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TELESCOPING METHOD, SUMMATION FORMULAS, AND INVERSION PAIRS

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Abstract. Based on Gosper's algorithm, we present an approach to the telescoping of general sequences. Along this approach, we propose a summation formula and a bibasic extension of Ma's inversion formula. From the formulas, we are able to derive several hypergeometric and elliptic hypergeometric identities.

1. **Introduction.** The telescoping method aims to solve the following problem: For a sequence $f(k)$ in some class $\mathcal{S}$, decide whether there exists $g(k) \in \mathcal{S}$ such that

$$f(k) = g(k+1) - g(k).$$

The sequence $g(k)$ is called the *anti-difference* of $f(k)$. Clearly, once we obtain $g(k)$, we will derive the indefinite summation formula

$$\sum_{k=a}^{b-1} f(k) = g(b) - g(a). \tag{1}$$

Gosper [7] solved the telescoping problem for hypergeometric terms by giving the so-called Gosper's algorithm. As mentioned in the book of Petkovše et al. [12], Gosper's algorithm is one of the landmarks in the history of symbolic summation. It not only fully solves the telescoping problem of hypergeometric terms, but also plays an important role in the creative telescoping algorithm developed by Wilf and Zeilberger [15].

Since the appearance of Gosper's algorithm and Zeilberger's algorithm, the telescoping method for more general sequences has been extensively studied. Chyzak [4] extended Zeilberger's algorithm to holonomic sequences. Inspired by Karr's summation algorithm [8], Schneider [13] presented an approach to summation in difference rings. Recently, Paule and Schneider [11] gave a symbolic summation theory for unspecified sequences.

This paper is motivated by considering the inversion formulas of basic and elliptic hypergeometric series. We find that the idea of Gosper's algorithm is an efficient mechanism for this purpose.

Let us recall the main steps of Gosper's algorithm. Suppose t_k is a hypergeometric term, i.e., the ratio t_{k+1}/t_k is a rational function of k. We first write the ratio

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t_{k+1}/t_k in the form

$$\frac{t_{k+1}}{t_k} = \frac{a(k)}{b(k)} \frac{c(k+1)}{c(k)}, \tag{2}$$

where $a(k), b(k), c(k)$ are polynomials in k such that $\gcd(a(k), b(k+h)) = 1$ for all nonnegative integers h. Then a hypergeometric telescoping z_k of t_k exists if and only if there is a polynomial solution $x(k)$ to the linear difference equation

$$a(k)x(k+1) - b(k-1)x(k) = c(k). \tag{3}$$

If $x(k)$ exists, the telescoping of t_k is

$$z_k = \frac{b(k-1)x(k)}{c(k)} t_k. \tag{4}$$

To deal with sequences which are not hypergeometric, we release the restriction that t_{k+1}/t_k is a rational function of k so that the $a(k), b(k), c(k)$ in (2) are arbitrary functions of k. It is straightforward to check that if $x(k)$ is a function satisfying (3), then the function z_k given by (4) is still a telescoping of t_k.

This observation motivates a new approach to the definite summation of sequences beyond the hypergeometric ones. Given a sequence t_k, we get (2) by spreading out the common factors of the numerator and the denominator of the ratio t_{k+1}/t_k up to a shift. Then we try to find a function $x(k)$ satisfying (3). Once we have (2) and (3), we will obtain the telescoping z_k and thus derive the summation formula (1).

Recall that two upper triangle matrices $(f_{n,k})_{n,k\geq 0}, (g_{n,k})_{n,k\geq 0}$ are called an *inversion pair* if

$$\sum_{k=0}^{n} f_{n,k} g_{k,l} = \delta_{n,l} = \begin{cases} 1, & \text{if } n = l, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

If (f, g) is an inversion pair, we have

$$\sum_{k=0}^{n} f_{n,k} F(k) = G(n) \iff \sum_{k=0}^{n} g_{n,k} G(k) = F(n).$$

There are many interesting applications of the inversion pairs. For example, Warnaar [14] established some new identities on multibasic theta hypergeometric series by elliptic analogue of the inverse relations. Ma gave an extension of Warnaar's matrix inversion and obtained a series of classical identities in [10].

Ma [10] showed that the inversion relations (5) can be derived by summation formulas of form (1). This enables us to study inversion pairs by solving the telescoping problem. We will see that the trivial case of $x(k) = 1$ leads to the key summation formula (Theorem 4) in [10]. Note that the assumption on f, g in Ma's formula (Equaiton (6) below) is nothing but the Gosper equation (3).

Along this approach, we construct a new summation formula which extends Ma's result. Based on it and the related inversion pairs, we are able to derive several basic hypergeometric identities and elliptic hypergeometric identities.

The paper is organized as follows. In Section 2, we give a simple proof of Ma's summation formula based on the telescoping method. Then in Section 3, we propose a bibasic extension of Ma's summation formula along the telescoping approach and give some applications. Then in Section 4, we present a summation formula corresponding to the non trivial case where $x(k) \neq 1$. Section 5 provides the inversion formulas related to the summation formulas.

Notation. Throughout this paper, we will employ the following standard notations for the theory of q-series and elliptic functions [6]. Assume $|q| < 1$, we define the q-shifted factorials by

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \quad \text{and} \quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

Write

$$(a_1,a_2,\ldots,a_m;q)_n = \prod_{j=1}^{m}(a_j;q)_n,$$

an basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s\left[\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ & b_1, & \ldots, & b_r \end{matrix}; q, z\right] = \sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_k}{(q,b_1\ldots,b_s;q)_k}[(-1)^k q^{\binom{k}{2}}]^{1+s-r}z^k.$$

and the balanced basic hypergeometric series ${}_{r+1}\phi_r$ is defined by $z = q$ and the parameters satisfy $b_1b_2\cdots b_r = qa_1a_2\cdots a_{r+1}$.

Assume further that $|p| < 1$. The elliptic function is defined by

$$\theta(x) = \theta(x;p) = (x;p)_\infty(p/x;p)_\infty.$$

The elliptic analogues of the q-shifed factorials are given by

$$(a;q,p)_\infty = \prod_{k=0}^{\infty}\theta(aq^k;p), \quad \text{and} \quad (a;q,p)_n = \frac{(a;q,p)_\infty}{(aq^n;q,p)_\infty}.$$

Also, we write

$$(a_1,a_2,\ldots,a_m;q,p)_n = \prod_{j=1}^{m}(a_j;q,p)_n, \quad \text{and} \quad (a;q,p)_{kn} = (a,aq,\ldots,aq^{k-1};q^k,p)_n.$$

Note that $\theta(x;0) = 1-x$ and hence $(a;q,0)_n = (a;q)_n$.

The balanced, very-well-poised, elliptic (or modular) hypergeometric series is defined by

$${}_{r+1}\omega_r(a_1;a_4,\ldots,a_{r+1};q,p) = \sum_{k=0}^{\infty}\frac{\theta(a_1q^{2k};p)}{\theta(a_1;p)}\frac{(a_1,a_4,\ldots,a_{r+1};q,p)_k}{(q,a_1q/a_4,\ldots,a_1q/a_{r+1};q,p)_k}q^k,$$

where the parameters satisfy $(a_4\cdots a_{r+1})^2 = a_1^{r-3}q^{r-5}$.

We follow the usual convention in defining produces as

$$\prod_{j=k}^{m}A_j := \begin{cases} A_kA_{k+1}\cdots A_m, & m \geq k, \\ 1, & m = k-1, \\ (A_{m+1}A_{m+2}\cdots A_{k-1})^{-1}, & m \leq k-2. \end{cases}$$

2. A simple proof of Ma's formula.

Ma used the following summation formula to prove the inversion formulas. We will give a simple proof by the idea of Gosper's algorithm.

Theorem 2.1 ([10, Theorem 4])**.** *Let $f(a,b)$ and $g(a,b)$ be two arbitrary nonzero functions over $\mathbb{C}$ in variables a,b such that*

$$f(a,c)g(b,d) - f(a,d)g(b,c) = f(a,b)g(c,d), \tag{6}$$

and a_i, b_i, c_i, d_i be arbitrary sequences such that none of the denominators in (7) vanishes. Then for any nonnegative integers m, n,

$$\sum_{k=-n}^{m} f(a_k, b_k) g(c_k, d_k) \frac{\prod_{j=1}^{k-1} f(a_j, c_j) g(b_j, d_j)}{\prod_{j=1}^{k} f(a_j, d_j) g(b_j, c_j)} = \frac{\prod_{j=1}^{m} f(a_j, c_j) g(b_j, d_j)}{\prod_{j=1}^{m} f(a_j, d_j) g(b_j, c_j)} - \frac{\prod_{j=-n}^{0} f(a_j, d_j) g(b_j, c_j)}{\prod_{j=-n}^{0} f(a_j, c_j) g(b_j, d_j)}. \quad (7)$$

Proof. Let

$$t_k = f(a_k, b_k) g(c_k, d_k) \frac{\prod_{j=1}^{k-1} f(a_j, c_j) g(b_j, d_j)}{\prod_{j=1}^{k} f(a_j, d_j) g(b_j, c_j)}.$$

We have

$$\frac{t_{k+1}}{t_k} = \frac{f(a_k, c_k) g(b_k, d_k)}{f(a_{k+1}, d_{k+1}) g(b_{k+1}, c_{k+1})} \frac{f(a_{k+1}, b_{k+1}) g(c_{k+1}, d_{k+1})}{f(a_k, b_k) g(c_k, d_k)}.$$

Set

$$a(k) = f(a_k, c_k) g(b_k, d_k),\ b(k-1) = f(a_k, d_k) g(b_k, c_k),\ c(k) = f(a_k, b_k) g(c_k, d_k).$$

Then by the assumption (6), we have

$$a(k) - b(k-1) = c(k).$$

Therefore, the telescoping of t_k is

$$z_k = \frac{b(k-1)}{c(k)} t_k = \frac{\prod_{j=1}^{k-1} f(a_j, c_j) g(b_j, d_j)}{\prod_{j=1}^{k-1} f(a_j, d_j) g(b_j, c_j)}.$$

we get the formula (7) by summing k from $-n$ to m. □

3. **A bibasic extension of Ma's formula.** Motivated by Ma's summation formula, we consider another special kind of $f(a, b)$ and derive a new summation formula.

Theorem 3.1. *Let $f(a, b)$ be an arbitrary nonzero functions over $\mathbb{C}$ in variables a, b such that*

$$f(a, b) f(a, c) f(a, d) f(a, e) - f(b, 1) f(c, 1) f(d, 1) f(e, 1) = b f(a, 1) f(a, bc) f(a, bd) f(a, be), \quad (8)$$

and a_i, b_i, c_i, d_i, e_i be arbitrary sequences such that none of the denominators in (9) vanishes. Then for any nonnegative integers m, n, we have

$$\sum_{k=-n}^{m} b_k f(a_k, 1) f(a_k, b_k c_k) f(a_k, b_k d_k) f(a_k, b_k e_k) \times \frac{\prod_{j=1}^{k-1} f(a_j, b_j) f(a_j, c_j) f(a_j, d_j) f(a_j, e_j)}{\prod_{j=1}^{k} f(b_j, 1) f(c_j, 1) f(d_j, 1) f(e_j, 1)}$$

$$= \frac{\prod_{j=1}^{m} f(a_j, b_j) f(a_j, c_j) f(a_j, d_j) f(a_j, e_j)}{\prod_{j=1}^{m} f(b_j, 1) f(c_j, 1) f(d_j, 1) f(e_j, 1)}$$

$$-\frac{\prod_{j=1}^{-n-1} f(a_j,b_j)f(a_j,c_j)f(a_j,d_j)f(a_j,e_j)}{\prod_{j=1}^{-n-1} f(b_j,1)f(c_j,1)f(d_j,1)f(e_j,1)}. \tag{9}$$

Proof. Let

$$t_k = b_k f(a_k,1)f(a_k,b_kc_k)f(a_k,b_kd_k)f(a_k,b_ke_k) \times \frac{\prod_{j=1}^{k-1} f(a_j,b_j)f(a_j,c_j)f(a_j,d_j)f(a_j,e_j)}{\prod_{j=1}^{k} f(b_j,1)f(c_j,1)f(d_j,1)f(e_j,1)}.$$

Then,

$$\frac{t_{k+1}}{t_k} = \frac{b_{k+1}f(a_{k+1},1)f(a_{k+1},b_{k+1}c_{k+1})f(a_{k+1},b_{k+1}d_{k+1})f(a_{k+1},b_{k+1}e_{k+1})}{b_kf(a_k,1)f(a_k,b_kc_k)f(a_k,b_kd_k)f(a_k,b_ke_k)} \times \frac{f(a_k,b_k)f(a_k,c_k)f(a_k,d_k)f(a_k,e_k)}{f(b_{k+1},1)f(c_{k+1},1)f(d_{k+1},1)f(e_{k+1},1)}.$$

Set

$$a(k) = f(a_k,b_k)f(a_k,c_k)f(a_k,d_k)f(a_k,e_k),$$

$$b(k) = f(b_{k+1},1)f(c_{k+1},1)f(d_{k+1},1)f(e_{k+1},1),$$

$$c(k) = b_kf(a_k,1)f(a_k,b_kc_k)f(a_k,b_kd_k)f(a_k,b_ke_k).$$

We see that (2) holds. By the assumption (8), we have

$$a(k) - b(k-1) = c(k).$$

Hence by the Gosper's algorithm, we obtain the telescoping of t_k:

$$z_k = \frac{b(k-1)}{c(k)}t_k = \frac{\prod_{j=1}^{k-1} f(a_j,b_j)f(a_j,c_j)f(a_j,d_j)f(a_j,e_j)}{\prod_{j=1}^{k-1} f(b_j,1)f(c_j,1)f(d_j,1)f(e_j,1)}.$$

we get the formula (9) by summing k from $-n$ to m, . □

Now we give some applications of Theorem 3.1. The first one is a summation formula for finite series.

Corollary 3.1. *Let a_i, b_i, c_i, d_i, e_i be arbitrary sequences with $a_i^2 = b_ic_id_ie_i$ such that none of the denominators in (10) vanishes. Then for any nonnegative integer m, we have*

$$\sum_{k=-n}^{m} b_k(1-a_k)(1-a_k/b_kc_k)(1-a_k/b_kd_k)(1-a_k/b_ke_k) \times \frac{\prod_{j=1}^{k-1}(1-a_j/b_j)(1-a_j/c_j)(1-a_j/d_j)(1-a_j/e_j)}{\prod_{j=1}^{k}(1-bj)(1-c_j)(1-d_j)(1-e_j)}$$

$$=\frac{\prod_{j=1}^{m}(1-a_j/b_j)(1-a_j/c_j)(1-a_j/d_j)(1-a_j/e_j)}{\prod_{j=1}^{m}(1-b_j)(1-c_j)(1-d_j)(1-e_j)} - \frac{\prod_{j=1}^{-n-1}(1-a_j/b_j)(1-a_j/c_j)(1-a_j/d_j)(1-a_j/e_j)}{\prod_{j=1}^{-n-1}(1-bj)(1-c_j)(1-d_j)(1-e_j)}. \tag{10}$$

Proof. Let $f(a,b) = 1 - a/b$. It is straightfoward to check that (8) holds by the assumption $a_i^2 = b_i c_i d_i e_i$. Hence by Theorem 3.1, we derive (10). □

Corollary 3.1 is equivalent to a result of Ian G. Macdonald, first published by Bhatnagar and Milne [2, Eq.(2.30)]. From this formula, we can derive several summation identities. Here are two examples.

Example 3.1. In Corollary 3.1, set

$$a_k = aq^{2k},\ b_k = q^k,\ c_k = aq^k/b,\ d_k = aq^k/c,\ e_k = aq^k/d,$$

where $a = bcd$. We derive that

$$\sum_{k=0}^{m} \frac{(a, qa^{1/2}, -qa^{1/2}, b, c, d; q)_k}{(q, a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d; q)_k} \left(\frac{aq}{bcd}\right)^k = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_m}{(aq/b, aq/c, aq/d, aq/bcd; q)_m}.$$

Let $m \to \infty$, we obtain a balanced ${}_6\phi_5$ series [6, Page 356, Eq. (II.20)]

$${}_6\phi_5 \left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d \\ & a^{1/2}, & -a^{1/2}, & aq/b, & aq/c, & aq/d \end{matrix} ; q, \frac{aq}{bcd} \right]$$
$$= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}.$$

Example 3.2. Setting

$$a_k = adq^k p^k,\ b_k = dq^k,\ c_k = adq^k/b,\ d_k = adp^k/x,\ e_k = bxp^k/d$$

in Corollary 3.1, we derive an indefinite bibasic summation formula [6, Eq. (II.36)]:

$$\sum_{k=-n}^{m} \frac{(1 - adp^k q^k)(1 - bp^k q^{-k}/d)}{(1 - ad)(1 - b/d)} \frac{(a, b; p)_k (x, ad^2/bx; q)_k}{(dq, adq/b; q)_k (adp/x, bpx/d; p)_k} q^k$$
$$= \frac{(1-a)(1-b)(1-x)(1-ad^2/bx)}{(1-ad)(1-b/d)(d-x)(1-ad/bx)} \left\{ \frac{(ap, bp; p)_m (xq, ad^2q/bx; q)_m}{(dq, adq/b; q)_m (adp/x, bpx/d; p)_m} \right.$$
$$\left. - \frac{(1/d, b/ad; q)_{n+1} (x/ad, d/bx; p)_{n+1}}{(1/a, 1/b; p)_{n+1} (1/x, bx/ad^2; q)_{n+1}} \right\}.$$

Gasper and Rahman used this identity to set up a series of quadratic and cubic summation and transformation formulas of basic hypergeometric series.

The second one is an elliptic hypergeometric identity.

Corollary 3.2. *Let a_i, b_i, c_i, d_i, e_i be arbitrary sequences with $a_i^2 = b_i c_i d_i e_i$ such that none of the denominators in (11) vanishes. Then for any nonnegative integer m, n, we have*

$$\sum_{k=-n}^{m} b_k \theta(a_k, a_k/b_k c_k, a_k/b_k d_k, a_k/b_k e_k; p) \frac{\prod_{j=1}^{k-1} \theta(a_j/b_j, a_j/c_j, a_j/d_j, a_j/e_j; p)}{\prod_{j=1}^{k} \theta(b_j, c_j, d_j, e_j; p)}$$
$$= \frac{\prod_{j=1}^{m} \theta(a_j/b_j, a_j/c_j, a_j/d_j, a_j/e_j; p)}{\prod_{j=1}^{m} \theta(b_j, c_j, d_j, e_j; p)} - \frac{\prod_{j=1}^{-n-1} \theta(a_j/b_j, a_j/c_j, a_j/d_j, a_j/e_j; p)}{\prod_{j=1}^{-n-1} \theta(b_j, c_j, d_j, e_j; p)}. \tag{11}$$

Proof. Let $f(a,b)=\theta(a/b;p)$. Noting that $a^2=bcde$, we have Weierstrass' identity

$$\theta(a/b,a/c,a/d,a/e;p)-\theta(b,c,d,e;p)=b\theta(a,a/bc,a/bd,a/be;p),$$

which means that $f(a,b)$ satisfies (8). Hence by Theorem 3.1, we derive (11). □

This formula has been stated in an equivalent form by Warnaar [14, Eq.(3.2)]. We list some applications of Corollary 3.2 on elliptic hypergeometric series.

Example 3.3. Setting

$$a_k=q^{2k}a,\ b_k=q^k,\ c_k=q^ka/b,\ d_k=q^ka/c,\ e_k=q^ka/d$$

and $n=0$ in Corollary 3.2 where $a=bcd$, and making some simplifications, we deduce that a special case of [6, Page 321, Eq.(11.4.4)] where $a=bcd$

$$\sum_{k=0}^{m+1}\frac{\theta(q^{2k}a;p)}{\theta(a;p)}\frac{(a,b,c,d,a^2q^{m+2}/bcd,q^{-m-1};q,p)_k}{(q,qa/b,qa/c,qa/d,bcdq^{-m-1}/a,aq^{m+2};q,p)_k}q^k$$
$$=\frac{(qa,qa/bc,qa/bd,qa/cd;q,p)_{m+1}}{(qa/b,qa/c,qa/d,qa/bcd;q,p)_{m+1}}.$$

Example 3.4. Setting

$$a_k=aq^kr^k,\ b_k=q^k/bd,\ c_k=adq^k,\ d_k=cr^k,\ e_k=abr^k/c$$

and $n=0$ in Corollary 3.2, we get

$$\sum_{k=0}^{m}q^k\frac{\theta(aq^kr^k,bq^{-k}r^k;p)}{\theta(a,b;p)}\frac{(abd,1/d;r,p)_k}{(cr,abr/c;r,p)_k}\frac{(a/c,c/b;q,p)_k}{(q/bd,adq;q,p)_k}$$
$$=\frac{\theta(c,ab/c,bd,ad;p)}{\theta(a,b,abd/c,cd;p)}\left(1-\frac{(abd,dr^{-m};r,p)_{m+1}}{(r^{-m}/c,ab/c)_{m+1}}\frac{(a/c,bq^{-m}/c;q,p)_{m+1}}{(bdq^{-m},ad;q,p)_{m+1}}\right).$$

It is an indefinite elliptic hypergeometric series of Warnaar [14]. Moreover, Warnaar used the specialization of this identity to set up a pair of inverse matrices.

Example 3.5. Setting

$$a_k=ad(rst/q)^k,\ b_k=dq^k,\ c_k=\frac{ad}{b}(st/q)^k,\ d_k=\frac{ad}{c}(rt/q)^k,\ e_k=\frac{bc}{d}(rs/q)^k$$

in Corollary 3.2, and making some simplifications, we deduce that [6, Page 326, Eq.(11.6.6)]

$$\sum_{k=-m}^{n}\frac{\theta(ad(rst/q)^k,br^k/dq^k,cs^k/dq^k,adt^k/bcq^k;p)}{\theta(ad,b/d,c/d,ad/bc;p)}$$
$$\times\frac{(a;rst/q^2,p)_k(b;r,p)_k(c;s,p)_k(ad^2/bc;t,p)_k}{(dq;q,p)_k(adst/bq;st/q,p)_k(adrt/cq;rt/q,p)_k(bcrs/dq;rs/q,p)_k}q^k$$
$$=\frac{\theta(a,b,c,ad^2/bc;p)}{d\theta(ad,b/d,c/d,ad/bc;p)}$$
$$\times\left\{\frac{(arst/q^2;rst/q^2,p)_n(br;r,p)_n(cs;s,p)_n(ad^2t/bc;t,p)_n}{(dq;q,p)_n(adst/bq;st/q,p)_n(adrt/cq;rt/q,p)_n(bcrs/dq;rs/q,p)_n}\right.$$
$$\left.-\frac{(c/ad;rt/q,p)_{m+1}(d/bc;rs/q,p)_{m+1}(1/d;q,p)_{m+1}(b/ad;st/q,p)_{m+1}}{(1/c;s,p)_{m+1}(bc/ad^2;t,p)_{m+1}(1/a;rst/q^2,p)_{m+1}(1/b;r,p)_{m+1}}\right\}.$$

4. **A general summation formula.** In this section, we consider the case of nontrivial $x(k)$, i.e., $x(k) \neq 1$. For an arbitrary function $g(a,b)$, we will find a function $x(k)$ satisfying (3) and derive a general summation formula.

Theorem 4.1. *Let $g(a,b)$ be an arbitrary function over the complex field $\mathbb{C}$ in variables a, b, and $a_i, b_i, c_i, p_i, q_i, r_i$ be arbitrary sequences such that none of the denominators in (13) vanishes. Suppose $x(k)$ is a function over the complex field $\mathbb{C}$ which satisfies*

$$\frac{x(k+1)}{x(k)} = \frac{g(a_k, p_k)g(a_k b_k c_k, p_k q_k r_k) - g(a_k b_k, p_k q_k)g(a_k c_k, p_k r_k)}{g(b_k, q_k)g(c_k, r_k)}. \tag{12}$$

Then for any nonnegative integers m, n,

$$\sum_{k=-n}^{m} x(k)g(a_k b_k, p_k q_k)g(a_k c_k, p_k r_k)\frac{\prod_{j=1}^{k-1} g(b_j, q_j)g(c_j, r_j)}{\prod_{j=1}^{k} g(a_j, p_j)g(a_j b_j c_j, p_j q_j r_j)}$$

$$= x(-n)\prod_{j=1}^{-n-1} \frac{g(b_j, q_j)g(c_j, r_j)}{g(a_j, p_j)g(a_j b_j c_j, p_j q_j r_j)} - x(m+1)\prod_{j=1}^{m} \frac{g(b_j, q_j)g(c_j, r_j)}{g(a_j, p_j)g(a_j b_j c_j, p_j q_j r_j)}. \tag{13}$$

Proof. Let

$$t_k = -x(k)g(a_k b_k, p_k q_k)g(a_k c_k, p_k r_k)\frac{\prod_{j=1}^{k-1} g(b_j, q_j)g(c_j, r_j)}{\prod_{j=1}^{k} g(a_j, p_j)g(a_j b_j c_j, p_j q_j r_j)}.$$

Then,

$$\frac{t_{k+1}}{t_k} = \frac{-x(k+1)g(a_{k+1}b_{k+1}, p_{k+1}q_{k+1})g(a_{k+1}c_{k+1}, p_{k+1}r_{k+1})}{-x(k)g(a_k b_k, p_k q_k)g(a_k c_k, p_k r_k)} \times \frac{g(b_k, q_k)g(c_k, r_k)}{g(a_{k+1}, p_{k+1})g(a_{k+1}b_{k+1}c_{k+1}, p_{k+1}q_{k+1}r_{k+1})}.$$

Set

$$a(k) = g(b_k, q_k)g(c_k, r_k),$$

$$b(k) = g(a_{k+1}, p_{k+1})g(a_{k+1}b_{k+1}c_{k+1}, p_{k+1}q_{k+1}r_{k+1}),$$

$$c(k) = -x(k)g(a_k b_k, p_k q_k)g(a_k c_k, p_k r_k).$$

We see that (2) holds. By the assumption (12), we have

$$a(k)x(k+1) - b(k-1)x(k) = c(k).$$

Hence the telescoping of t_k is given by

$$z_k = \frac{b(k-1)x(k)}{c(k)}t_k = x(k)\prod_{j=1}^{k-1} \frac{g(b_j, q_j)g(c_j, r_j)}{g(a_j, p_j)g(a_j b_j c_j, p_j q_j r_j)}.$$

We get the formula (13) by summing k from $-n$ to m and making some simplifications. □

Actually, given an arbitrarily function $g(a,b)$, we can derive an explicit formula for the function $x(k)$ by taking a nonzero initial value $x(0)$. This allows us to construct some new summation identities. In the following part of this section, we will give some applications of Theorem 4.1. We only consider the simple case $x(0) = 1$.

To derive hypergeometric identities, we take $g(a, b) = a - b$ and obtain

Corollary 4.1. *Let $a_i, b_i, c_i, p_i, q_i, r_i$ be arbitrary sequences such that none of the denominators in (14) vanishes. Then for any nonnegative integer m, n, we have*

$$\sum_{k=-n}^{m} (-1)^k (a_k b_k - p_k q_k)(a_k c_k - p_k r_k) \frac{\prod_{j=1}^{k-1} a_j p_j (b_j - q_j)(c_j - r_j)}{\prod_{j=1}^{k} (a_j - p_j)(a_j b_j c_j - p_j q_j r_j)}$$

$$= (-1)^n \prod_{j=1}^{-n-1} \frac{a_j p_j (b_j - q_j)(c_j - r_j)}{(a_j - p_j)(a_j b_j c_j - p_j q_j r_j)} + (-1)^m \prod_{j=1}^{m} \frac{a_j p_j (b_j - q_j)(c_j - r_j)}{(a_j - p_j)(a_j b_j c_j - p_j q_j r_j)}. \tag{14}$$

Proof. Let $g(a, b) = a - b$. Then by (12) we derive that

$$\frac{x(k+1)}{x(k)} = -a_k p_k.$$

Considering the initial value, we have

$$x(k) = (-1)^k \prod_{i=0}^{k-1} a_i p_i.$$

Hence by Theorem 4.1 we obtain (14). □

As examples, we derive the following hypergeometric and basic hypergeometric identities.

Example 4.1. Setting $n = 0$ and

$$a_i = i + 1, \; b_i = p + i + 1, \; c_i = p^2 + i + 1, \; p_i = 1, \; q_i = 1, \; r_i = 1$$

in (14) where $0 \le i \le m$, we see that

$$x(k) = (-1)^k k!.$$

After some simplifications, we derive that

$$\sum_{k=0}^{m} (-1)^k k \binom{p+k-1}{p} \binom{p^2+k-1}{p^2} \times \frac{((k+1)(p+k+1)-1)\big((k+1)(p^2+k+1)-1\big)}{\prod_{j=1}^{k} (j(p+j)(p^2+j)-1)}$$

$$= (-1)^m \binom{p+m}{p} \binom{p^2+m}{p^2} \frac{(m+1)}{\prod_{j=1}^{m} (j(p+j)(p^2+j)-1)}.$$

Example 4.2. Setting $n = 0$ and

$$a_i = (i+1)^2, \; b_i = (p+i+1)^2, \; c_i = (p^2+i+1)^2, \; p_i = 1, \; q_i = 1, \; r_i = 1$$

in (14), we see that

$$x(k) = (-1)^k (k!)^2.$$

After some simplifications, we derive that

$$\sum_{k=0}^{m}(-1)^k k^2 \binom{p+k-1}{p}\binom{p^2+k-1}{p^2}\binom{p+k+1}{p+2}\binom{p^2+k+1}{p^2+2}$$
$$\times \frac{\left((k+1)^2(p+k+1)^2-1\right)\left((k+1)^2(p^2+k+1)^2-1\right)}{\prod_{j=1}^{k}\left(j^2(p+j)^2(p^2+j)^2-1\right)}$$
$$=\binom{p+m}{p}\binom{p^2+m}{p^2}\binom{p+m+2}{p+2}\binom{p^2+m+2}{p^2+2}\frac{(-1)^m(m+1)^2}{\prod_{j=1}^{m}\left(j^2(p+j)^2(p^2+j)^2-1\right)}.$$

Example 4.3. Let

$$x_1 = cr^i,\ x_2 = bq^i,\ x_3 = ap^i,\ x_4 = ds^i$$

Setting $n = 0$ and

$$a_i = x_1 - x_2,\ b_i = x_1 - x_3,\ c_i = x_2 - x_4,$$
$$p_i = 1/x_2 - 1/x_1,\ q_i = 1/x_3 - 1/x_1,\ r_i = 1/x_4 - 1/x_2$$

in (14), we see that

$$x(k) = \frac{(b/c; q/r)_k (bc; qr)_k}{b^{2k} q^{2\binom{k}{2}}}.$$

After some simplifications, we derive that [1, Exapmle 6.2]

$$\sum_{k=0}^{m} q^{\binom{k+1}{2}}(1-abp^kq^k)(1-ap^k/bq^k)(1-cdr^ks^k)(1-ds^k/cr^k)f(k)$$
$$=(b-c)(1-1/bc)(a-d)(ad-1)$$
$$+ q^{\binom{m}{2}}\frac{(bdq^ms^m-1)(bq^m-ds^m)(cr^m-ap^m)(acp^mr^m-1)}{bcr^m}f(m),$$

where

$$f(k) = \frac{(a/c; p/r)_k (ac; pr)_k (d/b; s/q)_k (bd; qs)_k b^k}{a^k p^{\binom{k+1}{2}} (bq/cr; q/r)_k (bcqr; qr)_k (ds/ap; s/p)_k (adps; ps)_k}.$$

Now we consider another choice of the function $g(a, b)$.

Corollary 4.2. *Let $a_i, b_i, c_i, p_i, q_i, r_i$ be arbitrary sequences such that none of the denominators in (15) vanishes. Then for any nonnegative integers m, n, we have*

$$\sum_{k=-n}^{m}(-1)^k(1-a_kb_kp_kq_k)(1-a_kc_kp_kr_k)\frac{\prod_{j=1}^{k-1}a_jp_j(1-b_jq_j)(1-c_jr_j)}{\prod_{j=1}^{k}(1-a_jp_j)(1-a_jb_jc_jp_jq_jr_j)}$$
$$=(-1)^n\prod_{j=1}^{-n-1}\frac{a_jp_j(1-b_jq_j)(1-c_jr_j)}{(1-a_jp_j)(1-a_jb_jc_jp_jq_jr_j)} + (-1)^m\prod_{j=1}^{m}\frac{a_jp_j(1-b_jq_j)(1-c_jr_j)}{(1-a_jp_j)(1-a_jb_jc_jp_jq_jr_j)}. \tag{15}$$

Proof. Let $g(a, b) = 1 - ab$. Then by (12) we have

$$\frac{x(k+1)}{x(k)} = -a_kp_k.$$

Considering the initial value, we derive that

$$x(k) = (-1)^k \prod_{i=0}^{k-1} a_i p_i.$$

Hence by Theorem 4.1 we obtain (15). □

Example 4.4. Set $n = 0$ and

$$a_k = a,\ b_k = b,\ c_k = c,\ p_k = p^k,\ q_k = q^k,\ r_k = r^k$$

in (15), we see that

$$x(k) = (-a)^k p^{\binom{k}{2}}.$$

After some simplifications, we derive

$$\sum_{k=0}^{m} \frac{(-a)^k p^{\binom{k}{2}}(1-abp^kq^k)(1-acp^kr^k)(b;q)_k(c;r)_k}{(ap;p)_k(abcpqr;pqr)_k}$$

$$= (a-1)(abc-1) + \frac{(-a)^m ap^{\binom{m+1}{2}}(b;q)_{m+1}(c;r)_{m+1}}{(ap;p)_m(abcpqr;pqr)_m}.$$

We remark that this formula is a result of Gosper, first published by Bauer and Petkovšek [1, Eq.(6.39)].

5. **Inversion pairs.** In this section, we will derive an inversion pair via the summation formula (9) and present some applications.

To construct the inversion pair, we first take $n = 0$ and $m = n$ in (9) and multiply both sides by the common denominator. We thus have

Lemma 5.1. *Let $f(a, b)$ be an arbitrary function over the complex field $\mathbb{C}$ in variables a, b which satisfies (8), and a_i, b_i, c_i, d_i, e_i be arbitrary sequences. Then for any nonnegative integers n,*

$$\sum_{k=0}^{n} b_k f(a_k,1)f(a_k,b_kc_k)f(a_k,b_kd_k)f(a_k,b_ke_k)$$
$$\times \prod_{j=0}^{k-1} f(a_j,b_j)f(a_j,c_j)f(a_j,d_j)f(a_j,e_j) \prod_{j=k+1}^{n} f(b_j,1)f(c_j,1)f(d_j,1)f(e_j,1)$$

$$= \prod_{j=0}^{n} f(a_j,b_j)f(a_j,c_j)f(a_j,d_j)f(a_j,e_j) - \prod_{j=0}^{n} f(b_j,1)f(c_j,1)f(d_j,1)f(e_j,1). \quad (16)$$

Now we are ready to give the inversion pair.

Theorem 5.1. *Let z and x_i, y_i $(i \in \mathbb{Z})$ be indeterminate such that none of the denominators in (18) and (19) vanishes. Suppose $f(x, y)$ is a bivariate function over the complex field $\mathbb{C}$ in variables x, y which satisfies (8) and*

$$f(x,y) = -(x/y)f(y,x), \quad f(tx,ty) = f(x,y). \quad (17)$$

for any variable t. Set

$$f_{n,k} = \frac{f(z,z/x_ky_k)f(z^2x_k/y_k,z)}{f(zx_n,z/y_n)f(z^2x_n/y_n,z)} \frac{\prod_{j=k+1}^{n} f(zx_n/y_j,1)f(x_ny_j,1)}{\prod_{j=k}^{n-1} x_j f(x_n,x_j)f(zx_n,1/x_j)} \quad (18)$$

and

$$g_{n,k} = \frac{\prod_{j=k}^{n-1} f(x_k, 1/y_j) f(zx_k, y_j)}{\prod_{j=k+1}^{n} x_j f(x_k/x_j, 1) f(zx_k x_j, 1)}. \tag{19}$$

Then $F = (f_{n,k})_{n,k\in\mathbb{Z}}$ *and* $G = (g_{n,k})_{n,k\in\mathbb{Z}}$ *form an inversion pair.*

Proof. Since $f(tx, ty) = f(x, y)$, we have

$$f_{n,n} = \frac{f(z, z/x_n y_n) f(z^2 x_n/y_n, z)}{f(zx_n, z/y_n) f(z^2 x_n/y_n, z)} = \frac{f(z, z/x_n y_n)}{f(zx_n, z/y_n)} = 1.$$

Therefore it holds for $l = n$.

Now assume $n \geq l + 1$. Take $n \to n - l$ and

$$a_j = zx_l x_n,\; b_j = zx_n/y_{j+l},\; c_j = x_n y_{j+l},\; d_j = x_l/x_{j+l},\; e_j = zx_l x_{j+l}$$

in (16). The first product of the right hand side of (16) contains the term

$$f(a_{n-l}, e_{n-l}) = f(zx_l x_n, zx_l x_n)$$

and the second product of the right hand side of (16) contains the term

$$f(d_0, 1) = f(x_l/x_l, 1).$$

Noting that $f(x, y) = -(x/y) f(y, x)$ implies that $f(x, x) = 0$, we thus derive that the right hand side of (16) is zero.

By the relation (17), the left hand side becomes

$$\begin{aligned}
&\sum_{k=l}^{n} \frac{zx_n}{y_k} f(zx_l x_n, 1) f(zx_l, zx_n) f(z, z/x_k y_k) f(z, z^2 x_k/y_k) \\
&\qquad \times \prod_{j=k+1}^{n} f(zx_n/y_j, 1) f(x_n y_j, 1) f(x_l/x_j, 1) f(zx_l x_j, 1) \\
&\qquad \times \prod_{j=l}^{k-1} f(x_l, 1/y_j) f(zx_l, y_j) f(zx_n, 1/x_j) f(x_n, x_j) \\
&= \sum_{k=l}^{n} -\frac{x_n}{x_k} f(zx_l x_n, 1) f(zx_l, zx_n) f(z, z/x_k y_k) f(z^2 x_k/y_k, z) \\
&\qquad \times \prod_{j=k+1}^{n} f(zx_n/y_j, 1) f(x_n y_j, 1) f(x_l/x_j, 1) f(zx_l x_j, 1) \\
&\qquad \times \prod_{j=l}^{k-1} f(x_l, 1/y_j) f(zx_l, y_j) f(zx_n, 1/x_j) f(x_n, x_j).
\end{aligned}$$

Dividing the factor

$$\begin{aligned}
&- x_n f(zx_l x_n, 1) f(zx_l, zx_n) f(zx_n, z/y_n) f(z^2 x_n/y_n, z) \prod_{j=l}^{n} x_j \\
&\times \prod_{j=l}^{n-1} f(zx_n, 1/x_j) f(x_n, x_j) \prod_{j=l+1}^{n} f(x_l/x_j, 1) f(zx_l x_j, 1),
\end{aligned}$$

we could get it. □

It is straightforward to check that $f(x,y) = 1 - x/y$ satisfies (8) and (17). We thus obtain

Corollary 5.1. *Let*

$$f_{n,k} = \frac{(1-x_k y_k)(1-zx_k/y_k)}{(1-x_n y_n)(1-zx_n/y_n)} \frac{\prod_{j=k+1}^{n}(1-zx_n/y_j)(1-x_n y_j)}{\prod_{j=k}^{n-1} x_j(1-x_n/x_j)(1-zx_n x_j)},$$

and

$$g_{n,k} = \frac{\prod_{j=k}^{n-1}(1-x_k y_j)(1-zx_k/y_j)}{\prod_{j=k+1}^{n} x_j(1-x_k/x_j)(1-zx_k x_j)}.$$

Then $(f_{n,k})$ *and* $(g_{n,k})$ *form an inversion pair.*

We remark that this inversion pair was first given by Krattenthaler [9, Eq.(1.5)]. From this inversion pair, we can derive several classical inversion pairs.

Example 5.1. Set $z = a$, $x_j = bq^j$ and $y_j = q^j$ in Corollary 5.1. After some simplifications, we derive the inversion pair

$$f_{n,k} = \frac{(1-aq^{2k})(b;q)_{n+k}(ba^{-1};q)_{n-k}(ba^{-1})^k}{(1-a)(aq;q)_{n+k}(q;q)_{n-k}}$$

and

$$g_{n,k} = \frac{(1-bq^{2k})(a;q)_{n+k}(ab^{-1};q)_{n-k}(ab^{-1})^k}{(1-b)(bq;q)_{n+k}(q;q)_{n-k}}.$$

This pair is due to Bressoud [3], which was used to study the finite forms of Rogers-Ramanujan identities.

Example 5.2. Set $z = a$, $x_j = bp^j$ and $y_j = q^j$ in Corollary 5.1. After some simplifications, we derive the following inversion pair

$$f_{n,k} = p^{k-n} \frac{(1-bp^k q^k)(1-ap^{-k}q^k/b)(bp^n q^n;p^{-1})_{n-k}(bp^n q^{-n}/a;p^{-1})_{n-k}}{(1-bp^n q^n)(1-ap^{-n}q^n/b)(aq^{2n-1};q^{-1})_{n-k}(q^{-1};q^{-1})_{n-k}}$$

and

$$g_{n,k} = \frac{(bp^k q^k;q^{-1})_{n-k}(bp^k q^{-k};q^{-1})_{n-k}}{(aq^{2k+1};q)_{n-k}(q;q)_{n-k}}.$$

Notice that it is an inversion pair that involves rising q-factorials with two different bases, mentioned by Gasper [5]. They used this matrix inverse to derive numerous bibasic, cubic, and quartic summation formulas for basic hypergeometric series.

By the definition of theta functions, we see that $f(x,y) = \theta(x/y;p)$ also satisfies (8) and (17). Therefore,

Corollary 5.2. *Let*

$$f_{n,k} = \frac{\theta(x_k y_k;p)\theta(zx_k/y_k;p)}{\theta(x_n y_n;p)\theta(zx_n/y_n;p)} \frac{\prod_{j=k+1}^{n}\theta(zx_n/y_j;p)\theta(zx_n y_j;p)}{\prod_{j=k}^{n-1} x_j\theta(x_n/x_j;p)\theta(zx_n x_j;p)}$$

and

$$g_{n,k} = \frac{\prod_{j=k}^{n-1}\theta(x_k y_j)\theta(zx_k/y_j)}{\prod_{k+1}^{n} x_j\theta(x_k/x_j)\theta(zx_k x_j)}.$$

Then $(f_{n,k})$ *and* $(g_{n,k})$ *form an inversion pair.*

This is a pair of inverse matrices due to Warnaar [14] and the elliptic analogue of inverse matrices of Krattenthaler [9, Eq.(1.5)]. Moreover, we can derive the classical pairs of inverse matrices.

Example 5.3. Set $z = ab$, $x_j = aq^j$ and $y_j = r^j$ in Corollary 5.2, making some simplifications, we can derive a pair of inverse matrices

$$f_{n,k} = (-1)^{n-k} q^{\binom{n-k}{2}} \frac{\theta(aq^k r^k; p)\theta(q^k r^{-k}/b; p)}{\theta(aq^n r^n; p)\theta(q^n r^{-n}/b; p)} \frac{(aq^{k+1}r^n, q^{k+1}r^{-n}/b; q, p)_{n-k}}{(r, abr^{n+k}; r, p)_{n-k}}$$

then

$$g_{n,k} = \frac{(aq^k r^k, q^k r^{-k}/b; q, p)_{n-k}}{(r, abr^{2k+1}; r, p))_{n-k}}$$

We note that the case of $p = 0$ corresponds to [5, Eqs. (3.2) and (3.3)] and [9, Eq. (4.3)].

Example 5.4. Set $z = ab$, $x_j = aq^j$ and $y_j = q^{rj}$ in Corollary 5.2. After some simplifications, we derive the following inversion pair

$$\begin{aligned} f_{n,k} =& \frac{(b; q, p)_{rn}}{(aq; q, p)_{rn}} \frac{\theta(aq^{(r+1)k}; p)\theta(bq^{(r-1)k}; p)}{\theta(a; p)\theta(b; p)} \\ & \times \frac{(a, 1/b; q, p)_k}{(q^r, abq^r; q^r, p)_k} \frac{(abq^{rn}/b, q^{-rn}; q^r, p)_k}{(q^{1-rn}, aq^{rn+1}; q, p)_k} q^k \end{aligned}$$

and

$$g_{n,k} = \frac{\theta(abq^{2rk}; p)}{\theta(ab; p)} \frac{(aq^n; q, p)_{rk}}{(bq^{1-n}; q, p)_{rk}} \frac{(ab, q^{-rn}; q^r, p)_k}{(q^r, abq^{rn+r}; q^r, p)_k} q^{rk}.$$

It is due to Warnaar [14, Eq.(3.5)], who used it to set up a series of summation and transformation formulas for terminating, balanced, very-well poised, elliptic hypergeometric series.

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