unique $\mathcal{L}$-module homomorphism

$$\psi : V_{\mathcal{L}}(\ell_{123}, 0) \longrightarrow \langle U_W \rangle; \;\; \mathbf{1} \mapsto \mathbf{1}_W.$$

Then

$$\psi(\omega_n v) = L(z)_n \psi(v) = \psi(\omega)_n \psi(v),$$
$$\psi(I_n v) = I_{\sigma_t}(z)_n \psi(v) = \psi(I)_n \psi(v)$$

for all $v \in V_{\mathcal{L}}(\ell_{123}, 0)$, $n \in \mathbb{Z}$. Hence ψ is a vertex algebra homomorphism. Therefore, W is a weak σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-module with $Y_{\sigma_t}(L_{-2}\mathbf{1}, z) = L(z)$, $Y_{\sigma_t}(I_{-1}\mathbf{1}, z) = I_{\sigma_t}(z)$. □

Conversely, we have

Theorem 4.5. *If W is a σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$ ($\ell_2 = 0$)-module, then W is a restricted $\mathcal{L}_t$-module of level ℓ_{13} with $L(z) = Y_W(L_{-2}\mathbf{1}, z)$, $I_{\sigma_t}(z) = Y_W(I_{-1}\mathbf{1}, z)$.*

Proof. Let W be a σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)(\ell_2 = 0)$-module. Recall the following formula (c.f. (2.40) of [14])

$$[Y_W(a, z_1), Y_W(b, z_2)] = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\left(\frac{\partial}{\partial z_2} \right)^j z_1^{-1} \delta\left(\frac{z_2}{z_1} \right) \left(\frac{z_2}{z_1} \right)^{\frac{k}{t}} \right) Y_W(a_j b, z_2),$$

where k is determined by a. In our case, when $a = L_{-2}\mathbf{1}$, $k = 0$, when $a = I_{-1}\mathbf{1}$, $k = 1$. For $a = b = L_{-2}\mathbf{1}$, we have ((2.21) of [12])

$$(L_{-2}\mathbf{1})_j L_{-2}\mathbf{1} = (j+1) L_{j-3}\mathbf{1} + \delta_{j-3,0} \frac{(j-1)^3 - (j-1)}{12} c_1 \mathbf{1},$$

so

$$\begin{aligned}
&[Y_W(L_{-2}\mathbf{1}, z_1), Y_W(L_{-2}\mathbf{1}, z_2)] \\
&= Y_W(L_{-3}\mathbf{1}, z_2) z_2^{-1} \delta\left(\frac{z_1}{z_2} \right) \\
&\quad + 2 Y_W(L_{-2}\mathbf{1}, z_2) \left(\frac{\partial}{\partial z_1} \right) z_2^{-1} \delta\left(\frac{z_1}{z_2} \right) \\
&\quad + \frac{1}{12} \left(\frac{\partial}{\partial z_1} \right)^3 z_2^{-1} \delta\left(\frac{z_1}{z_2} \right) \ell_1 \mathbf{1}.
\end{aligned} \tag{4.7}$$

For $a = L_{-2}\mathbf{1}, b = I_{-1}\mathbf{1}$, we have ((2.22) of [12] with $\ell_2 = 0$)

$$(L_{-2}\mathbf{1})_j I_{-1}\mathbf{1} = I_{j-2}\mathbf{1},$$

so

$$\begin{aligned}
&[Y_W(L_{-2}\mathbf{1}, z_1), Y_W(I_{-1}\mathbf{1}, z_2)] \\
&= Y_W(I_{-2}\mathbf{1}, z_2) z_1^{-1} \delta\left(\frac{z_2}{z_1} \right) + Y_W(I_{-1}\mathbf{1}, z_2) \left(\frac{\partial}{\partial z_2} \right) z_1^{-1} \delta\left(\frac{z_2}{z_1} \right).
\end{aligned} \tag{4.8}$$

For $a = b = I_{-1}\mathbf{1}$, we have ((2.23) of [12])

$$(I_{-1}\mathbf{1})_j I_{-1}\mathbf{1} = j \delta_{j-1,0} c_3 \mathbf{1},$$

so

$$[Y_W(I_{-1}\mathbf{1}, z_1), Y_W(I_{-1}\mathbf{1}, z_2)] = \left(\frac{\partial}{\partial z_2} \right) z_1^{-1} \delta\left(\frac{z_2}{z_1} \right) \left(\frac{z_2}{z_1} \right)^{\frac{1}{t}} \ell_3 \mathbf{1}. \tag{4.9}$$

Note that for $t \neq 2$, we have to require $\ell_3 = 0$ (Theorem 3.2). Therefore, with (4.4), (4.5) and (4.6), W is a $\mathcal{L}_t$-module with $L(z) = Y_W(L_{-2}\mathbf{1}, z)$, $I_{\sigma_t}(z) = Y_W(I_{-1}\mathbf{1}, z)$, $k_i = \ell_i$, $i = 1, 3$. Then W is restricted of level ℓ_{13} is clear. □

Let

$$\mathcal{L}_t^{(0)} = \mathbb{C}L_0 \oplus \mathbb{C}k_1 \oplus \mathbb{C}k_3, \quad \mathcal{L}_t^{(n)} = \mathbb{C}L_{-n} \ \text{ for } 0 \neq n \in \mathbb{Z},$$
$$\mathcal{L}_t^{(-\frac{1}{t}+n)} = \mathbb{C}I_{-n+\frac{1}{t}}, \quad \mathcal{L}_t^{(k)} = 0 \ \text{ for } k \in \frac{1}{t}\mathbb{Z},\ k \notin \mathbb{Z}, -\frac{1}{t} + \mathbb{Z}.$$

Then $\mathcal{L} = \coprod_{n\in\mathbb{Z}} \mathcal{L}_t^{(\frac{n}{t})}$ is a $\frac{1}{t}\mathbb{Z}$-graded Lie algebra, and the grading is given by $\mathrm{ad}L_0$-eigenvalues.

By Theorem 4.4 and Theorem 4.5 we have the following result.

Theorem 4.6. *The σ_t-twisted modules for $V_{\mathcal{L}}(\ell_{123}, 0)$ ($\ell_2 = 0$) viewed as a vertex operator algebra (i.e. $\mathbb{C}$-graded by L_0-eigenvalues and with the two grading restrictions (TW6), (TW7)) are exactly those restricted modules for the Lie algebra $\mathcal{L}_t$ of level ℓ_{13} that are $\mathbb{C}$-graded by L_0-eigenvalues and with the two grading restrictions. Furthermore, for any σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-module W, the σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-submodules of W are exactly the submodules of W for $\mathcal{L}_t$, and these submodules are in particular graded.*

Hence irreducible restricted $\mathcal{L}_t$-modules of level ℓ_{13} corresponds to irreducible σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-modules. Recall that for any $h \in \mathbb{C}$, $L_{\mathcal{L}_t}(\ell_1, \ell_3, h)$ is an irreducible restricted $\mathcal{L}_t$-module of level ℓ_{13}, so it is an irreducible σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-module with $\ell_2 = 0$.

Now we give the complete list of irreducible σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-modules.

Theorem 4.7. *Let $\ell_2 = 0$, $\ell_1, \ell_3 \in \mathbb{C}$. Then $\{L_{\mathcal{L}_t}(\ell_1, \ell_3, h) \mid h \in \mathbb{C}\}$ is a complete list of irreducible σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-modules.*

Proof. Let $W = \coprod_{r\in\mathbb{C}} W_{(r)}$ be an irreducible σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-module. By Theorem 4.6, W is an irreducible restricted $\mathcal{L}_t$-module of level ℓ_{13}. So k_i acts on W as a scalar ℓ_i for $i = 1, 3$. From Remark 2.5, there exists $h \in \mathbb{C}$ such that $W_{(h)} \neq 0$ and $W_{(h-n)} = 0$ for all $n \in \frac{1}{t}\mathbb{Z}_{\geq 1}$. Let $0 \neq w \in W_{(h)}$. Then

$$L_0 w = hw, \ \ L_m w = 0, \ \ I_{n+\frac{1}{t}} w = 0$$

for $m \geq 1, n \geq 0$. In view of Remark 4.2, there is a unique $\mathcal{L}_t$-module homomorphism

$$\varphi : M_{\mathcal{L}_t}(\ell_1, \ell_3, h) \longrightarrow W$$

such that $\varphi(\mathbf{1}_{\Bbbk_{13},h}) = w$. By Proposition 2.8, φ is a σ-twisted $\langle U_W\rangle$-module homomorphism (since $\mathcal{L}_t$ generates the vertex algebra $\langle U_W\rangle$), where σ is an order t automorphism of the vertex algebra $\langle U_W\rangle$. Recall that $M_{\mathcal{L}_t}(\ell_1, \ell_3, h)$ is a weak σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-module via the vertex algebra homomorphism ψ in Theorem 4.4. So φ can be viewed as a σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-module homomorphism. Since W is irreducible and $T_{\mathcal{L}_t}(\ell_1, \ell_3, h)$ is the (unique) largest proper submodule of $M_{\mathcal{L}_t}(\ell_1, \ell_3, h)$, it follows that

$$\varphi(M_{\mathcal{L}_t}(\ell_1, \ell_3, h)) = W$$

and

$$\mathrm{Ker}\ \varphi = T_{\mathcal{L}_t}(\ell_1, \ell_3, h).$$

Thus φ reduces to a σ_t-twisted $V_{\mathcal{L}}(\ell_{123}, 0)$-module isomorphism from $L_{\mathcal{L}_t}(\ell_1, \ell_3, h)$ to W. □

It is interesting and important to classify the irreducible modules for the fixed point subalgebra $V_{\mathcal{L}}(\ell_{123},0)^{\sigma_t} := \{v \in V_{\mathcal{L}}(\ell_{123},0) \mid \sigma_t(v) = v\}$, $t \in \mathbb{Z}_{\geq 1}$. In the case of $\ell_2 = 0$ and $\ell_3 \neq 0$, we have $t = 2$. Then σ_t is the order 2 automorphism of $V_{\mathcal{L}}(\ell_{123},0)$ which is induced from its Heisenberg vertex operator subalgebra. Denote by $\sigma_2 = \sigma$. Precisely, the automorphism

$$\sigma : V_{\mathcal{L}}(\ell_{123},0) \longrightarrow V_{\mathcal{L}}(\ell_{123},0)$$

is defined on the basis elements by

$$I_{-k_1}\cdots I_{-k_s}L_{-m_1}\cdots L_{-m_r}\mathbf{1} \mapsto (-1)^s I_{-k_1}\cdots I_{-k_s}L_{-m_1}\cdots L_{-m_r}\mathbf{1},$$

and extended linearly, where $r, s \geq 1$, $m_1 \geq \cdots \geq m_r \geq 2$, $k_1 \geq \cdots \geq k_s \geq 1$. Let

$$V_{\mathcal{L}}(\ell_{123},0)^+ = \{v \in V_{\mathcal{L}}(\ell_{123},0) \mid \sigma(v) = v\}$$

be the fixed point subalgebra under σ.

For $\ell_3 \neq 0$, denote by

$$c_{\tilde{Vir}} = \ell_1 - 1 + 12\frac{\ell_2^2}{\ell_3}.$$

Let $V_{\mathcal{H}}(\ell_3,0)$ be the vertex operator algebra constructed from the Heisenberg subalgebra $\mathcal{H}$ which is equipped with the nonstandard conformal vector $\omega_H = \frac{1}{2\ell_3}I_{-1}I_{-1}\mathbf{1} + \frac{\ell_2}{\ell_3}I_{-2}\mathbf{1}$ (of central charge $1 - 12\frac{\ell_2^2}{\ell_3}$). Let $\tilde{Vir}$ be the Virasoro algebra constructed by $\tilde{\omega} = \omega - \omega_H$. Let $V_{\tilde{Vir}}(c_{\tilde{Vir}},0)$ be the corresponding Virasoro vertex operator algebra. Recall that when $\ell_3 \neq 0$, we have (cf. Theorem 3.16 of [12])

$$V_{\mathcal{L}}(\ell_{123},0) \cong V_{\mathcal{H}}(\ell_3,0) \otimes V_{\tilde{Vir}}(c_{\tilde{Vir}},0)$$

as vertex operator algebras.

It is well known that there exists an order 2 isomorphism of the Heisenberg vertex operator algebra $V_{\mathcal{H}}(\ell_3,0)$. Let again

$$\sigma : V_{\mathcal{H}}(\ell_3,0) \longrightarrow V_{\mathcal{H}}(\ell_3,0)$$

be the order 2 automorphism which is defined on the basis elements by

$$I_{-k_1}\cdots I_{-k_s}\mathbf{1} \mapsto (-1)^s I_{-k_1}\cdots I_{-k_s}\mathbf{1}$$

and extended linearly, where $s \geq 1$, $k_1 \geq \cdots \geq k_s \geq 1$. The fixed point subalgebra $V_{\mathcal{H}}(\ell_3,0)^+ = \{v \in V_{\mathcal{H}}(\ell_3,0) \mid \sigma(v) = v\}$ has beed extensively studied (cf. [6] etc.).

Then it is immediate to see that when $\ell_3 \neq 0$ and $\ell_2 = 0$, we have an isomorphism of vertex operator algebras

$$V_{\mathcal{L}}(\ell_{123},0)^+ \cong V_{\mathcal{H}}(\ell_3,0)^+ \otimes V_{\tilde{Vir}}(c_{\tilde{Vir}},0).$$

Up to isomorphism, irreducible modules for the vertex operator algebra $V_{\mathcal{H}}(\ell_3,0)^+$ are (cf. [6], etc.) $V_{\mathcal{H}}(\ell_3,0)^{\pm}, V_{\mathcal{H}}(\ell_3,0)(\sigma)^{\pm}, V_{\mathcal{H}}(\ell_3,\lambda) \cong V_{\mathcal{H}}(\ell_3,-\lambda)$ for any $0 \neq \lambda \in \mathbb{C}$. Up to isomorphism, irreducible modules for the Virasoro vertex operator algebra $V_{\tilde{Vir}}(c_{\tilde{Vir}},0)$ are $L_{\tilde{Vir}}(c_{\tilde{Vir}},h)$ for all $h \in \mathbb{C}$ (cf. [9], [16], etc.). Therefore, in the case of $\ell_2 = 0$ and $\ell_3 \neq 0$, irreducible modules of $V_{\mathcal{L}}(\ell_{123},0)^+$ are one-to-one correspond to the tensor product of irreducible modules of $V_{\mathcal{H}}(\ell_3,0)^+$ and irreducible modules of $V_{\tilde{Vir}}(c_{\tilde{Vir}},0)$ (cf. Proposition 4.7.2 and Theorem 4.7.4 of [7]).

For other automorphism σ_t, the complete list of irreducible $V_{\mathcal{L}}(\ell_{123},0)^{\sigma_t}$-modules remains to be investigated.

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