C^*-ALGEBRAS ASSOCIATED WITH ASYMPTOTIC EQUIVALENCE RELATIONS DEFINED BY HYPERBOLIC TORAL AUTOMORPHISMS

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Abstract. We study the C^*-algebras of the étale groupoids defined by the asymptotic equivalence relations for hyperbolic automorphisms on the two-dimensional torus. The algebras are proved to be isomorphic to four-dimensional non-commutative tori by an explicit numerical computation. The ranges of the unique tracial states of its K_0-groups of the C^*-algebras are described in terms of the hyperbolic matrices of the automorphisms on the torus.

1. **Introduction.** In [22] and [23], D. Ruelle has introduced the notion of Smale space. A Smale space is a hyperbolic dynamical system with local product structure. He has constructed groupoids and its operator algebras from the Smale spaces. After the Ruelle's initial study, I. Putnam in [14] (cf. [9], [15], [16], [17], [26], etc.) constructed various groupoids from Smale spaces and studied their C^*-algebras. The class of Smale spaces contain two important subclasses of topological dynamical systems as its typical examples. One is the class of shifts of finite type, which are sometimes called topological Markov shifts. The other one is the class of hyperbolic toral automorphisms. The study of the former class from the view point of C^*-algebras is closely related to the study of Cuntz-Krieger algebras as in [7], [8], [10], [11], [12], etc. That of the latter class is closely related to the study of the crossed product C^*-algebras of the homeomorphisms of the hyperbolic automorphisms on the torus.

In this paper, we will focus on the study of the latter class, the hyperbolic toral automorphisms from the view points of C^*-algebras constructed from the associated groupoids as Smale spaces. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ be a hyperbolic matrix. Let $q : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$ be the natural quotient map. We denote by $\mathbb{R}^2/\mathbb{Z}^2$ the two-dimensional torus $\mathbb{T}^2$ with metric d defined by

$$d(x, y) = \inf\{\|z - w\| : q(z) = x, q(w) = y, z, w \in \mathbb{R}^2\} \quad \text{for } x, y \in \mathbb{T}^2$$

where $\|\cdot\|$ is the Euclid norm on $\mathbb{R}^2$. Then the matrix A defines a homeomorphism on $\mathbb{T}^2$ which is called a hyperbolic toral automorphism. It is a specific example of an Anosov diffeomorphism on a compact Riemannian manifold (see [4], [25], etc.).

Let λ_u, λ_s be the eigenvalues of A such that $|\lambda_u| > 1 > |\lambda_s|$. They are both real numbers. Let $v_u = (u_1, u_2), v_s = (s_1, s_2)$ be the normalized eigenvectors for λ_u, λ_s, respectively. The direction along v_u expands by A, whereas the direction of v_s expands by A^{-1}. These directions determine local product structure which makes $\mathbb{T}^2$ a Smale space. The groupoid G_A^a introduced by D. Ruelle [22] of the asymptotic equivalence relation is defined by

$$G_A^a = \{(x, z) \in \mathbb{T}^2 \times \mathbb{T}^2 \mid \lim_{n \to \infty} d(A^n x, A^n z) = \lim_{n \to \infty} d(A^{-n} x, A^{-n} z) = 0\} \qquad (1)$$

with its unit space

$$(G_A^a)^{(0)} = \{(x, x) \in \mathbb{T}^2 \times \mathbb{T}^2\} = \mathbb{T}^2. \qquad (2)$$

The multiplication and the inverse operation on G_A^a are defined by

$$(x, z)(z, w) = (x, w), \qquad (x, z)^{-1} = (z, x) \qquad \text{for } (x, z), (z, w) \in G_A^a.$$

As in [14], the groupoid G_A^a has a natural topology defined by inductive limit topology, which makes G_A^a étale. The étale groupoid G_A^a is called the asymptotic groupoid for the hyperbolic toral automorphism $(\mathbb{T}^2, A)$. We will first see that the groupoid G_A^a is realized as a transformation groupoid $\mathbb{T}^2 \times_{\alpha^A} \mathbb{Z}^2$ by a certain action $\alpha^A : \mathbb{Z}^2 \longrightarrow \text{Homeo}(\mathbb{T}^2)$ associated to G_A^a, so that the C^*-algebra $C^*(G_A^a)$ of the groupoid G_A^a is isomorphic to the C^*-algebra of the crossed product $C(\mathbb{T}^2) \times_{\alpha^A} \mathbb{Z}^2$ by the induced action $\alpha^A : \mathbb{Z}^2 \longrightarrow \text{Aut}(C(\mathbb{T}^2))$. As the action $\alpha^A : \mathbb{Z}^2 \longrightarrow \text{Homeo}(\mathbb{T}^2)$ is free and minimal having a unique invariant ergodic measure, a general theory of C^*-crossed product ensures that $C(\mathbb{T}^2) \times_{\alpha^A} \mathbb{Z}^2$ is a simple AT-algebra having a unique tracial state (cf. [13], [14], [17]).

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2, \mathbb{Z})$ be a hyperbolic matrix which satisfies $\det(A) = \pm 1$, We denote by $\Delta(A) = (a + d)^2 - 4(ad - bc)$ the discriminant of the characteristic polynomial of the matrix A, which is positive. We will show the following result.

Theorem 1.1 (Theorem 2.10 and Proposition 3.1)**.** *The C^*-algebra $C^*(G_A^a)$ of the étale groupoid G_A^a for a hyperbolic matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a simple AT-algebra with unique tracial state τ that is isomorphic to the four-dimensional non-commutative torus generated by four unitaries U_1, U_2, V_1, V_2 satisfying the following relations:*

$$U_1 U_2 = U_2 U_1, \qquad V_1 V_2 = V_2 V_1,$$
$$V_1 U_1 = e^{2\pi i \theta_1} U_1 V_1, \qquad V_1 U_2 = e^{2\pi i \theta_2} U_2 V_1,$$
$$V_2 U_1 = e^{2\pi i \theta_3} U_1 V_2, \qquad V_2 U_2 = e^{2\pi i \theta_4} U_2 V_2,$$

where

$$\theta_1 = \frac{1}{2}(1 + \frac{a - d}{\sqrt{\Delta(A)}}), \quad \theta_2 = \frac{c}{\sqrt{\Delta(A)}}, \quad \theta_3 = \frac{b}{\sqrt{\Delta(A)}}, \quad \theta_4 = \frac{1}{2}(1 - \frac{a - d}{\sqrt{\Delta(A)}}).$$

The range $\tau_*(K_0(C^*(G_A^a)))$ of the tracial state τ of the K_0-group $K_0(C^*(G_A^a))$ of the C^*-algebra $C^*(G_A^a)$ is

$$\tau_*(K_0(C^*(G_A^a))) = \mathbb{Z} + \mathbb{Z}\theta_1 + \mathbb{Z}\theta_2 + \mathbb{Z}\theta_3 \quad in \ \mathbb{R}. \qquad (3)$$

We note that the slopes $\theta_i, i = 1, 2, 3, 4$ are determined by the formulas (10), (11) for the slopes of the eigenvectors $v_u = (u_1, u_2), v_s = (s_1, s_2)$.

Since the étale groupoid G_A^a is a flip conjugacy invariant and the C^*-algebra $C^*(G_A^a)$ has a unique tracial state written τ, we know that the trace value $\tau_*(K_0(C^*(G_A^a)))$ is a flip conjugacy invariant of the hyperbolic toral automorphism $(\mathbb{T}^2, A)$.

As commuting matrices have common eigenvectors, we know that if two matrices $A, B \in \mathrm{GL}(2, \mathbb{Z})$ commute with each other, then the C^*-algebras $C^*(G_A^a)$ and $C^*(G_B^a)$ are canonically isomorphic. Hence two matrices $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ have the isomorphic C^*-algebras. On the other hand, as the range $\tau_*(K_0(C^*(G_A^a)))$ of the tracial state of the K_0-group $K_0(C^*(G_A^a))$ is invariant under isomorphism class of the algebra $C^*(G_A^a)$, the C^*-algebra $C^*(G_{A_1})$ is not isomorphic to $C^*(G_{A_2})$ for the matrices $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ (Proposition 4.2).

2. **The groupoid G_A^a and its C^*-algebra $C^*(G_A^a)$.** For a vector $(m, n) \in \mathbb{R}^2$, we write the vector $(m, n)^t$ as $\begin{bmatrix} m \\ n \end{bmatrix}$ and sometimes identify (m, n) with $\begin{bmatrix} m \\ n \end{bmatrix}$. A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ with $\det(A) = \pm 1$ is said to be hyperbolic if A does not have eigenvalues of modulus 1. Let λ_u, λ_s be the eigenvalues of A such that $|\lambda_u| > 1 > |\lambda_s|$. They are eigenvalues for unstable direction, stable direction, respectively. We note that $b \neq 0, c \neq 0$ because of the conditions $ad - bc = \pm 1$ and $|\lambda_u| > 1 > |\lambda_s|$. Take nonzero eigenvectors v_u, v_s for the eigenvalues λ_u, λ_s such that $\|v_u\| = \|v_s\| = 1$. We set $v_u = (u_1, u_2), v_s = (s_1, s_2) \in \mathbb{T}^2$ as vectors. The numbers $\lambda_u, \lambda_s, u_1, u_2, s_1, s_2$ are all real numbers because of the hyperbolicity of the matrix A. It is easy to see that the slopes $\frac{u_1}{u_2}, \frac{s_1}{s_2}$ are irrational. We set

$$r_A := \langle v_u | v_s \rangle.$$

Define two vectors

$$v_1 := v_u - r_A v_s, \qquad v_2 := r_A v_u - v_s.$$

Lemma 2.1. *For two vectors $x, z \in \mathbb{T}^2$, the following three conditions are equivalent.*

(i) $(x, z) \in G_A^a$.
(ii) $z = x + \frac{1}{1-r_A^2} \langle (m, n) | v_1 \rangle v_u$ *for some $m, n \in \mathbb{Z}$.*
(iii) $z = x + \frac{1}{1-r_A^2} \langle (m, n) | v_2 \rangle v_s$ *for some $m, n \in \mathbb{Z}$.*

Proof. For two vectors $x, z \in \mathbb{T}^2$ regarding them as elements of $\mathbb{R}^2$ modulo $\mathbb{Z}^2$, we have $(x, z) \in G_A^a$ if and only if

$$z \equiv x + tv_u \equiv x + sv_s \pmod{\mathbb{Z}^2} \qquad \text{for some } t, s \in \mathbb{R}. \tag{4}$$

In this case, we see that $tv_u - sv_s = (m, n)$ for some $m, n \in \mathbb{Z}$ so that

$$\langle tv_u - sv_s | v_u \rangle = \langle (m, n) | v_u \rangle, \tag{5}$$

$$\langle tv_u - sv_s | v_s \rangle = \langle (m, n) | v_s \rangle \tag{6}$$

and we have

$$t = \frac{1}{1 - r_A^2} \langle (m, n) | v_1 \rangle, \qquad s = \frac{1}{1 - r_A^2} \langle (m, n) | v_2 \rangle. \tag{7}$$

This shows the implications (i) $\Longrightarrow$ (ii) and (iii).

Assume that (ii) holds. By putting $s = \frac{1}{1-r_A^2} \langle (m, n) | v_2 \rangle$, we have the equalities both (5) and (6), so that $tv_u - sv_s = (m, n)$. Hence the equality (4) holds and we see that (x, z) belongs to the groupoid G_A^a. This shows that the implication (ii) $\Longrightarrow$ (i) holds, and similarly (iii) $\Longrightarrow$ (i) holds. $\square$

Let us define an action $\alpha^A : \mathbb{Z}^2 \longrightarrow \mathrm{Homeo}(\mathbb{T}^2)$ in the following way. We set

$$\alpha^A_{(m,n)}(x) := x + \frac{1}{1 - r_A^2}\langle(m,n)|v_1\rangle v_u, \qquad (m,n) \in \mathbb{Z}^2,\, x \in \mathbb{T}^2.$$

For a fixed $(m,n) \in \mathbb{Z}^2$, the map $x \in \mathbb{T}^2 \longrightarrow \alpha^A_{(m,n)}(x) \in \mathbb{T}^2$ is the parallel transformation along the vector $\frac{1}{1-r_A^2}\langle(m,n)|v_1\rangle v_u$. Hence $\alpha^A_{(m,n)}$ defines a homeomorphism on the torus $\mathbb{T}^2$. It is clear to see that $\alpha^A_{(m,n)} \circ \alpha^A_{(k,l)} = \alpha^A_{(m+k,n+l)}$ for $(m,n),(k,l) \in \mathbb{Z}^2$.

Lemma 2.2. *Keep the above notation.*

  (i) *If $\alpha^A_{(m,n)}(x) = x$ for some $x \in \mathbb{T}^2$, then $(m,n) = (0,0)$. Hence the action $\alpha^A : \mathbb{Z}^2 \longrightarrow \mathrm{Homeo}(\mathbb{T}^2)$ is free.*
  (ii) *For $x \in \mathbb{T}^2$, the set $\{\alpha^A_{(m,n)}(x)| \in (m,n) \in \mathbb{Z}^2\}$ is dense in $\mathbb{T}^2$. Hence the action $\alpha^A : \mathbb{Z}^2 \longrightarrow \mathrm{Homeo}(\mathbb{T}^2)$ is minimal.*

Proof. (i) Suppose that $\alpha^A_{(m,n)}(x) = x$ for some $x \in \mathbb{T}^2$, so that $\frac{1}{1-r_A^2}\langle(m,n)|v_1\rangle v_u = (k,l)$ for some $(k,l) \in \mathbb{Z}^2$. As the slope of the vector v_u is irrational, we have $(k,l) = (0,0)$ and hence $(m,n) = (0,0)$.

(ii) Let $v_1 = (\gamma_1, \gamma_2)$. As the slope of v_s is irrational and $\langle v_s|v_1\rangle = 0$, the slope $\frac{\gamma_1}{\gamma_2}$ of v_1 is irrational, so that the set $\{m\gamma_1 + n\gamma_2|m,n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Since $\langle(m,n)|v_1\rangle v_u = (m\gamma_1 + n\gamma_2)v_u$ and the set $\{x + tv_u \in \mathbb{T}^2|t \in \mathbb{R}\}$ is dense in $\mathbb{T}^2$, we see that the set

$$\{x + \frac{1}{1 - r_A^2}\langle(m,n)|v_1\rangle v_u \mid (m,n) \in \mathbb{Z}^2\}$$

is dense in $\mathbb{T}^2$. $\square$

The action $\alpha^A : \mathbb{Z}^2 \longrightarrow \mathrm{Homeo}(\mathbb{T}^2)$ induces an action of $\mathbb{Z}^2$ to the automorphism group $\mathrm{Aut}(C(\mathbb{T}^2))$ of $C(\mathbb{T}^2)$ by $f \in C(\mathbb{T}^2) \longrightarrow f \circ \alpha^A_{(m,n)} \in C(\mathbb{T}^2)$. We write it still α^A without confusing.

If a discrete group Γ acts freely on a compact Hausdorff space X by an action $\alpha : \Gamma \longrightarrow \mathrm{Homeo}(X)$, the set $\{(x, \alpha_\gamma(x)) \in X \times X|x \in X, \gamma \in \Gamma\}$ has a groupoid structure in a natural way (cf. [2], [18], [19]). The groupoid is called a transformation groupoid written $X \times_\alpha \Gamma$.

Proposition 2.3. *The étale groupoid G_A^a is isomorphic to the transformation groupoid*

$$\mathbb{T}^2 \times_{\alpha^A} \mathbb{Z}^2 = \{(x, \alpha^A_{(m,n)}(x)) \in \mathbb{T}^2 \times \mathbb{T}^2|(m,n) \in \mathbb{Z}^2\}$$

defined by the action $\alpha^A : \mathbb{Z}^2 \longrightarrow \mathrm{Homeo}(\mathbb{T}^2)$. Hence the C^*-algebra $C^*(G_A^a)$ of the groupoid G_A^a is isomorphic to the crossed product $C(\mathbb{T}^2) \times_{\alpha^A} \mathbb{Z}^2$ of $C(\mathbb{T}^2)$ by the action α^A of $\mathbb{Z}^2$.

Proof. By the preceding discussions, a pair $(x,z) \in \mathbb{T}^2$ belongs to the groupoid G_A^a if and only if $z = \alpha^A_{(m,n)}(x)$ for some $(m,n) \in \mathbb{Z}^2$. Since the action $\alpha^A : \mathbb{Z}^2 \longrightarrow \mathrm{Homeo}(\mathbb{T}^2)$ is free, the groupoid G_A^a is identified with the transformation groupoid $\mathbb{T}^2 \times_{\alpha^A} \mathbb{Z}^2$ in a natural way. By a general theory of the C^*-algebras of groupoids ([2], [18]), the C^*-algebra $C^*(\mathbb{T}^2 \times_{\alpha^A} \mathbb{Z}^2)$ of the groupoid $\mathbb{T}^2 \times_{\alpha^A} \mathbb{Z}^2$ is isomorphic to the crossed product $C^*(\mathbb{T}^2) \times_{\alpha^A} \mathbb{Z}^2$. $\square$

Remark 2.4. Define a map $\alpha_A : \mathbb{Z}^2 \longrightarrow \mathbb{T}^2$ by

$$\alpha_A(m,n) := \frac{1}{1 - r_A^2} \langle (m,n)|v_1 \rangle v_u, \qquad (m,n) \in \mathbb{Z}^2. \tag{8}$$

It is easy to see that the étale groupoid G_A^a may be written

$$G_A^a = \mathbb{T}^2 \times \alpha_A(\mathbb{Z}^2) \tag{9}$$

as a transformation groupoid.

We set

$$\theta_1 := \frac{u_1 s_2}{u_1 s_2 - u_2 s_1}, \qquad \theta_2 := \frac{u_2 s_2}{u_1 s_2 - u_2 s_1}, \tag{10}$$

$$\theta_3 := \frac{-u_1 s_1}{u_1 s_2 - u_2 s_1}, \qquad \theta_4 := \frac{-u_2 s_1}{u_1 s_2 - u_2 s_1}. \tag{11}$$

Lemma 2.5. *The real numbers $\theta_i, i = 1, 2, 3, 4$ satisfy*

$$\frac{\theta_2}{\theta_1} = \frac{\theta_4}{\theta_3} = \frac{u_2}{u_1}, \qquad \frac{\theta_1}{\theta_3} = \frac{\theta_2}{\theta_4} = -\frac{s_2}{s_1}, \tag{12}$$

$$\theta_1 + \theta_4 = 1. \tag{13}$$

Conversely, if real numbers $\zeta_i, i = 1, 2, 3, 4$ satisfy

$$\frac{\zeta_2}{\zeta_1} = \frac{\zeta_4}{\zeta_3} = \frac{u_2}{u_1}, \qquad \frac{\zeta_1}{\zeta_3} = \frac{\zeta_2}{\zeta_4} = -\frac{s_2}{s_1}, \tag{14}$$

$$\zeta_1 + \zeta_4 = 1, \tag{15}$$

then we have $\zeta_i = \theta_i, i = 1, 2, 3, 4$.

Proof. The identities (12) and (13) are immediate. Conversely, suppose that real numbers $\zeta_i, i = 1, 2, 3, 4$ satisfy (14) and (15). As $\zeta_1 = \frac{u_2}{u_1}\zeta_2 = \frac{u_2}{u_1}(-\frac{s_2}{s_1})\zeta_4$, the equality (15) implies

$$\{\frac{u_2}{u_1}(-\frac{s_2}{s_1}) + 1\}\zeta_4 = 1,$$

so that

$$\zeta_4 = \frac{-u_2 s_1}{u_1 s_2 - u_2 s_1}$$

and hence

$$\zeta_1 = \frac{u_1 s_2}{u_1 s_2 - u_2 s_1}, \qquad \zeta_2 = \frac{u_2 s_2}{u_1 s_2 - u_2 s_1}, \qquad \zeta_3 = \frac{-u_1 s_1}{u_1 s_2 - u_2 s_1}.$$

$\square$

Proposition 2.6. *For $x = (x_1, x_2) \in \mathbb{T}^2$, we have*

$$\alpha_{(1,0)}^A(x_1, x_2) = (x_1 + \theta_1, x_2 + \theta_2), \qquad \alpha_{(0,1)}^A(x_1, x_2) = (x_1 + \theta_3, x_2 + \theta_4),$$

and hence

$$\alpha_{(m,n)}^A(x_1, x_2) = (x_1 + m\theta_1 + n\theta_3, x_2 + m\theta_2 + n\theta_4) \quad \text{for } (m,n) \in \mathbb{Z}^2.$$

Proof. We have

$$\alpha_{(m,n)}^A(x_1, x_2) = (x_1, x_2) + \frac{1}{1 - r_A^2} \langle (m,n)|v_u - r_A v_s \rangle v_u$$

$$= (x_1, x_2) + \frac{1}{1 - r_A^2} \langle (m,n)|(u_1 - r_A s_1, u_2 - r_A s_2) \rangle (u_1, u_2).$$

In particular, for $(m, n) = (1, 0), (0, 1)$, we have

$$\alpha_{(1,0)}^A(x_1, x_2) = (x_1 + \frac{1}{1 - r_A^2}(u_1 - r_A s_1)u_1, \; x_2 + \frac{1}{1 - r_A^2}(u_1 - r_A s_1)u_2),$$

$$\alpha_{(0,1)}^A(x_1, x_2) = (x_1 + \frac{1}{1 - r_A^2}(u_2 - r_A s_2)u_1, \; x_2 + \frac{1}{1 - r_A^2}(u_2 - r_A s_2)u_2).$$

We put $\xi_i = \frac{1}{1 - r_A^2}(u_i - r_A s_i)$ for $i = 1, 2$ so that

$$\alpha_{(1,0)}^A(x_1, x_2) = (x_1 + \xi_1 u_1, \; x_2 + \xi_1 u_2), \tag{16}$$

$$\alpha_{(0,1)}^A(x_1, x_2) = (x_1 + \xi_2 u_1, \; x_2 + \xi_2 u_2). \tag{17}$$

We then have

$$\xi_1 = \frac{1}{1 - r_A^2}\{u_1 - (u_1 s_1 + u_2 s_2)s_1\} = \frac{1}{1 - r_A^2}\{u_1(1 - s_1^2) - u_2 s_2 s_1\}$$

$$= \frac{1}{1 - r_A^2}(u_1 s_2 - u_2 s_1)s_2$$

and similarly

$$\xi_2 = \frac{1}{1 - r_A^2}\{u_2 - (u_1 s_1 + u_2 s_2)s_2\} = \frac{1}{1 - r_A^2}\{u_2(1 - s_2^2) - u_1 s_1 s_2\}$$

$$= \frac{1}{1 - r_A^2}(u_2 s_1 - u_1 s_2)s_1.$$

Hence we have $\frac{\xi_1}{\xi_2} = -\frac{s_2}{s_1}$. We also have

$$\xi_1 u_1 + \xi_2 u_2 = \frac{1}{1 - r_A^2}\{(u_1 - r_A s_1)u_1 + (u_2 - r_A s_2)u_2\}$$

$$= \frac{1}{1 - r_A^2}\{u_1^2 + u_2^2 - r_A(u_1 s_1 + u_2 s_2)\}$$

$$= \frac{1}{1 - r_A^2}(1 - r_A^2) = 1.$$

By Lemma 2.5, we have $\xi_1 u_1 = \theta_1$, $\xi_1 u_2 = \theta_2$, $\xi_2 u_1 = \theta_3$, $\xi_2 u_2 = \theta_4$, proving the desired assertion from the identities (16) and (17). □

We will next express $\theta_i, i = 1, 2, 3, 4$ in terms of the matrix elements a, b, c, d of A.

Lemma 2.7. *The following identities hold.*

(i)

$$a\theta_1 + b\theta_2 = \lambda_u \theta_1, \qquad a\theta_3 + b\theta_4 = \lambda_u \theta_3,$$

$$c\theta_1 + d\theta_2 = \lambda_u \theta_2, \qquad c\theta_3 + d\theta_4 = \lambda_u \theta_4,$$

and hence

$$a\theta_1 + b\theta_2 + c\theta_3 + d\theta_4 = \lambda_u. \tag{18}$$

(ii)

$$a\theta_3 - b\theta_1 = \lambda_s \theta_3, \qquad a\theta_4 - b\theta_2 = \lambda_s \theta_4,$$

$$c\theta_3 - d\theta_1 = -\lambda_s \theta_1, \qquad c\theta_4 - d\theta_2 = -\lambda_s \theta_2,$$

and hence

$$a\theta_4 - b\theta_2 - c\theta_3 + d\theta_1 = \lambda_s.$$

Proof. By the identities

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \frac{s_2}{u_1 s_2 - u_2 s_1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix} = \frac{-s_1}{u_1 s_2 - u_2 s_1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$\begin{bmatrix} \theta_3 \\ -\theta_1 \end{bmatrix} = \frac{-u_1}{u_1 s_2 - u_2 s_1} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \qquad \begin{bmatrix} \theta_4 \\ -\theta_2 \end{bmatrix} = \frac{-u_2}{u_1 s_2 - u_2 s_1} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix},$$

with $\theta_1 + \theta_4 = 1$, we see the desired assertions. $\square$

Lemma 2.8.
  (i) $(a\theta_1 + b\theta_2)\theta_4 = (c\theta_3 + d\theta_4)\theta_1$.
  (ii) $(a\theta_3 - b\theta_1)\theta_2 = (-c\theta_4 + d\theta_2)\theta_3$.
Hence we have

$$b\theta_2 = c\theta_3.$$

Proof. (i) By the first and the fourth identities in Lemma 2.7 (i), we know the identity (i). The identities of (ii) is similarly shown to those of (i). By (i) and (ii) with the identity $\theta_1\theta_4 = \theta_2\theta_3$, we get $b\theta_2 = c\theta_3$. $\square$

Recall that $\Delta(A)$ denotes the discriminant $(a+d)^2 - 4(ad-bc)$ of the characteristic polynomial of the matrix A. The real number $\Delta(A)$ is positive because of the hyperbolicity of A. By elementary calculations, we see the following lemma.

Lemma 2.9. *The identities*

$$\theta_1 \cdot \theta_4 = \theta_2 \cdot \theta_3, \qquad \theta_1 + \theta_4 = 1,$$
$$(a\theta_1 + b\theta_2)\theta_4 = (c\theta_3 + d\theta_4)\theta_1, \qquad (a\theta_3 - b\theta_1)\theta_2 = (-c\theta_4 + d\theta_2)\theta_3$$

imply

$$(\theta_1, \theta_2, \theta_3, \theta_4) \tag{19}$$

$$= \begin{cases} \left(\frac{1}{2}(1 + \frac{|a-d|}{\sqrt{\Delta(A)}}), \frac{|a-d|}{a-d}\frac{c}{\sqrt{\Delta(A)}}, \frac{|a-d|}{a-d}\frac{b}{\sqrt{\Delta(A)}}, \frac{1}{2}(1 - \frac{|a-d|}{\sqrt{\Delta(A)}}) \right) & or \\ \left(\frac{1}{2}(1 - \frac{|a-d|}{\sqrt{\Delta(A)}}), -\frac{|a-d|}{a-d}\frac{c}{\sqrt{\Delta(A)}}, -\frac{|a-d|}{a-d}\frac{b}{\sqrt{\Delta(A)}}, \frac{1}{2}(1 + \frac{|a-d|}{\sqrt{\Delta(A)}}) \right) & if\ a \neq d, \\ (\frac{1}{2}, \frac{1}{2}\sqrt{\frac{c}{b}}, \frac{1}{2}\sqrt{\frac{b}{c}}, \frac{1}{2}) & or \\ (\frac{1}{2}, -\frac{1}{2}\sqrt{\frac{c}{b}}, -\frac{1}{2}\sqrt{\frac{b}{c}}, \frac{1}{2}) & if\ a = d. \end{cases}$$

$$\tag{20}$$

We thus have the following theorem.

Theorem 2.10. *The C^*-algebra $C^*(G_A^a)$ of the groupoid G_A^a for a hyperbolic matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is isomorphic to the simple C^*-algebra generated by four unitaries U_1, U_2, V_1, V_2 satisfying the following relations:*

$$U_1 U_2 = U_2 U_1, \qquad V_1 V_2 = V_2 V_1,$$
$$V_1 U_1 = e^{2\pi i\theta_1} U_1 V_1, \qquad V_1 U_2 = e^{2\pi i\theta_2} U_2 V_1,$$
$$V_2 U_1 = e^{2\pi i\theta_3} U_1 V_2, \qquad V_2 U_2 = e^{2\pi i\theta_4} U_2 V_2,$$

where

$$\theta_1 = \frac{1}{2}(1 + \frac{a-d}{\sqrt{\Delta(A)}}), \quad \theta_2 = \frac{c}{\sqrt{\Delta(A)}}, \quad \theta_3 = \frac{b}{\sqrt{\Delta(A)}}, \quad \theta_4 = \frac{1}{2}(1 - \frac{a-d}{\sqrt{\Delta(A)}}).$$

$$\tag{21}$$

Hence the C^*-algebra $C^*(G_A^a)$ is isomorphic to the four-dimensional non-commutative torus.

Proof. As in Lemma 2.2, the action $\alpha^A : \mathbb{Z}^2 \longrightarrow \text{Homeo}(\mathbb{T}^2)$ is free and minimal, hence the C^*-crossed product $C(\mathbb{T}^2) \times_{\alpha^A} \mathbb{Z}^2$ is simple. The C^*-crossed product is canonically identified with the C^*-crossed product $((C(\mathbb{T}) \otimes C(\mathbb{T})) \times_{\alpha^A_{(1,0)}} \mathbb{Z}) \times_{\alpha^A_{(0,1)}} \mathbb{Z}$. Let U_1, U_2 be the unitaries in $C(\mathbb{T}) \otimes C(\mathbb{T})$ defined by $U_1(t, s) = e^{2\pi it}, U_2(t, s) = e^{2\pi is}$. Let V_1, V_2 be the implementing unitaries corresponding to the automorphisms $\alpha^A_{(1,0)}, \alpha^A_{(0,1)}$, respectively. By Proposition 2.6, we know the commutation relations among the unitaries U_1, U_2, V_1, V_2 for the slopes $\theta_1, \theta_2, \theta_3, \theta_4$ satisfying (20). The second values of (20) go to the first of (20) by substituting V_1, U_1 with V_2, U_2, respectively. The forth values of (20) go to the third of (20) by substituting V_1, U_1 with V_1^*, U_1^*, respectively. When $a = d$, we have $\Delta(A) = 4bc > 0$ so that $\pm\sqrt{\frac{c}{b}} = \frac{c}{\sqrt{\Delta(A)}}, \pm\sqrt{\frac{b}{c}} = \frac{b}{\sqrt{\Delta(A)}}$. Hence the first two of (20) include the second two of (20), so that we may unify (20) into (21). $\qquad\square$

Since the C^*-algebra $C^*(G_A^a)$ is isomorphic to a four-dimensional non-commutative torus, we know the following proposition by Slawny [24] (see also Putnam[14]).

Proposition 2.11 (Slawny [24], Putnam [14]). *The C^*-algebra $C^*(G_A^a)$ has a unique tracial state.*

Remark 2.12.   (i) We note that the simplicity of the algebra $C^*(G_A^a)$ comes from a general theory of Smale space C^*-algebras as in [14], [17] as well as a unique existence of tracial state on it. It also follows from a general theory of crossed product C^*-algebras because the action α^A of $\mathbb{Z}^2$ to $\text{Homeo}(\mathbb{T}^2)$ is free and minimal. It has been shown that a simple higher dimensional non-commutative torus is an AT-algebra by Phillips [13].

(ii) Suppose that two hyperbolic matrices $A, B \in GL(2, \mathbb{Z})$ commute each other. By (8) and (9), the equality $\alpha_A(\mathbb{Z}^2) = \alpha_B(\mathbb{Z}^2)$ holds for the commuting matrices A and B, because they have the same eigenvectors. Hence we know that $G_A^a = G_B^a$, so that the C^*-algebras $C^*(G_A^a)$ and $G_B^a)$ are isomorphic.

3. **The range** $\tau_*(K_0(C^*(G_A^a)))$. In this section, we will describe the trace values $\tau_*(K_0(C^*(G_A^a)))$ of the K_0-group of the C^*-algebra $C^*(G_A^a)$ in terms of the hyperbolic matrix A.

In [20], M. A. Rieffel studied K-theory for irrational rotation C^*-algebras A_θ with irrational numbers θ, which are called two-dimensional non-commutative tori, and proved that $\tau_*(K_0(A_\theta)) = \mathbb{Z} + \mathbb{Z}\theta$ in $\mathbb{R}$, where τ is the unique tracial state on A_θ. In [6], G. A. Elliott (cf. [3], [13], [21], [24], etc.) initiated to study higher-dimensional non-commutative tori. It is well-known the K-groups of the four-dimensional non-commutative torus as in [6] which says

$$K_0(C(\mathbb{T}^2) \times_{\alpha^A} \mathbb{Z}^2) \cong K_1(C(\mathbb{T}^2) \times_{\alpha^A} \mathbb{Z}^2) \cong \mathbb{Z}^8$$

([6], cf. [24]). For $g = (a_1, b_1, a_2, b_2), h = (c_1, d_1, c_2, d_2) \in \mathbb{Z}^4$, we define a wedge product $g \wedge h \in \mathbb{Z}^4$ by

$$(a_1, b_1, a_2, b_2) \wedge (c_1, d_1, c_2, d_2) = \left(\begin{vmatrix} a_1 & c_1 \\ b_1 & d_1 \end{vmatrix}, \begin{vmatrix} a_1 & c_1 \\ b_2 & d_2 \end{vmatrix}, \begin{vmatrix} a_2 & c_2 \\ b_1 & d_1 \end{vmatrix}, \begin{vmatrix} a_2 & c_2 \\ b_2 & d_2 \end{vmatrix} \right)$$

where $\begin{vmatrix} x & y \\ z & w \end{vmatrix} = xw - yz$. Let $\Theta = [\theta_{jk}]_{j,k=1}^4$ be a 4×4 skew symmetric matrix over $\mathbb{R}$. We regard the matrix Θ as a linear map from $\mathbb{Z}^4 \wedge \mathbb{Z}^4$ to $\mathbb{R}$ by $\Theta(x \wedge y) = \Theta x \cdot y$. Then $\Theta \wedge \Theta : (\mathbb{Z}^4 \wedge \mathbb{Z}^4) \wedge (\mathbb{Z}^4 \wedge \mathbb{Z}^4) = \wedge^4 \mathbb{Z}^4 \longrightarrow \mathbb{R}$ is defined by

$$(\Theta \wedge \Theta)(x_1 \wedge x_2) \wedge (x_3 \wedge x_4) = \frac{1}{2!2!} \sum_{\sigma \in \mathfrak{S}_4} \mathrm{sgn}(\sigma) \Theta(x_{\sigma(1)} \wedge x_{\sigma(2)}) \Theta(x_{\sigma(3)} \wedge x_{\sigma(4)})$$

for $x_1, x_2, x_3, x_4 \in \mathbb{Z}^4$. Although we may generally define $\wedge^n \Theta : \wedge^{2n} \mathbb{Z}^4 \longrightarrow \mathbb{R}$, the wedge product $\wedge^{2n} \mathbb{Z}^4 = 0$ for $n > 3$, so that

$$\exp_\wedge(\Theta) = 1 \oplus \Theta \oplus \frac{1}{2}(\Theta \wedge \Theta) \oplus \frac{1}{6}(\Theta \wedge \Theta \wedge \Theta) \oplus \cdots : \wedge^{\mathrm{even}} \mathbb{Z}^4 \longrightarrow \mathbb{R}$$

becomes

$$\exp_\wedge(\Theta) = 1 \oplus \Theta \oplus \frac{1}{2}(\Theta \wedge \Theta).$$

Let A_Θ be the C^*-algebra generated by four unitaries $u_j, j = 1, 2, 3, 4$ satisfying the commutation relations $u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j$, $j, k = 1, 2, 3, 4$. The C^*-algebra A_Θ is called the four-dimensional non-commutative torus ([6]). If Θ is non-degenerate, the algebra A_Θ has a unique tracial state written τ. By Elliott's result in [6], there exists an isomorphism $h : K_0(A_\Theta) \longrightarrow \wedge^{\mathrm{even}} \mathbb{Z}^4$ such that $\exp_\wedge(\Theta) \circ h = \tau_*$, so that we have

$$\exp_\wedge(\Theta)(\wedge^{\mathrm{even}} \mathbb{Z}^4) = \tau_*(K_0(A_\Theta)). \tag{22}$$

Proposition 3.1. *Let τ be the unique tracial state on $C^*(G_A^a)$. Then we have*

$$\tau_*(K_0(C^*(G_A^a))) = \mathbb{Z} + \mathbb{Z}\theta_1 + \mathbb{Z}\theta_2 + \mathbb{Z}\theta_3 \quad \textit{in } \mathbb{R}. \tag{23}$$

Proof. Take the unitaries U_1, U_2, V_1, V_2 and the real numbers $\theta_1, \theta_2, \theta_3, \theta_4$ as in Theorem 2.10. We set the real numbers $\theta_{jk}, j, k = 1, 2, 3, 4$ such as $\theta_{jj} = \theta_{12} = \theta_{21} = \theta_{34} = \theta_{43} = 0$ for $j = 1, 2, 3, 4$ and $\theta_{13} = \theta_4, \theta_{14} = \theta_3, \theta_{23} = \theta_2, \theta_{24} = \theta_1$. Let $u_1 = V_2, u_2 = V_1, u_3 = U_2, u_4 = U_1$ so that we have the commutation relations

$$u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j, \qquad j, k = 1, 2, 3, 4.$$

As $\theta_1 \cdot \theta_4 = \theta_2 \cdot \theta_3$, we have

$$\theta_{12}\theta_{34} - \theta_{13}\theta_{24} + \theta_{14}\theta_{23} = 0.$$

By (22) or [6] (cf. [3, 2.21], [13, Theorem 3.9]), we have

$$\tau_*(K_0(C^*(G_A^a))) = \mathbb{Z} + \mathbb{Z}(\theta_{12}\theta_{34} - \theta_{13}\theta_{24} + \theta_{14}\theta_{23}) + \sum_{1 \le j < k \le 4} \mathbb{Z}\theta_{jk}$$

$$= \mathbb{Z} + \mathbb{Z}\theta_1 + \mathbb{Z}\theta_2 + \mathbb{Z}\theta_3.$$

$\square$

Remark 3.2.   (i) It is straightforward to see that the skew symmetric matrix $\Theta = [\theta_{jk}]_{j,k=1}^4$ in our setting above is non-degenerate.

(ii) Suppose that two hyperbolic toral automorphisms $(\mathbb{T}^2, A)$ and $(\mathbb{T}^2, B)$ are topologically conjugate. We then know that both the C^*-algebras $C^*(G_A^a)$ and $C^*(G_B^a)$ are isomorphic. Since they have unique tracial states τ_A and τ_B respectively, we see that

$$\tau_{A*}(K_0(C^*(G_A^a))) = \tau_{B*}(K_0(C^*(G_B^a))).$$

We may also find a matrix $M \in GL(2, \mathbb{Z})$ such that $AM = MB$ by [1]. We then directly see that the ranges $\tau_{A*}(K_0(C^*(G_A^a)))$ and $\tau_{B*}(K_0(C^*(G_B^a)))$

coincide by using the formula (23). Similarly we may directly show that the equality $\tau_{A*}(K_0(C^*(G_A^a))) = \tau_{A^{-1}*}(K_0(C^*(G_{A^{-1}}^a)))$ by the formula (23).

4. **Examples.** In this section, we will present some examples.

1. $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Since $a = b = c = 1, d = 0$, we have by Theorem 2.10,

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (\frac{1}{2}(1 + \frac{1}{\sqrt{5}}), \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{2}(5 - \frac{1}{\sqrt{5}})). \tag{24}$$

By the formula (23), we have

$$\tau_*(K_0(C^*(G_A^a))) = \mathbb{Z} + \frac{5 + \sqrt{5}}{10}\mathbb{Z}.$$

Proposition 4.1. *Let A be the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Put $\theta = \frac{1}{2}(1 + \frac{1}{\sqrt{5}})$. Then the C^*-algebra $C^*(G_A^a)$ is isomorphic to the tensor product $A_\theta \otimes A_{5\theta}$ between the irrational rotation C^*-algebras A_θ and $A_{5\theta}$ with its rotation angles θ and 5θ respectively.*

Proof. Let U_1, U_2, V_1, V_2 be the generating unitaries in Theorem 2.10. Since

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (\theta, 2\theta - 1, 2\theta - 1, 1 - \theta)$$

by (24), we have

$$U_1 U_2 = U_2 U_1, \qquad V_1 V_2 = V_2 V_1,$$
$$V_1 U_1 = e^{2\pi i\theta} U_1 V_1, \qquad V_1 U_2 = e^{2\pi i 2\theta} U_2 V_1,$$
$$V_2 U_1 = e^{2\pi i 2\theta} U_1 V_2, \qquad V_2 U_2 = e^{-2\pi i\theta} U_2 V_2,$$

We set

$$u_1 = U_1 U_2^2, \qquad u_2 = U_2, \qquad v_1 = V_1 V_2^2, \qquad v_2 = V_2.$$

It is straightforward to see that the following equalities hold

$$u_1 u_2 = u_2 u_1, \qquad v_1 v_2 = v_2 v_1,$$
$$v_1 u_1 = e^{2\pi i 5\theta} u_1 v_1, \qquad v_1 u_2 = u_2 v_1,$$
$$v_2 u_1 = u_1 v_2, \qquad v_2 u_2 = e^{-2\pi i\theta} u_2 v_2.$$

Since the C^*-algebra $C^*(u_1, u_2, v_1, v_2)$ generated by u_1, u_2, v_1, v_2 coincides with $C^*(G_A^a)$, we have

$$C^*(G_A^a) \cong C^*(u_1, v_1) \otimes C^*(u_2, v_2) \cong A_{5\theta} \otimes A_\theta.$$

$\square$

2. $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$. Since $a = 3, b = d = 1, d = 2$, we have by Theorem 2.10,

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (\frac{3 + \sqrt{3}}{6}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{6}, \frac{3 - \sqrt{3}}{6})$$

and

$$\lambda_u = a\theta_1 + b\theta_2 + c\theta_3 + d\theta_4 = 2 + \sqrt{3}, \qquad \lambda_s = a\theta_4 - b\theta_2 - c\theta_3 + d\theta_1 = 2 - \sqrt{3}.$$

Since $\theta_4 = 1 - \theta_1, \theta_2 = 2\theta_3, \theta_1 = \frac{1}{2} + \theta_3$, the formula (23) tells us

$$\tau_*(K_0(C^*(G_A^a))) = \mathbb{Z} + \mathbb{Z}\theta_1 + \mathbb{Z}\theta_2 + \mathbb{Z}\theta_3 = \frac{1}{2}\mathbb{Z} + \frac{\sqrt{3}}{6}\mathbb{Z}.$$

Proposition 4.2. *Let* $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ *and* $A_2 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$. *Then the C^*-algebra $C^*(G^a_{A_1})$ is not isomorphic to $C^*(G^a_{A_2})$.*

Proof. Since the C^*-algebra $C^*(G^a_A)$ has the unique tracial state τ, the range $\tau_*(K_0(C^*(G^a_A)))$ of τ of the K_0-group $K_0(C^*(G^a_A))$ is invariant under isomorphism class of the C^*-algebra. As

$$\tau_*(K_0(C^*(G^a_{A_1}))) = \mathbb{Z} + \frac{5 + \sqrt{5}}{10}\mathbb{Z}, \qquad \tau_*(K_0(C^*(G^a_{A_1}))) = \frac{1}{2}\mathbb{Z} + \frac{\sqrt{3}}{6}\mathbb{Z},$$

we see that $\tau_*(K_0(C^*(G^a_{A_1}))) \neq \tau_*(K_0(C^*(G^a_{A_2})))$, so that the C^*-algebra $C^*(G_{A_1})$ is not isomorphic to $C^*(G_{A_2})$. $\square$

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