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ON THE UNIVERSAL α-CENTRAL EXTENSIONS OF THE SEMI-DIRECT PRODUCT OF HOM-PRELIE ALGEBRAS

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Abstract. We study Hom-actions, semidirect product and describe the relation between semi-direct product extensions and split extensions of Hom-preLie algebras. We obtain the functorial properties of the universal α-central extensions of α-perfect Hom-preLie algebras. We give that a derivation or an automorphism can be lifted in an α-cover with certain constraints. We provide some necessary and sufficient conditions about the universal α-central extension of the semi-direct product of two α-perfect Hom-preLie algebras.

1. **Introduction.** A Hom-preLie algebra was introduced by Makhlouf-Silvestrov [11]. Specifically, for a vector space L over a field $\mathbb{K}$ equipped with a bilinear map $\mu : L \times L \to L$ and a linear map $\alpha : L \to L$, we say that the triple (L, μ, α) is a *Hom-preLie algebra* if

$$\alpha(x)(yz) - (xy)\alpha(z) = \alpha(y)(xz) - (yx)\alpha(z).$$

for all $x, y, z \in L$. If the elements of L also satisfy the following equation

$$(xy)\alpha(z) = (xz)\alpha(y).$$

Then we call (L, μ, α) is a Hom-Novikov algebra. Clearly, a Hom-Novikov algebra is a Hom-preLie algebra. Moreover, Hom-preLie algebras generalizes the notation of pre-Lie algebras ($\alpha = \mathrm{Id}_L$), which has been extensively studied in the construction and Classification of Hom-Novikov algebras (Yau [14])etc. Since Hom-preLie

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algebras are a kind of Hom-Lie admissible algebras, there are some close connections between Hom-preLie algebra and Hom-Lie algebra theories. For example, a derivation of a Hom-preLie algebra with respect to a Hom-representation is an α-derivation which is introduced in[12].

In recent year, the universal central extension of a perfect Leibniz algebra was studied in several articles[2, 6, 3, 1, 8, 9, 10]. In [4, 5, 7], authors study universal (α)-central extension.

In [13], we study universal α-central extensions of Hom-preLie algebras. We define Hom-co-representations and low-dimensional chain complex, which derive a low-dimensional homology $\mathbb{K}$-vector space of a Hom-preLie algebra. We construct a right exact covariant functor $\mathfrak{uce}_\alpha$ of a Hom-preLie algebra which acts on a α-perfect Hom-preLie algebra L its universal α-central extension $\mathfrak{uce}_\alpha(L) = \frac{\alpha_L(L)\otimes\alpha_L(L)}{I_L}$, where $I_L = \langle \alpha_L(x_1)\otimes x_2x_3 - x_1x_2\otimes\alpha_L(x_3) - \alpha_L(x_2)\otimes x_1x_3 + x_2x_1\otimes\alpha_L(x_3)\rangle$.

The purpose of this paper is to study the universal α-central extension of semi-direct product of two α-perfect Hom-preLie algebras. We introduce a Hom-action between two perfect Hom-preLie algebras (Q, Id_Q) and (M, α_M), giving a semi-direct product between two perfect Hom-preLie algebras. We use an associative Hom-action of (Q, Id_Q) on (M, α_M) to induce a Hom-action of $(\mathfrak{uce}(Q), \mathrm{Id}_{\mathfrak{uce}(Q)})$ on $(\mathfrak{uce}_\alpha(M), \overline{\alpha_M})$. We obtain semi-direct product $(\mathfrak{uce}_\alpha(M), \overline{\alpha_M}) \rtimes (\mathfrak{uce}(Q), \mathrm{Id}_{\mathfrak{uce}(Q)})$ and define a linear map $\tau \rtimes \sigma$ on the semi-direct product. Casas and Pacheco Rego gave the linear map $\tau \rtimes \sigma$ is a homomorphism of Hom-leibniz algebras in [5]. We add a condition that Hom-action of (Q, Id_Q) on (M, α_M) is $\mathfrak{uce}$-associative, that is, $\{mm', q\} = \{m, m' \centerdot q\}$ $\forall m, m' \in M, q \in Q$. We also obtain a linear map $\tau \rtimes \sigma$ is a homomorphism of Hom-preLie algebras. We give a couple of necessary and sufficient conditions for the universal α-central extension of semi-direct product of two α-perfect Hom-preLie algebras by the above results.

The paper is organized as follows. Section 2 a preliminary section which contains Hom-actions and semidirect product of Hom-preLie algebras. We describe the relation between the semi-direct product extension and split extensions of Hom-preLie algebras. In section 3 we analyzing the functorial properties of the universal (α)-central extensions of (α)-perfect Hom-preLie algebras. In section 4 we obtain that an automorphism or a derivation can be lifted in an α-cover with certain constraints. In the final section we give some necessary and sufficient conditions about the universal α-central extension of the semi-direct product of two α-perfect Hom-preLie algebras.

Throughout this paper $\mathbb{K}$ denotes an arbitrary field.

2. **Hom-action.**

Definition 2.1. Let (M, α_M) and (L, α_L) be Hom-preLie algebras. A Hom-action of (L, α_L) over (M, α_M) consists of two bilinear maps, $\rho : M \otimes L \to M$, $\rho(m \otimes l) = m \centerdot l$ and $\lambda : L \otimes M \to M$, $\lambda(l \otimes m) = l \centerdot m$, the following identities hold.

a) $(xy) \centerdot \alpha_M(m) - \alpha_L(x) \centerdot (y \centerdot m) = (yx) \centerdot \alpha_M(m) - \alpha_L(y) \centerdot (x \centerdot m)$,
b) $(m \centerdot x) \centerdot \alpha_L(y) - \alpha_M(m) \centerdot (xy) = (x \centerdot m) \centerdot \alpha_L(y) - \alpha_L(x) \centerdot (m \centerdot y)$,
c) $(mm') \centerdot \alpha_L(x) - \alpha_M(m)(m' \centerdot x) = (m'm) \centerdot \alpha_L(x) - \alpha_M(m')(m \centerdot x)$,
d) $(x \centerdot m)\alpha_M(m') - \alpha_L(x) \centerdot (mm') = (m \centerdot x)\alpha_M(m') - \alpha_M(m)(x \centerdot m')$,
e) $\alpha_M(x \centerdot m) = \alpha_L(x) \centerdot \alpha_M(m)$,
f) $\alpha_M(m \centerdot x) = \alpha_M(m) \centerdot \alpha_L(x)$,

for all $x, y \in L$ and $m, m' \in M$.

If (M, α_M) is an abelian Hom-preLie algebra, then the Hom-action is said to be a Hom-representation.

Example 1. a) Let (K, α_K) be a subalgebra of a Hom-preLie algebra (L, α_L) and (H, α_H) a Hom-ideal of (L, α_L). There is a Hom-action of (K, α_K) over (H, α_H) by the multiplication in (L, α_L).

b) Let $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is an exact sequence of Hom-Lie algebras. If (M, α_M) is an abelian Hom-preLie algebra, then we call the sequence is abelian. An abelian sequence gives a Hom-representation of (L, α_L) over (M, α_M) by defining $\rho : M \otimes L \to M, \rho(m, l) = mk, \pi(k) = l,\ \lambda : L \otimes M \to M, \lambda(l, m) = km, \pi(k) = l$.

Proposition 1. *Let (M, α_M) and (L, α_L) be Hom-preLie algebras with a Hom-action of (L, α_L) over (M, α_M). Then $(M \rtimes L, \tilde{\alpha})$ is a Hom-preLie algebra, where $\tilde{\alpha} : M \rtimes L \to M \rtimes L$ is defined by $\tilde{\alpha}(m, l) = (\alpha_M(m), \alpha_L(l))$ and multiplication*

$$(m_1, l_1)(m_2, l_2) = (m_1 m_2 + \alpha_L(l_1) \centerdot m_2 + m_1 \centerdot \alpha_L(l_2), l_1 l_2).$$

Proof. It follows by the direct computation. $\square$

Definition 2.2. [13] A short exact sequence of Hom-preLie algebras $(K) : 0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is said to be split if there exists a Hom-preLie algebra homomorphism $\sigma : (L, \alpha_L) \to (K, \alpha_K)$ such that $\pi \circ \sigma = \mathrm{Id}_L$.

Let (M, α_M) and (L, α_L) be Hom-preLie algebras with a Hom-action of (L, α_L) over (M, α_M). We define two linear maps $i : M \to M \rtimes L, i(m) = (m, 0)$ and $\pi : M \rtimes L \to L, \pi(m, l) = l$. Then we obtain the following sequence

$$0 \to (M, \alpha_M) \xrightarrow{i} (M \rtimes L, \tilde{\alpha}) \xrightarrow{\pi} (L, \alpha_L) \to 0.$$

Furthermore, this sequence splits by $\sigma : L \to M \rtimes L, \sigma(l) = (0, l)$.

Definition 2.3. Let (M, α_M) and (L, α_L) be Hom-preLie algebras with a Hom-action of (L, α_L) over (M, α_M). Two extensions of (L, α_L) by (M, α_M), $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ and $0 \to (M, \alpha_M) \xrightarrow{i'} (K', \alpha_{K'}) \xrightarrow{\pi'} (L, \alpha_L) \to 0$, are equivalent if there is a homomorphism of Hom-preLie algebra $\varphi : (K, \alpha_K) \to (K', \alpha'_K)$ satisfies that the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (M, \alpha_M) \arrow[r, "i"] \arrow[d, equal] & (K, \alpha_K) \arrow[r, "\pi"] \arrow[d, "\varphi"] & (L, \alpha_L) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (M, \alpha_M) \arrow[r, "i'"] & (K', \alpha'_K) \arrow[r, "\pi'"] & (L, \alpha_L) \arrow[r] & 0.
\end{tikzcd}$$

Lemma 2.4. *Let (C, Id_C) and (A, α_A) be Hom-preLie algebras with a Hom-action of (C, Id_C) over (A, α_A). A sequence of Hom-preLie algebras $0 \to (A, \alpha_A) \xrightarrow{i} (B, \alpha_B) \xrightarrow{\pi} (C, \mathrm{Id}_C) \to 0$ is split if and only if it is equivalent to the semi-direct sequence $0 \to (A, \alpha_A) \xrightarrow{j} (A \rtimes C, \tilde{\alpha}) \xrightarrow{p} (C, \mathrm{Id}_C) \to 0$.*

Proof. If $0 \to (A, \alpha_A) \xrightarrow{i} (B, \alpha_B) \xrightarrow{\pi} (C, \mathrm{Id}_C) \to 0$ is split by $t : (C, \mathrm{Id}_C) \to (B, \alpha_B)$, then the Hom-action of (C, Id_C) over (A, α_A) is defined by

$$c \centerdot a = t(c)i(a); \quad a \centerdot c = i(a)t(c).$$

So we obtain the following split extension:

$$\begin{tikzcd} 0 \arrow[r] & (A,\alpha_A) \arrow[r, "k"] & (A\rtimes C,\tilde{\alpha}) \arrow[r, shift left, "q"] & (C,\mathrm{Id}_C) \arrow[l, shift left, "\tau"] \arrow[r] & 0 \end{tikzcd},$$

where $k: A\to A\rtimes C, k(a)=(a,0), p: A\rtimes C\to C, q(a,c)=c$ and $\tau: C\to A\rtimes C, \tau(c)=(0,c)$. Furthermore the Hom-action of (C,Id_C) over (A,α_A) induced by this extension coincides with the initial one:

$$c\bullet a=\sigma(c)j(a)=(0,c)(a,0)=(0a+\mathrm{Id}_C(c)\cdot a+0\cdot 0,c0)\equiv c\cdot a.$$

Since $\varphi:(A\rtimes C,\tilde{\alpha})\to(B,\alpha_B),\varphi(a,c)=i(a)+s(c)$ is a homomorphism of Hom-preLie algebras such that the following diagram commutative

$$\begin{tikzcd} 0 \arrow[r] & (A,\alpha_A) \arrow[r, "j"] \arrow[d, equal] & (A\rtimes C,\tilde{\alpha}) \arrow[r, shift left, "p"] \arrow[d, dashed, "\varphi"] & (C,\mathrm{Id}_C) \arrow[l, shift left, "\sigma"] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & (A,\alpha_A) \arrow[r, "i"] & (B,\alpha_B) \arrow[r, shift left, "\pi"] & (C,\mathrm{Id}_C) \arrow[l, shift left, "s"] \arrow[r] & 0, \end{tikzcd} \tag{1}$$

the extensions are equivalent.

Suppose that two extensions are equivalent, that is, there is a homomorphism of Hom-preLie algebras $\varphi:(A\rtimes C,\tilde{\alpha})\to(B,\alpha_B)$ such that diagram (1) is commutative, then $t:(C,\mathrm{Id}_C)\to(B,\alpha_B)$ given by $t(c)=\varphi(0,c)$, is a split extension. $\square$

Definition 2.5. [12] Let (M,α_M) be a Hom-representation of a Hom-preLie algebra (L,α_L). A derivation of (L,α_L) over (M,α_M) is a $\mathbb{K}$-linear map $d:L\to M$ such that:

a) $d(l_1l_2)=\alpha_L(l_1)\cdot d(l_2)+d(l_1)\cdot\alpha_L(l_2)$,
b) $d\circ\alpha_L=\alpha_M\circ d$,

for all $l_1,l_2\in L$.

Example 2. a) Let (M,α_M) be a Hom-representation of $(M\rtimes L,\tilde{\alpha})$ via π. Then the linear map $\theta: M\rtimes L\to M,\theta(m,l)=m$, is a derivation.

b) When $(M,\alpha_M)=(L,\alpha_L)$ is a representation follows from Example 1 *a*), then a derivation consists of a $\mathbb{K}$-linear map $d:L\to L$ such that $d(l_1l_2)=\alpha_L(l_1)d(l_2)+d(l_1)\alpha_L(l_2)$ and $d\circ\alpha_L=\alpha_M\circ d$.

Proposition 2. *Let (M,α_M) be a Hom-representation of a Hom-preLie algebra (L,α_L). For every f-derivation $d:(X,\alpha_X)\to(M,\alpha_M)$ $(d(x_1x_2)=d(x_1)\cdot\alpha_L(f(x_2))$ $+\alpha_L(f(x_1))\cdot d(x_2)\ \forall x_1,x_2\in X)$ and every homomorphism of Hom-preLie algebras $f:(X,\alpha_X)\to(L,\alpha_L)$ there is a unique homomorphism of Hom-preLie algebras $h:(X,\alpha_X)\to(M\rtimes L,\tilde{\alpha})$, such that the following diagram commute.*

$$\begin{tikzcd} & (X,\alpha_X) \arrow[dl, "d"'] \arrow[d, "h"] \arrow[dr, "f"] & \\ (M,\alpha_M) \arrow[r, shift left, "i"] & (M\rtimes L,\tilde{\alpha}) \arrow[l, shift left, "\theta"] \arrow[r, "\pi"] & (L,\alpha_L) \end{tikzcd}$$

Conversely, every homomorphism of Hom-preLie algebras $h:(X,\alpha_X)\to(M\rtimes L,\tilde{\alpha})$, decide a homomorphism of Hom-preLie algebras $f=\pi\circ h:(X,\alpha_X)\to(L,\alpha_L)$ and f-derivation $d=\theta\circ h:(X,\alpha_X)\to(M,\alpha_M)$, where $\theta(m,l)=x,\forall m\in M,l\in L$.

Proof. Let $h:X\to M\rtimes L,h(x)=(d(x),f(x))$ be a homomorphism. Then the homomorphism h satisfies all the conditions. $\square$

Corollary 1. *The set of all derivations from* (L, α_L) *to* (M, α_M) *is in one-to-one correspondence with the set of Hom-preLie algebra homomorphisms* $h : (L, \alpha_L) \to (M \rtimes L, \tilde{\alpha})$*, such that* $\iota\pi \circ h = \mathrm{Id}_L$*.*

Proof. Take $(X, \alpha_X) = (L, \alpha_L)$ in Proposition 2. $\square$

3. Functorial properties.

Definition 3.1. Let (L, α_L) be a perfect Hom-preLie algebra. It is said to be centrally closed if its universal central extension is

$$0 \to 0 \to (L, \alpha_L) \xrightarrow{\sim} (L, \alpha_L) \to 0,$$

i.e., $HL_2^{\alpha}(L) = 0$ and $(\mathfrak{uce}(L), \tilde{\alpha}) \cong (L, \alpha_L)$.

Corollary 2. *Let* (L, α_L) *be a* α*-perfect Hom-preLie algebra. If* $0 \to (\mathrm{Ker}(U_\alpha), \alpha_{K_|}) \to (K, \alpha_K) \xrightarrow{U_\alpha} (L, \alpha_L) \to 0$ *is the universal* α*-central extension of* (L, α_L)*, then* (L, α_L) *is centrally closed.*

Proof. $HL_1^{\alpha}(K) = HL_2^{\alpha}(K) = 0$ thanks to Corollary 4.12 a) in [13]. By the proof of Corollary 4.12 b) in [13], $HL_1^{\alpha}(K) = 0$ if and only if (K, α_K) is perfect. By Theorem 4.11 c) in [13], there exists a universal central extension $0 \to (HL_2^{\alpha}(K), \widetilde{\alpha}_|) \to (\mathfrak{uce}(K), \widetilde{\alpha}) \xrightarrow{u_K} (K, \alpha_K) \to 0$. Since $HL_2^{\alpha}(K) = 0$, u_K is an isomorphism. $\square$

Definition 3.2. A Hom-preLie algebra (L, α_L) is said to be simply connected if every central extension $\tau : (F, \alpha_F) \twoheadrightarrow (L, \alpha_L)$ splits uniquely as the product of Hom-preLie algebras $(F, \alpha_F) = (\mathrm{Ker}(\tau), \alpha_{F|}) \times (L, \alpha_L)$.

Proposition 3. *Let* (L, α_L) *be a perfect Hom-preLie algebra. Then the following conditions are equivalent:*

a) (L, α_L) *is simply connected.*
b) (L, α_L) *is centrally closed.*

Proof. *a)* $\Rightarrow$ *b)* Let $0 \to (HL_2^{\alpha}(L), \widetilde{\alpha}_|) \to (\mathfrak{uce}(L), \widetilde{\alpha}) \xrightarrow{u_L} (L, \alpha_L) \to 0$ be the universal central extension of (L, α_L), then it is split. Consequently $\mathfrak{uce}(L) \cong L$ and $HL_2^{\alpha}(L) = 0$.

b) $\Rightarrow$ *a)* Let $0 \to 0 \to (L, \alpha_L) \xrightarrow{\sim} (L, \alpha_L) \to 0$ be a universal central extension of (L, α_L). So every central extension splits uniquely follows from the universal property. $\square$

Proposition 4. *Let* (L, α_L) *be a perfect Hom-preLie algebra. If* $\mu : (L, \alpha_L) \twoheadrightarrow (M, \alpha_M)$ *is a central extension, then the following statements hold.*

a) Proposition 3 a) implies that $\mu : (L, \alpha_L) \twoheadrightarrow (M, \alpha_M)$ *is a universal central extension.*
b) If $\mu : (L, \alpha_L) \twoheadrightarrow (M, \alpha_M)$ *is a universal* α*-central extension, then statements a) and b) hold in Proposition 3.*

Proof. *a*) It follows from Theorem 4.11 b) in [13] that if (L, α_L) is perfect and every central extension splits, then $\mu : (L, \alpha_L) \twoheadrightarrow (M, \alpha_M)$ is a universal central extension. Note that every central extension of (L, α_L) splits by the simply connectivity and (L, α_L) is perfect by hypothesis. Then $\mu : (L, \alpha_L) \twoheadrightarrow (M, \alpha_M)$ is universal.

b) It follows from Lemma 4.10 in [13] that the composition of two central extensions is an α-central extensionon. Consider a central extension $0 \to (N, \alpha_N) \xrightarrow{j} (A, \alpha_A) \xrightarrow{\rho} (L, \alpha_L) \to 0$. Let (L, α_L) be perfect and $\mu : (L, \alpha_L) \twoheadrightarrow (M, \alpha_M)$ be a central extension. Note that $0 \to \mathrm{Ker}(\mu \circ \rho) \to (A, \alpha_A) \stackrel{\mu\circ\rho}{\to} (M, \alpha_M) \to 0$ is an α-central extension. Since $\mu : (L, \alpha_L) \twoheadrightarrow (M, \alpha_M)$ is a universal α-central extension, there exists a unique homomorphism of Hom-preLie algebras φ such that $\mu \circ \rho \circ \varphi = \mu$. By Lemma 4.7 in [13], we have $\rho \circ \varphi = \mathrm{Id}$. So (L, α_L) is simply connected, that is, it is centrally closed. $\square$

Let $f : (L', \alpha_{L'}) \to (L, \alpha_L)$ be a homomorphism of perfect Hom-preLie algebras. It induces a linear map $f \otimes f : L' \otimes L' \to L \otimes L$ given by $(f \otimes f)(x_1 \otimes x_2) = f(x_1) \otimes f(x_2)$, which maps $I_{L'}$ to I_L. So $f \otimes f$ induces a homomorphism of Hom-preLie algebras $\mathfrak{uce}(f) : \mathfrak{uce}(L') \to \mathfrak{uce}(L)$, given by $\mathfrak{uce}(f)\{x_1, x_2\} = \{f(x_1), f(x_2)\}$. From the above conditions, the following diagram commutate.

$$\begin{tikzcd}
HL_2^{\alpha}(L') \arrow[d, hook] & HL_2^{\alpha}(L) \arrow[d, hook] \\
(\mathfrak{uce}(L'), \widetilde{\alpha'}) \arrow[r, "\mathfrak{uce}(f)"] \arrow[d, two heads, "u_{L'}"] & (\mathfrak{uce}(L), \tilde{\alpha}) \arrow[d, two heads, "u_L"] \\
(L', \alpha_{L'}) \arrow[r, "f"] & (L, \alpha_L)
\end{tikzcd} \tag{2}$$

From diagram (2), there exists a covariant right exact functor $\mathfrak{uce}$: Hom-preLie$^{\mathrm{perf}}$ $\to$ Hom-preLie$^{\mathrm{perf}}$ between the category of perfect Hom-preLie algebras. So an automorphism f of (L, α_L) induces an automorphism $\mathfrak{uce}(f)$ of $(\mathfrak{uce}(L), \tilde{\alpha})$. $\mathfrak{uce}(f)$ leaves $HL_2^{\alpha}(L)$ invariant since diagram (2) is commutative. Consequently, we obtain the Hom-group homomorphism

$$\begin{aligned}
Aut(L, \alpha_L) &\to \{g \in Aut(\mathfrak{uce}(L), \tilde{\alpha}) : g(HL_2^{\alpha}(L)) = HL_2^{\alpha}(L)\}. \\
f &\mapsto \mathfrak{uce}(f)
\end{aligned}$$

Similar to the above discussion, we also obtain the functorial properties of α-perfect Hom-preLie algebras. In other words, consider a homomorphism of α-perfect Hom-preLie algebras $f : (L', \alpha_{L'}) \to (L, \alpha_L)$. Let I_L the vector subspace of $\alpha_L(L) \otimes \alpha_L(L)$ spanned by $\alpha_L(x_1) \otimes x_2x_3 - x_1x_2 \otimes \alpha_L(x_3) - \alpha_L(x_2) \otimes x_1x_3 + x_2x_1 \otimes \alpha_L(x_3), x_1, x_2, x_3 \in L$, respectively $I_{L'}$. f induces a linear map $f \otimes f : (\alpha_{L'}(L') \otimes \alpha_{L'}(L'), \alpha_{L'\otimes L'}) \to (\alpha_L(L) \otimes \alpha_L(L), \alpha_{L\otimes L})$, given by $f \otimes f(\alpha_{L'}(x_1') \otimes \alpha_{L'}(x_2')) = \alpha_L(f(x_1')) \otimes \alpha_L(f(x_2'))$ such that $f \otimes f(I_{L'}) \subseteq I_L$. Hence, it induces a homomorphism of Hom-preLie algebras $\mathfrak{uce}_{\alpha}(f) : (\mathfrak{uce}_{\alpha'}(L'), \overline{\alpha'}) \to (\mathfrak{uce}_{\alpha}(L), \overline{\alpha})$ given by $\mathfrak{uce}_{\alpha}(f)\{\alpha_{L'}(x_1'), \alpha_{L'}(x_2')\} = \{\alpha_L(f(x_1')), \alpha_L(f(x_2'))\}$ such that the following diagram

$$\begin{tikzcd}
\mathrm{Ker}(U_{\alpha'}) \arrow[d, hook] & \mathrm{Ker}(U_{\alpha}) \arrow[d, hook] \\
(\mathfrak{uce}_{\alpha}(L'), \overline{\alpha'}) \arrow[r, "\mathfrak{uce}_{\alpha}(f)"] \arrow[d, two heads, "U_{\alpha'}"] & (\mathfrak{uce}_{\alpha}(L), \overline{\alpha}) \arrow[d, two heads, "U_{\alpha}"] \\
(L', \alpha_{L'}) \arrow[r, "f"] & (L, \alpha_L)
\end{tikzcd} \tag{3}$$

is commutative.

From diagram (3), there exists a covariant right exact functor $\mathfrak{uce}_{\alpha}$: Hom-preLie$^{\alpha-\mathrm{perf}}$ $\to$ Hom-preLie$^{\alpha-\mathrm{perf}}$ between the category of α-perfect Hom-preLie algebras. So an automorphism f of (L, α_L) induces an automorphism $\mathfrak{uce}_\alpha(f)$ of $(\mathfrak{uce}_\alpha(L), \overline{\alpha})$. $\mathfrak{uce}_\alpha(f)$ leaves $\mathrm{Ker}(U_\alpha)$ invariant since diagram (3) is commutative. So we obtain the Hom-group homomorphism

$$\begin{aligned} Aut(L, \alpha_L) &\to \{g \in Aut(\mathfrak{uce}_\alpha(L), \overline{\alpha}) : g(\mathrm{Ker}(U_\alpha)) = \mathrm{Ker}(U_\alpha)\} \\ f &\mapsto \mathfrak{uce}_\alpha(f) \end{aligned}$$

Next we consider a derivation d of the α-perfect Hom-preLie algebra (L, α_L). The linear map $d : \alpha_L(L) \otimes \alpha_L(L) \to \alpha_L(L) \otimes \alpha_L(L)$ given by $d(\alpha_L(x_1) \otimes \alpha_L(x_2)) = d(\alpha_L(x_1)) \otimes \alpha_L^2(x_2) + \alpha_L^2(x_1) \otimes d(\alpha_L(x_2))$, keeps invariant the subspace I_L of $\alpha_L(L) \otimes \alpha_L(L)$ spanned by $\alpha_L(x_1) \otimes x_2x_3 - x_1x_2 \otimes \alpha_L(x_3) - \alpha_L(x_2) \otimes x_1x_3 + x_2x_1 \otimes \alpha_L(x_3), x_1, x_2, x_3 \in L$. Indeed,

$$\begin{aligned} & d(\alpha_L(x_1) \otimes x_2x_3 - x_1x_2 \otimes \alpha_L(x_3) - \alpha_L(x_2) \otimes x_1x_3 + x_2x_1 \otimes \alpha_L(x_3)) \\ =& d(\alpha_L(x_1)) \otimes \alpha_L(x_2x_3) + \alpha_L^2(x_1) \otimes d(x_2x_3) - d(x_1x_2) \otimes \alpha_L^2(x_3) - \alpha_L(x_1x_2) \otimes d(\alpha_L(x_3)) \\ & - d(\alpha_L(x_2)) \otimes \alpha_L(x_1x_3) - \alpha_L^2(x_2) \otimes d(x_1x_3) + d(x_2x_1) \otimes \alpha_L^2(x_3) + \alpha_L(x_2x_1) \otimes d(\alpha_L(x_3)) \\ =& \alpha_L(d(x_1)) \otimes \alpha_L(x_2)\alpha_L(x_3) + \alpha_L^2(x_1) \otimes d(x_2)\alpha_L(x_3) + \alpha_L^2(x_1) \otimes \alpha_L(x_2)d(x_3) \\ & - d(x_1)\alpha_L(x_2) \otimes \alpha_L^2(x_3) - \alpha_L(x_1)d(x_2) \otimes \alpha_L^2(x_3) - \alpha_L(x_1)\alpha_L(x_2) \otimes \alpha_L(d(x_3)) \\ & - \alpha_L(d(x_2)) \otimes \alpha_L(x_1)\alpha_L(x_3) - \alpha_L^2(x_2) \otimes d(x_1)\alpha_L(x_3) - \alpha_L^2(x_2) \otimes \alpha_L(x_1)d(x_3) \\ & + d(x_2)\alpha_L(x_1) \otimes \alpha_L^2(x_3) + \alpha_L(x_2)d(x_1) \otimes \alpha_L^2(x_3) + \alpha_L(x_2)\alpha_L(x_1) \otimes \alpha_L(d(x_3)) \\ \in & I_L. \end{aligned}$$

So it induces a linear map $\mathfrak{uce}_\alpha(d) : (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \to (\mathfrak{uce}_\alpha(L), \overline{\alpha})$, given by

$$\mathfrak{uce}_\alpha(d)(\{\alpha_L(x_1), \alpha_L(x_1)\}) = \{d(\alpha_L(x_1)), \alpha_L^2(x_2)\} + \{\alpha_L^2(x_1), d(\alpha_L(x_2))\},$$

such that the following diagram

$$\begin{tikzcd} (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \arrow[r, "\mathfrak{uce}_\alpha(d)"] \arrow[d, two heads, "U_\alpha"] & (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \arrow[d, two heads, "U_\alpha"] \\ (L, \alpha_L) \arrow[r, "d"] & (L, \alpha_L) \end{tikzcd} \tag{4}$$

is commutative. Hence, a derivation d of (L, α_L) induces a derivation $\mathfrak{uce}_\alpha(d)$ of $(\mathfrak{uce}_\alpha(L), \overline{\alpha})$. $\mathfrak{uce}_\alpha(d)$ maps $\mathrm{Ker}(U_\alpha)$ on itself since diagram (4) is commutative. Consequently, we obtain the homomorphism of Hom-$\mathbb{K}$-vector spaces

$$\begin{aligned} \mathfrak{uce}_\alpha : \mathrm{Der}(L, \alpha_L) &\to \{\delta \in \mathrm{Der}(\mathfrak{uce}_\alpha(L), \overline{\alpha}) : \delta(\mathrm{Ker}(U_\alpha)) \subseteq \mathrm{Ker}(U_\alpha)\} \\ d &\mapsto \mathfrak{uce}_\alpha(d), \end{aligned}$$

Lemma 3.3. *Let $f : (L', \alpha_{L'}) \to (L, \alpha_L)$ be a homomorphism of α-perfect Hom-preLie algebras. If $d,\ d' \in \mathrm{Der}(L)$ satisfies $f \circ d' = d \circ f$, then $\mathfrak{uce}_\alpha(f) \circ \mathfrak{uce}_\alpha(d') = \mathfrak{uce}_\alpha(d) \circ \mathfrak{uce}_\alpha(f)$.*

Proof. For any $x_1', x_2' \in L'$, we have

$$\begin{aligned} & \mathfrak{uce}_\alpha(f) \circ \mathfrak{uce}_\alpha(d')(\{\alpha_{L'}(x_1), \alpha_{L'}(x_2)\}) \\ =& \mathfrak{uce}_\alpha(f)(\{d'(\alpha_{L'}(x_1)), \alpha_{L'}^2(x_2)\} + \{\alpha_{L'}^2(x_1), d'(\alpha_{L'}(x_2))\}) \\ =& \{\alpha_{L'}(d(f(x_1))), \alpha_{L'}^2(f(x_2))\} + \{\alpha_{L'}^2(f(x_1)), \alpha_{L'}(d(f(x_2)))\}. \end{aligned}$$

On the other hand

$$\begin{aligned}
&\mathfrak{uce}_\alpha(d)\circ\mathfrak{uce}_\alpha(f)(\{\alpha_{L'}(x_1),\alpha_{L'}(x_2)\})\\
=&\mathfrak{uce}_\alpha(d)(\{\alpha_{L'}(f(x_1)),\alpha_{L'}(f(x_2))\})\\
=&\{\alpha_{L'}(d(f(x_1))),\alpha^2_{L'}(f(x_2))\}+\{\alpha^2_{L'}(f(x_1)),\alpha_{L'}(d(f(x_2)))\}.
\end{aligned}$$

Hence we prove the lemma. □

4. **Lifting automorphisms and derivations.**

Definition 4.1. Let $(L',\alpha_{L'})$ be a Hom-preLie algebra. A central extension of Hom-preLie algebras $f:(L',\alpha_{L'})\twoheadrightarrow(L,\alpha_L)$ is said to be an α-cover if $(L',\alpha_{L'})$ is α-perfect.

Lemma 4.2. *Let $f:(L',\alpha_{L'})\to(L,\alpha_L)$ be a surjective homomorphism of Hom-preLie algebras. If $(L',\alpha_{L'})$ is α-perfect, then (L,α_L) is also α-perfect.*

Proof. Routine checking. □

Let $f:(L',\alpha_{L'})\twoheadrightarrow(L,\alpha_L)$ be an α-cover. By Lemma 4.2, (L,α_L) is an α-perfect Hom-preLie algebra. By Theorem 4.19 in [13], it has a universal α-central extension. By means of diagram (3), we obtain the following diagram:

$$\begin{tikzcd}
\mathrm{Ker}(U_{\alpha'}) \arrow[d, hook] & \mathrm{Ker}(U_\alpha) \arrow[d, hook]\\
(\mathfrak{uce}_\alpha(L'),\overline{\alpha}') \arrow[r, "\mathfrak{uce}_\alpha(f)"] \arrow[d, two heads, "U_{\alpha'}"] & (\mathfrak{uce}_\alpha(L),\overline{\alpha}) \arrow[d, two heads, "U_\alpha"]\\
(L',\alpha_{L'}) \arrow[r, "f"] & (L,\alpha_L)
\end{tikzcd}$$

By Remark 4.4 in [13], $U_{\alpha'}:(\mathfrak{uce}_{\alpha'}(L'),\overline{\alpha'})\twoheadrightarrow(L',\alpha_{L'})$ is a universal central extension. Since $f:(L',\alpha_{L'})\twoheadrightarrow(L,\alpha_L)$ is a central extension and $U_{\alpha'}:(\mathfrak{uce}_{\alpha'}(L'),\overline{\alpha'})\twoheadrightarrow(L',\alpha_{L'})$ is a universal central extension, by Proposition 4.15 a) in [13], the extension $f\circ U_{\alpha'}:(\mathfrak{uce}_{\alpha'}(L'),\overline{\alpha'})\to(L,\alpha_L)$ is α-central which is universal in the sense of Definition 4.13 in [13].

In addition, since $U_\alpha:(\mathfrak{uce}_\alpha(L),\overline{\alpha})\twoheadrightarrow(L,\alpha_L)$ is a universal α-central extension, there is a unique homomorphism $\varphi:(\mathfrak{uce}_\alpha(L),\overline{\alpha})\twoheadrightarrow(\mathfrak{uce}_{\alpha'}(L'),\overline{\alpha'})$ satifies $f\circ U_{\alpha'}\circ\varphi=U_\alpha$. So we have

$$f\circ U_{\alpha'}\circ\varphi\circ\mathfrak{uce}_\alpha(f)=U_\alpha\circ\mathfrak{uce}_\alpha(f)=f\circ U_{\alpha'},$$

that is to say the following diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{Ker}(f\circ U_{\alpha'}),\overline{\alpha'_|}) \arrow[r] & (\mathfrak{uce}_{\alpha'}(L'),\overline{\alpha'}) \arrow[r, "f\circ U_{\alpha'}"] \arrow[d, shift right, "\varphi\circ\mathfrak{uce}_\alpha(f)"'] \arrow[d, shift left, "\mathrm{Id}"] & (L,\alpha_L) \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & (\mathrm{Ker}(f\circ U_{\alpha'}),\overline{\alpha'_|}) \arrow[r] & (\mathfrak{uce}_{\alpha'}(L'),\overline{\alpha'}) \arrow[r, "f\circ U_{\alpha'}"] & (L,\alpha_L) \arrow[r] & 0
\end{tikzcd}$$

is commutative. Since $f\circ U_{\alpha'}$ is an α-central extension which is universal in the sense of Definition 4.13 in [13], we have $\varphi\circ\mathfrak{uce}_\alpha(f)=\mathrm{Id}$.

Conversely, $\mathfrak{uce}_\alpha(f)\circ\varphi = \mathrm{Id}$ since the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{Ker}(U_\alpha), \bar{\alpha}_|) \arrow[r] & (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \arrow[r, "U_\alpha"] \arrow[d, shift right, "\mathfrak{uce}_\alpha(f)\circ\varphi"'] \arrow[d, shift left, "\mathrm{Id}"] & (L, \alpha_L) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\mathrm{Ker}(U_\alpha), \overline{\alpha_|}) \arrow[r] & (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \arrow[r, "U_\alpha"] & (L, \alpha_L) \arrow[r] & 0
\end{tikzcd}$$

whose horizontal rows are central extensions and $(\mathfrak{uce}_\alpha(L), \overline{\alpha})$ is α-perfect, the uniqueness of the vertical homomorphism is guaranteed by Lemma 4.18 in [13]. Consequently $\mathfrak{uce}_\alpha(f)$ is an isomorphism and we will denote the notation $\mathfrak{uce}_\alpha(f)^{-1}$ by φ.

Moreover, $U_{\alpha'}\circ\mathfrak{uce}_\alpha(f)^{-1} : (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \twoheadrightarrow (L', \alpha_{L'})$ is an α-cover. In the sequel, we will denote its kernel by

$$C := \mathrm{Ker}(U_{\alpha'}\circ\mathfrak{uce}_\alpha(f)^{-1}) = \mathfrak{uce}_\alpha(f)(\mathrm{Ker}(U_{\alpha'})).$$

In fact, for any $x \in \mathrm{Ker}(U_{\alpha'}\circ\mathfrak{uce}_\alpha(f)^{-1})$, we have $U_{\alpha'}\circ\mathfrak{uce}_\alpha(f)^{-1}(x) = 0$. Hence $x \in \mathfrak{uce}_\alpha(f)(\mathrm{Ker}(U_{\alpha'}))$.

Conversely, for any $x \in \mathfrak{uce}_\alpha(f)(\mathrm{Ker}(U_{\alpha'}))$, there exists a $y \in \mathrm{Ker}(U_{\alpha'})$ such that $x = \mathfrak{uce}_\alpha(f)(y)$. So $U_{\alpha'}\circ\mathfrak{uce}_\alpha(f)^{-1}(x) = U_{\alpha'}(y) = 0$.

Theorem 4.3. *Let $f : (L', \alpha_{L'}) \twoheadrightarrow (L, \alpha_L)$ be an α-cover. For any automorphism h on (L, α_L), there is a unique $\theta_h \in Aut(L', \alpha'_L)$ such that the following diagram is commutative:*

$$\begin{tikzcd}
(L', \alpha_{L'}) \arrow[r, "f"] \arrow[d, two heads, "\theta_h"'] & (L, \alpha_L) \arrow[d, two heads, "h"] \\
(L', \alpha_{L'}) \arrow[r, "f"] & (L, \alpha_L)
\end{tikzcd} \tag{5}$$

if and only if the automorphism $\mathfrak{uce}_\alpha(h)$ of $(\mathfrak{uce}_\alpha(L), \overline{\alpha})$ such that $\mathfrak{uce}_\alpha(h)(C) = C$. Futhermore, we obtain a group isomorphism:

$$\Theta : \{h \in Aut(L, \alpha_L) : \mathfrak{uce}_\alpha(h)(C) = C\} \to \{g \in Aut(L', \alpha_{L'}) : g(\mathrm{Ker}(f)) = \mathrm{Ker}(f)\}.$$
$$h \mapsto \theta_h$$

Proof. Let $h \in Aut(L, \alpha_L)$. Suppose that there is an automorphism θ_h on $(L', \alpha_{L'})$ such that diagram (5) commute. Apply the functor $\mathfrak{uce}_\alpha(-)$ to diagram (5), the following commutative diagram holds:

$$\begin{tikzcd}
(\mathfrak{uce}_\alpha(L'), \overline{\alpha_{L'}}) \arrow[r, "\mathfrak{uce}_\alpha(f)"] \arrow[d, two heads, "\mathfrak{uce}_\alpha(\theta_h)"'] & (\mathfrak{uce}_\alpha(L), \overline{\alpha_L}) \arrow[d, two heads, "\mathfrak{uce}_\alpha(h)"] \\
(\mathfrak{uce}_\alpha(L'), \overline{\alpha_{L'}}) \arrow[r, "\mathfrak{uce}_\alpha(f)"] & (\mathfrak{uce}_\alpha(L), \overline{\alpha_L}).
\end{tikzcd}$$

So $\mathfrak{uce}_\alpha(h)(C) = \mathfrak{uce}_\alpha(h)\circ\mathfrak{uce}_\alpha(f)(Ker(U_{\alpha'})) = \mathfrak{uce}_\alpha(f)\circ\mathfrak{uce}_\alpha(\theta_h)(\mathrm{Ker}(U_{\alpha'})) = \mathfrak{uce}_\alpha(f)(\mathrm{Ker}(U_{\alpha'})) = C$.

Conversely, diagram (3) implies that $U_\alpha = f \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}$, so we have the following diagram:

$$\begin{tikzcd}
C \arrow[r, tail] \arrow[d] & (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \arrow[r, two heads, "U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}"] \arrow[d, "\mathfrak{uce}_\alpha(h)"] & (L', \alpha_{L'}) \arrow[r, two heads, "f"] \arrow[d, dashed, "\theta_h"] & (L, \alpha_L) \arrow[d, "h"] \\
C \arrow[r, tail] & (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \arrow[r, two heads, "U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}"] & (L', \alpha_{L'}) \arrow[r, two heads, "f"] & (L, \alpha_L).
\end{tikzcd}$$

If $\mathfrak{uce}_\alpha(h)(C) = C$, then $U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} \circ \mathfrak{uce}_\alpha(h)(C) = U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}(C) = 0$, so there is a unique $\theta_h : (L', \alpha_{L'}) \to (L', \alpha_{L'})$ satisfies $\theta_h \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} = U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} \circ \mathfrak{uce}_\alpha(h)$.

On the other side, $h \circ f \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} = f \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} \circ \mathfrak{uce}_\alpha(h) = f \circ \theta_h \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}$. Since $(L', \alpha_{L'})$ is an α-perfect Hom-preLie algebra and $\mathfrak{uce}_\alpha(f)^{-1}$ is an isomorphism, we have $h \circ f = f \circ \theta_h$. Moreover, $h \circ f$ is an α-cover since $\mathrm{Ker}(h \circ f) \subseteq \mathrm{Ker}(f) \subseteq Z(L')$. Hence, θ_h is uniquely by Lemma 4.18 in [13]. Finally, $\theta_h(\mathrm{Ker}(f)) = \mathrm{Ker}(f)$. Indeed, we have $f \circ \theta_h(\mathrm{Ker}(f)) = h \circ f(\mathrm{Ker}(f)) = 0$. Conversely, for $x \in \mathrm{Ker}(f)$, there exists a $y \in L'$ such that $x = \theta_h(y)$. Hence, $f(y) \in \mathrm{Ker}(h) = 0$.

We know that Θ is well-defined, it is a monomorphism follows from the uniqueness of θ_h. Θ is an epimorphism, since any $g \in Aut(L', \alpha_{L'})$ with $g(\mathrm{Ker}(f)) = \mathrm{Ker}(f)$, gives rise to a unique homomorphism $h : (L, \alpha_L) \to (L, \alpha_L)$ satisfies $h \circ f = f \circ g$. Consequently, $g = \theta_h$ and $\mathfrak{uce}_\alpha(h)(C) = C$. □

Corollary 3. *Let (L, α_L) be an α-perfect Hom-preLie algebra. Then there exists a group isomorphism:*

$$\begin{aligned} Aut(L, \alpha_L) &\to \{g \in Aut(\mathfrak{uce}_\alpha(L), \overline{\alpha}) : g(\mathrm{Ker}(U_\alpha)) = \mathrm{Ker}(U_\alpha)\}. \\ h &\mapsto \mathfrak{uce}_\alpha(h) \end{aligned}$$

Proof. By Theorem 4.3, $U_\alpha : (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \twoheadrightarrow (L, \alpha_L)$ is an α-cover. Let $C = 0$ and $\mathfrak{uce}_\alpha(f)(0) = 0$ in Theorem 4.3. □

Theorem 4.4. *Let $f : (L', \alpha_{L'}) \twoheadrightarrow (L, \alpha_L)$ be an α-cover. Denote $C = \mathfrak{uce}_\alpha(f)$ $\mathrm{Ker}(U_{\alpha'}) \subseteq \mathrm{Ker}(U_\alpha)$. Then the following statements hold:*

a) For any $d \in \mathrm{Der}(L, \alpha_L)$, there exists a $\delta_d \in \mathrm{Der}(L', \alpha_{L'})$ such that the following diagram is commutative

$$\begin{tikzcd}
(L', \alpha_{L'}) \arrow[r, "f"] \arrow[d, "\delta_d"'] & (L, \alpha_L) \arrow[d, "d"] \\
(L', \alpha_{L'}) \arrow[r, "f"] & (L, \alpha_L)
\end{tikzcd} \tag{6}$$

if and only if the derivation $\mathfrak{uce}_\alpha(d)$ of $(\mathfrak{uce}_\alpha(L), \overline{\alpha_L})$ satisfies $\mathfrak{uce}_\alpha(d)(C) \subseteq C$.

b) There exists an isomorphism of Hom-vector spaces

$$\begin{aligned} \Delta : \{d \in \mathrm{Der}(L, \alpha_L) : \mathfrak{uce}_\alpha(d)(C) \subseteq C\} &\to \{\rho \in \mathrm{Der}(L', \alpha_{L'}) : \rho(\mathrm{Ker}(f)) \subseteq \mathrm{Ker}(f)\}. \\ d &\mapsto \delta_d \end{aligned}$$

c) Let $U_\alpha : (\mathfrak{uce}_\alpha(L), \overline{\alpha_L}) \twoheadrightarrow (L, \alpha_L)$ be an α-cover. Then there exists an isomorphism of Hom-vector spaces

$$\mathfrak{uce}_\alpha : \mathrm{Der}(L, \alpha_L) \to \{\delta \in \mathrm{Der}(\mathfrak{uce}_\alpha(L), \overline{\alpha_L}) : \delta(\mathrm{Ker}(U_\alpha)) \subseteq \mathrm{Ker}(U_\alpha)\}.$$

Proof. a) Let $d \in \mathrm{Der}(L, \alpha_L)$. Suppose that there exists a $\delta_d \in \mathrm{Der}(L', \alpha_{L'})$, which makes diagram (6) commute. We have that the following diagram is commutative thanks to Lemma 3.3.

$$\begin{tikzcd}
(\mathfrak{uce}_\alpha(L'), \overline{\alpha_{L'}}) \arrow[r, "\mathfrak{uce}_\alpha(f)"] \arrow[d, "\mathfrak{uce}_\alpha(\delta_d)"'] & (\mathfrak{uce}_\alpha(L), \overline{\alpha_L}) \arrow[d, "\mathfrak{uce}_\alpha(d)"] \\
(\mathfrak{uce}_\alpha(L'), \overline{\alpha_{L'}}) \arrow[r, "\mathfrak{uce}_\alpha(f)"] & (\mathfrak{uce}_\alpha(L), \overline{\alpha_L}).
\end{tikzcd}$$

Hence, by diagram (4), we obtain $\mathfrak{uce}_\alpha(d)(C) = \mathfrak{uce}_\alpha(d) \circ \mathfrak{uce}_\alpha(f)(Ker(U_{\alpha'})) = \mathfrak{uce}_\alpha(f) \circ \mathfrak{uce}_\alpha(\delta_d)(\mathrm{Ker}(U_{\alpha'})) \subseteq \mathfrak{uce}_\alpha(f)(\mathrm{Ker}(U_{\alpha'})) = C$.

Conversely, we have that $U_\alpha = f \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}$ follows from diagram (3), hence we obtain the following diagram:

$$\begin{tikzcd}
C \arrow[r, tail] \arrow[d] & (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \arrow[r, two heads, "U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}"] \arrow[d, "\mathfrak{uce}_\alpha(h)"] & (L', \alpha_{L'}) \arrow[r, two heads, "f"] \arrow[d, dashed, "\theta_h"] & (L, \alpha_L) \arrow[d, "h"] \\
C \arrow[r, tail] & (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \arrow[r, two heads, "U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}"] & (L', \alpha_{L'}) \arrow[r, two heads, "f"] & (L, \alpha_L).
\end{tikzcd}$$

If $\mathfrak{uce}_\alpha(d)(C) \subseteq C$, then $U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} \circ \mathfrak{uce}_\alpha(d)(C) \subseteq U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}(C) = 0$, so there exists a unique $\delta_d : (L', \alpha_{L'}) \to (L', \alpha_{L'})$ such that $\delta_d \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} = U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} \circ \mathfrak{uce}_\alpha(d)$.

On the other side, $d \circ f \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} = d \circ U_\alpha = U_\alpha \circ \mathfrak{uce}_\alpha(d) = f \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} \circ \mathfrak{uce}_\alpha(d) = f \circ \delta_d \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}$. Since $(L', \alpha_{L'})$ is an α-perfect Hom-preLie algebra and $\mathfrak{uce}_\alpha(f)^{-1}$ is an isomorphism, we have $d \circ f = f \circ \delta_d$. Moreover, $d \circ f$ is an α-cover since $\mathrm{Ker}(d \circ f) \subseteq \mathrm{Ker}(f) \subseteq Z(L')$. Hence, δ_d is uniquely determined by Lemma 4.2. At last, we have that δ_d is a derivation of L' a direct verify.

b) It well known that the map Δ is a homomorphism of Hom-vector spaces, it is a monomorphism thanks to the uniqueness of δ_d and it is an epimorphism, since any $\rho \in \mathrm{Der}(L', \alpha_{L'})$ with $\rho(\mathrm{Ker}(f)) \subseteq \mathrm{Ker}(f)$, gives rise to a unique homomorphism $d : (L, \alpha_L) \to (L, \alpha_L)$ satisfies the following diagram

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, tail] \arrow[d] & (L', \alpha_{L'}) \arrow[r, "f"] \arrow[d, "\rho"] & (L, \alpha_L) \arrow[d, dashed, "d"] \\
\mathrm{Ker}(f) \arrow[r, tail] & (L', \alpha_{L'}) \arrow[r, "f"] & (L, \alpha_L)
\end{tikzcd}$$

is commutative, where $d : (L, \alpha_L) \to (L, \alpha_L)$ is a derivation such that $\mathfrak{uce}_\alpha(d)(C) = \mathfrak{uce}_\alpha(d) \circ \mathfrak{uce}_\alpha(f)(\mathrm{Ker}(U_{\alpha'})) = \mathfrak{uce}_\alpha(f) \circ \mathfrak{uce}_\alpha(\rho)(\mathrm{Ker}(U_{\alpha'})) \subseteq \mathfrak{uce}_\alpha(f)(\mathrm{Ker}(U_{\alpha'})) = C$.

c) Let $C = \mathfrak{uce}_\alpha(U_\alpha)(\mathrm{Ker}(U_\alpha)) = 0$ in statement b). □

5. **Universal α-central extension of a semi-direct product.** Now, we give a split extension of α-perfect Hom-preLie algebras as follow

$$\begin{tikzcd}
0 \arrow[r] & (M, \alpha_M) \arrow[r, "t"] & (G, \alpha_G) \arrow[r, shift left, "p"] & (Q, \mathrm{Id}_Q) \arrow[l, shift left, "s"] \arrow[r] & 0 .
\end{tikzcd}$$

By Lemma 2.4, $(G,\alpha_G)\cong(M,\alpha_M)\rtimes(Q,\mathrm{Id}_Q)$, where the Hom-action of (Q,Id_Q) on (M,α_M) is given by $q\bullet m=s(q)t(m)$ and $m\bullet q=t(m)s(q)$, $q\in Q, m\in M$. Sometimes, it is necessary to assume that the previous action is associative, i.e., $(mm')\bullet q=m(m'\bullet q)$, $q\in Q, m, m'\in M$.

If (M,α_M) is an α-perfect Hom-preLie algebra and Q is a Hom-preLie algebra (Q,Id_Q). Then the direct product $(G,\alpha_G)=(M,\alpha_M)\times(Q,\mathrm{Id}_Q)=(M\times Q,\alpha_M\times\mathrm{Id}_Q)$ satisfies the above situation. Applying the functorial properties of $\mathfrak{uce}_\alpha(-)$ given by diagram (3) and (Q,Id_Q) is perfect, we obtain that the following diagram

$$\begin{tikzcd}
 & \mathrm{Ker}(U_\alpha^M) \arrow[d, tail] & \mathrm{Ker}(U_\alpha^G) \arrow[d, tail] & HL_2(Q) \arrow[d, tail] & \\
 & (\mathfrak{uce}_\alpha(M),\overline{\alpha_M}) \arrow[r, "\tau"] \arrow[d, "U_\alpha^M"] & (\mathfrak{uce}_\alpha(G),\overline{\alpha_G}) \arrow[r, shift left, "\pi"] \arrow[d, "U_\alpha^G"] & (\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)}) \arrow[l, shift left, "\sigma"] \arrow[d, "u_Q"] & \\
0 \arrow[r] & (M,\alpha_M) \arrow[r, "t"] & (G,\alpha_G) \arrow[r, shift left, "p"] & (Q,\mathrm{Id}_Q) \arrow[l, shift left, "s"] \arrow[r] & 0
\end{tikzcd}$$

is commutative. Here $\tau=\mathfrak{uce}_\alpha(t), \pi=\mathfrak{uce}_\alpha(p), \sigma=\mathfrak{uce}_\alpha(s)$. Since $p\circ s=\mathrm{Id}_Q$, the sequence

$$\begin{tikzcd}
(\mathfrak{uce}_\alpha(M),\overline{\alpha_M}) \arrow[r, "\tau"] & (\mathfrak{uce}_\alpha(G),\overline{\alpha_G}) \arrow[r, shift left, "\pi"] & (\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)}) \arrow[l, shift left, "\sigma"]
\end{tikzcd}$$

is split. So $\mathfrak{uce}_\alpha(p)\circ\mathfrak{uce}_\alpha(s)=\mathfrak{uce}_\alpha(\mathrm{Id}_Q)$, i.e., $\pi\circ\sigma=\mathrm{Id}_{\mathfrak{uce}(Q)}$. Hence π is an epimorphism and there is a Hom-action of $(\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)})$ on $(\mathrm{Ker}(\pi),\alpha_{G|})$ given by:

$\lambda:\mathfrak{uce}(Q)\otimes\mathrm{Ker}(\pi)\to\mathrm{Ker}(\pi)$,

$$\begin{aligned}
&\lambda(\{q_1,q_2\}\otimes\{\alpha_G(g_1),\alpha_G(g_2)\})\\
=&\{q_1,q_2\}\bullet\{\alpha_G(g_1),\alpha_G(g_2)\}=\sigma(\{q_1,q_2\})i(\{\alpha_G(g_1),\alpha_G(g_2)\})\\
=&\{s(q_1),s(q_2)\}\{\alpha_G(g_1),\alpha_G(g_2)\}=\{s(q_1q_2),\alpha_G(g_1g_2)\}
\end{aligned}$$

and $\rho:\mathrm{Ker}(\pi)\otimes\mathfrak{uce}(Q)\to\mathrm{Ker}(\pi)$,

$$\begin{aligned}
&\rho(\{\alpha_G(g_1),\alpha_G(g_2)\}\otimes\{q_1,q_2\})\\
=&\{\alpha_G(g_1),\alpha_G(g_2)\}\bullet\{q_1,q_2\}=(\{\alpha_G(g_1),\alpha_G(g_2)\})\sigma(\{q_1,q_2\})\\
=&\{\alpha_G(g_1),\alpha_G(g_2)\}\{s(q_1),s(q_2)\}=\{\alpha_G(g_1g_2),s(q_1q_2)\}.
\end{aligned}$$

By Lemma 2.4, the split sequence

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{Ker}(\pi),\overline{\alpha_{G|}}) \arrow[r, "i"] & (\mathfrak{uce}_\alpha(G),\overline{\alpha_G}) \arrow[r, shift left, "\pi"] & (\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)}) \arrow[l, shift left, "\sigma"] \arrow[r] & 0
\end{tikzcd}$$

is equivalent to the semi-direct product sequence, i.e.,

$$(\mathfrak{uce}_\alpha(G),\overline{\alpha_G})\cong(\mathrm{Ker}(\pi),\overline{\alpha_{G|}})\rtimes(\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)}).$$

Let $q\in Q$ and $\alpha_M(m_1),\alpha_M(m_2)\in\alpha_M(M)$. In $(\mathfrak{uce}_\alpha(G),\overline{\alpha_G})$, we have

$$\begin{aligned}
&\{\alpha_G(s(q)),t(\alpha_M(m_1))t(\alpha_M(m_2))\}=\{s(q)t(\alpha_M(m_1)),\alpha_G(t(\alpha_M(m_2)))\}\\
&+\{\alpha_G(t(\alpha_M(m_1))),s(q)t(\alpha_M(m_2))\}-\{t(\alpha_M(m_1))s(q),\alpha_G(t(\alpha_M(m_2)))\}
\end{aligned}$$

and

$$\{t(\alpha_M(m_1))t(\alpha_M(m_2)), \alpha_G(s(q))\} = \{\alpha_G(t(\alpha_M(m_1))), t(\alpha_M(m_2))s(q)\}$$
$$- \{\alpha_G(t(\alpha_M(m_2))), t(\alpha_M(m_1))s(q)\} + \{t(\alpha_M(m_2))t(\alpha_M(m_1)), \alpha_G(s(q))\}.$$

These above equalities and the α-perfection of (M, α_M) imply:

$$\begin{aligned}
&\{s(Q), M\} = \{s \circ \mathrm{Id}_Q(Q), M\} = \{\alpha_G \circ s(Q), \alpha_M(M)\alpha_M(M)\} \\
\subseteq &\{s(Q)\alpha_M(M), \alpha_G(\alpha_M(M))\} + \{\alpha_G(\alpha_M(M)), s(Q)\alpha_M(M)\} - \{\alpha_M(M)s(Q), \alpha_G(\alpha_M(M))\} \\
\subseteq &\{\alpha_G(M), \alpha_G^2(M)\} + \{\alpha_G^2(M), \alpha_G(M)\} - \{\alpha_G(M), \alpha_G^2(M)\} \\
\subseteq &\{\alpha_M(M), \alpha_M(M)\}
\end{aligned}$$

and

$$\begin{aligned}
\{M, s(Q)\} = &\{\alpha_M(M)\alpha_M(M), \alpha_G \circ s(Q)\} \subseteq \{\alpha_G(\alpha_M(M)), \alpha_M(M)s(Q)\} \\
&\subseteq \{\alpha_M(M), \alpha_M(M)\}.
\end{aligned}$$

Futhermore,

$$\tau(\mathfrak{uce}_\alpha(M), \overline{\alpha_M}) \equiv (\{\alpha_M(M), \alpha_M(M)\}, \alpha_{G|}) \tag{7}$$

and

$$\sigma(\mathfrak{uce}(Q)) = \{s(Q), s(Q)\} = \{\alpha_G(s(Q)), \alpha_G(s(Q))\}$$

since $\tau\{\alpha_M(m_1), \alpha_M(m_2)\} = \{t(\alpha_M(m_1)), t(\alpha_M(m_2))\} \equiv \{\alpha_M(m_1), \alpha_M(m_2)\}$, and $\sigma(\{q_1, q_2\}) = \{s(q_1), s(q_2)\} = \{\alpha_G(s(q_1)), \alpha_G(s(q_2))\}$.

Lemma 5.1. *With the above notations, we have*

$$(\mathfrak{uce}_\alpha(G), \overline{\alpha_G}) = (\{s(Q), s(Q)\} + \{\alpha_M(M), \alpha_M(M)\}, \overline{\alpha_G}). \tag{8}$$

Proof. For any $\alpha_G(g) \in G$, there exists an $\alpha_M(m) \in \alpha_M(M)$ such that $\alpha_G(g) = s(p(\alpha_G(g))) + \alpha_M(m)$. Hence

$$\begin{aligned}
\{\alpha_G(g_1), \alpha_G(g_2)\} = &\{s(p(\alpha_G(g_1))) + \alpha_M(m_1), s(p(\alpha_G(g_2))) + \alpha_M(m_2)\} \\
= &\{s(p(\alpha_G(g_1))), s(p(\alpha_G(g_2)))\} + \{s(p(\alpha_G(g_1))), \alpha_M(m_2)\} \\
&+ \{\alpha_M(m_1), s(p(\alpha_G(g_2)))\} + \{\alpha_M(m_1), \alpha_M(m_2)\} \\
\subseteq &\{s(Q), s(Q)\} + \{s(Q), M\} + \{M, s(Q)\} + \{\alpha_M(M), \alpha_M(M)\} \\
\subseteq &\{s(Q), s(Q)\} + \{\alpha_M(M), \alpha_M(M)\}.
\end{aligned}$$

Conversely,

$$\{s(Q), s(Q)\} + \{\alpha_M(M), \alpha_M(M)\} \subseteq \{\alpha_G(G), \alpha_G(G)\} = \mathfrak{uce}_\alpha(G).$$

Hence, we prove the lemma. □

Proposition 5. *With the above notations, we have*

$$(\mathrm{Ker}(\pi), \overline{\alpha_{G|}}) = (\{\alpha_M(M), \alpha_M(M)\}, \overline{\alpha_{G|}}) = \tau(\mathfrak{uce}_\alpha(M), \overline{\alpha_M}).$$

Proof. Let $\{g_1, g_2\} \in \mathrm{Ker}(\pi)$. Then by Eq.(8), we have

$$\{g_1, g_2\} = \{s(q_1), s(q_2)\} + \{\alpha_M(m_1), \alpha_M(m_2)\} \in \mathfrak{uce}_\alpha(G).$$

Hence $0 = \pi\{g_1, g_2\} = \{p(s(q_1)), p(s(q_2))\} + \{p(\alpha_M(m_1)), p(\alpha_M(m_2))\} = \{q_1, q_2\}$, i.e., $q_1 \otimes q_2 \in I_Q$. So $\sigma\{q_1, q_2\} = \{s(q_1), s(q_2)\} = 0$. Consequently, $\mathrm{Ker}(\pi)$ has elements of the form $\{\alpha_M(m_1), \alpha_M(m_2)\}$. It is easy to prove the reverse inclusion. Eq.(7) gives a proof of the second equality. □

Theorem 5.2. *Consider a split extension of α-perfect Hom-preLie algebras*

$$0 \longrightarrow (M,\alpha_M) \xrightarrow{t} (G,\alpha_G) \underset{s}{\overset{p}{\rightleftarrows}} (Q,\mathrm{Id}_Q) \longrightarrow 0\ .$$

Then the following statements hold

1) $(\mathfrak{uce}_\alpha(G),\alpha_G) = \tau(\mathfrak{uce}_\alpha(M),\alpha_M) \rtimes \sigma(\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)})$.

2) $\sigma(\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)}) \cong (\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)})$.

3) $(\mathrm{Ker}(U^G_\alpha),\overline{\alpha_{G|}}) \cong \tau(\mathrm{Ker}(U^M_\alpha),\overline{\alpha_{M|}}) \oplus \sigma(HL_2(Q),\mathrm{Id}_{\mathfrak{uce}(Q)})$.

Proof. 1) and 2) Since $\pi\circ\sigma = \mathrm{Id}$, we have

$$(\mathfrak{uce}_\alpha(G),\alpha_G) = (\mathrm{Ker}(\pi),\overline{\alpha_G}) \rtimes \sigma(\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)}).$$

Moreover, $(\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)}) \cong \sigma(\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)})$. By proposition 5, 1) and 2) hold.

3) Let $(\tau(m),\sigma(q)) \in (\mathfrak{uce}_\alpha(G),\overline{\alpha_G})$ from 1), where $m \in (\mathfrak{uce}_\alpha(M),\alpha_M)$ and $q \in (\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)})$. So $(\tau(m),\sigma(q)) \in \mathrm{Ker}(U^G_\alpha) \Leftrightarrow U^G_\alpha(\tau(m),\sigma(q)) = 0 \Leftrightarrow t\circ U^M_\alpha(m) = U^G_\alpha(\tau(m)) = 0, s\circ u_Q(q) = U^G_\alpha(\sigma(q)) = 0 \Leftrightarrow m \in \mathrm{Ker}(U^M_\alpha), q \in HL_2(Q)$. □

Suppose that there is an associative Hom-action of (Q,Id_Q) on (M,α_M), we have a Hom-action of $(\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)})$ on $(\mathfrak{uce}_\alpha(M),\alpha_M)$ given by:

$$\begin{aligned}
\lambda : \mathfrak{uce}(Q)\otimes\mathfrak{uce}_\alpha(M) &\to \mathfrak{uce}_\alpha(M)\\
\{q_1,q_2\}\otimes\{\alpha_M(m_1),\alpha_M(m_2)\} &\mapsto \{q_1,q_2\}\centerdot\{\alpha_M(m_1),\alpha_M(m_2)\}\\
&= \{(q_1q_2)\centerdot\alpha_M(m_1),\alpha_M^2(m_2)\}\\
&\quad - \{\alpha_M(m_1)\centerdot(q_1q_2),\alpha_M^2(m_2)\}\\
&\quad + \{\alpha_M^2(m_1),(q_1q_2)\centerdot\alpha_M(m_2)\}
\end{aligned}$$

and

$$\begin{aligned}
\rho : \mathfrak{uce}_\alpha(M)\otimes\mathfrak{uce}(Q) &\to \mathfrak{uce}_\alpha(M)\\
\{\alpha_M(m_1),\alpha_M(m_2)\}\otimes\{q_1,q_2\} &\mapsto \{\alpha_M(m_1),\alpha_M(m_2)\}\centerdot\{q_1,q_2\}\\
&= \{\alpha_M^2(m_1),\alpha_M(m_2)\centerdot(q_1q_2)\}.
\end{aligned}$$

When it is need, the Hom-action of (Q,Id_Q) on (M,α_M) is $\mathfrak{uce}$-associative, i.e., $\{mm',q\} = \{m,m'\centerdot q\}$.

So we define the following homomorphism of Hom-preLie algebras

$$\begin{aligned}
\tau\rtimes\sigma : (\mathfrak{uce}_\alpha(M),\overline{\alpha_M}) \rtimes (\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)}) &\to (\mathfrak{uce}_\alpha(G),\overline{\alpha_G}) \cong\\
&\qquad \tau(\mathfrak{uce}_\alpha(M),\alpha_M)\rtimes\sigma(\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)})\\
(\{\alpha_M(m_1),\alpha_M(m_2)\},\{q_1,q_2\}) &\mapsto (\{t(\alpha_M(m_1)),t(\alpha_M(m_2))\},\{s(q_1),s(q_2)\}),
\end{aligned}$$

where the Hom-action of (Q,Id_Q) on (M,α_M) is $\mathfrak{uce}$-associative.

We obtain that $\tau\rtimes\sigma$ is an epimorphism since

$$(\mathfrak{uce}_\alpha(G),\overline{\alpha_G}) \cong \tau(\mathfrak{uce}_\alpha(M),\alpha_M)\rtimes\sigma(\mathfrak{uce}(Q),\mathrm{Id}_{\mathfrak{uce}(Q)}).$$

Next we define a surjective homomorphism of Hom-preLie algebras

$$\begin{aligned}
\Phi := (t\circ U^M_\alpha)\rtimes(s\circ u_Q) : (\mathfrak{uce}_\alpha(M)\rtimes\mathfrak{uce}(Q),\overline{\alpha_M}\rtimes\mathrm{Id}_{\mathfrak{uce}(Q)}) &\to (G,\alpha_G)\\
(\{\alpha_M(m_1),\alpha_M(m_2)\},\{q_1,q_2\}) &\mapsto (t(\alpha_M(m_1)\alpha_M(m_2)),s(q_1q_2)),
\end{aligned}$$

such that the following diagram

$$
\begin{tikzcd}
(\mathfrak{uce}_\alpha(M) \rtimes \mathfrak{uce}(Q), \overline{\alpha_M} \rtimes \mathrm{Id}_{\mathfrak{uce}(Q)}) \arrow[rr, "\tau \rtimes \sigma"] \arrow[dr, dashed, "\Phi"'] & & (\mathfrak{uce}_\alpha(G), \overline{\alpha_G}) \arrow[dl, "U_\alpha^G"] \\
& (G, \alpha_G) &
\end{tikzcd}
\tag{9}
$$

is commutative. We have that

$$\mathfrak{uce}(Q) \centerdot \mathrm{Ker}(U_\alpha^M) \oplus \mathrm{Ker}(U_\alpha^M) \centerdot \mathfrak{uce}(Q) \subseteq \mathrm{Ker}(\tau) \subseteq \mathrm{Ker}(U_\alpha^M).$$

Second inclusion holds since $t \circ U_\alpha^M = U_\alpha^G \circ \tau$ and t is injective. Since the following diagram is commutative

$$
\begin{tikzcd}
& (\mathrm{Ker}(U_\alpha^M)), \overline{\alpha_{M|}} \arrow[r, dashed] \arrow[d, tail] & (\mathrm{Ker}(U_\alpha^M), \overline{\alpha_{G|}}) \arrow[d, tail] \\
\mathrm{Ker}(\tau) \arrow[r, tail] \arrow[ur, dashed] & (\mathfrak{uce}_\alpha(M), \overline{\alpha_M}) \arrow[r, "\tau"] \arrow[d, two heads, "U_\alpha^M"] & (\mathfrak{uce}_\alpha(G), \overline{\alpha_G}) \arrow[d, two heads, "U_\alpha^G"] \\
& (M, \alpha_M) \arrow[r, "t"] & (G, \alpha_G),
\end{tikzcd}
$$

$U_\alpha^G \circ \tau(\mathrm{Ker}(U_\alpha^M)) = t \circ U_\alpha^M(\mathrm{Ker}(U_\alpha^M)) = 0$, then $\tau(\mathrm{Ker}(U_\alpha^M)) \subseteq \mathrm{Ker}(U_\alpha^G) \subseteq Z(\mathfrak{uce}_\alpha(G))$, so

$$\tau(\mathfrak{uce}(Q) \centerdot \mathrm{Ker}(U_\alpha^M)) = \sigma(\mathfrak{uce}(Q))\tau(\mathrm{Ker}(U_\alpha^M)) = 0$$

and

$$\tau(\mathrm{Ker}(U_\alpha^M) \centerdot \mathfrak{uce}(Q)) = \tau(\mathrm{Ker}(U_\alpha^M))\sigma(\mathfrak{uce}(Q)) = 0.$$

Consequently, $\mathfrak{uce}(Q) \centerdot \mathrm{Ker}(U_\alpha^M) \oplus \mathrm{Ker}(U_\alpha^M) \centerdot \mathfrak{uce}(Q) \subseteq \mathrm{Ker}(\tau)$.

On the other side, we have that $\mathfrak{uce}(Q) \centerdot \mathrm{Ker}(U_\alpha^M) \oplus \mathrm{Ker}(U_\alpha^M) \centerdot \mathfrak{uce}(Q)$ is an ideal of $(\mathfrak{uce}_\alpha(M), \alpha_M)$. Then the Hom-action of $(\mathfrak{uce}(Q), \mathrm{Id}_{\mathfrak{uce}(Q)})$ on $(\mathfrak{uce}_\alpha(M), \alpha_M)$ gives rise to a Hom-action of $(\mathfrak{uce}(Q), \mathrm{Id}_Q)$ on

$$\left(\overline{\mathfrak{uce}_\alpha(M)}, \overline{\overline{\alpha_M}}\right) = \left(\frac{\mathfrak{uce}_\alpha(M)}{\mathfrak{uce}(Q) \centerdot \mathrm{Ker}(U_\alpha^M) \oplus \mathrm{Ker}(U_\alpha^M) \centerdot \mathfrak{uce}(Q)}, \overline{\overline{\alpha_M}}\right).$$

Since τ vanishes on $\mathfrak{uce}(Q) \centerdot \mathrm{Ker}(U_\alpha^M) \oplus \mathrm{Ker}(U_\alpha^M) \centerdot \mathfrak{uce}(Q)$, it gives rise to $\overline{\tau} : \overline{\mathfrak{uce}_\alpha(M)} \to \tau(\mathfrak{uce}_\alpha(M))$. Put $I = \mathfrak{uce}(Q) \centerdot \mathrm{Ker}(U_\alpha^M) \oplus \mathrm{Ker}(U_\alpha^M) \centerdot \mathfrak{uce}(Q)$, we have the following diagram

$$
\begin{tikzcd}
(I, \overline{\alpha_{M|}}) \arrow[d, tail] \arrow[dr, bend left, "0"] & & \\
(\mathfrak{uce}_\alpha(M), \overline{\alpha_M}) \arrow[rr, "\tau"] \arrow[r] \arrow[dd] & & (\mathfrak{uce}_\alpha(G), \overline{\alpha_G}) \\
& \tau(\mathfrak{uce}_\alpha(M), \overline{\alpha_M}) \arrow[ur, tail] & \\
\left(\overline{\mathfrak{uce}_\alpha(M)}, \overline{\overline{\alpha_M}}\right). \arrow[ur, dashed, "\overline{\tau}"] & &
\end{tikzcd}
$$

We obtain the following commutative diagram:

$$\begin{tikzcd}
I \arrow[r, tail] \arrow[d, tail] & I \rtimes 0 \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\
\mathrm{Ker}(\tau \rtimes \sigma) \arrow[r, tail] \arrow[d, two heads] & (\mathfrak{uce}_{\alpha}(M) \rtimes \mathfrak{uce}(Q), \overline{\alpha_M} \rtimes \mathrm{Id}_{\mathfrak{uce}(Q)}) \arrow[r, two heads, "\tau \rtimes \sigma"] \arrow[d, two heads] & (\mathfrak{uce}_{\alpha}(G), \overline{\alpha_G}) \arrow[d, equal] \\
\frac{\mathrm{Ker}(\tau \rtimes \sigma)}{I} \arrow[r, tail] & \left(\overline{\mathfrak{uce}_{\alpha}(M)} \rtimes \mathfrak{uce}(Q), \overline{\overline{\alpha_M}} \rtimes \mathrm{Id}_{\mathfrak{uce}(Q)}\right) \arrow[r, dashed, "\Psi"] & (\mathfrak{uce}_{\alpha}(G), \overline{\alpha_G}),
\end{tikzcd}$$

whose bottom row is a central extension. Since $(\mathfrak{uce}_{\alpha}(G), \overline{\alpha_G})$ is an α-perfect Hom-preLie algebra, by Theorem 4.19 in [13], it has a universal α-central extension. So $\mathfrak{uce}_{\alpha}(G)$ is centrally closed by Corollary 2, that is, $\mathfrak{uce}(\mathfrak{uce}_{\alpha}(G)) \cong \mathfrak{uce}_{\alpha}(G)$. Hence, we have the following diagram

$$\begin{tikzcd}
\left(\overline{\mathfrak{uce}_{\alpha}(M)} \rtimes \mathfrak{uce}(Q), \overline{\overline{\alpha_M}} \rtimes \mathrm{Id}_{\mathfrak{uce}(Q)}\right) \arrow[r, two heads, "\Psi"] \arrow[d, two heads, "\Psi"] \arrow[dd, bend right=60, "\mathrm{Id}"'] & (\mathfrak{uce}_{\alpha}(G), \overline{\alpha_G}) \arrow[d, equal] \\
(\mathfrak{uce}_{\alpha}(G), \overline{\alpha_G}) \arrow[r, two heads, "\mathrm{Id}"] \arrow[d, tail, "\mu"] & (\mathfrak{uce}_{\alpha}(G), \overline{\alpha_G}) \arrow[d, equal] \\
\left(\overline{\mathfrak{uce}_{\alpha}(M)} \rtimes \mathfrak{uce}(Q), \overline{\overline{\alpha_M}} \rtimes \mathrm{Id}_{\mathfrak{uce}(Q)}\right) \arrow[r, two heads, "\Psi"] & (\mathfrak{uce}_{\alpha}(G), \overline{\alpha_G}).
\end{tikzcd}$$

Since $(\mathfrak{uce}_{\alpha}(G), \overline{\alpha_G})$ is centrally closed and Ψ is a central extension, $\mathrm{Id} : (\mathfrak{uce}_{\alpha}(G), \overline{\alpha_G}) \to (\mathfrak{uce}_{\alpha}(G), \overline{\alpha_G})$ is a universal central extension. Then there is a unique homomorphism of Hom-preLie algebras $\mu : (\mathfrak{uce}_{\alpha}(G), \overline{\alpha_G}) \to \left(\overline{\mathfrak{uce}_{\alpha}(M)} \rtimes \mathfrak{uce}(Q), \overline{\overline{\alpha_M}} \rtimes \mathrm{Id}_{\mathfrak{uce}(Q)}\right)$ satisfies $\Psi \circ \mu = \mathrm{Id}$. Since $\Psi \circ \mu \circ \Psi = \mathrm{Id} \circ \Psi = \Psi \circ \mathrm{Id}$ and $\left(\overline{\mathfrak{uce}_{\alpha}(M)} \rtimes \mathfrak{uce}(Q), \overline{\overline{\alpha_M}} \rtimes \mathrm{Id}_{\mathfrak{uce}(Q)}\right)$ is α-perfect, then $\mu \circ \Psi = \mathrm{Id}$ follows from Lemma 4.18 in [13]. Hence, Ψ is an isomorphism, then $\mathrm{Ker}(\Psi) = \frac{\mathrm{Ker}(\tau \rtimes \sigma)}{I} = 0$. Consequently, $\mathrm{Ker}(\tau \rtimes \sigma) \subseteq I$.

The above discussion can be summarized in:

$$\mathrm{Ker}(\tau \rtimes \sigma) \cong \mathfrak{uce}(Q) \centerdot \mathrm{Ker}(U_{\alpha}^{M}) \oplus \mathrm{Ker}(U_{\alpha}^{M}) \centerdot \mathfrak{uce}(Q).$$

We can obtain the following theorem from the above results.

Theorem 5.3. *Let the Hom-action of* (Q, Id_Q) *on* (M, α_M) *be* $\mathfrak{uce}$*-associative and the extension*

$$0 \longrightarrow (M, \alpha_M) \xrightarrow{t} (G, \alpha_G) \underset{s}{\overset{p}{\rightleftarrows}} (Q, \mathrm{Id}_Q) \longrightarrow 0$$

split. Then the following conditions hold

1) The homomorphism of Hom-preLie algebras

$$\Phi : (\mathfrak{uce}_{\alpha}(M) \rtimes \mathfrak{uce}(Q), \overline{\alpha_M} \rtimes \mathrm{Id}_{\mathfrak{uce}(Q)}) \to (G, \alpha_G)$$

defined by $\Phi(\{\alpha_M(m_1), \alpha_M(m_2)\}, \{q_1, q_2\}) = (t(\alpha_M(m_1)\alpha_M(m_2)), s(q_1 q_2))$ *is an epimorphism whose kernel is* $\mathrm{Ker}(U_{\alpha}^{M}) \oplus HL_2(Q)$*. Moreover, diagram (9) is commutative.*

2) $\mathrm{Ker}(\tau \rtimes \sigma) \cong \mathfrak{uce}(Q) \centerdot \mathrm{Ker}(U_{\alpha}^{M}) \oplus \mathrm{Ker}(U_{\alpha}^{M}) \centerdot \mathfrak{uce}(Q)$.

Theorem 5.4. *The following statements are equivalent:*

a) $\Phi := (t \circ U_\alpha^M) \rtimes (s \circ u_Q) : (\mathfrak{uce}_\alpha(M) \rtimes \mathfrak{uce}(Q), \overline{\alpha_M} \rtimes \mathrm{Id}_{\mathfrak{uce}(Q)}) \to (G, \alpha_G)$ *is a central extension. So it is an α-cover.*

b) The Hom-action of $(\mathfrak{uce}(Q), \mathrm{Id}_{\mathfrak{uce}(Q)})$ *on* $(\mathrm{Ker}(U_\alpha^M), \overline{\alpha_{M|}})$ *is trivial.*

c) $\tau \rtimes \sigma$ *is an isomorphism. Hence* $\mathfrak{uce}_\alpha(M) \rtimes \mathfrak{uce}(Q)$ *is the universal α-central extension of* (G, α_G).

d) τ *is injective. In particular,*

$$(\mathfrak{uce}_\alpha(M \times Q), \overline{\alpha_M \times \mathrm{Id}_Q}) \cong (\mathfrak{uce}_\alpha(M) \times \mathfrak{uce}(Q), \overline{\alpha_M} \times \mathrm{Id}_{\mathfrak{uce}(Q)}).$$

Proof. *a)* $\Leftrightarrow$ *b)* If $\Phi := (t \circ U_\alpha^M) \rtimes (s \circ u_Q) : (\mathfrak{uce}_\alpha(M) \rtimes \mathfrak{uce}(Q), \overline{\alpha_M} \rtimes \mathrm{Id}_{\mathfrak{uce}(Q)}) \to (G, \alpha_G)$ is a central extension and $\mathrm{Ker}(\Phi) = \mathrm{Ker}(U_\alpha^M) \oplus HL_2(Q)$, then the Hom-action of $(\mathfrak{uce}(Q), \mathrm{Id}_{\mathfrak{uce}(Q)})$ on $(\mathrm{Ker}(U_\alpha^M), \overline{\alpha_{M|}})$ is trivial. It is easy to prove the converse. Furthermore, the Hom-action is trivial, we have $(\mathfrak{uce}_\alpha(M) \rtimes \mathfrak{uce}(Q), \overline{\alpha_M} \rtimes \mathrm{Id}_{\mathfrak{uce}(Q)})$ is α-perfect. Consequently, the extension is an α-cover.

b) $\Leftrightarrow$ *c)* By Theorem 5.3, we have that $\mathrm{Ker}(\tau \rtimes \sigma) \cong \mathfrak{uce}(Q) \centerdot \mathrm{Ker}(U_\alpha^M) \oplus \mathrm{Ker}(U_\alpha^M) \centerdot \mathfrak{uce}(Q)$, then $\tau \circ \sigma$ is injective if and only if the Hom-action is trivial.

By diagram (9), we have that $\mathfrak{uce}_\alpha(M) \rtimes \mathfrak{uce}(Q)$ is the universal α-central extension of (G, α_G).

c) $\Leftrightarrow$ *d)* It suffices to verify that $\tau\{\alpha_M(m_1), \alpha_M(m_2)\} = (\tau \circ \sigma)(\{\alpha_M(m_1), \alpha_M(m_2)\}, 0)$, since $\mathrm{Ker}(\tau) = \mathrm{Ker}(\tau \circ \sigma)$. So the conclusion holds.

Since the Hom-action of (Q, Id_Q) on (M, α_M) is trivial, the Hom-action of $(\mathfrak{uce}(Q), \mathrm{Id}_{\mathfrak{uce}(Q)})$ on $(\mathfrak{uce}_\alpha(M), \overline{\alpha_M})$ is also trivial. Consequently, $(\mathfrak{uce}_\alpha(M) \times \mathfrak{uce}(Q), \overline{\alpha_M} \times \mathrm{Id}_{\mathfrak{uce}(Q)}) = (\mathfrak{uce}_\alpha(M) \rtimes \mathfrak{uce}(Q), \overline{\alpha_M} \times \mathrm{Id}_{\mathfrak{uce}(Q)})$. The proof of the particular case is completed by statement c). □

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REFERENCES

[1] J. M. Casas and N. Corral, On universal central extensions of Leibniz algebras, *Comm. Algebra*, **37** (2009), 2104–2120.

[2] J. M. Casas and M. Ladra, Stem extensions and stem covers of Leibniz algebras, *Georgian Math. J.*, **9** (2002), 659–669.

[3] J. M. Casas and M. Ladra, Computing low dimensional Leibniz homology of some perfect Leibniz algebras, *Southeast Asian Bull. Math.*, **31** (2007), 683–690.

[4] J. M. Casas, M. A. Insua and N. P. Rego, On universal central extensions of Hom-Leibniz algebras, *J. Algebra Appl.*, **13** (2014), 1450053, 22pp.

[5] J. M. Casas and N. P. Rego, On the universal α-central extension of the semi-direct product of Hom-Leibniz algebras, *Bull. Malays. Math. Sci. Soc.*, **39** (2016), 1579-1602.

[6] J. M. Casas and A. M. Vieites, Central extensions of perfect of Leibniz algebras, *Recent Advances in Lie Theory*, **25** (2002), 189–196.

[7] X. García-Martínez, E. Khmaladze and M. Ladra, Non-abelian tensor product and homology of Lie superalgebras, *J. Algebra*, **440** (2015), 464–488.

[8] A. V. Gnedbaye, Third homology groups of universal central extensions of a Lie algebra, *Afrika Mat.*, **10** (1999), 46–63.

[9] A. V. Gnedbaye, A non-abelian tensor product of Leibniz algebras, *Ann. Inst. Fourier (Grenoble)*, **49** (1999), 1149–1177.

[10] R. Kurdiani and T. Pirashvili, A Leibniz algebra structure on the second tensor power, *J. Lie Theory*, **12** (2002), 583–596.

[11] A. Makhlouf and S. D. Silvestrov, Hom-algebra structures, *J. Gen. Lie Theory Appl.*, **2** (2008), 51–64.

[12] Y. Sheng, Representations of hom-Lie algebras, *Algebr. Represent. Theory*, **15** (2012), 1081–1098.

[13] B. Sun, L. Y. Chen and X. Zhou, On universal α-central extensions of Hom-preLie algebras, arXiv:1810.09848.

[14] D. Yau, Hom-Novikov algebras, *J. Phys. A*, **44** (2011), 085202, 20 pp.

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